

# The Alignment-Distribution Graph

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## Abstract

Implementing a data-parallel language such as Fortran 90 on a distributed-memory parallel computer requires distributing aggregate data objects (such as arrays) among the memory modules attached to the processors. The mapping of objects to the machine determines the amount of residual communication needed to bring operands of parallel operations into alignment with each other. We present a program representation called the *alignment-distribution graph* that makes these communication requirements explicit. We describe the details of the representation, show how to model communication cost in this framework, and outline several algorithms for determining object mappings that approximately minimize residual communication.

## 1 Introduction

When a data-parallel language such as Fortran 90 is implemented on a distributed-memory parallel computer, the aggregate data objects (arrays) have to be distributed among the multiple memory units of the machine. The mapping of objects to the machine determines the amount of residual communication needed to bring operands of parallel operations into alignment with each other. A common approach is to break the mapping into two stages: first, an *alignment* that maps all objects to an abstract Cartesian grid called a *template*, and then a *distribution* that maps the template to the processors. This two-phase approach separates language issues from machine issues; it is used in Fortran D [6], High Performance Fortran [8], and CM-Fortran [13].

A compiler for a data-parallel language attempts to produce data and work mappings that reduce completion time. Completion time has two components: computation and communication. Communication can be separated into intrinsic and residual communication. *Intrinsic communication* arises from operations such as reductions that must move data as an integral part of the operation. *Residual communication* arises from nonlocal data references in an operation whose operands are not mapped to the same processors. We use the term *realignment* to refer to residual communication due to misalignment, and *redistribution* to refer to residual communication due to changes in distribution.

In this paper, we describe a representation of array-based data-parallel programs called the *Alignment-Distribution Graph*, or ADG for short. We show how to model residual communication cost using the ADG, and discuss algorithms [2, 3, 4] for analyzing the alignment requirements of a program. The ADG is closely related to the *static single assignment* (SSA) form of programs developed by Cytron *et al.* [5], but is tailored for alignment and distribution analysis. In particular, it uses new techniques to represent the residual communication due to loop-carried dependences, assignment to sections of arrays, and transformational array operations such as reductions and spreads. The ADG provides a detailed and realistic model of residual communication cost that accounts for a wide variety of program features; it allows us to handle alignment and distribution as discrete optimization problems.

The remainder of the paper is organized as follows. Section 2 motivates and formally describes the ADG representation of programs. Section 3 describes how the ADG is used to model the communication cost of a program. Section 4 discusses approximations that we make in the model presented in Section 2 for reasons of practicality.

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Section 5 outlines some algorithms for alignment analysis that use the ADG. Our previous papers [2, 3, 4] contain more details about the algorithms. Section 6 compares the ADG representation with SSA form and with the *preference graph*, another representation used in alignment and distribution analysis. Section 7 discusses open problems and future work.

## 2 The ADG representation of data-parallel programs

The ADG is a directed graph representation of data flow in a program. Nodes in the ADG represent computation; edges represent flow of data.

Alignments are associated with endpoints of edges, which we call *ports*. A node constrains the relative alignments of the ports representing its operands and its results. Realignment occurs whenever the ports of an edge have different alignments. The goal of alignment analysis is to determine alignments for the ports that satisfy the node constraints and minimize the total realignment cost, which is a sum over edges of the cost of all realignment that occurs on that edge during program execution.

Similar mechanisms can be used for determining distributions. In the distribution analysis method that we foresee, the template has a distribution at each node of the ADG; the alignment of an object to the template and the distribution of the template jointly define the distribution of the object onto processors at that node. The distributions are chosen to minimize an execution time model that accounts for redistribution cost (again associated with edges of the ADG) and computation cost, associated with nodes of the ADG and dependent on the distribution as well. For the remainder of this paper, however, we focus on alignment analysis.

The ADG distinguishes between program array variables and array-valued objects in order to separate names from values. An *array-valued object* (object for short) is created by every array operation and by every assignment to a section of an array. Assignment to a whole array, on the other hand, names an object. The algorithms presented in Section 5 determine an alignment for each object in the program rather than for each program variable.

### 2.1 A summary of SSA form

A program is in *SSA form* if each variable is the target of exactly one assignment statement in the program text [5]. Any program can be translated to SSA form by renaming variables and introducing a pseudo-assignment called a  $\phi$ -function at some of the join nodes in the control flow graph of the program. Cytron *et al.* [5] present an efficient algorithm to compute minimal SSA form (*i.e.*, SSA form with the smallest number of  $\phi$ -functions inserted) for programs with arbitrary control flow graphs. Johnson and Pingali [9] have recently improved the complexity of SSA-conversion.

In common usage, SSA form is usually defined for sequential scalar languages, but this is not a fundamental limitation. It can be used for array languages if care is taken to properly model references and updates to individual array elements and array sections [5, §3.1]. The ADG uses SSA form in this manner.

A major contribution of SSA form is the separation between the values manipulated by a program and the storage locations where the values are kept. This separation, which allows greater opportunities for optimization, is the primary reason for basing the ADG on SSA form. After optimization, the program must be translated from SSA form to object code. Cytron *et al.* discuss two optimizations (dead code elimination and storage allocation by coloring) that produce efficient object code [5, §7].

### 2.2 Ports and alignment

The ADG has a *port* for each textual definition or use of an object. Ports are joined by edges as described below. The ports that represent the inputs and outputs of a program operation are grouped together to form a *node*. Some ports are named and correspond to program variables; others are anonymous and correspond to intermediate values produced by the computation.

The principal attribute of a port is its *alignment*, which is a one-to-one mapping of the elements of the object into the cells of a template. We use the notation

$$A(\theta_1, \dots, \theta_a) \boxplus [g_1(\theta_1, \dots, \theta_a), \dots, g_t(\theta_1, \dots, \theta_a)]$$

**Fortran 90**

```

real A(100,100), V(200)
do k = 1, 100
  A(k,1:100) = A(k,1:100) + V(k:k+99)
enddo

```

**SSA form**

```

real A1(100,100), A2(100,100), A3(100,100)
real V(200)
do k = 1, 100
  A2 = phi(A1, A3)
  A3 = Update(A2, (k,1:100),
             A2(k,1:100) + V1(k:k+99))
enddo

