# Dynamic Consumption and Portfolio Choice with Stochastic Volatility in Incomplete Markets

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## Abstract

This paper examines the optimal consumption and portfolio choice problem of long-horizon investors who have access to a riskless asset with constant return and a risky asset ("stocks") with constant expected return and time varying precision—the reciprocal of volatility. Markets are incomplete, and investors have recursive preferences defined over intermediate consumption. The paper obtains a solution to this problem which is exact for investors with unit elasticity of intertemporal substitution of consumption, and approximate otherwise. The optimal portfolio demand for stocks includes an intertemporal hedging component that is negative when investors have coefficients of relative risk aversion larger than one, and the instantaneous correlation between volatility and stock returns is negative, as typically estimated from stock return data. Our estimates of the joint process for stock returns and precision (or volatility) using US data confirm this finding. But we also find that stock return volatility does not appear to be variable and persistent enough to generate large intertemporal hedging demands.



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# 1 Introduction

There is strong empirical evidence that the conditional variance of asset returns, particularly stock market returns, is not constant over time. Bollerslev, Chou and Kroner (1992), Campbell, Lo and MacKinlay (1997, Chapter 12), Campbell, Lettau, Malkiel and Xu (2001) and others review the main findings of the ample econometric research on stock return volatility: Stock return volatility is serially correlated, and shocks to volatility are negatively correlated with unexpected stock returns. Changes in volatility are persistent (French, Schwert and Stambaugh 1987, Campbell and Hentschel 1992). Large negative stock returns tend to be associated with increases in volatility that persist over long periods of time. Stock return volatility appears to be correlated across markets over the world (Engle, Ito and Lin 1990, Ang and Bekaert 1999).

While there is an abundant literature exploring the pricing of assets when volatility is time varying, there is not much research exploring optimal dynamic portfolio choice with volatility risk. This situation is unfortunate, because Samuelson (1969) and Merton (1969, 1971, 1973) have shown that time variation in investment opportunities imply optimal portfolio strategies for multi-period investors that can be different from those of single-period, or myopic, investors. Multi-period investors value assets not only for their short-term risk-return characteristics, but also for their ability to hedge consumption against adverse shifts in future investment opportunities. Thus these investors have an extra demand for risky assets that reflects intertemporal hedging.

Intertemporal hedging is not only conceptually interesting; it is also empirically relevant. Recent research summarized in Campbell and Viceira (2002) has found that intertemporal hedging is quantitatively important in light of the observed predictable variation in both interest rates and equity premia in the US (Balduzzi and Lynch 1997, Barberis 2000, Brandt 1998, Brennan, Schwartz and Lagnado 1996, 1997, Campbell and Viceira 1999, 2001, Campbell, Chan and Viceira 2002).

This paper explores systematically optimal portfolio choice with volatility risk in a continuous-time setting. We consider the optimal consumption and investment problem of investors with Duffie and Epstein (1992) recursive utility over consumption. Investors have two assets available for investment, a riskless asset with constant return and a risky asset ("stocks") with constant expected return and time-varying return volatility.[1] (In an extension of the model, we allow the expected excess return on stocks to be an affine function of volatility.)

For mathematical convenience, we work with precision, the reciprocal of volatility, and assume that it follows a mean-reverting, square-root process which is instantaneously correlated with stock returns.[2] This implies a process for volatility that inherits the properties of the process for precision and captures the main stylized empirical facts about stock market volatility. In particular, we allow for imperfect instantaneous correlation between volatility and stock returns in the model, and work in an incomplete markets setting.

Under these assumptions, we derive analytic expressions for the optimal consumption and portfolio policies which are exact when investors have unit elasticity

[1] The term "volatility" is somewhat vague, and it is used in the literature sometimes as meaning "variance" and sometimes as meaning "standard deviation." Throughout this paper, though, when we use the term "volatility," we mean "variance."

