

Rigidly Supersymmetric Gauge Theories on Curved Superspace

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Abstract

In this note we construct rigidly supersymmetric gauged sigma models and gauge theories on certain Einstein four-manifolds, and discuss constraints on these theories. In work elsewhere, it was recently shown that on some nontrivial Einstein four-manifolds such as AdS_4 , $N = 1$ rigidly supersymmetric sigma models are constrained to have target spaces with exact Kähler forms. Similarly, in gauged sigma models and gauge theories, we find that supersymmetry imposes constraints on Fayet-Iliopoulos parameters, which have the effect of enforcing that Kähler forms on quotient spaces be exact. We discuss the ‘background principle’ in this context. We also discuss general aspects of universality classes of gauged sigma models, as encoded by stacks, and also discuss affine bundle structures implicit in these constructions. In an appendix, we discuss how anomalies in four-dimensional gauge theories, such as those which play an important role in our analysis, can be recast in the language of stacks.

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1 Introduction

The idea of supersymmetry has been around for nearly forty years, which generated numerous discussions ranging from model building in particle physics to pure theoretical investigations, and proved to be a powerful tool for understanding quantum field theory. Historically, most discussions of four-dimensional rigidly supersymmetric nonlinear sigma models have focused on Minkowski spacetimes. Recently, rigidly supersymmetric nonlinear sigma models on some nontrivial four-dimensional spacetime manifolds have been discussed by several groups [1, 2, 3]. (See also references contained therein for older literature on this subject.) These have interesting new properties, different from the traditional Minkowski spacetime models, essentially because one must add additional terms to the action to take into account the curvature of the spacetime manifold.

One way to derive those extra terms in the action is to manually add extra terms consistent with the requirements imposed by (rigid) supersymmetry [1, 3]. Another approach [1, 2] is to start with a supergravity theory in four dimensions, then decouple gravity in order to obtain a theory that is rigidly supersymmetric. Demanding that the supersymmetry variation of the gravitino vanishes then constrains the possible spacetime four-manifolds. The solutions of these constraining equations generate two classes of spacetime geometries, including AdS_4 and S^4 (after Wick rotation to a Euclidean metric) in one class, and $S^3 \times \mathbb{R}$ in a second class that requires a covariantly constant vector field.

These theories have many interesting properties that are different from the Minkowski spacetime case. For instance [1, 2], the target spaces of the supersymmetric nonlinear sigma models on spacetimes such as AdS_4 and S^4 must be noncompact Kähler manifolds, with exact Kähler forms. Furthermore, the Lagrangian depends only on certain combination of the Kähler potential and the superpotential – neither alone is physically meaningful.

In this paper we construct $N = 1$ rigidly supersymmetric gauged nonlinear sigma models and gauge theories on nontrivial four-dimensional spacetime manifolds, by starting with $N = 1$ supergravity and decoupling gravity. Just as target spaces of rigidly supersymmetric ungauged theories are constrained, we find analogous constraints in gauge theories. For example, just as the Fayet-Iliopoulos parameter is constrained in $N = 1$ supergravity [4, 5, 6], we find a constraint on the Fayet-Iliopoulos parameter in rigidly supersymmetric theories, which guarantees that the Kähler form on the quotient space is exact. Just as in $N = 1$ supergravity in four dimensions, the superpotential is a section of a line bundle [7], we interpret the superpotential in these rigidly supersymmetric theories as a section of an affine bundle. Just as in $N = 1$ supergravity [4, 5], where the Fayet-Iliopoulos parameter was determined by the group action on the Bagger-Witten line bundle, here too the Fayet-Iliopoulos parameter can be understood in terms of lifts to the affine bundle.

We should mention that some analogous results were obtained in linearized supergravity theories obtained by coupling a rigidly supersymmetric theory to gravity. In such theories,

for *e.g.* couplings involving the Ferrara-Zumino multiplet, one also often sees that Kähler forms are exact and Fayet-Iliopoulos parameters vanish [8, 9], just as we describe here for rigidly supersymmetric theories on *e.g.* AdS₄. (As observed in [6], however, one should distinguish supergravities obtained by coupling a rigid theory to gravity, from more general supergravity theories. For example, in generic heterotic Calabi-Yau compactifications to four dimensions and N=1 supergravity, it is widely believed that the Bagger-Witten line bundle is nontrivial, and so such supergravities cannot be obtained by coupling a rigid theory in the fashion above.)

We start in section 2 with a review of those rigidly supersymmetric nonlinear sigma models constructed in [1, 2, 3], using the superspace formulation to make the story more compact. We also give the interpretation of the superpotential as a section of certain affine bundle over the target space. In section 3 we construct supersymmetric gauged sigma models and gauge theories, from which we derive some constraints on the theory. We find that the Fayet-Iliopoulos parameter has to vanish in these theories, which has the effect of enforcing that Kähler forms on quotient spaces be exact. In section 5 we discuss some general aspects of universality classes of gauged sigma models, as encoded by stacks, and theories defined by restrictions on nonperturbative sectors. We also provide some mathematical background about affine bundles and equivariant structures on affine bundles in an appendix.

In passing, since one of the spaces we study will be AdS₄, we should mention that, just as in previous papers [1, 2], we shall ignore the role of boundary conditions. See *e.g.* [10, 11, 12, 13] for an overview of boundary conditions in AdS₄, and *e.g.* [14, 15, 16] for information on how such boundary conditions can restrict chiral matter representations.

2 Review of rigidly susy sigma models on curved superspace

There are several ways of deriving rigid supersymmetric nonlinear sigma model from supergravity. For example, we can decouple gravity in the weak coupling limit to get supersymmetric nonlinear sigma model on AdS₄ [1]. On the other hand, it was noted in [2] that the auxiliary fields b_μ and M from the $N = 1$ supergravity multiplet could be used to determine the geometry of spacetime, therefore generating two classes of spacetime geometries. The idea is to start with $N = 1$ supergravity Lagrangian, then set the gravitino to zero to completely remove the dynamics of gravity, and make the auxiliary fields b_μ and M from the supergravity multiplet satisfy certain constraining equations to make sure we have $N = 1$ supersymmetry, as well as the ability to perform a modified Kähler transformation with the resulting Lagrangian invariant.

Let us review the approach of [2], as we shall apply it to gauge theories in the next

section. We start with the $N = 1$ chiral supergravity Lagrangian in superspace [17]:

$$\mathcal{L} = \frac{1}{\kappa^2} \int d^2\Theta \, 2\mathcal{E} \left[\frac{3}{8}(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R) \exp\left(-\frac{\kappa^2}{3}K(\Phi^i, \bar{\Phi}^{\bar{i}})\right) + \kappa^2 W(\Phi^i) \right] + h.c. \quad (1)$$

Then we remove the effect of gravity. First, we need to expand in κ^2 , then only keep the terms that are independent of κ . We get

$$\mathcal{L} = \int d^2\Theta \, 2\mathcal{E} \left[-\frac{1}{8}(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R)K(\Phi^i, \bar{\Phi}^{\bar{i}}) + W(\Phi^i) \right] + h.c. \quad (2)$$

As observed in [2], in order for consistency of the method above, we must demand that the spacetime be such that the supersymmetry variation of the gravitino vanishes, as we have truncated it. The off-shell supersymmetry variation of the gravitino is of the form [2, 17]

$$\begin{aligned} \delta\Psi_\mu^\alpha &= -2\nabla_\mu\zeta^\alpha + \frac{i}{3} \left(M(\epsilon\sigma_\mu\bar{\zeta})^\alpha + 3b_\mu\zeta^\alpha + 2b^\nu(\zeta\sigma_{\nu\mu})^\alpha \right), \\ \delta\bar{\Psi}_{\mu\dot{\alpha}} &= -2\nabla_\mu\bar{\zeta}_{\dot{\alpha}} - \frac{i}{3} \left(\bar{M}(\zeta\sigma_\mu)_{\dot{\alpha}} + 3b_\mu\bar{\zeta}_{\dot{\alpha}} + 2b^\nu(\bar{\zeta}\bar{\sigma}_{\nu\mu})_{\dot{\alpha}} \right), \end{aligned} \quad (3)$$

and demanding that it vanishes implies constraints of the form [2][equ'n (2.11)]

$$\begin{aligned} Mb_\mu = \bar{M}b_\mu = 0, \quad \nabla_\mu b_\nu = 0, \quad \partial_\mu M = \partial_\mu \bar{M} = 0, \quad W_{\mu\nu\kappa\lambda} = 0, \\ R_{\mu\nu} = -\frac{2}{9}(b_\mu b_\nu - g_{\mu\nu}b_\rho b^\rho) + \frac{1}{3}g_{\mu\nu}M\bar{M}, \end{aligned}$$

where M , \bar{M} , b_μ are auxiliary fields in the $N=1$ supergravity multiplet, and $W_{\mu\nu\kappa\lambda}$ is the Weyl tensor. According to [2], there are two classes of solutions to the equations above, namely:

1. $b_\mu = 0$, constant M , \bar{M} ,
2. $M = \bar{M} = 0$, b_μ a covariantly-constant vector.