```

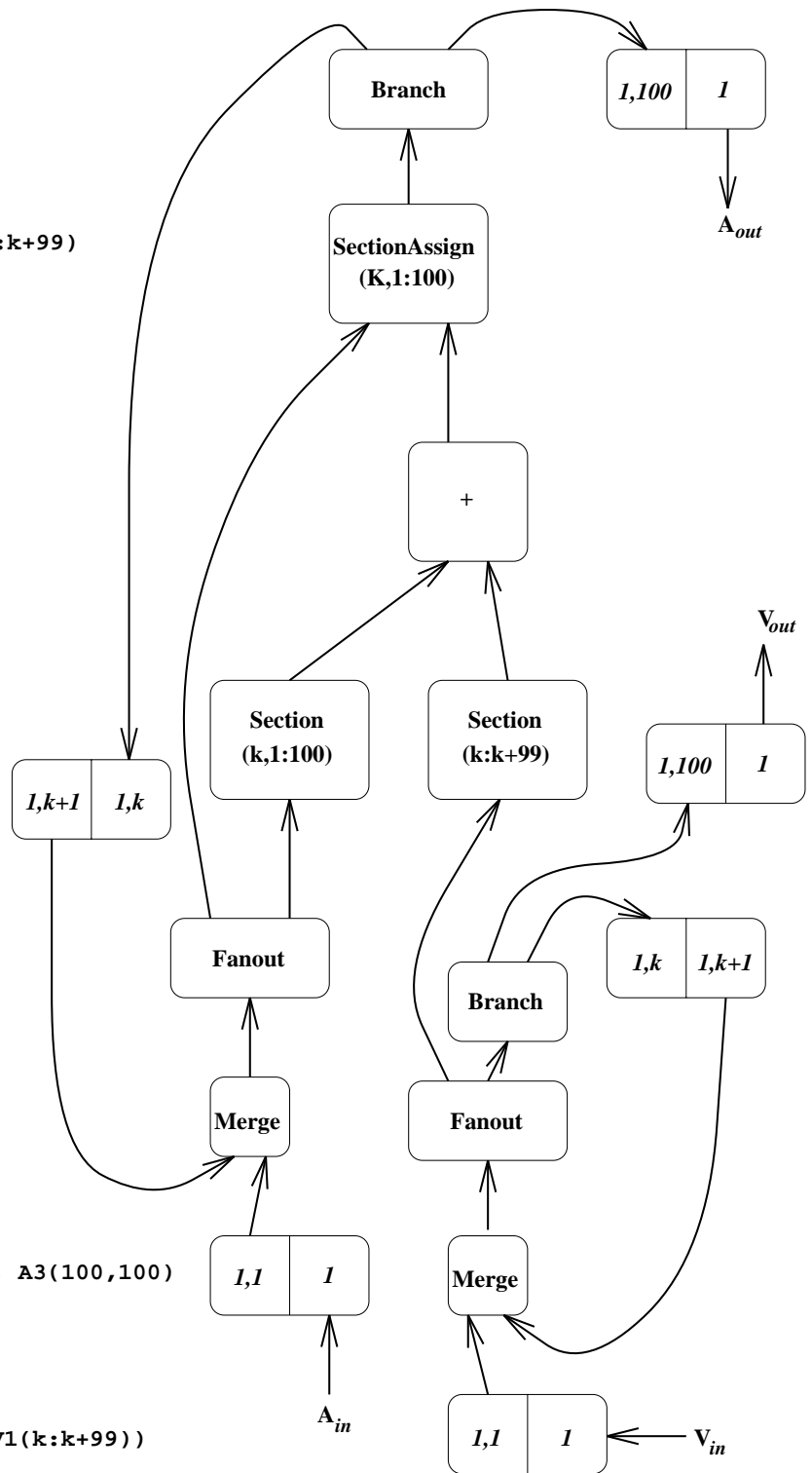


Figure 1: A Fortran 90 program fragment, its SSA form, and its ADG.

to indicate the alignment of the  $d$ -dimensional object  $A$  to the  $t$ -dimensional (unnamed) template. The formula above has  $d$  implicit universal quantifiers to its left, one for each of the index variables  $\theta_1$  through  $\theta_d$ .

When we consider an object defined or used in a nest of  $do$  loops with induction variables (LIVs)  $\xi_1, \dots, \xi_k$ , we extend the notation to

$$A(\theta_1, \dots, \theta_d) \boxplus_{\xi} [g_1(\theta_1, \dots, \theta_d), \dots, g_t(\theta_1, \dots, \theta_d)],$$

where  $\xi = (1, \xi_1, \dots, \xi_k)$ , and each  $g_k$  is now a function of  $\xi$ . The additional 1 at the beginning of  $\xi$  signifies that an object outside any loop nests has a position independent of any loop iteration variables. (See Section 4.2 for more details.) In this notation, the index variables are universally quantified, but the induction variables are free. Such an alignment is said to be *mobile*.

High Performance Fortran allows a program to use more than one template. We have extended our theory to use multiple templates, but in this paper, for simplicity, we assume that all array objects are aligned to a single template.

We restrict our attention to alignments in which each axis of the object maps to a different axis of the template, and elements are evenly spaced along template axes. Such an alignment has three components: *axis* (the mapping of object axes to template axes), *stride* (the spacing of successive elements along each template axis), and *offset* (the position of the object origin along each template axis). Each  $g_j$  is thus either a constant  $f_j$  (in which case the axis is called a *space axis*), or a function of a single array index of the form  $s_j \theta_{a_j} + f_j$  (in which case it is called a *body axis*). There are  $d$  body axes and  $(t - d)$  space axes. We allow the stride and offset components to be functions of the induction variables in the mobile case. In matrix notation, the alignment  $g_A(\theta)$  of object  $A$  can be written as  $g_A(\theta) = L_A \theta + f_A$ , where  $L_A$  is a  $t \times d$  matrix whose columns are orthogonal and contain exactly one nonzero element each,  $f_A$  is a  $t$ -vector, and  $\theta = (\theta_1, \dots, \theta_d)^T$ . The elements of  $L_A$  and  $f_A$  are expressions in  $\xi$ . The nonzero structure of  $L_A$  gives the axis alignment, its nonzero values give the stride alignment, and  $f_A$  gives the offset alignment.

We also allow replication of objects. The offset of an object in a space axis of the template, rather than being a scalar, may be a set of values. We restrict our attention to sets are arithmetic sequences representable by triplets  $\ell : h : s$ .

## 2.3 Edges, iteration spaces, and control weights

An edge in the ADG connects the definition of an object with a use of the object. Multiple definitions or uses are handled with merge, branch, and fanout nodes as described below. Thus every edge has exactly two ports. The purpose of the alignment phase is to label each port with an alignment. All communication necessary for realignment is associated with edges; if the two ports of an edge have different alignments, then the edge incurs a cost that depends on the alignments and the total amount of data that flows along the edge during program execution. An edge has three attributes: data weight, iteration space, and control weight.