[2] Portfolio problems require very often working with precision rather than with volatility itself. One example is the mean-variance allocation to risky assets, which is linear in precision.

of intertemporal substitution of consumption, and approximate otherwise. We use this model to empirically evaluate the importance of volatility risk for intertemporal hedging in the US stock market, using estimates of the process for stock returns and volatility based on monthly returns from 1926 to 2000, and annual returns from 1871 to 2000.

Our solution contributes to recent research that has expanded significantly the set of known exact analytical solutions to continuous-time intertemporal portfolio choice problems with time-varying investment opportunities. This research has provided solutions in settings where markets are incomplete, but constraining utility to be defined over terminal wealth (Kim and Omberg 1996, Wachter 2002); and in settings where investors have utility over intermediate consumption, but constraining markets to be complete (Brennan and Xia 2001, Wachter 2000, Schroder and Skiadas 1999, and Fisher and Gilles 1998). This paper provides an exact solution for the case of utility defined over intermediate consumption which does not require assuming that markets are complete.

This exact solution requires though that investors have unit elasticity of intertemporal substitution of consumption. This assumption is difficult to justify on empirical grounds, because the existing estimates of this elasticity from aggregate and disaggregate data are well below one (Hall 1988, Campbell and Mankiw 1989, Campbell 1999, Vissin-Jorgensen 2001). However, our calibration exercise suggests that this assumption is not particularly constraining if one is interested only in dynamic portfolio choice. This exercise shows that optimal portfolio allocations are very similar across a wide range of values for the elasticity of intertemporal substitution of consumption. Working in discrete time, Campbell and Viceira (1999, 2001, 2002) and Campbell, Chan and Viceira (2002) also reach similar conclusions in their analysis of optimal consumption and portfolio choice with time variation in expected returns and interest rates.[3]

In two papers closely related to ours, Liu (2000, 2001) examines the optimal allocation to stocks when stock return volatility is stochastic.[4] Both papers provide exact analytical solutions in an incomplete markets setting for investors with power utility defined over terminal wealth, and specifications of stochastic volatility which are slightly different from the ones in this paper. Liu (2000) considers the Heston (1993) and Stein and Stein (1991) models of stochastic volatility, in which volatility follows a mean-reverting process and stock returns are a linear function of volatility. These models imply a Sharpe ratio of stocks that is increasing in the square root of volatility, and a ratio between expected stock excess returns and stock return volatility—the mean-variance allocation to stocks—that is constant. Our model where we assume that expected stock returns are an affine function of volatility have similar implications for the Sharpe ratio and the mean-variance allocation to stocks in the special case where we constrain the intercept of the affine function to be zero. Liu (2000) also considers a model that includes both interest rate risk and volatility risk. A calibration of this model to US data arrives at conclusions similar to ours regarding the relatively modest size of intertemporal hedging demands generated by volatility risk. Finally, Liu (2001) considers a general class of stochastic volatility models that

[3]Their analytical solutions are also exact for investors with unit elasticity of intertemporal substitution of consumption, up to a discrete-time approximation to the log return on wealth.

[4]Lynch and Balduzzi (2000) have also addressed tangentially the implications of time-varying volatility for portfolio choice in their study of optimal portfolio rebalancing with stock return predictability and transaction costs. They find that allowing for return heteroskedasticity can have important effects on the optimal portfolio rebalancing behavior of long-horizon investors.

nests our basic specification with constant expected returns.

The paper is organized as follows. Section 2 states the dynamic optimization problem, Section 3 presents an exact solution to the problem in the case with unit elasticity of intertemporal substitution. Section 3 also presents some comparative statics results. Section 4 explains the continuous-time approximate solution method that allows us to solve the problem when the elasticity of intertemporal substitution differs from unity, and states the solution implied by the method. Section 5 explores the solution to the problem when expected excess returns are an affine function of volatility. Section 6 calibrates the model to monthly U.S. stock market data and explores the empirical implications of stochastic volatility for portfolio choice. Section 7 discusses some alternative approximate solution and issues related to the accuracy of the approximate analytical solution. Finally, Section 8 concludes.

