VT-IPNAS 10-12
TCC-022-10
UTTG-08-10

Quantization of Fayet-Iliopoulos Parameters in Supergravity

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In this short note we discuss quantization of the Fayet-Iliopoulos parameter in supergravity theories. We argue that in supergravity, the Fayet-Iliopoulos parameter determines a lift of the group action to a line bundle, and such lifts are quantized. Just as D-terms in rigid $\mathcal{N} = 1$ supersymmetry are interpreted in terms of moment maps and symplectic reductions, we argue that in supergravity the quantization of the Fayet-Iliopoulos parameter has a natural understanding in terms of linearizations in geometric invariant theory (GIT) quotients, the algebro-geometric version of symplectic quotients.

July 2010

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1. Introduction

The recent paper [1] discussed quantization of Fayet-Iliopoulos parameters in four-dimensional supergravity theories in which the group action on scalars was realized linearly. In this short note, we observe that that quantization condition has a more general understanding, as a choice of lift of the group action to a line bundle over the moduli space. Such a lift is precisely a inearization in the sense of geometric invariant theory (GIT) quotients, the algebro-geometric analogue of symplectic quotients.

After reviewing the result that Kähler classes on moduli spaces in supergravity are integral forms in section 2, we discuss the quantization of the Fayet-Iliopoulos parameter in section 3. In rigid supersymmetry, the Fayet-Iliopoulos parameter is interpreted in terms of symplectic quotients, and is not quantized. We argue that in $\mathcal{N} = 1$ supergravity in four dimensions, the Fayet-Iliopoulos parameter should instead be interpreted in terms of a choice of lift of the G action to a holomorphic line bundle, and such choices are quantized. In section 4, we discuss how to interpret the supergravity quotient in terms of the algebraic-geometry version of symplectic quotients, known as geometric invariant theory quotients. In section 5 we briefly comment on implications of this work for discussions of supersymmetry breaking in supergravity. Finally in section 6 we conclude with some observations on analogues in $\mathcal{N} = 2$ supergravity in four dimensions. In appendices we discuss pertinent sigma model anomalies, conditions for bundles to admit lifts of group actions, and work through a simple example of a geometric invariant theory quotient.

The recent paper [1] also discussed two-dimensional theories defined by restricting sums over instantons to a subset of all instantons. Such theories are the same as strings on gerbes, special kinds of stacks, as is discussed in the physics literature in for example [2–9] and reviewed in conference proceedings including [10–12]. (There is also a significant mathematics literature on Gromov-Witten invariants of stacks and gerbes; see for example [13–16] for a few representative examples of that literature.) A more direct description of a string on a gerbe is as the (RG endpoint of) a gauged sigma model in which the group acts ineffectively, meaning a subgroup acts trivially. More globally, gauging ineffective group actions and restricting nonperturbative sectors go hand-in-hand.

In the special case of stacks that are gerbes, *i.e.* the theories discussed in [1], such theories in two dimensions are equivalent to nonlinear sigma models on disjoint unions of spaces [8], a result named the "decomposition conjecture." We can understand the decomposition conjecture schematically as follows. Consider a nonlinear sigma model on a space X, for simplicity with $H^2(X, \mathbf{Z}) = \mathbf{Z}$, with a restriction on worldsheet instantons to degrees divisible by k. We can realize that restriction in the path integral by inserting a projection operator

$$\frac{1}{k} \sum_{n=0}^{k-1} \exp\left(i \int \phi^* \left(\frac{2\pi n}{k} \omega \right) \right)$$

where ω is the de Rham image of a generator of $H^2(X, \mathbf{Z})$. Inserting this operator into a partition function is equivalent to working with a sum of partition functions with rotating B fields, and this is the essence of the decomposition conjecture.

One of the applications of the result above is to Gromov-Witten theory, where it has been checked and applied to simplify computations of Gromov-Witten invariants of gerbes, see [17–22]. Another application is to gauged linear sigma models [9], where it answers old questions about the meaning of the Landau-Ginzburg point in a GLSM for a complete intersection of quadrics, gives a physical realization of Kuznetsov's homological projective duality [23–25], and updates old lore on GLSM's.

A more detailed discussion of gerbes, including four-dimensional theories (which have somewhat different properties from the two-dimensional ones reviewed above), examples of gerby moduli 'spaces' in field and string theory, and a discussion of the Fayet-Iliopoulos quantization condition for gerby moduli spaces in supergravity, will appear in [26].

2. Review of Bagger-Witten

Bagger and Witten [27] discussed how the Kähler class of the moduli space[1] of scalars of a supergravity theory is quantized, resulting from the fact that ultimately the Kähler class must be the first Chern class of a line bundle over the moduli space. In this section we will describe an analogous argument for quantization of the Fayet-Iliopoulos term in supergravity.

Let us briefly begin by reviewing the quantization of Newton's constant in ungauged supergravity theories, following [27]. First, across coordinate patches on the moduli space, the Kähler potential K transforms as

$$K \mapsto K + f + \overline{f}$$

where f is a holomorphic function of moduli. To be a symmetry of the theory, this must be accompanied by a rotation of the gravitino ψ_μ and the superpartners χ^i of the scalar fields on the moduli space:

$$\chi^i \mapsto \exp\left(+ \frac{i}{2} \mathrm{Im}\, f \right) \chi^i, \quad \psi_\mu \mapsto \exp\left(- \frac{i}{2} \mathrm{Im}\, f \right) \psi_\mu \tag{2.1}$$

(Since the χ^i and ψ_μ are chiral fermions, these are chiral rotations, hence there are potential anomalies – see for example [28] or appendix A for a discussion.)

Consistency of the rotations (2.1) across triple overlaps (even within classical physics) implies that the f's define a line bundle with even c_1, to which the fermions χ^i, ψ_μ couple.

[1] The arguments of [27], and our own arguments here, all assume that the moduli space of the supergravity theory is a smooth manifold.