The *data weight* of an edge is the size of the object whose definition and use it connects. As the objects in our programs are rectangular arrays, the size of an object is the product of its extents. If an object is within a loop nest, we allow its extents, and hence its size, to be functions of the LIVs. We write the data weight of edge  $(x, y)$  at iteration  $\xi$  as  $w_{xy}(\xi)$ .

The ADG is a static representation of the data flow in a program. However, to model communication cost accurately, we must take control flow into account. The *branch* and *merge* nodes in the ADG are a static representation, in a data-oriented model, of the forks and joins in control flow. Control flow has two effects: data may not always flow along an edge during program execution (due to conditional constructs), and data may flow along an edge multiple times during program execution (due to iterative constructs). An *activation* of an edge is a instance of data flowing along the edge during program execution. To model the communication cost correctly, we attach *iteration space* and *control weight* attributes to edges.

First consider a singly nested  $do$ -loop, as in Figure 1. Data flows once along the edges from the preceding computation into the loop, along the forward edges of the loop at every iteration, along the loop-back edges after all but the last iteration, and once, after this last iteration, out of the loop to the following computation. Summing the contribution of each edge over its set of iterations correctly accounts for the realignment cost of an execution of the loop construct. In general, an edge  $(x, y)$  inside a nest of  $k$   $do$ -loops is labeled with an *iteration space*  $\mathcal{I}_{xy} \subset \mathbb{Z}^{k+1}$ , whose elements are the vectors  $\xi$  of values taken by the LIVs. As explained above, both the size of the object on an edge and the alignment of the object at a port can be functions of the LIVs. The realignment and redistribution cost attributed to an edge is the sum of these costs over all iterations in its iteration space.

```

do i = 1,5
  if (cond(i)) then
    A = A + B(i,:)
  else
    A = A + B(:,i)
  endif
enddo

```

```

do i = 1,5
  A1 = phi(A0, A4)
  if (cond(i)) then
    A2 = A1 + B(i,:)
  else
    A3 = A1 + B(:,i)
  endif
  A4 = phi(A2, A3)
enddo

```

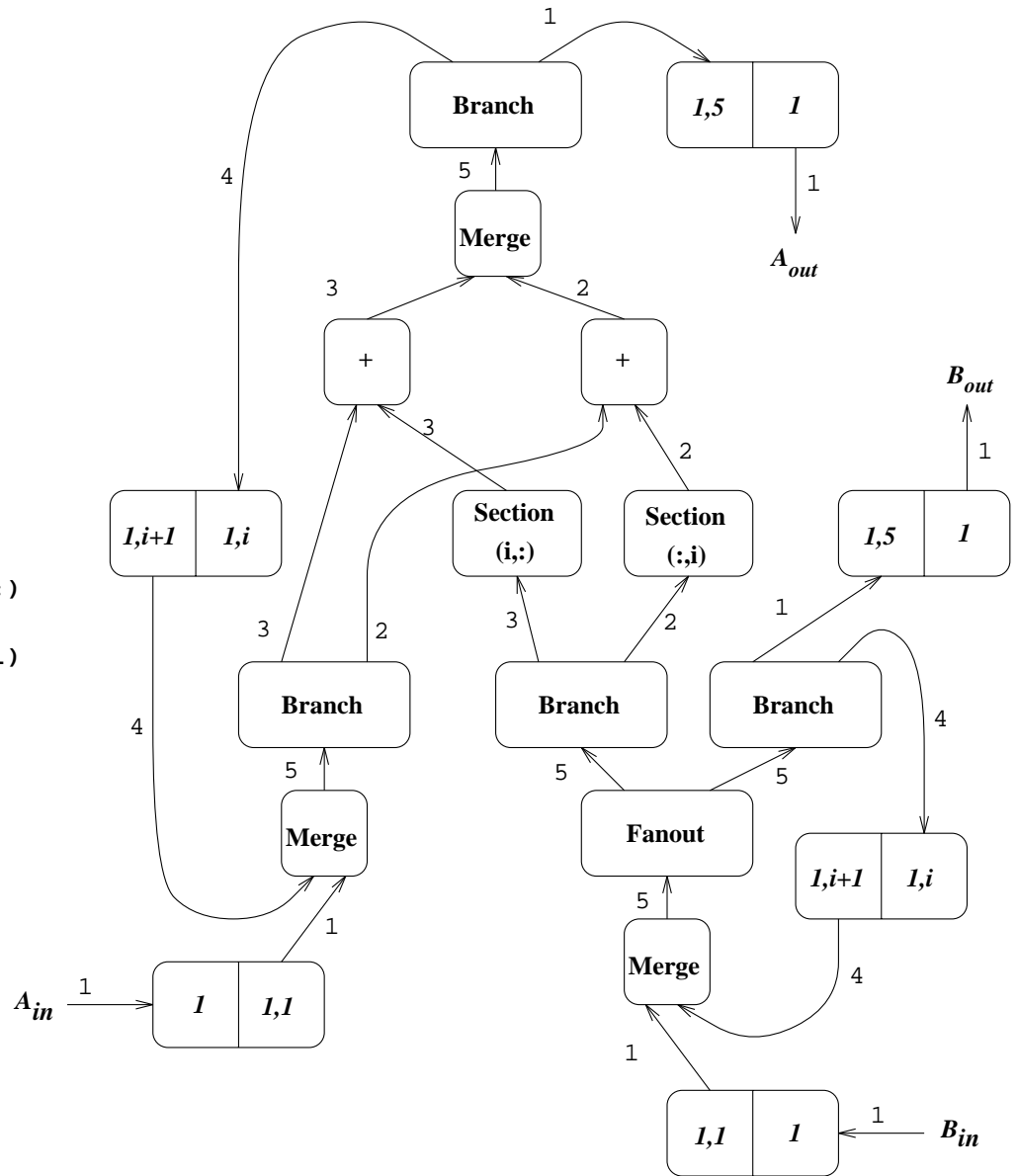


Figure 2: The ADG for a program with conditional branches. The labels on the edges are their control weights, assuming that the *then* branch of the conditional is taken 60% of the time, and that the *else* branch is taken 40% of the time.