In the first case, if we Wick rotate to Euclidean space, it can be argued from the existence of spinors ζ in the gravitino variation [19] that the spacetime metric either has constant sectional curvature, or is Ricci-flat and self-dual or anti-self-dual. To see this, note in this case we have Killing spinor equations

$$\begin{aligned} \nabla_\mu\zeta^\alpha &= \frac{i}{6}M(\epsilon\sigma_\mu\bar{\zeta})^\alpha, \\ \nabla_\mu\bar{\zeta}_{\dot{\alpha}} &= \frac{-i}{6}\bar{M}(\zeta\sigma_\mu)_{\dot{\alpha}}, \end{aligned} \quad (4)$$

Taking the covariant derivative of the first equation, then together with the second equation we can get

$$\nabla_\nu\nabla_\mu\zeta^\alpha = \frac{1}{36}M\bar{M}(\sigma_\mu\zeta\bar{\sigma}_\nu)^\alpha, \quad (5)$$

then it follows that

$$[\nabla_\mu, \nabla_\nu]\zeta^\alpha = R_{\mu\nu\rho\sigma}(\zeta\sigma^{\rho\sigma})^\alpha = \frac{1}{9}M\bar{M}(\zeta\sigma_{\mu\nu})^\alpha, \quad (6)$$

Similarly, we can find an analogous equation from the second Killing spinor equation above

$$[\nabla_\mu, \nabla_\nu]\bar{\zeta}_{\dot{\alpha}} = R_{\mu\nu\rho\sigma}(\bar{\zeta}\bar{\sigma}^{\rho\sigma})_{\dot{\alpha}} = \frac{1}{9}M\bar{M}(\bar{\zeta}\bar{\sigma}_{\mu\nu})_{\dot{\alpha}}, \quad (7)$$

Suppose we have $\zeta^\alpha \neq 0$ as well as $\bar{\zeta}_{\dot{\alpha}} \neq 0$, then we see

$$R_{\mu\nu\rho\sigma}\sigma^{\rho\sigma} \propto g_{\mu\rho}g_{\nu\sigma}\sigma^{\rho\sigma}, \quad (8)$$

which is equivalent to

$$R_{\mu\nu\rho\sigma}\sigma^{\rho\sigma} \propto g_{\mu\sigma}g_{\nu\rho}\sigma^{\sigma\rho} = -g_{\mu\sigma}g_{\nu\rho}\sigma^{\rho\sigma}, \quad (9)$$

Therefore, from linear independence of the σ 's, we see that

$$R_{\mu\nu\rho\sigma} \propto g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}, \quad (10)$$

which means the spacetime has constant sectional curvature, *i.e.* it is a space form. Similarly, if $\zeta^\alpha = 0$ (or $\bar{\zeta}_{\dot{\alpha}} = 0$), it follows that the spacetime is Ricci-flat and self-dual (or anti-self-dual). In the second case, it can be similarly argued [19] that the spacetime metric is a product of a line and a metric of nonnegative constant sectional curvature.

2.1 $M = \bar{M} = \mathbf{constant}$, $b_\mu = 0$

Now, let us specialize to the first case, in which M, \bar{M} are nonzero constants, and b_μ vanishes. With this choice the above superspace Lagrangian can be written in an interesting form

$$\mathcal{L} = \int d^2\Theta \, 2\mathcal{E} \left[-\frac{1}{8}(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R)(K(\Phi^i, \bar{\Phi}^{\bar{i}}) - \frac{3}{2M}W(\Phi^i) - \frac{3}{2\bar{M}}\bar{W}(\bar{\Phi}^{\bar{i}})) \right] + h.c. \quad (11)$$

from which we can clearly see that this nonlinear sigma model Lagrangian depends explicitly on the combination of the Kähler potential and the superpotential

$$K - \frac{3}{2M}W - \frac{3}{2\bar{M}}\bar{W}, \quad (12)$$

which suggests that K and W alone are not physically meaningful; it is only the combination above that is physically meaningful. This makes the modified Kähler transformation mentioned above apparent, which can be derived as following: if we perform super-Kähler transformation to the superspace Kähler potential

$$K(\Phi, \bar{\Phi}) \mapsto K(\Phi, \bar{\Phi}) + F(\Phi) + \bar{F}(\bar{\Phi}) \quad (13)$$

then in order to make the superspace Lagrangian (11) invariant, we must transform the superspace superpotential $W(\Phi)$ accordingly, which leads to the following transformation of the superpotential

$$W(\Phi) \mapsto W(\Phi) + \frac{2}{3}MF(\Phi). \quad (14)$$

The Lagrangian is invariant under the combination of these two transformations (the modified Kähler transformation)¹. This superspace transformation leads exactly to what was observed by others [1, 2, 3] that the nonlinear sigma model action is invariant under the Kähler transformation of the target space X

$$K(\phi, \bar{\phi}) \mapsto K(\phi, \bar{\phi}) + f(\phi) + \bar{f}(\bar{\phi}) \quad (15)$$

supplemented by the following transformation of the superpotential

$$W(\phi) \mapsto W(\phi) + \frac{2}{3}Mf(\phi). \quad (16)$$

Let U_α and U_β be two open subsets of the target space X . Then across $U_\alpha \cap U_\beta$ we have

$$\begin{aligned} K_\alpha &\mapsto K_\beta + f_{\alpha\beta} + \bar{f}_{\alpha\beta}, \\ W_\alpha &\mapsto W_\beta + \frac{2}{3}Mf_{\alpha\beta}, \end{aligned} \quad (17)$$

which is a clear indication that the superpotential W is not a function globally on X , but rather is a section of a rank 1 affine bundle² (\mathcal{O}, A) over X , whose line bundle part is trivially \mathcal{O} , while the \mathcal{O} -torsor A is determined by the geometry of the spacetime and the Kähler transformation of the target space. Then the combination (12) should be interpreted as a pairing between sections of affine bundles and their dual bundles, which is globally well-defined and invariant under Kähler transformation of the target space.

Physically, the transformations above mean that there is not a well-defined global function W that we can think of as the superpotential, as suggested by the appearance of the combination (12). We can combine (K_α, W_α) on patches into

$$\left(K'_\alpha \equiv K_\alpha - \frac{3}{2M}W_\alpha - \frac{3}{2M}\bar{W}_\alpha, 0 \right),$$

and then perform another Kähler transformation to

$$\left(K'_\alpha - \frac{3}{2M}f_\alpha - \frac{3}{2M}\bar{f}_\alpha, f_\alpha \right),$$

¹Note in supergravity, the superspace Lagrangian is invariant under the combined super-Kähler and super-Weyl transformations, the latter of which is a transformation of the superspace superpotential [17] which indicates the fact that the superpotential is a holomorphic section of a line bundle over the target space [7, 17]

² See appendix A for a discussion of affine bundles.

thus replacing W_α by f_α , for any holomorphic function f_α on the patch. Only the combination of K and W is physically meaningful.

One consequence of this phenomenon is that the target space X is necessarily noncompact [1, 2, 3]. Equation (17) not only requires the Čech cocycle $(\delta f)_{\alpha\beta\gamma} = 0$ on all triple overlaps, but also requires that the Čech cocycle be trivial ($f_{\alpha\beta}$ is a Čech coboundary), therefore the Kähler form of X must be cohomologically trivial, which leads to the noncompactness of X , as well as the existence of a globally defined Kähler potential.

Example 1: AdS₄

To describe the spacetime AdS₄ spacetime, we set

$$\begin{aligned} M = \bar{M} &= -\frac{3}{2r}, \\ b_\mu &= 0, \end{aligned} \quad (18)$$

where r can be interpreted as the radius of the AdS₄ curvature, with the scalar curvature given by $\mathcal{R} = \frac{15}{2r^2}$.³ The resulting superspace Lagrangian is

$$\mathcal{L} = \int d^2\Theta \, 2\mathcal{E} \left[-\frac{1}{8}(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R)(K(\Phi^i, \bar{\Phi}^{\bar{i}}) + rW(\Phi^i) + r\bar{W}(\bar{\Phi}^{\bar{i}})) \right] + h.c. \quad (19)$$

When expanded in components, we get the off-shell Lagrangian on AdS₄ described in [1, 2], namely

$$\begin{aligned} \mathcal{L} = & -g_{i\bar{j}}\partial_\mu\phi^i\partial^\mu\bar{\phi}^{\bar{j}} - ig_{i\bar{j}}\bar{\chi}^{\bar{j}}\bar{\sigma}^\mu\mathcal{D}_\mu\chi^i + g_{i\bar{j}}F^i\bar{F}^{\bar{j}} - F^i\left(\frac{1}{2}g_{i\bar{j},\bar{k}}\bar{\chi}^{\bar{j}}\bar{\chi}^{\bar{k}} - \frac{1}{r}(K_i + rW_i)\right) \\ & - \bar{F}^{\bar{i}}\left(\frac{1}{2}g_{\bar{i},k}\chi^j\chi^k - \frac{1}{r}(K_{\bar{i}} + r\bar{W}_{\bar{i}})\right) - \frac{1}{2r}(K_{ij} + rW_{ij})\chi^i\chi^j - \frac{1}{2r}(K_{\bar{i}\bar{j}} + r\bar{W}_{\bar{i}\bar{j}})\bar{\chi}^{\bar{i}}\bar{\chi}^{\bar{j}} \\ & + \frac{1}{4}g_{i\bar{j},k\bar{l}}\chi^i\chi^k\bar{\chi}^{\bar{j}}\bar{\chi}^{\bar{l}} + \frac{3}{r^2}(K + rW + r\bar{W}), \end{aligned} \quad (20)$$

where $\mathcal{D}_\mu\chi^i = \partial_\mu\chi^i - \omega_\mu\chi^i + \Gamma_{jk}^i\mathcal{D}_\mu\phi^j\chi^k$, with ω_μ the spin connection on the AdS₄ spacetime. Note in this way we can also recover the supersymmetry transformation of the chiral multiplet on AdS₄, simply by setting the gravitino to zero in the supergravity transformation of the chiral multiplet from chiral supergravity, which leads to

$$\begin{aligned} \delta_\zeta\phi^i &= \sqrt{2}\zeta\chi^i, \\ \delta_\zeta\chi^i &= \sqrt{2}F^i\zeta + i\sqrt{2}\sigma^\mu\bar{\zeta}\partial_\mu\phi^i, \\ \delta_\zeta F^i &= -\frac{\sqrt{2}}{2r}\zeta\chi^i + i\sqrt{2}\bar{\zeta}\bar{\sigma}^\mu\mathcal{D}_\mu\chi^i, \end{aligned} \quad (21)$$