In more formal language, if we let that line bundle be $\mathcal{L}^{\otimes 2}$, we can summarize (2.1) by saying that the gravitino is a spinor-valued section of $TX \otimes \phi^* \mathcal{L}^{-1}$, where X is the four-dimensional low-energy effective spacetime and $\phi : X \to M$ the boson of the four-dimensional nonlinear sigma model on the compactification moduli space M, and that the fermions χ^i are spinor-valued sections of $\phi^*(TM \otimes \mathcal{L})$. In the same language, the Kähler form on M is a (de Rham representative of) $c_1(\mathcal{L}^{-2})$ (and hence an even integral form). Moreover, given how the χ^i transform, the superpotential W transforms as a holomorphic section of $\mathcal{L}^{\otimes 2}$. Because the Kähler form determines the metric on the fermi kinetic terms, which must be positive-definite, [27] argues that $\mathcal{L}^{\otimes 2}$ must be a negative bundle – so if the moduli space M is a smooth compact manifold, then the superpotential must vanish.

3. Quantization of the Fayet-Iliopoulos parameter

Now, let us imagine gauging a group action on the target space M of the nonlinear sigma model above. We will argue that in supergravity, one must lift the group action on the base M to the line bundle $\mathcal{L}$, and that the Fayet-Iliopoulos parameter corresponds to such a choice of lift. As there are integrally-many choices of lifts, possible values of the Fayet-Iliopoulos parameter are quantized.

Let us begin by reviewing how one gauges group actions in nonlinear sigma models in general. To preserve supersymmetry (see *e.g.* [29, 30]), the group action must be generated by holomorphic Killing vectors

$$X^{(a)} \;\equiv\; X^{(a)i} \frac{\partial}{\partial \phi^i}$$

where (a) denotes a Lie algebra index, and ϕ a map in the nonlinear sigma model. To be holomorphic Killing means they must satisfy

$$\begin{aligned} \nabla_i X^{(a)}_j \;+\; \nabla_j X^{(a)}_i &= 0 \\ \nabla_{\bar{\imath}} X^{(a)}_j \;+\; \nabla_j X^{(a)}_{\bar{\imath}} &= 0 \end{aligned}$$

On a Kähler manifold, the first equation holds automatically. The second equation implies that there exist real scalar functions $D^{(a)}(\phi^i, \phi^{\bar{\imath}})$ such that

$$\begin{aligned} g_{i\bar{\jmath}} X^{(a)\bar{\jmath}} &= \; i \frac{\partial}{\partial \phi^i} D^{(a)} \\ g_{i\bar{\jmath}} X^{(a)i} &= \; -i \frac{\partial}{\partial \phi^{\bar{\jmath}}} D^{(a)} \end{aligned} \tag{3.1}$$

These conditions only determine the $D^{(a)}$ up to additive constants.

Quantities $D^{(a)}$ solving the equations above are known as “Killing potentials,” and are moment maps for the group action [29, 30]. Because only their derivatives are defined, they are ambiguous up to a constant shift, and such constant shifts are Fayet-Iliopoulos parameters.

In rigid supersymmetry, we interpret the gauging mathematically as a symplectic quotient in symplectic geometry. The $D^{(a)}$ define moment maps, and the constant shifts, the Fayet-Iliopoulos parameters, define the coadjoint orbit on which the symplectic reduction takes place. For a gauged $U(1)$, say, there is a single Fayet-Iliopoulos parameter which can take any real value, defining symplectic quotients with symplectic forms in real-valued cohomology.

In supergravity, however, that picture is problematic, as can be seen from the following quick and slighty sloppy argument. The value of the Fayet-Iliopoulos parameter determines the Kähler form on the quotient, but as we just outlined, [27] have argued that in an ungauged moduli space, the Kähler form is integral (and even). To get an integral Kähler form on the quotient, the Fayet-Iliopoulos parameter must be quantized.

The simplest example is the construction of $\mathbb{P}^n$ as a symplectic quotient of $\mathbb{C}^{n+1}$ by $U(1)$. One begins on $\mathbb{C}^{n+1}$ with an integral (in fact trivial) Kähler form, but by varying the image of the moment map, one can recover $\mathbb{P}^n$ with any real Kähler class, not necessarily integral. To get an integral Kähler class, the image of the moment map must also be integral. More generally [31], for abelian G, the moment map takes values in t^*, but only if one reduces on points in $T^* \subset t^*$ can one hope to get an integral Kähler form on the quotient.

Hence, to get an integral Kähler form on the quotient, the Fayet-Iliopoulos parameter must be quantized.

This argument is a little too slick; it is not immediately obvious, from the arguments of [27], that the Kähler class on the symplectic quotient must also be quantized. In the case of linearly-realized group actions, as was recently discussed in [1][section 3], it is easy to make the argument precise. The supergravity actions contains

$$-3 \int d^4\Theta \, E \exp(-K - rV)/3$$

where r is the Fayet-Iliopoulos parameter. As a result, gauge transformations

$$V \;\mapsto\; V \,+\, \Lambda \,+\, \overline{\Lambda}$$

act as Kähler transformations with $f = -r\Lambda$. Thus, the gauge symmetry acts as an R-symmetry under which the superpotential has charge $-r$. As a result, if a superfield Φ^j has charge q_j, then the fermion χ^j has charge $q_j + r/2$, and so charge quantization implies that $r/2$ must be an integer, *i.e.* the Fayet-Iliopoulos parameter must be an even integer.

However, it remains to understand this problem more generally. What is clear from the geometrical discussion of §2 is that, for the purposes of the supergravity theory, it does not suffice to define the action of G on M; we also need an equivariant lift[2] of the G-action

[2] Given $\{g \in G\}$, such a lift is a set $\{\tilde{g}\}$ acting on the bundle such that $\tilde{g}\tilde{h} = \widetilde{gh}$. These are known

to the line bundle $\mathcal{L}$, and as we shall discuss later, such equivariant lifts, when they exist, are quantized. In particular, we will identify Fayet-Iliopoulos parameters with, in essence, a choice of equivariant lift, and this is the ultimate reason for their quantization in supergravity.