For a program where the only control flow occurs in nests of *do*-loops, iteration spaces exactly capture the number of activations of an edge. However, programming languages allow a wider variety of control flow constructs—*while*- and *repeat*-loops, *if-then-else* constructs, conditional *gotos*, and so on. Iteration spaces can both underestimate and overestimate the effect of such control flow on communication. First, consider a *do*-loop nested within a *repeat*-loop. In this case, the iteration space indicated by the *do*-loop may underestimate the actual number of activations of the edges in the loop body. Second, because of *if-then-else* constructs in a loop, an edge may be activated on only a subset of its iteration space. For this reason, we associate a control weight  $c_{xy}(\xi)$  with every edge  $(x, y)$  and every iteration  $\xi$  in its iteration space. One may think of  $c_{xy}(\xi)$  as the expected number of activations of the edge  $(x, y)$  on an iteration with LIVs equal to  $\xi$ . Control weights enter multiplicatively into our estimate of communication cost.

Consider the *if-then-else* construct in the code of Figure 2. In the ADG, we have introduced two *branch* nodes, since the values A1 and B can flow to one, but not both, of two alternative uses, depending on the outcome of the conditional. If the outcomes were known, we could simply partition the iteration space accordingly, and assign to each edge leaving these branch nodes the exact set of iterations on which data flows over the edge. Since this is impractical, we label these edges with the whole iteration space  $\{1, 2, 3, 4, 5\}$ . The control weights of these edges shown in Figure 2 capture the dynamic behavior of the program.

Iteration spaces and control weights are alternative models of multiple activations of an ADG edge. The iteration space approach, when it is usable, enables a more accurate model of communication cost. When an exact iteration space can be determined statically, as in the case of *do*-loops, it should be used to characterize control flow. Control weights should be used only when an exact iteration space cannot be determined statically, *e.g.*, for conditional *gotos*, multiway branches, *while*-loops, and breaks.

## 2.4 Nodes

Every array operation is a node of the ADG, with one port for each operand and result. Figure 1 contains examples of a “+” node representing elementwise addition, a *section* node whose input is an array and whose output is a section of the array, and a *section-assign* node whose inputs are an array and a new object to replace a section of the array, and whose output is the modified array. (The terms *section* and *section-assign* are related to the terms *Access* and *Update* used by Cytron *et al.* [5].)

When a single use of a value can be reached by multiple definitions, the ADG contains a *merge node* with one port for each definition and one port for the use. (This node corresponds to the  $\phi$ -function of Cytron *et al.* [5].) Conversely, when a single definition can reach at most one of several possible uses, the ADG contains a *branch node*. Figures 1 and 2 contain examples of merge and branch nodes.

Now consider the situation where a single definition actually reaches multiple uses. This is different from the branching situation, in which a definition has several *alternative* uses. Given alignments for the definition and for all the uses, the optimal way to make the object available at the positions where it is used is through a *Steiner tree* [15] in the metric space of possible alignments, spanning the alignment at the definition, the alignments at the uses, and additional positions as required to minimize the sum of the edge lengths. Determining the best Steiner tree is NP-hard for most metric spaces. We therefore approximate the Steiner tree as a star, adding one additional node, called a *fanout node*, at the center of the star. Figures 1 and 2 contain examples of fanout nodes. There remains the possibility of replacing this star by a true Steiner tree in a later pass.

Finally, for a program with *do*-loops, we need to characterize the introduction, removal, and update of LIVs, as we intend to let data weights and alignments be functions of these LIVs. Accordingly, for every edge that carries data into, out of, or around a loop, we insert a *transformer node* that enforces a relationship between the iteration spaces at its two ports. Figures 1 and 2 contain examples.

ADG nodes define relations among the alignments of their ports, as well as among the data weights, control weights, and iteration spaces of their incident edges. The relations on alignments constrain the solution provided by alignment analysis. They must be satisfied if the computation performed at the nodes is to be done without further realignment. An alignment (of all ports, at all iterations of their iteration spaces) that satisfies the node constraints is said to be *feasible*.

The constraints force all realignment communication onto the edges of the ADG. By suitably choosing the node constraints, the apparently “intrinsic” communication of such operations as transpose and spread is exposed as

realignment; this enables optimization of realignment.

Only intrinsic communication and computation happens within the nodes. In our current language model, the only program operations with intrinsic communication are reduction and vector-valued subscripting (scatter and gather), which access values from different parts of an object as part of the computation.

## 2.5 Nodal relations and constraints

We now list the constraints on alignment (the matrix  $L$  and the vector  $f$  introduced in Section 2.2) and the relations on iteration spaces and control weights that hold at each type of node. The relations on data weights are a simple consequence of language semantics and are not stated here.

**Elementwise operations and fanout nodes** An elementwise arithmetic or logical operation on congruent objects  $A_1$  through  $A_k$  produces an object  $R$  of the same shape and size. Fanout nodes behave the same way. Alignments are identical at all ports:

$$L_{A_1} = \dots = L_{A_k} = L_R$$

and

$$f_{A_1} = \dots = f_{A_k} = f_R.$$

**Array sectioning** Let  $A$  be a  $d$ -dimensional object, and  $S$  a section specifier. Then  $A(S)$  is the object corresponding to the section of that object. A section specifier is a  $d$ -vector  $(\sigma_1, \dots, \sigma_d)$ , where each  $\sigma_i$  is either a scalar  $\ell_i$  or a triplet  $\ell_i : h_i : s_i$ . Array axes corresponding to positions where the section specifier is a scalar are projected away, while the axes where the specifier is a triplet form the axes of  $A(S)$ . Let the elements of  $S$  that are triplets be in positions  $\lambda_1, \dots, \lambda_c$  in ascending order. Let  $e_i$  be a column vector of length  $d$  whose only nonzero entry is 1 at position  $i$ .

The axis alignment of  $A(S)$  is inherited from the dimensions of  $A$  that are preserved (not projected away) by the sectioning operation. Strides are multiplied by the sectioning strides. The offset alignment of  $A(S)$  is equal to the position of  $A(\ell_1, \dots, \ell_d)$ :

$$L_{A(S)} = L_A \cdot [s_{\lambda_1} e_{\lambda_1}, \dots, s_{\lambda_c} e_{\lambda_c}]$$

and

$$f_{A(S)} = g_A((\ell_1, \dots, \ell_d)^T) = f_A + L_A(\ell_1, \dots, \ell_d)^T$$

where  $\cdot$  denotes matrix multiplication.