³Two notes on conventions. First, we are working in mostly-plus metric conventions in this paper, same as [17], and in those conventions, typically the AdS₄ curvature is negative, not positive. The reason it is positive above is that we are following the conventions of [17], in which the spin connection has an atypical sign [17][equ'n (17.12)], and which results in the AdS₄ curvature being positive instead of negative. We would like to thank the authors of [1] for explaining this to us. Second, the curvature is related to other descriptions as follows. If we describe AdS₄ as the hypersurface $-u^2 - v^2 + x^2 + y^2 + z^2 = -\alpha^2$ in $\mathbb{R}^{2,3}$, then its curvature is $\mathcal{R} = -\frac{12}{\alpha^2}$.

where the supersymmetry parameter ζ should satisfy the Killing spinor equations

$$\begin{aligned}(\nabla_\mu \zeta)^\alpha &= \frac{i}{2r} (\bar{\zeta} \bar{\sigma}_\mu)^\alpha, \\ (\nabla_\mu \bar{\zeta})_{\dot{\alpha}} &= \frac{i}{2r} (\zeta \sigma_\mu)_{\dot{\alpha}}.\end{aligned}\tag{22}$$

Example 2: S^4

Next, let us Wick rotate to a Euclidean spacetime. Consider the case of S^4 , where

$$\begin{aligned}M &= \bar{M} = -\frac{3i}{2r}, \\ b_\mu &= 0.\end{aligned}\tag{23}$$

The resulting Euclidean Lagrangian in components is

$$\begin{aligned}\mathcal{L} &= g_{i\bar{j}} \partial_\mu \phi^i \partial^\mu \bar{\phi}^{\bar{j}} + i g_{i\bar{j}} \bar{\chi}^{\bar{j}} \bar{\sigma}^\mu \mathcal{D}_\mu \chi^i - g_{i\bar{j}} F^i \bar{F}^{\bar{j}} + F^i \left(\frac{1}{2} g_{i\bar{j},\bar{k}} \bar{\chi}^{\bar{j}} \bar{\chi}^{\bar{k}} - \frac{i}{r} (K_i - irW_i) \right) \\ &+ \bar{F}^{\bar{i}} \left(\frac{1}{2} g_{j\bar{i},k} \chi^j \chi^k - \frac{i}{r} (K_{\bar{i}} - ir\bar{W}_{\bar{i}}) \right) + \frac{i}{2r} (K_{ij} - irW_{ij}) \chi^i \chi^j + \frac{i}{2r} (K_{\bar{i}\bar{j}} - ir\bar{W}_{\bar{i}\bar{j}}) \bar{\chi}^{\bar{i}} \bar{\chi}^{\bar{j}} \\ &- \frac{1}{4} g_{i\bar{j},k\bar{l}} \chi^i \chi^k \bar{\chi}^{\bar{j}} \bar{\chi}^{\bar{l}} + \frac{3}{r^2} (K - irW - ir\bar{W}).\end{aligned}\tag{24}$$

(Note in this case the pertinent combination of K , W , is $K - irW - ir\bar{W}$, and because the two terms with W are not complex conjugates, one could debate whether the symmetry mixing K and W should properly be termed a Kähler transformation.) As discussed in [2], this action is not real.

In an analogous fashion as of the AdS_4 case, we can find the supersymmetry transformations of this Euclidean theory on S^4

$$\begin{aligned}\delta_\zeta \phi^i &= \sqrt{2} \zeta \chi^i, \\ \delta_\zeta \chi^i &= \sqrt{2} F^i \zeta + i\sqrt{2} \sigma^\mu \bar{\zeta} \partial_\mu \phi^i, \\ \delta_\zeta F^i &= -\frac{\sqrt{2}i}{2r} \zeta \chi^i + i\sqrt{2} \bar{\zeta} \bar{\sigma}^\mu \mathcal{D}_\mu \chi^i,\end{aligned}\tag{25}$$

Now the Killing spinor equations become

$$\begin{aligned}(\nabla_\mu \zeta)^\alpha &= -\frac{1}{2r} (\bar{\zeta} \bar{\sigma}_\mu)^\alpha, \\ (\nabla_\mu \bar{\zeta})_{\dot{\alpha}} &= -\frac{1}{2r} (\zeta \sigma_\mu)_{\dot{\alpha}}.\end{aligned}\tag{26}$$

2.2 $M = \bar{M} = 0, b_\mu \neq 0$

So far we have only reviewed spacetimes corresponding to nonzero M and vanishing b_μ . The second class of solutions of the auxiliary fields M and b_μ found in [2], corresponding to a different class of spacetime geometries, are given by $M = \bar{M} = 0$ with b_μ a covariantly constant vector. For example, the spacetime $S^3 \times \mathbb{R}$ is consistent with the choices

$$\begin{aligned} M = \bar{M} = b_i &= 0, \\ b_0 &= -\frac{3}{r}. \end{aligned} \tag{27}$$

The corresponding component Lagrangian is

$$\begin{aligned} \mathcal{L} = & -g_{i\bar{j}}\partial_\mu\phi^i\partial^\mu\bar{\phi}^{\bar{j}} - ig_{i\bar{j}}\bar{\chi}^{\bar{j}}\bar{\sigma}^\mu\mathcal{D}_\mu\chi^i + g_{i\bar{j}}F^i\bar{F}^{\bar{j}} - F^i\left(\frac{1}{2}g_{i\bar{j},\bar{k}}\bar{\chi}^{\bar{j}}\bar{\chi}^{\bar{k}} - W_i\right) \\ & - \bar{F}^{\bar{i}}\left(\frac{1}{2}g_{j\bar{i},k}\chi^j\chi^k - \bar{W}_{\bar{i}}\right) - \frac{1}{2}W_{ij}\chi^i\chi^j - \frac{1}{2}\bar{W}_{\bar{i}\bar{j}}\bar{\chi}^{\bar{i}}\bar{\chi}^{\bar{j}} + \frac{1}{4}g_{i\bar{j},k\bar{l}}\chi^i\chi^k\bar{\chi}^{\bar{j}}\bar{\chi}^{\bar{l}} \\ & + \frac{i}{r}(K_i\partial_0\phi^i - K_{\bar{i}}\partial_0\bar{\phi}^{\bar{i}}) + \frac{1}{2r}g_{i\bar{j}}\chi^i\sigma_0\bar{\chi}^{\bar{j}}, \end{aligned} \tag{28}$$

where the last line contains some new terms that are different from the familiar Minkowski spacetime model. Note these extra terms vanishes at the limit $r \rightarrow \infty$, so this theory reduces to the Minkowski case as expected. (See for example [20, 21] for further discussions of rigidly supersymmetric theories on this spacetime.)

In this model the supersymmetry transformations are

$$\begin{aligned} \delta_\zeta\phi^i &= \sqrt{2}\zeta\chi^i, \\ \delta_\zeta\chi^i &= \sqrt{2}F^i\zeta + i\sqrt{2}\sigma^\mu\bar{\zeta}\partial_\mu\phi^i, \\ \delta_\zeta F^i &= \sqrt{2}\bar{\zeta}^{\dot{\alpha}}(i\mathcal{D}_{\alpha\dot{\alpha}}\chi^\alpha - \frac{1}{6}b_{\alpha\dot{\alpha}}\chi^\alpha), \end{aligned} \tag{29}$$

where the supersymmetry parameter ζ must satisfy [2]

$$\begin{aligned} (\nabla_0\zeta)_\alpha + \frac{i}{r}\zeta_\alpha &= 0, \\ 2(\nabla_i\zeta)_\alpha - \frac{i}{r}(\sigma_i\bar{\sigma}_0\zeta)_\alpha &= 0. \end{aligned} \tag{30}$$

In these cases there is no shift symmetry combining the Kähler potential and superpotential into a single quantity; the resulting Lagrangian is already Kähler invariant. Consequently, many of the conventional powerful methods from theories on Minkowski spacetime, such as holomorphy arguments, can be applied here.

3 Rigidly susy gauge theory on curved superspace

Now we apply the method of the last section to the $N = 1$ gauged supergravity Lagrangian in superspace [17]

$$\begin{aligned} \mathcal{L} = \frac{1}{\kappa^2} \int d^2\Theta \, 2\mathcal{E} \left[\frac{3}{8}(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R) \exp \left(-\frac{\kappa^2}{3}[K(\Phi^i, \bar{\Phi}^{\bar{i}}) + \Gamma(\Phi^i, \bar{\Phi}^{\bar{i}}, V)] \right) \right. \\ \left. + \frac{\kappa^2}{16g^2}W^{(a)}W^{(a)} + \kappa^2W(\Phi^i) \right] + h.c., \end{aligned} \quad (31)$$

where in Wess-Zumino gauge

$$\begin{aligned} \Gamma &= V^{(a)}D^{(a)} + \frac{1}{2}g_{i\bar{j}}X^{i(a)}X^{\bar{j}(b)}V^{(a)}V^{(b)}, \\ W_\alpha &= -\frac{1}{4}(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R)(\mathcal{D}_\alpha V - \frac{1}{2}[V, \mathcal{D}_\alpha V]). \end{aligned} \quad (32)$$

$X^{(a)}$ are the holomorphic Killing vectors on the target space X extended to superfields. We now study the two different classes of spacetime geometries separately.