We can see Fayet-Iliopoulos parameters as lifts explicitly in the supergravity lagrangians of [32][chapter 25]. In general, since the fermions χ^i, ψ_μ couple to $\mathcal{L}$, $\mathcal{L}^{-1}$, a group action on M must be lifted to an action on $\mathcal{L}$, $\mathcal{L}^{-1}$ in order to uniquely define the theory. (A group action on either of $\mathcal{L}$, $\mathcal{L}^{-1}$ defines a group action on the other, so henceforth we will only speak about group actions on $\mathcal{L}$.) We can see infinitesimal lifts explicitly in the infinitesimal group actions for real $\epsilon^{(a)}$ [32][(25.14)]:

$$\begin{aligned}
\delta\phi^i &= \epsilon^{(a)} X^{(a)i} \\
\delta A^{(a)}_\mu &= \partial_\mu \epsilon^{(a)} \: + \: f^{abc}\epsilon^{(b)} A^{(c)}_\mu \\
\delta\chi^i &= \epsilon^{(a)} \left(\frac{\partial X^{(a)i}}{\partial \phi^j} \chi^j \: + \: \frac{i}{2} \mathrm{Im}\, F^{(a)} \chi^i \right) \\
\delta\lambda^{(a)} &= f^{abc}\epsilon^{(b)}\lambda^{(c)} \: - \: \frac{i}{2}\epsilon^{(a)} \mathrm{Im}\, F^{(a)} \lambda^{(a)} \\
\delta\psi_\mu &= -\frac{i}{2}\epsilon^{(a)} \mathrm{Im}\, F^{(a)} \psi_\mu
\end{aligned}$$

where $F^{(a)} = X^{(a)}K + iD^{(a)}$ (K the Kähler potential), and $F^{(a)}$ is easily checked to be holomorphic. For real $\epsilon^{(a)}$, the Kähler potential undergoes a standard Kähler transformation

$$\delta K \: = \: \epsilon^{(a)} F^{(a)} \: + \: \epsilon^{(a)} \overline{F}^{(a)}$$

hence in the gauge transformations above, terms proportional to Im $F^{(a)}$ are precisely encoding the Kähler transformations on fermions given in equation (2.1). Thus, the gauge-transformation terms proportional to Im $F^{(a)}$ (also known as super-Weyl transformations) appear to encode an infinitesimal lift of the group action to $\mathcal{L}$. Strictly speaking, infinitesimal lifts are required to obey the Lie algebra:

$$\left[\delta^{(a)}, \delta^{(b)} \right] \psi_\mu \: = \: \frac{i}{2}\epsilon^{(a)}\epsilon^{(b)} f^{abc} \mathrm{Im}\, F^{(c)} \psi_\mu \tag{3.2}$$

(for real $\epsilon^{(a)}$). The $D^{(a)}$ can be chosen to obey [32][equ'n (24.6)]

$$\left[X^{(a)i}\partial_i \: + \: X^{(a)\bar{\imath}}\partial_{\bar{\imath}} \right] D^{(b)} \: = \: -f^{abc} D^{(c)}$$

and it is straightforward to check that with this choice, the $F^{(a)}$ do indeed satisfy equation (3.2), and hence define an infinitesimal lift of G (equivalently, an equivariant lift of the Lie algebra). Shifts in the imaginary part of $F^{(a)}$ are precisely Fayet-Iliopoulos parameters, hence, Fayet-Iliopoulos parameters encode a choice of equivariant lift.

technically as a G-equivariant structure or linearization. See appendix B for technical details and remarks on their existence.

Next, we shall show that the allowed values of the Fayet-Iliopoulos parameter are constrained (in fact, quantized) by the condition that the infinitesimal lifts integrate to honest (global) lifts. As noted above, the infinitesimal group action on $\mathcal{L}$, the infinitesimal lift, is given by

$$+\frac{i}{2}\epsilon^{(a)}\mathrm{Im}\,F^{(a)}$$

so that the lift of the group element

$$g \;\equiv\; \exp\left(i\epsilon^{(a)}T^{a}\right)$$

(T^a generators of the Lie algebra) is

$$\tilde{g} \;\equiv\; \exp\left(\frac{i}{2}\epsilon^{(a)}\mathrm{Im}\,F^{(a)}\right)$$

We require that the group be represented honestly, not projectively, *i.e.* $\tilde{g}\tilde{h} = \widetilde{gh}$, in order to define an honest lift of the group G (known technically as a G-equivariant structure or in this case, a G-linearization). Shifting the Fayet-Iliopoulos parameters (translating $D^{(a)}$) corresponds to a g-dependent rescaling of $\tilde{g}$:

$$\tilde{g} \;\mapsto\; \tilde{g}\exp\left(i\theta_g\right)$$

since $F^{(a)} = X^{(a)}K + iD^{(a)}$. We can use such shifts to try to produce an honest representation if the $\tilde{g}$'s do not already form an honest representation; nevertheless, we might not be able to do so. Let the group formed by the $\tilde{g}$ be denoted $\tilde{G}$, then for real[3] Lie groups we have a short exact sequence

$$1 \longrightarrow U(1) \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1 \tag{3.3}$$

We can shift Fayet-Iliopoulos parameters to get an honest representation if and only if the extension $\tilde{G}$ splits as $G \times U(1)$. In general, this will not always be the case – equivariant structures lifting group actions do not always exist. As we explain in appendix B, an equivariant moment map (3.2) suffices to guarantee an equivariant lift of the Lie algebra. For G connected and simply-connected, we show that this suffices to guarantee that (3.3) splits and gives an euivariant lift of G.