**Assignment to array sections** An assignment to a section of an array, as in the Fortran 90 statement `A(1:100:2, 1:100:2) = B`, is treated in SSA form as taking an input object  $A$ , a section specifier  $S$ , and a replacement object  $B$  conformable with  $A(S)$ , and producing a result object  $R$  that agrees with  $B$  on  $A(S)$  and with  $A$  elsewhere. The result aligns with  $A$ , and the alignment of  $B$  must match that of  $A(S)$ :

$$L_R = L_A, \quad f_R = f_A$$

and

$$L_B = L_{A(S)}, \quad f_B = f_{A(S)}.$$

**Transposition** Let  $A$  be a  $d$ -dimensional object, and let  $\rho$  be a permutation of  $(1, \dots, d)$ . The array object  $\rho A$  (produced by an ADG *transpose* node) is the array  $\rho A(\theta_{\rho_1}, \dots, \theta_{\rho_d}) = A(\theta_1, \dots, \theta_d)$ . (Fortran 90 uses the `reshape` and `transpose` intrinsics to perform general transposition.) The offset of the transposed array is unchanged, but its axes are a permutation of those of  $A$ :

$$L_{\rho A} = L_A \cdot [e_{\rho_1}, \dots, e_{\rho_d}], \quad f_{\rho A} = f_A.$$

**Reduction** Let  $A$  be a  $d$ -dimensional object. Then the program operation  $\text{sum}(A, \text{dim}=k)$  produces the  $(d - 1)$ -dimensional object  $R$  by reducing along axis  $k$  of  $A$ . (The operation used for reduction is of no importance in determining alignments.) Let  $n_k$  be the extent of  $A$  in axis  $k$ . Then  $R$  is aligned identically with  $A$  except that the template axis to which axis  $k$  of  $A$  was aligned is a space axis of  $R$ . The offset of  $R$  in this axis may be any of the positions occupied by  $A$ .

$$L_R = L_A \cdot [e_1, \dots, e_{k-1}, e_{k+1}, \dots, e_d]$$

and

$$f_R = f_A + \beta L_A e_k, \text{ where } 0 \leq \beta < n_k.$$

**Spread** Let  $A$  be a  $d$ -dimensional object. Then the program operation  $\text{spread}(A, \text{dim}=k, \text{ncopies}=n)$  produces a  $(d + 1)$ -dimensional object  $R$  with the new axis in position  $k$ , and with extent  $n$  along that axis. The alignment constraints are the converse of those for the dual operation, reduction. The new axis of  $R$  aligns with an unused template axis, and the other axes of  $R$  inherit their alignments from  $A$ .

$$L_A = L_R \cdot [e_1, \dots, e_{k-1}, e_{k+1}, \dots, e_{d+1}].$$

In order to make the communication required to replicate  $A$  residual rather than intrinsic, we require the offset alignment of  $A$  in dimension  $k$  to be replicated. This condition sounds strange, but it correctly assigns the required communication to the input edge of the *spread* node. In this view, a *spread* node performs neither computation nor communication, but transforms a replicated object into a higher-dimensional non-replicated object. Thus,

$$f_A = f_R + f_r$$

where the vector  $f_r$  has one nonzero component, a triplet in the axis spanned by the replicated dimension:

$$f_r = (0 : n - 1)L_R e_k.$$

The multiplication of a vector by a triplet is defined by  $(0 : h)(x_1, \dots, x_t)^T \equiv ((0 : x_1 h : x_1), \dots, (0 : x_t h : x_t))$ .

**Merge nodes** Merge nodes occur when multiple definitions of a value converge. This occurs on entry to a loop and as a result of conditional transfers. Merge nodes enforce identical alignment at their ports.

The iteration space of the outedges is the union of the iteration spaces of inedges. The expected number of activations of the outedge with LIVs  $\xi$  is just the sum of the expected number of activations with LIVs  $\xi$  of the inedges. Therefore, the control weight of the outedge is the sum of the control weights of the inedges. Let the iteration spaces of the inedges be  $\mathcal{I}_1$  through  $\mathcal{I}_m$ , and the corresponding control weights  $c_1(\xi)$  through  $c_m(\xi)$ . Let the iteration space of the outedge be  $\mathcal{I}_R$  and its control weight be  $c_R(\xi)$ . Extend the  $c_i$  for each input edge to  $\mathcal{I}_r$  by defining  $c_i(\xi)$  to be 0 for all  $\xi \in \mathcal{I}_r - \mathcal{I}_i$ . Then

$$\mathcal{I}_R = \bigcup_{i=1}^m \mathcal{I}_i$$

and

$$\forall \xi \in \mathcal{I}_R, \quad c_R(\xi) = \sum_{i=1}^m c_i(\xi).$$

**Branch nodes** Branch nodes occur when multiple mutually exclusive uses of a value diverge. Following the activation of the inedges, one of the outedges activates, with the selection made through program control flow. Branch nodes enforce identical alignment at their ports.

The relations satisfied by iteration spaces and control weights of the incident edges are dual to those of merge nodes. Let the iteration spaces of the outedges be  $\mathcal{I}_1$  through  $\mathcal{I}_m$ , and the corresponding control weights be  $c_1(\xi)$  through  $c_m(\xi)$ . Let the iteration space of the inedge be  $\mathcal{I}_A$  and its control weight be  $c_A(\xi)$ . Then

$$\mathcal{I}_A = \bigcup_{i=1}^m \mathcal{I}_i$$



and

$$\forall \xi \in \mathcal{I}_A, c_A(\xi) = \sum_{i=1}^m c_i(\xi).$$

**Transformer nodes** Transformer nodes are of two types: those that relate iterations at the same nesting level, and those that relate iterations at different nesting levels.

Transformer nodes of the first kind (*loop-back* transformer nodes) have the form

$$(1, \xi_1, \dots, \xi_k | 1, \xi_1, \dots, \xi_k + s),$$

corresponding to a change by the loop stride  $s$  in the value of the LIV  $\xi_k$ , and no change in any of the other LIVs in the loop nest. Let  $\xi = (1, \xi_1, \dots, \xi_k)$ ; define  $\widehat{\xi}(\xi) = (1, \xi_1, \dots, \xi_k + s)$ . Let the alignment on the input (“ $\xi$ ”) port be  $L\theta + f$ , and let the alignment on the output (“ $\widehat{\xi}$ ”) port be  $\widehat{L}\theta + \widehat{f}$ , where  $L, f, \widehat{L}$ , and  $\widehat{f}$  are all functions of  $\xi$ . Let  $\mathcal{I}$  be the iteration space of the input port and  $\widehat{\mathcal{I}}$  be the iteration space of the output port, Then the alignment constraints are

$$\forall \xi \in \mathcal{I}, L(\xi) = \widehat{L}(\widehat{\xi})$$

and

$$\forall \xi \in \mathcal{I}, f(\xi) = \widehat{f}(\widehat{\xi}).$$

The relation between the iteration spaces is

$$\widehat{\mathcal{I}} = \mathcal{I} + s(0, e_k)^T.$$

Consider one of the  $(1, k | 1, k + 1)$  transformer nodes in Figure 1. An offset alignment  $f = 2k + 3$  and  $\widehat{f} = 2k + 1$  satisfies the node’s alignment constraints. If the input iteration space is  $\mathcal{I} = \{(1, 1)^T, \dots, (1, n - 1)^T\}$ , then the output iteration space is  $\widehat{\mathcal{I}} = \{(1, 2)^T, \dots, (1, n)^T\}$ .