3.1 $M = \bar{M} = \text{constant}$, $b_\mu = 0$

Removing the dynamics of gravity and setting the background fields to $M = \bar{M} = \text{constant}$, $b_\mu = 0$ to generate the first class of spacetime geometries, we find in superspace

$$\begin{aligned} \mathcal{L} &= \int d^2\Theta \, 2\mathcal{E} \left[-\frac{1}{8}(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R)(K(\Phi^i, \bar{\Phi}^{\bar{i}}) + \Gamma(\Phi^i, \bar{\Phi}^{\bar{i}}, V)) + \frac{1}{16g^2}W^{(a)}W^{(a)} + W(\Phi^i) \right] + h.c. \\ &= \int d^2\Theta \, 2\mathcal{E} \left[-\frac{1}{8}(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R) \left[(K(\Phi^i, \bar{\Phi}^{\bar{i}}) - \frac{3}{2M}W(\Phi^i) - \frac{3}{2\bar{M}}\bar{W}(\bar{\Phi}^{\bar{i}})) + \Gamma(\Phi^i, \bar{\Phi}^{\bar{i}}, V) \right] \right. \\ &\quad \left. + \frac{1}{16g^2}W^{(a)}W^{(a)} \right] + h.c. \end{aligned} \quad (33)$$

To obtain a gauge invariant Lagrangian from (33), we need to impose some constraints. Since in both superspace and in components the Lagrangian only depends on the combination of the Kähler potential and the superpotential in (12), namely,

$$K - \frac{3}{2M}W - \frac{3}{2\bar{M}}\bar{W},$$

it is natural to start with the gauge transformation of this globally well-defined combination. We apply the superspace gauge transformations [17]

$$\begin{aligned} \delta \left(K - \frac{3}{2M}W - \frac{3}{2\bar{M}}\bar{W} \right) &= \Lambda^{(a)}F^{(a)} + \bar{\Lambda}^{(a)}\bar{F}^{(a)} - i[\Lambda^{(a)} - \bar{\Lambda}^{(a)}]D^{(a)}, \\ \delta\Gamma &= i[\Lambda^{(a)} - \bar{\Lambda}^{(a)}]D^{(a)}, \end{aligned} \quad (34)$$

where $\Lambda^{(a)}$ are the gauge transformation parameters extended to superfields, and

$$F^{(a)} = X^{(a)} \left(K - \frac{3}{2M} W - \frac{3}{2M} \bar{W} \right) + iD^{(a)} \quad (35)$$

is a holomorphic function of the superfields Φ^i . Applying these gauge transformations to the superspace Lagrangian, we get

$$\delta\mathcal{L} = \int d^2\Theta \, 2\mathcal{E} \left[-\frac{1}{8}(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R)(\Lambda^{(a)}F^{(a)} + \bar{\Lambda}^{(a)}\bar{F}^{(a)}) \right] + h.c. \quad (36)$$

We demand that the superspace Lagrangian (33) be invariant under these gauge transformations. Using the fact that when the gravitational fields are decoupled, the vielbein superfield $2\mathcal{E}$ has the form

$$2\mathcal{E} = (1 - \Theta\Theta\bar{M})e,$$

where e is the veilbein, we are led to the following constraining equations by setting the lowest component, the Θ component and the $\Theta\Theta$ component of the superspace $\delta\mathcal{L}$ to zero

$$\begin{aligned} M^2\epsilon^{(a)}F^{(a)}(\phi) &= 0 \\ MF^i\epsilon^{(a)}\partial_i F^{(a)}(\phi) &= 0 \\ M\chi^i\chi^j\epsilon^{(a)}\partial_i\partial_j F^{(a)}(\phi) &= 0 \\ M\partial_i\epsilon^{(a)}F^i F^{(a)}(\phi) &= 0 \\ M\chi^i\chi^j\partial_i\epsilon^{(a)}\partial_j F^{(a)}(\phi) &= 0 \\ M\chi^i\chi^j\partial_i\partial_j\epsilon^{(a)}F^{(a)}(\phi) &= 0 \end{aligned} \quad (37)$$

which can be solved if and only if

$$MF^{(a)}(\phi) = 0. \quad (38)$$

In all our examples in this class of spacetime geometries, we have $M \sim \frac{1}{r}$, where r is some constant characteristic radius of spacetime. Therefore the constraint is really

$$F^{(a)}(\phi) = 0. \quad (39)$$

Note that this is well defined globally, since the combination (12) is well defined globally. Also note that these constraints reduce to the flat Minkowski spacetime case in the limit $r \rightarrow \infty$, in which we have no constraint on $F^{(a)}$ and the superpotential is gauge invariant.

We should point out that we are implicitly giving up gauge-invariance of W , since it is not physically meaningful. Only the linear combination (12), namely,

$$K - \frac{3}{2M}W - \frac{3}{2M}\bar{W},$$

is physically meaningful. If we were to separately demand that W be gauge-invariant, then the resulting constraint we would obtain would only make sense for those special Kähler

transformations that leave W invariant – which is to say, none of them. More explicitly, if we were to treat K and W separately to analyze their individual gauge transformations using the first line of (33), then the gauge invariance of (33) leads us to

$$\begin{aligned}\delta W(\phi) &= -M\epsilon^{(a)}F'^{(a)}, \\ MF'^{(a)} &= 0,\end{aligned}\tag{40}$$

where $F'^{(a)} = X^{(a)}K + iD^{(a)}$ which is not invariant under Kähler transformations. Now the superpotential is gauge invariant, and we have the constraint $F'^{(a)} = 0$. However, in this case to make sense of the constraint $F'^{(a)} = 0$ globally, we need to use the globally defined Kähler potential (whose existence is guaranteed by the trivial Kähler class on the target space) and demand that no Kähler transformation is allowed, which is exactly what we have been seeing. Therefore, physically we should work with the combination (12). We should note that in this case, in the limit $r \rightarrow \infty$ which leads to the flat Minkowski spacetime, we recover the familiar gauge invariance of the superpotential as expected, since there will be no appearance of the combination (12) in the Lagrangian (33), which is reduced to the flat Minkowski spacetime Lagrangian.

Mathematically, there is another way of understanding the constraint $F'^{(a)} = 0$. Recall that the superpotential is a section of an affine bundle (\mathcal{O}, A) over the target space X , therefore we must lift the action of the gauge group to an action on this affine bundle. Comparing gauge transformations of the superpotential (40) with equation (62) in the Appendix, we see that we can describe the infinitesimal group action as an infinitesimal lift to the affine bundle, described by

$$\begin{aligned}\lambda &= 1, \\ \mu &= -M\epsilon^{(a)}F'^{(a)}.\end{aligned}\tag{41}$$

Thus the lifting property (63) requires for example $2F'^{(a)} = F'^{(a)}$ or simply $F'^{(a)} = 0$, *i.e.* the fact that the superpotential is a section of the affine bundle (\mathcal{O}, A) puts exactly the same constraint on the geometry of X as derived from gauge invariance.

Now let us discuss the implications of the constraint

$$F'^{(a)} = X^{(a)}\left(K - \frac{3}{2M}W - \frac{3}{2M}\bar{W}\right) + iD^{(a)} = 0\tag{42}$$

in detail. Let us begin with the definition of $D^{(a)}$, namely

$$\begin{aligned}\partial_i D^{(a)} &= -iX^{(a)\bar{j}}\partial_{\bar{j}}\partial_i K, \\ \partial_{\bar{j}} D^{(a)} &= iX^{(a)i}\partial_i\partial_{\bar{j}} K.\end{aligned}$$

Integrating the equations above, we find that the most general solution for $D^{(a)}$ is given by

$$D^{(a)} = -iX^{(a)\bar{j}}\partial_{\bar{j}}K' + C,$$

where K' is any Kähler potential (*i.e.* $K' = K + f + \bar{f}$ for any holomorphic function f), and C is a constant. Thus the constraint that $F^{(a)} = 0$ is fixing $C = 0$ (and also partially fixing K'). Physically, this is setting the Fayet-Iliopoulos parameter to zero.

Let us outline some examples, to understand the implication of this.

Example: Let the target space X be \mathbb{C}^n , with the standard Kähler potential $K = \sum_i |z_i|^2$, and consider an isometry group $U(1)^k$, in which each $U(1)$ acts by phases on the z_i as

$$z_i \mapsto \lambda^{Q_i^a} z_i.$$

Then the holomorphic Killing vectors are given by

$$X^{i(a)} = iQ_i^a z_i, \quad X^{\bar{i}(a)} = -iQ_i^a \bar{z}_i. \quad (43)$$

Then the constraining equations tell us

$$D^{(a)} = - \sum_i Q_i^a |z_i|^2. \quad (44)$$

For example, if there is only one $U(1)$ and all the $Q_i = 1$, then we are describing a projective space of zero radius. If there is only one $U(1)$, $n = 4$, two charges are $+1$ and two charges are -1 , then we are describing a conifold with zero-size small resolution.⁴

In the examples above, we saw that the quotient had Kähler form of trivial cohomology class, as expected – after all, the ungauged theory is only defined on spaces with trivial Kähler class, so one expects the moduli spaces of the gauge theories to have the same property.