Assuming that we can find an honest representation, *i.e.* assuming an honest lift exists, we still have a little freedom left in the Fayet-Iliopoulos parameters: we can deform $\tilde{g}$'s by θ_g's that represent G. In other words, if

$$\theta_g \;+\; \theta_h \;=\; \theta_{gh}$$

[3] For algebraic groups, we have the nearly identical sequence

$$1 \longrightarrow \mathbb{C}^{\times} \longrightarrow \tilde{G}_{\mathbb{C}} \longrightarrow G_{\mathbb{C}} \longrightarrow 1.$$

for all $g, h \in G$, then we can shift the Fayet-Iliopoulos parameters to give phases as

$$\tilde{g} \mapsto \tilde{g} \exp(i\theta_g)$$

while maintaining an honest representation:

$$\begin{aligned} (\tilde{g} \exp(i\theta_g)) \left(\tilde{h} \exp(i\theta_h)\right) &= \tilde{g}\tilde{h} \exp(i(\theta_g + \theta_h)) \\ &= \widetilde{gh} \exp(i\theta_{gh}) \end{aligned}$$

Such shifts θ (i.e., the *difference* between two splittings of (3.3)) are classified by $\mathrm{Hom}(G, U(1))$ (for real Lie groups G) or $\mathrm{Hom}(G_{\mathbf{C}}, \mathbf{C}^{\times})$ (for algebraic groups $G_{\mathbf{C}}$). These are, clearly, the only remaining allowed Fayet-Iliopoulos shifts.

For example, if the gauge group is $U(1)$, then we can shift

$$\frac{1}{2} \mathrm{Im}\, F^{(a)} \pm \text{(integer)}$$

and leave the group representation invariant. This quantized shift is the Fayet-Iliopoulos parameter.

The fact that honest lifts, when they exist, are quantized in the fashion above, is a standard result in the mathematics literature (see *e.g.* [33][prop. 1.13.1]). Since it also forms the intellectual basis for the central point of this paper, let us give a second explicit argument that differences between lifts are quantized, following[4] [34]. Assume the space is connected, and let $\{U_\alpha\}$ be an open cover, that is 'compatible' with the group action[5]. At the level of Cech cohomology, a G-equivariant line bundle is defined by transition functions $g_{\alpha\beta}$, a gauge field A_α, and related data such that

$$\begin{aligned} g^* A_\alpha &= A_\alpha + d \ln h_\alpha^g \\ g^* g_{\alpha\beta} &= (h_\alpha^g)(g_{\alpha\beta})(h_\beta^g)^{-1} \\ h_\alpha^{g_1 g_2} &= (g_2^* h_\alpha^{g_1})(h_\alpha^{g_2}) \end{aligned}$$

Now, suppose we have two distinct equivariant structures, two lifts, defined by h_α^g and $\overline{h}_\alpha^g$. Define

$$\phi_\alpha^g \equiv \frac{h_\alpha^g}{\overline{h}_\alpha^g}$$

From the consistency condition on $g^* g_{\alpha\beta}$ for each equivariant, we find that $\phi_\alpha^g = \phi_\beta^g$, *i.e.* ϕ_α^g is the restriction to U_α of a function we shall call ϕ^g. From the consistency condition

[4] Essentially the same argument, in a different context, is responsible for understanding discrete torsion as a choice of equivariant structure on the B field.

[5] We omit details concerning covers. The result whose derivation we are sloppily outlining here is standard.

on $g^* A_\alpha$ for each equivariant structure, we find that ϕ^g is a locally constant function, and finally from the remaining consistency condition we find that

$$\phi^{g_1 g_2} \;=\; \phi^{g_2} \phi^{g_1}$$

i.e. ϕ defines a homomorphism $G \to U(1)$. In other words, on a connected manifold, the difference[6] between any two G-lifts is an element of $\mathrm{Hom}(G, U(1))$.

If the gauge group is $U(1)$, then as $\mathrm{Hom}(U(1), U(1)) = \mathbb{Z}$, we see that the difference between any two lifts is an integer.

Applying to the present case, we find that the difference between any two versions of

$$\frac{1}{2} \mathrm{Im}\, F^{(a)}$$

is an integer, and since $\mathrm{Im}\, F = D + \cdots$, we see that the difference between any two allowed values of the Fayet-Iliopoulos parameter must be an even integer. (Any *e.g.* one-loop counterterm would merely product an overall shift; the difference between any two allowed values would still be an integer.)

4. Interpretation in geometric invariant theory

In rigid supersymmetry, the D terms and Fayet-Iliopoulos parameters are interpreted in terms of symplectic quotients and symplectic reduction. In supergravity, however, there are some key differences:

1. As we saw in the previous section, in supergravity the Fayet-Iliopoulos parameter is quantized, because it acts as a lift of the group action on M to a line bundle, $\mathcal{L}$. These structures have no analogue in rigid supersymmetry.
2. A more obscure but also important point is that $\mathcal{L}$ defines a projective embedding of M. The quantization of the Kähler form ω described in [27] means that M is a Hodge manifold. By the Kodaira Embedding Theorem [35], every Hodge manifold is projective, and $\mathcal{L}^{-n}$, for some $n \gg 0$ is the ample line bundle that provides the projective embedding.

These features are not characteristic of symplectic quotients, but they are characteristic of the algebro-geometric analogue of symplectic quotients, known as geometric invariant theory (GIT) quotients (see *e.g.* [36–38]) instead.

[6] Note that the lifts themselves cannot be canonically identified with elements of $\mathrm{Hom}(G, U(1))$ – unless the line bundle is trivial, there is in general no canonical "zero" lift — *i.e.*, there is no natural "zero" for the Fayet-Iliopoulos parameter. Instead, the *set* of linearizations is acted upon freely by this group. Technically, we say the set of linearizations is a torsor under this group.

In a GIT quotient, the analogue of the image of the moment map is quantized. Briefly, when both are defined, GIT quotients are essentially equivalent to symplectic quotients, except that GIT quotients are constructed to always have integral Kähler classes, whereas symplectic quotients can have arbitrary real Kähler classes.