Transformer nodes of the second kind (*entry/exit* transformer nodes) have the form

$$(1, \xi_1, \dots, \xi_{k-1} | 1, \xi_1, \dots, \xi_{k-1}, \xi_k = v)$$

or

$$(1, \xi_1, \dots, \xi_{k-1}, \xi_k = v | 1, \xi_1, \dots, \xi_{k-1}),$$

corresponding to the introduction or removal of the LIV  $\xi_k$  in a loop nest. Let  $\xi = (1, \xi_1, \dots, \xi_{k-1})$  and define  $\widehat{\xi}(\xi) = (1, \xi_1, \dots, \xi_{k-1}, v)$ . Let the alignment on the input (“ $\xi$ ”) port be  $L\theta + f$ , and let the alignment on the output (“ $\widehat{\xi}$ ”) port be  $\widehat{L}\theta + \widehat{f}$ . Let  $\mathcal{I}$  be the iteration space of the input port and  $\widehat{\mathcal{I}}$  be the iteration space of the output port, Then the alignment constraints are

$$\forall \xi \in \mathcal{I}, L(\xi) = \widehat{L}(\widehat{\xi})$$

and

$$\forall \xi \in \mathcal{I}, f(\xi) = \widehat{f}(\widehat{\xi}).$$

The relation satisfied by the iteration spaces is

$$\widehat{\mathcal{I}} = \mathcal{I} \times \{v\},$$

where  $\times$  denotes the Cartesian product. Thus, the  $(1 | 1, 1)$  transformer node in Figure 1 constrains its input position (which does not depend on  $k$ ) to equal its output position for  $k = 1$ . An offset alignment of  $f = 1$  and  $\widehat{f} = 2i - 1$  satisfies the node’s constraints.

### 3 Modeling residual communication cost using the ADG

The ADG describes the structural properties of the program that we need for alignment and distribution analysis. Residual communication occurs on the edges of the ADG, but so far we have not indicated how to estimate this cost. This missing piece is the *distance function*  $d$ , where the distance  $d(p, q)$  between two alignments  $p$  and  $q$  is a nonnegative number giving the cost per element to change the alignment of an array from  $p$  to  $q$ . The set of all alignments is normally a metric space under the distance function  $d$  [3]. We discuss the structure of  $d$  in Section 4.

We model the communication cost of the program as follows. Let  $E$  be the edge set of the ADG  $G$ , and let  $\mathcal{I}_{xy}$  be the iteration space of edge  $(x, y)$ . For a vector  $\xi$  in  $\mathcal{I}_{xy}$ , let  $w_{xy}(\xi)$  be the data weight, and let  $c_{xy}(\xi)$  be the control weight of the edge. Finally, let  $\pi$  be a feasible alignment for the program. Then the realignment cost of edge  $(x, y)$  at iteration  $\xi$  is  $c_{xy}(\xi) \cdot w_{xy}(\xi) \cdot d(\pi_x(\xi), \pi_y(\xi))$ , and the total realignment cost of the ADG is

$$K(G, d, \pi) = \sum_{(x,y) \in E} \sum_{\xi \in \mathcal{I}_{xy}} c_{xy}(\xi) \cdot w_{xy}(\xi) \cdot d(\pi_x(\xi), \pi_y(\xi)). \quad (1)$$

Our goal is to choose  $\pi$  to minimize this cost, subject to the node constraints. An analogous framework can be used to model redistribution cost.

## 4 Approximations

The definition of the ADG in Section 2 assumed complete knowledge of control flow, and also ignored the effect of the parameters of the model on the complexity of the optimization problem. In this section, we discuss approximations to the model to address questions of practicality. The approximations are of two kinds: those that make it possible to compute the parameters of the model, and those that make the optimization problem tractable.

### 4.1 Control weights

Our model of control weights as a function of LIVs is formally correct but difficult to evaluate in practice. We therefore approximate the control weight  $c_{xy}(\xi)$  as an averaged control weight  $c'_{xy}$  that does not depend on  $\xi$ . We now relate this averaged control weight to the execution counts of the basic blocks of the program.

Assume that we have a control flow graph (CFG) of the program (whose nodes are basic blocks of the program, and edges are transfers of control) and an estimate of the branching probabilities of the program. (These probabilities can be estimated using heuristics, profile information, or user input.) Let  $p_{ij}$  be this estimate for edge  $(i, j)$  of the CFG;  $p_{ij} = 0$  if  $(i, j)$  is not an edge of the CFG. Let there be  $B$  basic block nodes in the CFG, plus the two distinguished nodes *ENTER* and *EXIT*. We first determine the execution counts of the basic blocks by setting up and solving “conservation of control flow” equations at each basic block. The conservation equation for basic block  $i$  with execution count  $u_i$  is  $u_i = \sum_{j \neq i} p_{ji} u_j$ . In matrix notation, we solve the linear system  $Au = v$ , where  $u = [u_1, \dots, u_B]^T$  is the vector of execution counts,  $v = [p_{ENTER,1}, \dots, p_{ENTER,B}]^T$ , and  $A$  is a  $B \times B$  matrix with  $A(i, i) = 1$  and  $A(i, j) = -p_{ji}$  for  $i \neq j$ . The (averaged) control weight  $c'_{xy}$  of the ADG edge  $(x, y)$  coming from a computation in basic block  $b$  is then  $u_b / |\mathcal{I}_{xy}|$ .

### 4.2 Mobile alignment

So far we have not constrained the form that mobile alignments may take. In principle, they could be arbitrary functions of the LIVs. To keep the analysis tractable, we restrict mobile alignments to be affine functions of the LIVs. Thus, the alignment function for an object within a  $k$ -deep loop nest with LIVs  $\xi_1, \dots, \xi_k$  is of the form  $a_0 + a_1 \xi_1 + \dots + a_k \xi_k$ , where the coefficient vector  $a = (a_0, \dots, a_k)$  is what we must determine. We write this alignment succinctly in vector notation as  $a\xi^T$ . Both  $a$  and  $\xi$  are  $(k+1)$ -vectors. This reduces to the constant term  $a_0$  for an object outside any loops.