# 2 The Intertemporal Consumption and Portfolio Choice Problem

## 2.1 Investment opportunity set

We assume that wealth consists of only tradable assets. Moreover, to keep the analysis simple, we assume in this paper that there are only two tradable assets. One of the assets is riskless, with instantaneous return

$$\frac{dB_t}{B_t} = r dt.$$

The second asset is risky, with instantaneous total return dynamics given by

$$\frac{dS_t}{S_t} = \mu dt + \sqrt{\frac{1}{y_t}} dW_s, \tag{1}$$

where $S_t$ is the value of a fund fully invested in the asset that reinvests all dividends, and $y_t$ is the instantaneous precision of the risky asset return process—and $1/y_t$ is the instantaneous variance.

Equation (1) implies that the expected excess return on the risky asset over the riskless asset $(\mu - r)$ is constant over time—we relax this assumption in Section 5. However, the conditional precision of the risky asset return varies stochastically over time, and this induces time variation in investment opportunities. We assume the following dynamics for instantaneous precision:

$$dy_t = \kappa(\theta - y_t)dt + \sigma\sqrt{y_t}dW_y. \tag{2}$$

Precision follows a mean-reverting, square-root process with reversion parameter $\kappa > 0$, with $\mathrm{E}[y_t] = \theta$ and $\mathrm{Var}(y_t) = \sigma^2\theta/2\kappa$ (Cox, Ingersoll, and Ross, 1985). In order to satisfy standard integrability conditions, we assume that $2\kappa\theta > \sigma^2$.

The stochastic process for precision implies a mean-reverting process for the instantaneous volatility $v_t \equiv 1/y_t$. Applying Ito's Lemma to (2) we find that proportional changes in volatility follow a mean-reverting, square-root process:

$$\frac{dv_t}{v_t} = \kappa_v \left(\theta_v - v_t\right) dt - \sigma\sqrt{v_t}dW_y, \tag{3}$$

where $\theta_v = (\theta - \sigma^2/\kappa)^{-1}$ and $\kappa_v = \kappa(\theta - \sigma^2/\kappa) \equiv \kappa/\theta_v$. It is convenient to note here that the unconditional mean of instantaneous volatility is approximately equal to:

$$\mathrm{E}[v_t] \approx \frac{1}{\theta} + \frac{1}{2}\frac{\sigma^2}{\theta^2\kappa} = \frac{1}{\theta} + \frac{\mathrm{Var}\left(y_t\right)}{\theta^3}. \tag{4}$$

This follows from taking expectations of a second-order Taylor expansion of $v_t \equiv 1/y_t$ around $\theta$. Since we have assumed that the expected return on the risky asset is constant, equation (4) is also the unconditional variance of the risky asset return.[5]

We also assume throughout the paper that the shocks to precision are correlated with the instantaneous return on the risky asset, with $dW_y dW_S = \rho dt$. This in turn implies that proportional changes in volatility are correlated with stock returns, with instantaneous correlation given by

$$\mathrm{Corr}_t\left(\frac{dv_t}{v_t}, \frac{dS_t}{S_t}\right) = -\,\mathrm{Corr}_t\left(dy_t, \frac{dS_t}{S_t}\right) = -\rho dt.$$

This model for stock returns and precision or volatility can capture the main stylized empirical facts about stock return volatility, in particular its mean-reversion and negative correlation with stock returns. It also implies that proportional changes in volatility are more pronounced in times of high volatility than in times of low volatility.