In a GIT quotient, instead of quotienting the inverse image of the moment map by a real Lie group, one considers instead a quotient of a complex manifold by an algebraic group (typically the complexification of a real Lie group, which is very natural from the point of view of $\mathcal{N} = 1$ supersymmetry). The effect of quotienting by a complex algebraic group turns out to be functionally equivalent to first taking the inverse image of the moment map then quotienting by a real Lie group (a subgroup of the algebraic group). In GIT quotients, the quotient is built via an explicit embedding into a projective space (technically, the quotient is constructed as Proj of a graded ring of group invariants), and the Kähler class on the quotient is the pullback along that embedding of the first Chern class of $\mathcal{O}(1)$. To build a GIT quotient, we must specify a lift (technically, a 'linearization') of the (algebraic) group action to a line bundle L on the space being quotiented, whose first Chern class is the Kähler class upstairs. As this technology may not be widely familiar to physicists, in appendix C we show explicitly how projective spaces can be constructed in this form.

We have seen these structures in $\mathcal{N} = 1$ supergravity – the moduli space M has a projective embedding by virtue of $\mathcal{L}^{-1}$, and in order to define the quotient we must pick a G-linearization of $\mathcal{L}^{-1}$. In fact, the Fayet-Iliopoulos parameter is understood in terms of such a choice of lift of the group action. Thus, the structure of gaugings in $\mathcal{N} = 1$ supergravity closely parallels the key features of GIT quotient constructions.

That said, some of the technical details of GIT quotient constructions are rather different. In a GIT quotient, for example, quotients are build via embeddings into projective spaces constructed from invariant rings (thus the name), whereas symplectic reductions are built as G-quotients of fibers of the moment map. In the present case, although we see projective embeddings and Fayet-Iliopoulos parameters as G-linearizations of an ample line bundle, the rest of the quotient construction more nearly follows the standard symplectic story (G-quotients of fibers of a moment map) rather than that of geometric invariant theory (as invariant coordinate rings are not completely central, modulo the discussion above).

The construction of the GIT quotient, in terms of the ring of invariant functions, is closely reminiscent of the approach to four-dimensional gauge theories, where one describes the moduli space in terms of its (invariant) chiral rings (see for example [39][section 12.3] and references therein, though also see [40] for a different perspective).

Perhaps the best interpretation of the D-terms in supergravity is that the Fayet-Iliopoulos parameter is defined by a choice of linearization, though the rest of the construction should still be interpreted in terms of symplectic quotients. In particular, a choice of linearization directly defines a moment map. We can see this as follows. Let G act on a space X, which is lifted to a linearization on a line bundle L over X. Suppose G preserves a connection one-form A on L, whose curvature is the symplectic form ω. Then pairing vector fields from $\mathrm{Lie}(G)$ with A gives real-valued functions on X, in the usual fashion:

$$i_{V_g}\omega \;=\; d\mu_g$$

for a function $\mu : X \times \mathrm{Lie}(G) \to \mathbb{R}$. Such a pairing is equivalent to a moment map $X \to \mathrm{Lie}(G)^*$,

5. Supersymmetry breaking

A sufficient condition for supersymmetry breaking in supergravity is that $\langle D^{(a)} \rangle \neq 0$. One result of our analysis is that, in principle, for some moduli spaces and line bundles $\mathcal{L}$, there may not exist an allowed translation of $D^{(a)}$, for which $\langle D^{(a)} \rangle = 0$. In such a case, supersymmetry breaking would be inevitable.

An example of this phenomenon is discussed in [41][section 5]. There, $M = \mathbb{P}^1$. The group of isometries is $SO(3)$, but when $\mathcal{L}$ is an odd power of the tautological line bundle, the group that has an equivariant lift is actually $G = SU(2)$. Since $\mathrm{Hom}(SU(2), U(1))$ is trivial, the equivariant lift is unique.

Moreover for $\mathcal{L} = \mathcal{O}(-n)$,

$$(D^{(1)})^2 + (D^{(2)})^2 + (D^{(3)})^2 = \left(\frac{n}{2\pi} \right)^2$$

independent of the location on $\mathbb{P}^1$. Supersymmetry is always broken.

As another example, consider gauging just a $U(1)$ subgroup of the isometry group of $M = \mathbb{P}^1$. The allowed moment maps are

$$D = -\frac{1}{2\pi} \left(\frac{n}{1 + |\phi|^2} + k \right)$$

for any $k \in \mathbb{Z}$. Different choices of k correspond to different allowed values of the Fayet-Iliopoulos coefficient. There are two fixed points of the $U(1)$ action, the north pole ($\phi = 0$) and the south pole ($\phi' = 1/\phi = 0$). Both are extrema of the scalar potential (minima, for an appropriate range of k). For generic choice of k, supersymmetry is broken at both points. For $k = -n$, supersymmetry is unbroken at the north pole, and broken at the south pole. For $k = 0$, supersymmetry is broken at the north pole, and unbroken at the south pole. Exchanging the roles of north and south pole requires shifting $k \to -n - k$, reflecting the fact that there's no canonical "zero" for the FI coefficient; rather, they form a torsor for $\mathrm{Hom}(G, U(1))$.

6. Higher supersymmetry

We have not investigated higher supersymmetry cases thoroughly, though we will make some basic observations regarding $\mathcal{N} = 2$ supergravity in four dimensions. For example, consider the hypermultiplet moduli space. In rigid $\mathcal{N} = 2$ supersymmetry, that moduli space is a hyperKähler manifold, but in $\mathcal{N} = 2$ supergravity it is a quaternionic Kähler manifold [42]. It was argued in [43][equ'n (5.16)] that in $\mathcal{N} = 2$ supergravity in four dimensions, the curvature scalar on the quaternionic Kähler moduli manifold is uniquely determined, so that there is not even an integral ambiguity. Similarly, it seems to be a standard result that in quaternionic Kähler reduction, unlike hyperKähler reduction, there is no Fayet-Iliopoulos ambiguity in the moment map, but rather the moment map is uniquely defined[7] [44–46].