Likewise, we restrict the extents of objects to be affine in the LIVs, so that the size of an object is polynomial in the LIVs.

### 4.3 Replicated alignments

In Section 2, we introduced triplet offset positions to represent the possibility of replication. In practice, we treat the replication component of offset separately from the scalar component. The alignment space for replication has two elements, called  $\mathbf{R}$  (for replicated) and  $\mathbf{N}$  (for non-replicated). The extent of replication for an  $\mathbf{R}$  alignment is the entire template extent in that dimension. In this approximate model of replication, communication is required only when changing from a non-replicated alignment to a replicated one. Thus, the distance function is given by  $d(\mathbf{N}, \mathbf{R}) = 1$  and  $d(\mathbf{R}, \mathbf{R}) = d(\mathbf{R}, \mathbf{N}) = d(\mathbf{N}, \mathbf{N}) = 0$ . We call the process of determining these restricted replicated alignments *replication labeling*.

### 4.4 Distance functions

In introducing the distance function in Section 3, we defined it to be the cost per element of changing from one alignment to another. Now consider the various kinds of such changes, and their communication costs on a distributed-memory machine. A change in axis or stride alignment requires unstructured communication. Such communication is hard to model accurately as it depends critically on the topological properties of the network (bisection bandwidth), the interactions among the messages (congestion), and the software overheads. Offset realignment can be performed using shift communication, which is usually substantially cheaper than unstructured communication. Replicating an object involves broadcasting or multicasting, which typically uses some kind of spanning tree. Such broadcast communication is likely to cost more than shift communication but less than unstructured communication.

We could conceivably construct a single distance function capturing all these various effects and their interactions, but this would almost certainly make the analysis intractable. We therefore split the determination of alignments into several phases based on the relative costs of the different kinds of communication, and introduce simpler distance functions for each of these phases.

We determine axis and stride alignments (or the matrix  $L$  of Section 2.2) in one phase, using the *discrete metric* to model axis and stride realignment. This metric, in which  $d(p, q) = 0$  if  $p = q$  and  $d(p, q) = 1$  otherwise, is a crude but reasonable approximation for unstructured communication.

We determine scalar offset alignment in a separate phase, using the *grid metric* to model shift realignment. In this metric, alignments are the vectors  $f$  of Section 2.2, and  $d(f, f')$  is the Manhattan distance between them, *i.e.*,  $d(f, f') = \sum_{i=1}^t |f_i - f'_i|$ . Note that the distance between  $f$  and  $f'$  is the sum of the distances between their individual components. This property of the metric, called *separability*, allows us to solve the offset alignment problem independently for each axis [3].

Finally, we use yet another phase to determine replicated offsets, using the alignments and distance function described in Section 4.3.

The ordering of these phases is as follows: we first perform axis and stride alignment, then replication labeling, and finally offset alignment.

The various kinds of communication interact with one another. For instance, shifting an object in addition to changing its axis or stride alignment does not increase the communication cost, since the shift can be incorporated into the unstructured communication needed for the change of axis or stride. We model such effects in a simple manner, by introducing a *presence weight* of an edge for each phase of the alignment process. The presence weight multiplies the contribution of the edge to the total realignment cost for that phase. Initially, all presence weights are 1. Edges that end up carrying residual axis or stride realignment have their presence weights set to zero for the replication and offset alignment phases. Similarly, edges carrying replication communication have their presence weights set to zero for the offset alignment phase.

## 5 Determining alignments using the ADG

In this section, we briefly describe our algorithms for determining alignment. Full descriptions of these algorithms are in our previous papers [2, 3, 4].

## 5.1 Axis and stride alignment

To determine axis and stride alignment, we minimize the communication cost  $K(G, d, \pi)$  using the discrete metric as the distance function. The position of a port in this context is the matrix  $L$  of the alignment function. The algorithm we use, called *compact dynamic programming* [3], works in two phases.

In the first phase, we traverse the ADG from sources to sinks, building for each port a cost table indexed by position. The cost corresponding to position  $p$  gives the minimum cost of evaluating the subcomputation up to that port and placing the result in position  $p$ . The cost table at a node can be computed from the cost tables of its inputs and the distance functions. This recurrence is used to build the cost tables efficiently. At the end of the first phase, we examine the cost tables of the sinks of the ADG and place the sinks at their minimum-cost positions. In the second phase, we traverse the ADG from sinks to sources, placing the ports so as to achieve the minimum costs calculated in the first phase.

Compact dynamic programming is based on the dynamic programming approach of Mace [12]. The “compact” in the name refers to the way we exploit properties of the distance function to simplify the computation of costs and to compactly represent the cost tables. The method is exact if the ADG is a tree, but in general it is an approximation.

## 5.2 Offset alignment

For offset alignment, a position is the vector  $f$  of the alignment function, and the Manhattan metric is the distance function. As the distance function is separable, we can solve independently for each component of the vector. For code where the positions do not depend on any LIVs, the constrained minimization can be solved exactly using linear programming [2, 4]. With mobile alignments, the residual communication cost can be approximated and this approximate cost minimized exactly using linear programming.

## 5.3 Replication labeling

For replication labeling, the position space has two positions for each template axis, called **R** (for replicated) and **N** (for non-replicated). The distance function is as given in Section 4.3. Note that this distance function is not a metric. The sources of replication are *spread* operations, certain vector-valued subscripts, and read-only objects with mobile offset alignment. As in the offset alignment case, each axis can be treated independently.

A minimum-cost replication labeling can be computed efficiently using network flow [2].

# 6 Comparison with other work

In this section, we compare the ADG representation with SSA form and with the *preference graph*, another representation that has been used for automatic determination of alignments.

## 6.1 SSA form

While our ADG representation is based on SSA form, it has a number of extra features. All of the differences stem from the fact that the ADG representation was designed to manipulate positions of objects, while SSA form was designed to manipulate values. In fact, an SSA form based on a nonstandard “position semantics” would probably look exactly like the ADG formalism.

The ADG representation annotates the ports and edges with data weights, control weights, and iteration spaces, none of which are present in SSA form. However, the substantive difference between the two representations lies in the nodes. Every  $\phi$ -function of SSA corresponds to a merge node in ADG, but certain merge nodes (*e.g.*, for read-only objects within a loop) do not correspond to any  $\phi$ -functions in SSA. Similarly, fanout and branch nodes have no analog in SSA form.