More generally, it is straightforward to check that the constraint $F^{(a)} = 0$ ensures that the cohomology class of the Kähler form on the quotient is always trivial. Briefly, the point is that $D^{(a)} = 0$ if and only if

$$X^{(a)\bar{j}} \partial_{\bar{j}} K' = 0$$

which ensures that K' is gauge-invariant⁵ and so descends to the symplectic quotient, where it becomes a globally defined Kähler potential, whose second derivative is (manifestly) the descent of the restriction of the Kähler form on the original space.

⁴Note such zero-size effects imply strong coupling in the nonlinear sigma model. In this paper we only consider classical actions, not quantum physics.

⁵Gauge-invariance of a form ω , at least infinitesimally, is the statement that for a vector field

$$X^{(a)} = X^{(a)i} \partial_i + X^{(a)\bar{i}} \partial_{\bar{i}}$$

the Lie derivative $L_{X^{(a)}} \omega = 0$. For the function K' ,

$$L_{X^{(a)}} K' = X^{(a)i} \partial_i K' + X^{(a)\bar{i}} \partial_{\bar{i}} K'$$

whose vanishing follows immediately from $F^{(a)} = 0$. For the Kähler form ω , gauge-invariance $L_{X^{(a)}} \omega = 0$ is easily checked to be a consequence of the fact that the $X^{(a)}$ are Killing vectors.

Thus, we see that the constraint $F^{(a)} = 0$ forces the quotient space to admit a globally-defined Kähler potential, as we would naively expect from properties of ungauged sigma models.

Let us now apply these general argument to some examples from section 2.

Example 1: AdS₄

In components, the superspace Lagrangian (33) gives the Lagrangian on AdS₄ spacetime

$$\begin{aligned}
\mathcal{L} = & -g_{i\bar{j}}\mathcal{D}_\mu\phi^i\mathcal{D}^\mu\bar{\phi}^{\bar{j}} - ig_{i\bar{j}}\bar{\chi}^{\bar{j}}\bar{\sigma}^\mu\mathcal{D}_\mu\chi^i - i\bar{\lambda}^{(a)}\bar{\sigma}^\mu\mathcal{D}_\mu\lambda^{(a)} - \frac{1}{4}F_{\mu\nu}^{(a)}F^{\mu\nu(a)} - \frac{1}{2}D^{(a)2} \\
& + \sqrt{2}g_{i\bar{j}}(X^{i(a)}\bar{\chi}^{\bar{j}}\bar{\lambda}^{(a)} + \bar{X}^{\bar{j}(a)}\chi^i\lambda^{(a)}) - \frac{1}{2r}\mathcal{D}_i(K_j + rW_j)\chi^i\chi^j - \frac{1}{2r}\bar{\mathcal{D}}_{\bar{i}}(K_{\bar{j}} + r\bar{W}_{\bar{j}})\bar{\chi}^{\bar{i}}\bar{\chi}^{\bar{j}} \\
& - \frac{1}{r^2}g^{i\bar{j}}(K_j + rW_j)(K_{\bar{j}} + r\bar{W}_{\bar{j}}) + \frac{3}{r^2}(K + rW + r\bar{W}) + \frac{1}{4}\mathcal{R}_{i\bar{j}k\bar{l}}\chi^i\chi^k\bar{\chi}^{\bar{j}}\bar{\chi}^{\bar{l}},
\end{aligned} \tag{45}$$

where we have set the gauge coupling to one, and

$$\begin{aligned}
\mathcal{D}_\mu\phi^i &= \partial_\mu\phi^i - A_\mu^{(a)}X^{i(a)}, \\
\mathcal{D}_\mu\chi^i &= \partial_\mu\chi^i - \omega_\mu\chi^i + \Gamma_{jk}^i\mathcal{D}_\mu\phi^j\chi^k - A_\mu^{(a)}\partial_jX^{i(a)}\chi^j, \\
\mathcal{D}_\mu\lambda^{(a)} &= \partial_\mu\lambda^{(a)} - \omega_\mu\lambda^{(a)} - f^{abc}A_\mu^{(b)}\lambda^{(c)},
\end{aligned} \tag{46}$$

are the gauge-covariant derivatives, with f^{abc} being the structure constants of the gauge group G , and ω_μ being the spin connection on the AdS₄ spacetime. Similar to the ungauged theory, the supersymmetry transformations can be derived using the gauged supergravity transformations in [17]

$$\begin{aligned}
\delta_\zeta\phi^i &= \sqrt{2}\zeta\chi^i, \\
\delta_\zeta\chi^i &= \sqrt{2}F^i\zeta + i\sqrt{2}\sigma^\mu\bar{\zeta}\mathcal{D}_\mu\phi^i, \\
\delta_\zeta F^i &= -\frac{\sqrt{2}}{2r}\zeta\chi^i + i\sqrt{2}\bar{\zeta}\bar{\sigma}^\mu\mathcal{D}_\mu\chi^i + 2iT^{(a)}\phi^i\bar{\zeta}\bar{\lambda}^{(a)}, \\
\delta_\zeta A_\mu^{(a)} &= i(\zeta\sigma_\mu\bar{\lambda}^{(a)} + \bar{\zeta}\bar{\sigma}_\mu\lambda^{(a)}), \\
\delta_\zeta\lambda^{(a)} &= F_{\mu\nu}^{(a)}\sigma^{\mu\nu}\zeta - iD^{(a)}\zeta \\
\delta_\zeta D^{(a)} &= -\zeta\sigma^\mu\mathcal{D}_\mu\bar{\lambda}^{(a)} - \mathcal{D}_\mu\lambda^{(a)}\sigma^\mu\bar{\zeta}.
\end{aligned} \tag{47}$$

Note that this Lagrangian reduces to the flat Minkowski spacetime case when $r \rightarrow \infty$ as expected. Also note that in this Lagrangian the gaugino is not coupled to any part of the affine bundle (\mathcal{O}, A) , or equivalently, the lifting of the gauge group action to this line bundle is trivial. This should be compared to the case of $N = 1$ supergravity: there, the gaugino is a section of the Bagger-Witten line bundle, so that the gauge group action lifts to this line

bundle nontrivially, which leads to the quantization of the Fayet-Iliopoulos parameter [4, 5].

Example 2: S^4

Let us now Wick rotate to Euclidean space, and consider the case that the spacetime is S^4 , as a related example. Using the values of M , \bar{M} , b_μ in equation (23), the Euclidean Lagrangian is

$$\begin{aligned}
\mathcal{L} = & g_{i\bar{j}} \mathcal{D}_\mu \phi^i \mathcal{D}^\mu \bar{\phi}^{\bar{j}} + i g_{i\bar{j}} \bar{\chi}^{\bar{j}} \bar{\sigma}^\mu \mathcal{D}_\mu \chi^i + i \bar{\lambda}^{(a)} \bar{\sigma}^\mu \mathcal{D}_\mu \lambda^{(a)} + \frac{1}{4} F_{\mu\nu}^{(a)} F^{\mu\nu(a)} + \frac{1}{2} D^{(a)2} \\
& - \sqrt{2} g_{i\bar{j}} (X^{i(a)} \bar{\chi}^{\bar{j}} \bar{\lambda}^{(a)} + \bar{X}^{\bar{j}(a)} \chi^i \lambda^{(a)}) + \frac{i}{2r} \mathcal{D}_i (K_j - ir W_j) \chi^i \chi^j + \frac{i}{2r} \bar{\mathcal{D}}_{\bar{i}} (K_{\bar{j}} - ir \bar{W}_{\bar{j}}) \bar{\chi}^{\bar{i}} \bar{\chi}^{\bar{j}} \\
& - \frac{1}{r^2} g^{i\bar{j}} (K_j - ir W_j) (K_{\bar{j}} - ir \bar{W}_{\bar{j}}) + \frac{3}{r^2} (K - ir W - ir \bar{W}) - \frac{1}{4} \mathcal{R}_{i\bar{j}k\bar{l}} \chi^i \chi^k \bar{\chi}^{\bar{j}} \bar{\chi}^{\bar{l}}.
\end{aligned} \tag{48}$$

The supersymmetry transformations are

$$\begin{aligned}
\delta_\zeta \phi^i &= \sqrt{2} \zeta \chi^i, \\
\delta_\zeta \chi^i &= \sqrt{2} F^i \zeta + i \sqrt{2} \sigma^\mu \bar{\zeta} \mathcal{D}_\mu \phi^i, \\
\delta_\zeta F^i &= -\frac{\sqrt{2} i}{2r} \zeta \chi^i + i \sqrt{2} \bar{\zeta} \bar{\sigma}^\mu \mathcal{D}_\mu \chi^i + 2i T^{(a)} \phi^i \bar{\zeta} \bar{\lambda}^{(a)}, \\
\delta_\zeta A_\mu^{(a)} &= i (\zeta \sigma_\mu \bar{\lambda}^{(a)} + \bar{\zeta} \bar{\sigma}_\mu \lambda^{(a)}), \\
\delta_\zeta \lambda^{(a)} &= F_{\mu\nu}^{(a)} \sigma^{\mu\nu} \zeta - i D^{(a)} \zeta \\
\delta_\zeta D^{(a)} &= -\zeta \sigma^\mu \mathcal{D}_\mu \bar{\lambda}^{(a)} - \mathcal{D}_\mu \lambda^{(a)} \sigma^\mu \bar{\zeta}.
\end{aligned} \tag{49}$$

Using our method above, the gauge invariance of the Lagrangian leads to the following constraining equation

$$F^{(a)} = X^{(a)} (K - ir W - ir \bar{W}) + i D^{(a)} = 0. \tag{50}$$

(As in the ungauged theory, since the W terms in $K - ir W - ir \bar{W}$ are not complex conjugates, one might debate whether the symmetry transformation relating K , W should be called a Kähler transformation.) This constraint effectively makes $D^{(a)}$ complex, in line with the general observations in [2] on how terms breaking superconformal invariance on S^4 are complex. The real part of the constraint implies the Fayet-Iliopoulos parameter should vanish.