One can consider also the moduli space of vector multiplets in $\mathcal{N} = 2$ supergravity. Such moduli spaces are described by special geometry, and in this case (see *e.g.* [47][equ'n (10)]) the Kähler form on the moduli space arising in Calabi-Yau compactifications is identified with

$$\partial\overline{\partial} \ln\langle\Omega|\overline{\Omega}\rangle$$

and hence unique (as this is invariant under rescalings of the holomorphic top-form Ω). All of these results tell us that triplets of D terms in $\mathcal{N} = 2$ supergravity have no Fayet-Iliopoulos ambiguity.

We leave a careful analysis of $\mathcal{N} = 2$ supergravity to future work.

7. Conclusions

In this paper we have reviewed recent discussions of quantization of the Fayet-Iliopoulos parameter in supergravity theories. We argued that, In general, that quantization can be understood formally via a choice of linearization on a line bundle appearing in the theory, linking gauging in supergravity models with 'geometric invariant theory' quotients.

The recent paper [1] went one step further to consider *e.g.* $U(1)$ gaugings with non-minimal charges, which (as argued in the introduction) correspond mathematically to sigma models on gerbes. The claims of [1] regarding such models can be understood as arising from the fact that there are more ('fractional') line bundles over gerbes than exist over the underlying spaces. We will consider such models in the upcoming work [26].

[7] Our intuition for this is that in $\mathcal{N} = 2$ supergravity, there is a triplet of Fayet-Iliopoulos parameters, which (from the discussion of this section) must all be integral, and yet can also be rotated under the action of an $SU(2)_R$. The only triple of integers consistent with $SU(2)_R$ rotation is $(0, 0, 0)$.

8. Acknowledgements

We would like to thank D. Freed, S. Hellerman, A. Knutson, T. Pantev, and M. Verbitsky for useful conversations. E.S. would also like to thank the University of Pennsylvania for hospitality while this work was completed.

J.D. was partially supported by the US-Israel Binational Science Foundation and NSF grant PHY-0455649. E.S. was partially supported by NSF grants DMS-0705381 and PHY-0755614.

A. Four-dimensional sigma model anomalies

In section 2 we described the classical physics of the fermions in an $\mathcal{N} = 1$ supergravity theory in four dimensions. Because those fermions are chiral, and hence (necessarily) undergo chiral rotations across coordinate patches, there is the potential for an anomaly.

In this appendix we will briefly outline how the resulting anomalies, following [28] (see also [48–50] for background information on sigma model anomalies).

Globally, the fact that the fermions undergo chiral rotations across coordinate patches is encoded in an anomaly, given by the six-form piece of $\hat{A}(X) \wedge \mathrm{ch}(E)$, where

$$E \;=\; \phi^* \left(TM \otimes \mathcal{L} \right) \;\ominus\; \left(TX \ominus 1 \right) \otimes \phi^* \mathcal{L}^{-1}$$

(The first term is from the superpartners χ^i of the chiral superfields, the second from the gravitino ψ_μ.) In the notation above, M is the target space of the nonlinear sigma model (the moduli space of the supergravity theory), X is the four-dimensional spacetime, $\phi : X \to M$ the boson of the nonlinear sigma model, and $\mathcal{L}$ is the line bundle encoding the chiral rotations across overlaps.

The six-form piece above (for the E appropriate for supergravity) is given by

$$\phi^* \mathrm{ch}_3(M) \;-\; \frac{1}{24} p_1(X) \phi^* c_1(M) \;+\; \phi^* c_1(\mathcal{L}) \left(\phi^* \mathrm{ch}_2(M) \;+\; \frac{21-n}{24} p_1(X) \right) \\ \;+\; \frac{1}{2} \phi^* \left(c_1(\mathcal{L})^2 c_1(M) \right) \;+\; \frac{n+3}{6} \phi^* c_1(\mathcal{L})^3 \quad \text{(A.1)}$$

where n is the number of chiral superfields, the dimension of the moduli space M.

The first two terms are independent of $\mathcal{L}$, and are present even in the case of rigid supersymmetry. They rule out many classically-sensible supersymmetric sigma models, for instance projective spaces of dimension greater than two. Of course, for the phenomenologically-interesting case of the sigma model which arises for a spontaneously-broken global symmetry, $G \to H$, $M = T^*(G/H)$, the anomaly, in the rigid case, vanishes.

When coupled to supergravity, with nontrivial $\mathcal{L}$, the anomaly polynomial takes the above, more complicated, form. The moduli spaces which arise in string theory are typically noncompact but can, nonetheless, have quite complicated topology. So (A.1) seems to provide a nontrivial constrain. Unfortunately, these moduli spaces are typically not smooth varieties, but rather are stacks. The extension of these considerations, to the case of M a stack, will be pursued elsewhere.

For present purposes, we would like to extend (A.1) to the case where we gauge some global symmetry, G, of M. Two things change, when we do this. First of all, ϕ is no longer a map from X to a fixed manifold, M. Rather, let $P \to X$ be a G-principal bundle. We form the associated bundle

$$\mathcal{M} = (P \times M)/G$$

with fiber M. Now, ϕ is a *section*

$$\begin{array}{c} \mathcal{M} \\ \phi \left(\Big\downarrow \right. \\ X \end{array}$$

The first effect of this change is to replace TM in the above expression, with $T_{\mathrm{vert}}\mathcal{M}$. The second change is that, in supergravity, the gaugini transform as sections of $\phi^* \mathcal{L}^{-1}$. The net effect is to modify (A.1) to

$$\begin{aligned} \phi^* \mathrm{ch}_3(T_{\mathrm{vert}}\mathcal{M}) \; - \; & \frac{1}{24} p_1(X) \phi^* c_1(T_{\mathrm{vert}}\mathcal{M}) \\ & + \; \phi^* c_1(\mathcal{L}) \left(\phi^* \mathrm{ch}_2(T_{\mathrm{vert}}\mathcal{M}) + \frac{21 \; - \; n \; + \; \dim(G)}{24} p_1(X) \right) \\ & \qquad + \; \frac{1}{2} \phi^* \left(c_1(\mathcal{L})^2 c_1(T_{\mathrm{vert}}\mathcal{M}) \right) \; + \; \frac{n \; + \; 3 \; - \; \dim(G)}{6} \phi^* c_1(\mathcal{L})^3 \end{aligned} \tag{A.2}$$

Even when (A.2) does not vanish, it is possible to contemplate cancelling the anomaly by adding to the action "Wess-Zumino"-type terms, whose classical variation is anomalous [51, 52], but we will not pursue that here.