The fanout and branch nodes of the ADG resemble similar nodes in the Program Dependence Web representation developed by Ballance *et al.* [1]. However, the motivations behind them are very different.

## 6.2 The preference graph

Another representation that has been used in alignment analysis is the *preference graph*, which has several variants [7, 10, 11, 14]. The preference graph is an undirected, weighted graph constructed from the reference patterns in the program. The nodes of the preference graph correspond to dimensions of array occurrences in the program, the edges reflect beneficial alignment relations, and the weights reflect the relative importance of the edges. The edges of the preference graph encode axis alignment, while additional node attributes are used to encode stride and offset alignment.

The preference graph has two kinds of edges, corresponding to the two sources of alignment decisions. Each statement considered in isolation provides some relations among nodes that avoid residual communication in computing the given statement. The edges corresponding to these relations are called *conformance preference edges*. A conformance preference edge between two array occurrences indicates that if they are not aligned, residual communication will be needed to align them in preparation for the operation.

The second kind of alignment preference comes from relating definitions and uses of the same array variable. The edges corresponding to these relations are called *identity preference edges*. An identity preference edge between two array occurrences indicates that if they are not positioned identically, residual communication will be needed to make the values from the definition available at the use.

Alignment decisions are made by contracting graph edges. At each step, an edge is chosen, and if the two nodes connected by the edge satisfy certain conditions, the edge is contracted and one of the nodes is merged into the other. When an edge is contracted, we say that the alignment preference it carries has been *honored*. The contraction process stops when no more edges can be contracted. An acyclic preference graph can be contracted to a single node, guaranteeing a communication-free implementation. If, however, there are cycles in the preference graph, some alignment preferences may remain unhonored and will result in residual communication. In this case, we must decide which preferences to honor and which to break. There are two principal variants of this general framework.

**The Knobe-Lukas-Steele algorithm** Knobe, Lukas, and Steele [10] call the nodes of the preference graph *cells*. They introduce an additional attribute of cells (called *independence anti-preference*) to model the constraint that a dimension occurrence has sufficient parallelism and should not be serialized.

An alignment conflict can only occur within a cycle of the preference graph. We therefore need to locate a cycle, determine whether it causes a conflict, and resolve the conflict if it exists. Actually, we only need to examine and resolve conflicts in a set of fundamental cycles of the preference graph, since if such a cycle basis is conflict-free, then all cycles in the graph are conflict-free. A cycle basis can be easily determined from a spanning tree of the graph: each non-tree edge determines a unique fundamental cycle of the graph with respect to the spanning tree. This ignores the fact that the preference graph is weighted. Given a choice between honoring two preferences, we would choose to honor the preference with higher weight. The Knobe-Lukas-Steele algorithm uses the nesting depth of an edge as its weight and constructs a maximum-weight spanning tree.

An unhonored identity preference implies that an object will live in distinct location in different parts of the program. An unhonored conformance preference implies that not all of the operations in a statement are necessarily performed in the location of the “owner” of the LHS. In other words, breaking a conformance preference is the mechanism for improving the “owner-computes rule”.

**The Li-Chen algorithm** Li and Chen [11] use the preference graph model to determine axis alignment. Their algorithm was developed in the context of the Crystal language, which allows more general reference patterns than we have considered, for instance, a pattern like  $A(i, j) = B(i, i)$ . This indicates conformance preferences between the first axis of  $A$  and both axes of  $B$ ; such sets of preferences that are generated from the same reference pattern and are incident upon a common node are said to be *competing*.

Li and Chen call the preference graph the *Component Affinity Graph* (CAG), and call preferences *affinities*. Competing edges have weight  $\epsilon$ , while noncompeting edges are unit-weight. The quantities  $\epsilon$  and 1 are incommensurate, with  $\epsilon \ll 1$ . The axis alignment problem is framed as the following graph partitioning problem:

Partition the node set of the CAG into  $n$  disjoint subsets  $V_1, \dots, V_n$  (where  $n$  is the maximum dimensionality of any array) with the restriction that no two axes of the same array may be in the same partition, so as to minimize the sum of the weights of edges connecting nodes in different subsets of the partition.

The idea here is to align array dimensions in the same subset of the partition, with edges between nodes in different partitions corresponding to residual communication. Hence the condition that we minimize the sum of the weights of such edges.

The graph partitioning problem stated above is NP-complete [11]. Li and Chen therefore solve it heuristically by finding maximum-weight matchings of a sequence of bipartite subgraphs of the CAG.

**Discussion** We now compare the ADG approach with the preference graph approach with regard to semantics, cost modeling, and the treatment of control flow.

- The conformance preferences, while similar to the node constraints of our approach, are not clearly related to the data flow and the intermediate results of the computation. Intermediate results have no explicit representation in the conformance graph model. They could, however, be made explicit by preprocessing the source.
- The preference graph model distinguishes between objects at different levels of a loop nest, but does not consider the size of an object, or the shift distance, in computing the cost of moving it. In fact, it can be thought of as using the unweighted discrete metric to model all communication cost.
- Knobe *et al.* [10] discuss handling control flow by introducing alignments for arrays at merge points in the program. Our use of static single assignment form provides a sound theoretical basis regarding the placement of merge nodes.
- To the best of our knowledge, the preference graph method has not been used to determine when objects should be replicated.

## 7 Conclusions

We have motivated and described a representation of data-parallel programs called the alignment-distribution graph, which provides a mechanism for explicitly representing and optimizing residual communication cost. The representation is based on a separation of variables and values, extended from the static single assignment form of programs. The communication requirements of the program are faithfully modeled by carefully representing realignment communication due to control flow, loop-carried dependences, assignment to sections of arrays, and transformational array operations.

The ADG provides a more detailed model of residual communication than the preference graph, another representation that has been used for the same analysis. We have developed algorithms in the ADG framework to determine axis alignment, mobile stride and offset alignment, and replication. We have devised a scheme for generating the ADG from program text, which we are presently refining and implementing. We are also implementing the algorithms mentioned in Section 5 to test the validity of this approach.

We are currently studying several interesting problems that remain unsolved. These include developing algorithms for distribution analysis using the ADG framework, understanding the interactions between the alignment and distribution phases, developing interprocedural optimization techniques in this framework, finding better algorithms for solving the alignment problem in the discrete metric, and allowing for the possibility of skew alignments.

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