3.2 $M = \bar{M} = 0$, $b_\mu \neq 0$

For the other class of spacetime geometries which are determined by having nonzero b_μ and $M = \bar{M} = 0$, the situation is quite different. Take the $S^3 \times \mathbb{R}$ which is given by (27) as an

example: the superspace Lagrangian in this case is

$$\mathcal{L} = \int d^2\Theta \left[-\frac{1}{8}(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R)(K(\Phi^i, \bar{\Phi}^{\bar{i}}) + \Gamma(\Phi^i, \bar{\Phi}^{\bar{i}}, V)) + \frac{1}{16g^2}W^{(a)}W^{(a)} + W(\Phi^i) \right] + h.c., \quad (51)$$

where now the ‘‘chiral density’’ superfield \mathcal{E} has the property $2\mathcal{E} = 1$. Demanding (51) be invariant under the superspace gauge transformations

$$\begin{aligned} \delta K &= \Lambda^{(a)}F^{(a)} + \bar{\Lambda}^{(a)}\bar{F}^{(a)} - i[\Lambda^{(a)} - \bar{\Lambda}^{(a)}]D^{(a)}, \\ \delta \Gamma &= i[\Lambda^{(a)} - \bar{\Lambda}^{(a)}]D^{(a)}, \end{aligned} \quad (52)$$

where $F^{(a)} = X^{(a)}K + iD^{(a)}$ (same as in $N = 1$ supergravity), we are led to the result that the superpotential is gauge invariant with no further constraints on the theory, just as ordinary supersymmetric gauge theories on Minkowski spacetime. After eliminating the auxiliary fields, we find the component Lagrangian

$$\begin{aligned} \mathcal{L} &= -g_{i\bar{j}}\mathcal{D}_\mu\phi^i\mathcal{D}^\mu\bar{\phi}^{\bar{j}} - ig_{i\bar{j}}\bar{\chi}^{\bar{j}}\bar{\sigma}^\mu\mathcal{D}_\mu\chi^i - i\bar{\lambda}^{(a)}\bar{\sigma}^\mu\mathcal{D}_\mu\lambda^{(a)} - \frac{1}{4}F_{\mu\nu}^{(a)}F^{\mu\nu(a)} - \frac{1}{2}D^{(a)2} \\ &+ \sqrt{2}g_{i\bar{j}}(X^{i(a)}\bar{\chi}^{\bar{j}}\bar{\lambda}^{(a)} + \bar{X}^{\bar{j}(a)}\chi^i\lambda^{(a)}) - \frac{1}{2}(\mathcal{D}_iW_j)\chi^i\chi^j - \frac{1}{2}(\bar{\mathcal{D}}_{\bar{i}}\bar{W}_{\bar{j}})\bar{\chi}^{\bar{i}}\bar{\chi}^{\bar{j}} \\ &- g^{i\bar{j}}W_j\bar{W}_{\bar{j}} + \frac{1}{4}\mathcal{R}_{i\bar{j}k\bar{l}}\chi^i\chi^k\bar{\chi}^{\bar{j}}\bar{\chi}^{\bar{l}} + \frac{i}{r}(K_i\partial_0\phi^i - K_{\bar{i}}\partial_0\bar{\phi}^{\bar{i}}) + \frac{1}{2r}g_{i\bar{j}}\chi^i\sigma_0\bar{\chi}^{\bar{j}}, \end{aligned} \quad (53)$$

which, just as the ungauged case in (28), is almost the same as the Minkowski spacetime case, with some extra terms coming from the fact that the spacetime is curved. Note these extra terms vanish in the limit $r \rightarrow \infty$, so this theory reduces to the Minkowski case as expected. (See for example [20, 21] for further discussions of rigidly supersymmetric theories on this spacetime.)

The supersymmetry transformations of this model on $S^3 \times \mathbb{R}$ are

$$\begin{aligned} \delta_\zeta\phi^i &= \sqrt{2}\zeta\chi^i, \\ \delta_\zeta\chi^i &= \sqrt{2}F^i\zeta + i\sqrt{2}\sigma^\mu\bar{\zeta}\partial_\mu\phi^i, \\ \delta_\zeta F^i &= \sqrt{2}\bar{\zeta}^{\dot{\alpha}}(i\mathcal{D}_{\alpha\dot{\alpha}}\chi^\alpha - \frac{1}{6}b_{\alpha\dot{\alpha}}\chi^\alpha) \\ \delta_\zeta A_\mu^{(a)} &= i(\zeta\sigma_\mu\bar{\lambda}^{(a)} + \bar{\zeta}\bar{\sigma}_\mu\lambda^{(a)}), \\ \delta_\zeta\lambda^{(a)} &= F_{\mu\nu}^{(a)}\sigma^{\mu\nu}\zeta - iD^{(a)}\zeta, \\ \delta_\zeta D^{(a)} &= -\zeta\sigma^\mu\mathcal{D}_\mu\bar{\lambda}^{(a)} - \mathcal{D}_\mu\lambda^{(a)}\sigma^\mu\bar{\zeta}. \end{aligned} \quad (54)$$

4 Background principle

The authors of [1] proposed a ‘‘background principle’’: if a rigid $N = 1$ theory on Minkowski spacetime can be quantum-mechanically coupled to $N = 1$ supergravity in a consistent way,

then it should behave smoothly under deformation from Minkowski spacetime to AdS as classical theories. If a theory can be consistently coupled to gravity, then it should also be possible to consistently formulate it in a nontrivial background metric. In particular, cancellation of (quantum) anomalies in supergravity couplings is often tied to (classical) consistency conditions in rigid theories.

In this section we will observe that the same ideas also apply to gauge theories, and also trivially extend them to all four-manifolds of the first type discussed in this paper (for which $M = \overline{M}$ constant, $b_\mu = 0$), not just AdS₄.

Let us begin by reviewing the ungauged case, discussed in [1]. It was observed in [1] that in ungauged theories, the purely classical constraint on AdS of having a cohomologically trivial Kähler form prevents the appearance of gravitational anomaly when one couples the rigid theory to supergravity. In detail, start with the with the six-form anomaly polynomial of $N = 1$ supergravity coupled to an ungauged nonlinear sigma model [4]:

$$\begin{aligned}
P_{local} = & \phi^* ch_3(X) - \frac{1}{24} p_1(\Sigma) \phi^* c_1(X) \\
& + \phi^* c_1(L) \left(\phi^* ch_2(X) + \frac{21-n}{24} p_1(\Sigma) \right) \\
& + \frac{1}{2} \phi^* (c_1(L)^2 c_1(X)) + \frac{n+3}{6} \phi^* c_1(L)^3.
\end{aligned} \tag{55}$$

where Σ denotes the four-dimensional spacetime, L denotes the Kähler (Bagger-Witten) line bundle over the target space X (the moduli space of the supergravity), $\phi : \Sigma \rightarrow X$ denotes the map defining a vev of the bosons of the theory, and n is the complex dimension of the target space X . This anomaly polynomial decomposes as a sum

$$P_{local} = P_{global} + \Delta P, \tag{56}$$

where

$$P_{global} = \phi^* ch_3(X) - \frac{1}{24} p_1(\Sigma) \phi^* c_1(X), \tag{57}$$

is the anomaly polynomial of the rigid nonlinear sigma model, and

$$\begin{aligned}
\Delta P = & \phi^* c_1(L) \left[\left(\phi^* ch_2(X) + \frac{21-n}{24} p_1(\Sigma) \right) \right. \\
& \left. + \frac{1}{2} \phi^* (c_1(L) c_1(X)) + \frac{n+3}{6} \phi^* c_1(L)^2 \right].
\end{aligned} \tag{58}$$

If the Kähler form is cohomologically trivial, then $c_1(L) = 0$. Thus, if the rigid theory on AdS₄ is classically consistent, then coupling to supergravity does not change the anomaly: if the rigid theory is anomaly-free, then so is the theory coupled to supergravity.

In passing, let us make the trivial observation that the computation above, which [1] originally only applied to AdS₄, also applies to the other four-manifolds of the first type discussed in this paper (in which $M = \overline{M}$ is constant, $b_\mu = 0$).