B. Existence of equivariant structures

As noted in the text, G-equivariant structures on line bundles do not always exist. In this appendix, we shall work out conditions for finding such equivariant structures.

In general, if a group G acts on a manifold M, then a G-equivariant structure on a line bundle L is a lift of G to the total space of that line bundle, *i.e.* for all $g \in G$, a map

$\tilde{g} : \mathrm{Tot}(L) \to \mathrm{Tot}(L)$ such that

$$\pi(gy) \; = \; g\pi(y)$$

for all $y \in \mathrm{Tot}(L)$, and such that $\tilde{g}\tilde{h} = \widetilde{gh}$. Furthermore, one often imposes additional constraints, *e.g.* an equivariant lift that preserves a holomorphic structure or a connection.

It is a standard result in the mathematics literature that equivariant structures do not always exist. The obstruction is typically finding a lift such that $\tilde{g}\tilde{h} = \widetilde{gh}$ – often one can find a projective representation of the group, but it may not be possible to find an honest representation of the group.

One necessary condition for equivariant structures to exist is that characteristic classes be invariant under group actions, but this is not sufficient. Examples of non-equivariant line and vector bundles, with invariant characteristic classes, can be found in *e.g.* [53, 54] and references therein. Another example is as follows[8]. Let E be an elliptic curve with a marked point $\sigma \in E$. Let $L = \mathcal{O}_E(2\sigma)$. Let $x \in E$, and let $t_x : E \to E$ be the translation by x in the group law. Then for any x, the automorphism t_x preserves $c_1(L)$ simply because t_x is homotopic to the identity and so acts trivially on $H^2(E, \mathbb{Z})$. However, if x is a general point, $t_x^* L$ is not isomorphic to L as a holomorphic bundle. Hence this group of translations on E clearly can not have an equivariant lift[9]. However, this sort of group action is not of interest, even in globally-supersymmetric sigma models (let alone locally-supersymmetric). There is no moment map for the translation action. (In the language of (B.1), below, the vector field generating this symmetry is not in the kernel of s.)

In §3, we noted the condition for an equivariant lift (or linearization[10], in the nomenclature of GIT quotient constructions) of the infinitesimal G-action. Here, we will lay out those conditions more carefully, and consider the existence of an equivariant lift for finite G transformations.

The Kähler form, ω, endows M with a symplectic structure. (To be precise, we will use $\omega' = \frac{1}{2}\omega$ as the symplectic structure.) At the Lie algebra level, $\mathfrak{g} \subset \mathcal{X}_H$, the Lie algebra of Hamiltonian vector fields on M; in other words, the $\mathfrak{g}$-action arises from globally-defined moment maps (3.1). There is an exact sequence,

$$0 \longrightarrow H^0(M, \mathbb{R}) \longrightarrow C^\infty(M) \longrightarrow \mathcal{X}_\omega \stackrel{s}{\longrightarrow} H^1(M, \mathbb{R}) \longrightarrow 0 \tag{B.1}$$

where $\mathcal{X}_{\omega'}$ is the Lie algebra of symplectic vector fields (those preserving ω'), and $\mathcal{X}_H \subset \mathcal{X}_{\omega'}$

[8] We would like to thank T. Pantev for providing this example.

[9] In fact, $t_x^* L \cong L$ if and only if x is a point of order two on E. Now, even if we restrict to points of order two, we will not have an equivariant structure. The points of order two are an abelian group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ which preserves L, but L does not have an equivariant structure.

[10] Technically, a 'linearization' is an equivariant structure in which the group acts linearly on the fibers of vector bundles: $L_x \to L_{gx}$. All of the equivariant structures appearing in this paper are examples of linearizations.

is the subalgebra which is the kernel of s. So we naturally have a central extension

$$0 \longrightarrow H^0(M,\mathbb{R}) \longrightarrow \tilde{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow 0 \tag{B.2}$$

where $\tilde{\mathfrak{g}}$ acts equivariantly on $\mathcal{L}$ (an infinitesimal lift is the same as an equivariant action of the Lie algebra). Next, let us determine when the extension (B.2) splits. Let

$$\mu\colon\ M \longrightarrow \mathfrak{g}^*$$

be the moment map. In the notation of (3.1), the functions

$$D^{(a)} \ = \ 2\langle \mu, t^a \rangle$$

for $t^a \in \mathfrak{g}$. The condition for (B.2) to split is that μ be equivariant, *i.e.* that

$$\langle \mu, [t^a, t^b] \rangle \ = \ \{ \langle \mu, t^a \rangle, \langle \mu, t^b \rangle \} \tag{B.3}$$

where $\{\cdot,\cdot\}$ is the Poisson-bracket, defined using ω'. Given two splittings of (B.2), the *difference* is an element of $\mathrm{Hom}(\mathfrak{g}, H^0(M,\mathbb{R}))$.

Having found an infinitesimal lift, an equivariant lift of the Lie algebra, it remains to find a lift of the group G itself, that is, a splitting of the exact sequence of groups

$$1 \to U(1) \to \tilde{G} \to G \to 1$$

For G connected and simply-connected, there is no further obstruction. Given a lift at the level of the Lie algebra, every path in G lifts (uniquely!) to a path in $\tilde{G}$. Moreover, two paths in G, which are homotopic, lift to homotopic paths in $\tilde{G}$. If G is not simply-connected, there's no guarantee that a *closed* path in G lifts to a closed path in $\tilde{G}$. But for G simply-connected, every closed path is homotopic to the trivial path, and hence lifts to a closed path in $\tilde{G}$.