Another important implication of this model of changing risk is that the ratio of the expected excess return on the risky asset to its variance is a linear function of the state variable. This model assumption greatly facilitates solving the dynamic optimization problem that we present below. It is important however, to remark that the Sharpe ratio of the risky asset in this model is not a linear function of the

[5]We have performed a Monte Carlo experiment to corroborate this assertion and the quality of the approximation (4). Using the monthly parameter estimates of this process shown in Table I, we have generated 10,000 time series of the process (1)-(2), each 30 years in length, with a time step $dt = 0.01$ (or about 3 days). This experiment shows that the unconditional variance of stock returns is indeed given by the unconditional mean of volatility, and that the approximation (4) is fairly precise—in our experiment, it underestimates the true variance by 0.27%.

state variable, but a square-root function. Thus this model is not mathematically equivalent to a model where volatility is constant and the expected excess return on the risky asset changes stochastically in a mean-reverting fashion, as in Kim and Omberg (1996) or Campbell and Viceira (1999).

## 2.2 Investor preferences and dynamic optimization problem

Investor's preferences are described by a recursive utility function, a generalization of the standard, time-separable power utility model that separates relative risk aversion from the elasticity of intertemporal substitution of consumption. Epstein and Zin (1989, 1991) derive a parameterization of recursive utility in a discrete-time setting, while Duffie and Epstein (1992a, 1992b) and Fisher and Gilles (1998) offer a continuous-time analogue. We adopt the Duffie and Epstein (1992b) parameterization:

$$J = \mathrm{E}_t \left[ \int_t^{\infty} \mathrm{f}\left(C_s, J_s\right) ds \right], \tag{5}$$

where $\mathrm{f}(C_s, J_s)$ is a normalized aggregator of current consumption and continuation utility that takes the form

$$\mathrm{f}\left(C, J\right) = \frac{\beta}{1-\frac{1}{\psi}} \left(1-\gamma\right) J \left[ \left( \frac{C}{\left(\left(1-\gamma\right) J\right)^{\frac{1}{1-\gamma}}} \right)^{1-\frac{1}{\psi}} - 1 \right], \tag{6}$$

$\beta > 0$ is the rate of time preference, $\gamma > 0$ is the coefficient of relative risk aversion and $\psi > 0$ is the elasticity of intertemporal substitution. Power utility obtains from (6) by setting $\psi = 1/\gamma$.

The normalized aggregator $\mathrm{f}(C_s, J_s)$ takes the following form when $\psi \rightarrow 1$:

$$\mathrm{f}\left(C, J\right) = \beta \left(1-\gamma\right) J \left[ \log\left(C\right) - \frac{1}{1-\gamma} \log\left(\left(1-\gamma\right) J\right) \right]. \tag{7}$$

The investor maximizes (5) subject to the intertemporal budget constraint

$$dX_t = [\pi_t(\mu - r)X_t + rX_t - C_t]dt + \pi_t X_t \sqrt{\frac{1}{y_t}} dW_s, \tag{8}$$

where $X_t$ represents the investor's wealth, $\pi_t$ is the fraction of wealth invested in the risky asset and $C_t$ represents the investor's instantaneous consumption.

# 3 An Exact Solution with Unit Elasticity of Intertemporal Substitution of Consumption

Building on the work of Merton (1969, 1971, 1973), Giovannini and Weil (1989), Campbell and Viceira (1999, 2001), and Campbell, Chan, and Viceira (2002), we show in this section that it is possible to find an exact solution to the intertemporal optimization problem (5)-(8) when investors have unit elasticity of intertemporal substitution of consumption. In Section 4 we present an approximate analytic solution for the general case in which $\psi$ is not restricted to one.

## 3.1 Bellman equation

The optimization problem given by (5)-(8) has one state variable, the precision of the risky asset return or, equivalently, the volatility of the risky asset return. Therefore, the value function of the problem ($J$) depends on financial wealth ($X_t$) and this state variable.