It remains to ask whether our construction of $N = 1$ gauge theory satisfies this principle. To show this, let's start with the six-form anomaly polynomial of a $N = 1$ supergravity coupled to a gauged nonlinear sigma model, in which we gauge some global symmetry G of the target space X . We denote this anomaly polynomial as P_{local}^G to distinguish from the ungauged case above. From [4],

$$\begin{aligned}
P_{local}^G = & \phi^* ch_3(T_{vert}\mathcal{M}) - \frac{1}{24} p_1(\Sigma) \phi^* c_1(T_{vert}\mathcal{M}) \\
& + \phi^* c_1(\mathcal{L}) \left(\phi^* ch_2(T_{vert}\mathcal{M}) + \frac{21 - n + \dim(G)}{24} p_1(\Sigma) \right) \\
& + \frac{1}{2} \phi^* (c_1(\mathcal{L})^2 c_1(T_{vert}\mathcal{M})) + \frac{n + 3 - \dim(G)}{6} \phi^* c_1(\mathcal{L})^3.
\end{aligned} \tag{59}$$

In the expression above, ϕ is no longer a map $\Sigma \rightarrow X$. Instead, to define ϕ , we first pick a principal G bundle over our four-dimensional spacetime Σ , call it P . (The path integral of the gauge theory sums over P 's.) Define

$$\mathcal{M} \equiv (P \times X)/G$$

which is a bundle over Σ with fiber X . Then, ϕ is a section of \mathcal{M} , *i.e.* a map $\phi : \Sigma \rightarrow \mathcal{M}$ behaving well with respect to the projection $\mathcal{M} \rightarrow \Sigma$. Finally, the line bundle \mathcal{L} is defined by taking the pullback of $L \rightarrow X$ to $P \times X$, and then using the G -equivariant structure to descend to $\mathcal{M} = (P \times X)/G$, *i.e.* schematically, $\mathcal{L} = (\pi_X^* L)/G$.

It is worth emphasizing at this point that the Fayet-Iliopoulos parameters of the supergravity theory are encoded implicitly in the expression above. Specifically, they are encoded in $c_1(\mathcal{L})$. That Chern class manifestly contains information about $c_1(L)$ on X , and in addition, it also contains information about the choice of G -equivariant structure on L . That equivariant structure encodes the Fayet-Iliopoulos parameters in the supergravity, as described in [4].

It will be useful later to understand this in more detail, so let us consider a simple example. Suppose that the supergravity moduli space X is a point, so that any Bagger-Witten line bundle L is automatically trivial, and $c_1(L) = 0$. The choice of G -equivariant structure is then simply a one-dimensional representation of G , *i.e.* an action of G on the one-dimensional fiber \mathbb{C} . In this case, $\mathcal{M} = \Sigma$ and \mathcal{L} is then the line bundle associated to the principal bundle P via that representation. If that representation is nontrivial, then that associated bundle $\mathcal{L} \rightarrow \Sigma$ will vary as P varies, for general Σ .

In particular, it will be important later to note that the only way to ensure that $c_1(\mathcal{L}) = 0$ for all choices of Σ and P 's is if both $L \rightarrow X$ is trivial, *and* the G -equivariant structure on L is also trivial⁶ (Fayet-Iliopoulos parameters vanish).

⁶ For a nontrivial bundle, there is no meaningful notion of a 'trivial' equivariant structure, but in the special case that the bundle is trivial, there is a canonical 'trivial' equivariant structure.

In passing, we should note that the fact that in supergravity, Fayet-Iliopoulos parameters appear in anomalies, has been discussed elsewhere in the literature, see for example [24, 25] for an excellent description and overview.

Now, let us return to the background principle and the discussion of anomalies. Much as in [1], in the gauged case the supergravity anomaly decomposes as

$$\begin{aligned}
P_{local}^G &= P_{global}^G + \Delta P^G \\
P_{global}^G &= \phi^* ch_3(T_{vert}\mathcal{M}) - \frac{1}{24}p_1(\Sigma)\phi^* c_1(T_{vert}\mathcal{M}) \\
\Delta P^G &= \phi^* c_1(\mathcal{L}) \left[\left(\phi^* ch_2(T_{vert}\mathcal{M}) + \frac{21 - n + \dim(G)}{24}p_1(\Sigma) \right) \right. \\
&\quad \left. + \frac{1}{2}\phi^* (c_1(\mathcal{L})c_1(T_{vert}\mathcal{M})) + \frac{n + 3 - \dim(G)}{6}\phi^* c_1(\mathcal{L})^2 \right].
\end{aligned} \tag{60}$$

where P_{global}^G is the anomaly of the rigid G -gauged nonlinear sigma model.

As before, ΔP^G is proportional to $c_1(\mathcal{L})$. As we noted earlier, to guarantee that $c_1(\mathcal{L})$ vanish, we must require not only that the $L \rightarrow X$ be trivial (*i.e.* that the target space X has a cohomologically trivial Kähler form), but also that the Fayet-Iliopoulos parameters vanish.

We saw earlier in section 3.1 that in rigidly supersymmetric gauge theories on four-manifolds of the first type, including AdS_4 , we must classically require both that X have a cohomologically trivial Kähler form and that the Fayet-Iliopoulos parameters vanish. Thus, these classical constraints prevent anomalies when coupling to supergravity, consistent with the background principle.

Example: $\mathbb{C}P^n$ model

The $\mathbb{C}P^n$ model can be described either by a nonlinear sigma model with target space $\mathbb{C}P^n$, or by a $U(1)$ gauge theory with $n + 1$ chiral superfields of charge 1. Since $\mathbb{C}P^n$ is compact with a cohomologically nontrivial Kähler form, this theory cannot be coupled to gravity in a consistent way, as discussed above. Similarly, the four-dimensional gauge theory is anomalous. Note, on the other hand, in the example from section 3.1 we showed that our constraint $F^{(a)} = 0$ requires the target space to have zero radius, *i.e.* not really a $\mathbb{C}P^n$ anymore. Therefore the constraint $F^{(a)} = 0$ is consistent with the background principle, albeit perhaps trivially so.

5 Sigma models on stacks and restrictions on nonperturbative sectors

Physically, in supergravity theories we often want to work with moduli spaces possessing finite group symmetries [6, 5, 18]. These are described mathematically by stacks [4, 5], certain generalizations of spaces. (In fact, mathematically most moduli ‘spaces’ are actually stacks, so to really make contact between the supergravity literature and string compactifications, one must consider supergravities containing sigma models on stacks).

Two-dimensional sigma models on stacks have been extensively studied in the past, and four-dimensional sigma models have been considered more recently (see [5] for a discussion of the four-dimensional case, and references therein on two dimensions).

As discussed in [5] and references therein, one way to understand sigma models on stacks concretely is to use the fact that a smooth Deligne-Mumford stack over the complex numbers can be presented as a quotient $[X/G]$, for X a space and G a group (which need not be finite, and need not act effectively). To such a presentation we associate a G -gauged sigma model on X .

A stack does not uniquely determine such a presentation, but rather can often be described as $[X/G]$ for several X 's and G 's. Thus, one wants to associate stacks to universality classes of renormalization group flow, rather than to particular quotients $[X/G]$. In two dimensions, there are now extensive checks that renormalization group flow does indeed identify different presentations of the same stack (see *e.g.* [28, 29, 30, 31, 32]), though in four dimensions there are suggestions [5] that the same program might not work. (As an extreme example, since the gauge kinetic term is not fixed, some references consider Chern-Simons theories to be examples of sigma models on stacks, see *e.g.* [26] though the inclination of the authors is to only consider Yang-Mills-type kinetic terms in such language.)

In any event, as also observed in [5], even if stacks do not uniquely determine physics, nevertheless they can give some general insights. For example, when we describe a stack in terms of a quotient $[X/G]$, the Kähler form on the stack is determined by both the Kähler form on the covering space and also the Fayet-Iliopoulos parameter. To get an exact Kähler form on the stack, the naive generalization of the constraints of [1], requires both an exact Kähler form on the covering space X as well as vanishing Fayet-Iliopoulos parameter – the two notions are linked. (Similarly, in the case of $N = 1$ supergravity in four dimensions, integrality of the cohomology class of the Kähler form on the stack requires both integrality of the Kähler form on the covering space X [7] as well as an integrality constraint on the Fayet-Iliopoulos parameter [4]; again, the two notions are linked.) In appendix B we will give another example, and discuss how anomalies in supergravities contained gauged nonlinear sigma models can be described using stacks in a presentation-independent fashion.

In passing, although the Ferrara-Zumino multiplet has not played any direct role in this paper, this does seem an appropriate spot to mention that it transforms as a section of the Bagger-Witten line bundle over the quotient stack, which summarizes both its transformations across coordinate patches and its transformation under gauge transformations. For the Ferrara-Zumino multiplet to exist globally, requires not only that the Bagger-Witten line bundle on the covering space be trivial, but also that the Fayet-Iliopoulos parameter vanish⁷.

Another perspective on stacks is also sometimes useful. Stacks can be defined via their incoming maps. In particular, sigma models with restrictions on nonperturbative sectors are precisely examples of sigma models on stacks. Specifically, the stacks describing restrictions on nonperturbative sectors are known as gerbes. Over the last decade, there has been extensive work on sigma models with restrictions on nonperturbative sectors (sigma models on gerbes) in both the physics (see [4, 5, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38]) and especially the math communities (see for example [39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49] for a sample). Such theories have also been discussed implicitly in a number of other places in the physics literature (see for example [50, 51, 52, 53, 54, 55, 56]).

6 Conclusions and outlook

Supersymmetric nonlinear sigma models and gauge theories on curved spacetime, as we have seen in this paper, have a lot of interesting properties that are absent in the usual Minkowski spacetime case. We have discussed some of these properties in this paper, such as the appearance of the affine bundle structures, the combination of the Kähler potential and the superpotential, the constraints that are imposed by gauge invariance, the vanishing of the Fayet-Iliopoulos parameter which enforces the noncompactness of the target space, the general aspects of universality classes of gauged sigma models as encoded by stacks, and so on. These theories provide many examples of supersymmetric models on general spacetime. There is no doubt that there should be many other important new phenomena that yet to be discovered.