For G not simply-connected, we may need to go to a finite cover $\hat{G} \to G$, in order to find an equivariant lift. As an example, consider $M = \mathbb{C}P^1$, with $G = SO(3)$. As discussed in [41], when $\mathcal{L}$ is an odd power of the tautological line bundle, it is $\hat{G} = SU(2)$ that lifts to an equivariant action on $\mathcal{L}$; $\mathcal{L}$ does not admit an $SO(3)$-equivariant structure.

As noted earlier, when equivariant structures do exist, they are not unique. The set of equivariant structures on C^∞ line bundles preserving the connection form torsors under $\mathrm{Hom}(G, U(1))$; the set of equivariant structures on holomorphic line bundles preserving the holomorphic structure form torsors under $\mathrm{Hom}(G, \mathbb{C}^\times)$. As noted elsewhere in this paper, the Fayet-Iliopoulos parameters correspond precisely to such choices (and hence are quantized).

C. An example of a GIT quotient

In order to clarify some of the claims made in the text, and since the technology is not widely familiar to physicists, in this section we shall work through a very basic example of a geometric invariant theory (GIT) quotient. (See [36–38] for more information on GIT quotients, and [55][appendix C] for additional examples.)

In principle, given a complex manifold X with very ample line bundle $L \to X$, and the action of some group G on X which has been lifted to a linearization on L, then the GIT quotient is defined to be

$$\mathrm{Proj} \bigoplus_{n \geq 0} H^0\left(X, L^{\otimes n}\right)^G$$

The resulting quotient is sometimes denoted $X//G$, and depends upon the choice of linearization. (As discussed in the text, in 'typical' cases the result will be equivalent to a symplectic quotient, for reductions on special ('integral') coadjoint orbits.)

To clarify, let us describe a projective space $\mathbb{P}^{n-1}$ as $\mathbb{C}^n//\mathbb{C}^\times$, where the $\mathbb{C}^\times$ acts with weights 1. We write

$$\mathbb{C}^n//\mathbb{C}^\times \;=\; \mathrm{Proj} \bigoplus_{p \geq 0} H^0\left(\mathbb{C}^n, L^{\otimes p}\right)^{\mathbb{C}^\times} \tag{C.1}$$

where the line bundle L is necessarily $\mathcal{O}$. The choice of linearization, the choice of equivariant structure, can be encoded in the $\mathbb{C}^\times$ action on a generator, call it α of the module corresponding to L. Since $L \cong \mathcal{O}$,

$$H^0\left(\mathbb{C}^n, L^{\otimes p}\right) \;=\; \mathbb{C}[x_1, \cdots, x_n]$$

for all p, but the $\mathbb{C}^\times$ action varies. For example, if $s \in H^0\left(\mathbb{C}^n, L^{\otimes p}\right)$ is homogeneous of degree d, and the generator α is of weight r under $\mathbb{C}^\times$, then under the $\mathbb{C}^\times$ action,

$$s \;\mapsto\; \lambda^{d+pr} s$$

where $\lambda \in \mathbb{C}^\times$ – the λ^d because s is a degree d polynomial, the λ^{pr} because of the action on the generator.

If $r = 0$, then for all p the only $\mathbb{C}^\times$-invariant sections are constants, so we have

$$\mathrm{Proj} \bigoplus_{p \geq 0} \mathbb{C} \;=\; \mathrm{Proj}\, \mathbb{C}[y]$$

where y is taken to have degree 1. This is just a point.

If $r = -1$, then the $\mathbb{C}^\times$-invariant sections of $L^{\otimes p}$ are homogeneous polynomials of degree p. In this case, (C.1) becomes

$$\mathrm{Proj} \bigoplus_{p \geq 0} (\text{degree } p \text{ polynomials}) \;=\; \mathrm{Proj}\, \mathbb{C}[x_1, \cdots, x_n]$$

where the x_i all have degree 1, which is $\mathbb{P}^{n-1}$.

If $r < -1$, then (C.1) becomes

$$\mathrm{Proj} \bigoplus_{p \geq 0} (\text{ degree } -rp \text{ polynomials})$$

The map from $\mathbb{C}[x_1, \cdots, x_n]$ into the graded ring above defines the Veronese embedding of degree $-r$ of $\mathbb{P}^{n-1}$ into $\mathbb{P}^{(n-1+r) \text{ choose} (n-1)-1}$. This is a degree $-r$ map.

If $r > 0$, then there are no invariant sections except for constant sections in the special case that $p = 0$. In this case, (C.1) is the empty set.

Physically, the integer r corresponds to the Fayet-Iliopoulos parameter in the supergravity theory.

What distinguishes the various values of r is the Kähler class of the resulting space. In the GIT quotient construction, the Kähler class is the first Chern class of some line bundle, obtained by pulling back $\mathcal{O}(1)$ along the canonical embedding into a projective space defined by the Proj construction. For the linearization defined by $-r$, we have seen that the GIT quotient (defined by the Proj of the invariant subring) is given by the projective space $\mathbb{P}^{n-1}$ together with a natural embedding of degree $-r$ into a higher-dimensional projective space. Pulling back $\mathcal{O}(1)$ along such a map gives $\mathcal{O}(-r)$. Thus, the linearization defined by r ($r < 0$) corresponds to a Kähler class $-r$ on $\mathbb{P}^{n-1}$.

In principle, one would like a gerby analogue of the construction above, that produces a closed substack of a gerbe on a projective space, rather than a closed subvariety of a projective space. However, we do not know of a precise analogue – meaning, the Proj construction always builds spaces, not stacks, and we do not know of a stacky analogue of Proj.

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