One class of possible new phenomenon should arise as instanton effects in these supersymmetric theories on general spacetime manifolds. Instantons and their contribution to nonperturbative physics have been studied extensively in conventional theories in Minkowski spacetime, therefore one would expect interesting instanton effects could provide many new nonperturbative effects to quantum field theories on curved spacetime. We leave the details of these discussion to future work.

⁷ It was observed in [4] that the Fayet-Iliopoulos parameter in $N = 1$ supergravity in four dimensions is really a choice of lift of G action to the Bagger-Witten line bundle, and as such, it is not always possible to set it to zero. However, when the Bagger-Witten line bundle is itself trivial, there is a canonical ‘zero’ lift, and the Fayet-Iliopoulos parameter can vanish.

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A Affine bundles and equivariant structures

In this section we introduce the concept of affine bundle, as well as the equivariant structure on affine bundles. First we give the definition of affine space and affine space morphism [22]

Definition Let V be a vector space over some field k . An *affine space* modeled on V is a set A , together with a map $t : V \times A \rightarrow A$ defined by $t(v, a) = v + a$, which is a free transitive action of V (as an Abelian group under addition) on A .

Intuitively, an affine space is a vector space without origin. Clearly if we fix an element $a \in A$ to be the origin, the above definition makes A into a vector space over the field k , which is isomorphic to V .

Example A vector space V is naturally an affine space modeled over itself, with the map $t : V \times V \rightarrow V$ given by the natural addition operation.

Definition Let V and V' be two vector spaces over the same field k . Let A and A' be affine spaces modeled on V and V' respectively, with corresponding maps $t : V \times A \rightarrow A$ and $t' : V' \times A' \rightarrow A'$. An *affine space morphism* between A and A' is a map $\varphi : A \rightarrow A'$ and a linear transformation $\tau : V \rightarrow V'$ such that

$$t'(\tau(v), \varphi(a)) = \varphi(t(v, a)), \forall a \in A, v \in V. \quad (61)$$

Then an affine bundle on a topological space X can be defined as follows.

Definition Let A be an affine space modeled on a vector space V . An *affine bundle* (V, A) over X is a fiber bundle $\pi : (V, A) \rightarrow X$, defined by the following data: each point $x \in X$ has a neighborhood U and a U -isomorphism $\varphi : U \times A \rightarrow \pi^{-1}(U)$ such that the restriction $x \times A \rightarrow \pi^{-1}(x)$ is an affine space isomorphism.

In other words, V is an ordinary bundle and A is a V -torsor.

Next we consider the equivariant structure on affine bundles. We restrict our attention to the case of trivial affine line bundle $\pi : X \times \mathbb{A}^1 \rightarrow X$, which shows up in our discussion of

supersymmetric theories on AdS_4 . Let G be a group acting on X , then the G -action on the affine bundle $\pi : X \times \mathbb{A}^1 \rightarrow X$ is given by [23]

$$g(x, a) = (gx, \lambda_g a + \mu_g(x)), \forall g \in G, x \in X, a \in \mathbb{A}^1, \quad (62)$$

where λ and μ are functions on $G \times X$ such that for any $g, h \in G$

$$\begin{aligned} \lambda_{gh} &= \lambda_g \cdot \lambda_h, \lambda_e = 1, \\ \mu_{gh}(x) &= \lambda_g \cdot \mu_h(x) + \mu_g(hx), \mu_e = 0. \end{aligned} \quad (63)$$

Let $s : X \rightarrow X \times \mathbb{A}^1$ be an G -equivariant section of this affine bundle, defined by $x \rightarrow (x, \sigma(x))$ where $\sigma \in \mathcal{O}(X)$. Then from the above result we can see σ satisfies [23]

$$\sigma(gx) = \lambda_g \cdot \sigma(x) + \mu_g(x). \quad (64)$$

In this paper we see that the superpotential of supersymmetric theories on AdS_4 is a section of the trivial affine bundle $\pi : X \times \mathbb{C}^1 \rightarrow X$. Hence if we consider supersymmetric gauge theory on AdS_4 , we will have to lift the gauge group action to this affine bundle, which is given by the gauge transformation of the superpotential.

B Anomalies and stacks

In the text, we briefly outlined open questions involving the application of stacks to four-dimensional physics. Briefly, there is an issue of presentation-dependence, and although this issue has been resolved in two dimensions, in four dimensional theories it is a more significant issue.

In this appendix, we will briefly outline how the anomalies in four-dimensional supergravity theories described in [4] can be rewritten in a presentation-independent fashion in terms of stacks. This discussion is somewhat irrelevant to the bulk of the paper, but is timely and pertinent to the discussion of stacks, so we have placed it in this appendix.

First, let us recall the setup from [4]. Instead of a map $\phi : \Sigma \rightarrow X$ from the four-dimensional spacetime Σ to the moduli space X of the supergravity, we have a section of a bundle $(P \times X)/G$, where P is a principal G bundle. P is somewhat arbitrary, in the sense that the path integral of the gauge theory will sum over choices.

To compare, a map into the quotient stack $\Sigma \rightarrow [X/G]$ is determined by a principal G bundle P (with connection), together with a G -equivariant map $\tilde{\phi} : \text{Tot}(P) \rightarrow X$ (by definition of maps into stacks). Given such P and $\tilde{\phi}$, define $\phi^\# : \text{Tot}(P) \rightarrow P \times X$ by, $p \mapsto (p, \tilde{\phi}(p))$. Note that the G action commutes:

$$g \cdot p \mapsto (g \cdot p, \tilde{\phi}(g \cdot p)) = (g \cdot p, g \cdot \tilde{\phi}(p)) = g \cdot (p, \tilde{\phi}(p))$$

hence $\phi^\#$ descends to a map $\text{Tot}(P)/G(= \Sigma) \rightarrow (P \times X)/G$, which is precisely the map $\phi : \Sigma \rightarrow \mathcal{M} = (P \times X)/G$ of [4].

Moreover, it is straightforward to verify that the map $\phi : \Sigma \rightarrow \mathcal{M}$ which we have just derived, is a section of \mathcal{M} . Let $\pi : \mathcal{M} \rightarrow \Sigma$ denote the projection, then note

$$(\pi \circ \phi) : p/G \mapsto (p, \tilde{\phi}(p))/G = p/G$$

hence $\pi \circ \phi$ is the identity.

To make the rest of the identification of the anomaly polynomial, it is convenient to work with $\tilde{\phi}$ instead of ϕ . This gives an anomaly polynomial on $\text{Tot}(P)$ instead of Σ .

In this language, $T_{\text{vert}}\mathcal{M} = T[X/G]$, \mathcal{L} lifts to the line bundle L interpreted as a line bundle on $[X/G]$, and we can use the fact that

$$\dim [X/G] = \dim X - \dim G = n - \dim G$$

(as $n \equiv \dim X$). Then, the anomaly polynomial discussed earlier,

$$\begin{aligned} P_{local}^G &= \phi^* ch_3(T_{\text{vert}}\mathcal{M}) - \frac{1}{24} p_1(\Sigma) \phi^* c_1(T_{\text{vert}}\mathcal{M}) \\ &+ \phi^* c_1(\mathcal{L}) \left(\phi^* ch_2(T_{\text{vert}}\mathcal{M}) + \frac{21 - n + \dim(G)}{24} p_1(\Sigma) \right) \\ &+ \frac{1}{2} \phi^* (c_1(\mathcal{L})^2 c_1(T_{\text{vert}}\mathcal{M})) + \frac{n + 3 - \dim(G)}{6} \phi^* c_1(\mathcal{L})^3. \end{aligned} \quad (65)$$

can be rewritten as

$$\begin{aligned} P_{local}^G &= \tilde{\phi}^* ch_3(T[X/G]) - \frac{1}{24} p_1(\Sigma) \tilde{\phi}^* c_1(T[X/G]) \\ &+ \tilde{\phi}^* c_1(L) \left(\tilde{\phi}^* ch_2(T[X/G]) + \frac{21 - \dim [X/G]}{24} p_1(\Sigma) \right) \\ &+ \frac{1}{2} \tilde{\phi}^* (c_1(L)^2 c_1(T[X/G])) + \frac{3 + \dim [X/G]}{6} \tilde{\phi}^* c_1(L)^3. \end{aligned} \quad (66)$$

(Strictly speaking, on stacks one should consider an extension of ordinary Chern classes, with components on the extra components of the inertia stack. The application of such components to anomalies is currently under investigation; here, to be conservative, we only give the part on the identity component.) This should be compared to the expression for the anomaly in the ungauged theory:

$$\begin{aligned} P_{local} &= \phi^* ch_3(X) - \frac{1}{24} p_1(\Sigma) \phi^* c_1(X) \\ &+ \phi^* c_1(L) \left(\phi^* ch_2(X) + \frac{21 - n}{24} p_1(\Sigma) \right) \\ &+ \frac{1}{2} \phi^* (c_1(L)^2 c_1(X)) + \frac{n + 3}{6} \phi^* c_1(L)^3. \end{aligned} \quad (67)$$

Note that if we replaced all occurrences of X in the expression above with $[X/G]$, then formally we would have recovered the expression for the anomaly in the gauge theory. As spaces are special cases of stacks, this is a good consistency check of the applicability of stacks to anomalies.

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