The Bellman equation for this problem is

$$
\begin{aligned}
0 &= \sup_{\pi,C}\Bigg\{ \mathrm{f}\left(C_s,J_s\right)+\left[\pi_t(\mu-r)X_t+rX_t-C_t\right]J_X+\frac{1}{2}\pi^2X_t^2J_{XX}\frac{1}{y_t}+\kappa(\theta-y_t)J_y \\
&\quad +\frac{1}{2}\sigma^2J_{yy}y_t+\rho\sigma\pi_tX_tJ_{Xy}\Bigg\}, \qquad (9)
\end{aligned}
$$

where $\mathrm{f}\left(C,J\right)$ is given in (7) and subscripts on $J$ denote partial derivatives.

The first-order conditions for this equation are

$$
\begin{aligned}
C_t &= \beta\left(1-\gamma\right)\frac{J}{J_X}, \qquad (10)\\
\pi_t &= -\frac{J_X}{X_t\,J_{XX}}\left(\mu-r\right)y_t-\frac{J_{Xy}}{X_tJ_{XX}}\rho\sigma y_t. \qquad (11)
\end{aligned}
$$

Equation (10) shows the optimal consumption rule. It results from the envelope condition, $\mathrm{f}_C=J_X$, from which the optimal consumption rule obtains once the value function is known. Equation (11) shows the optimal portfolio share in the risky asset. Note, however, that equations (10) and (11) do not represent a complete solution to the model until we solve for $J(X_t,y_t)$.

Substituting the first-order conditions into (9) and rearranging gives the Bellman equation:

$$
\begin{aligned}
0 &= \mathrm{f}\left(C\left(J\right),J\right)-J_XC\left(J\right)-\frac{1}{2}\frac{\left(J_X\right)^2}{J_{XX}}\left(\mu-r\right)^2y_t-\frac{J_XJ_{Xy}}{J_{XX}}\rho\sigma\left(\mu-r\right)y_t \\
&\quad +J_XX_tr-\frac{1}{2}\frac{\left(J_{Xy}\right)^2}{J_{XX}}\rho^2\sigma^2y_t+J_y\kappa(\theta-y_t)+\frac{1}{2}J_{yy}\sigma^2y_t, \qquad (12)
\end{aligned}
$$

where $C(J)$ denotes the expression for consumption resulting from (10).

We now guess a solution of the form

$$
J(X_t,y_t)=I\left(y_t\right)\frac{X_t^{1-\gamma}}{1-\gamma}. \qquad (13)
$$

Substituting this solution into the Bellman equation and simplifying yields an ordinary differential equation (ODE) that has a solution of the form

$$I = \exp\left\{Ay_t + B\right\}. \tag{14}$$

Substitution of (14) into the simplified ODE leads to two algebraic equations for $A$ and $B$:

$$aA^2 + bA + c = 0, \tag{15}$$

$$(1-\gamma)\left(\beta \log \beta + r - \beta\right) - \beta B + \kappa\theta A = 0, \tag{16}$$

where

$$a = \frac{\sigma^2}{2\gamma\left(1-\gamma\right)}\left[\gamma\left(1-\rho^2\right)+\rho^2\right], \tag{17}$$

$$b = \frac{\rho\sigma(\mu - r)}{\gamma} - \frac{\beta+\kappa}{1-\gamma}, \tag{18}$$

$$c = \frac{(\mu-r)^2}{2\gamma}. \tag{19}$$

The first equation is a quadratic equation in $A$, and the second equation is linear in $B$ given $A$. For general parameter values, the equation for $A$ has two roots. We show in Appendix A that these roots are always real and have opposite signs when $\gamma > 1$, and they are real and have the same sign when $\gamma < 1$ provided that

$$\gamma > \frac{\sigma(\mu-r)\left(2\rho+\sigma(\mu-r)/\left(\beta+\kappa\right)\right)}{\left(\beta+\kappa\right)+\sigma(\mu-r)\left(2\rho+\sigma(\mu-r)/\left(\beta+\kappa\right)\right)}. \tag{20}$$

This condition also implies that both roots are positive.

The existence of real roots is a necessary (but not sufficient) condition for the existence of a solution to the problem. We still need to determine which root delivers the correct solution to the model. We show in Appendix A that only one of these roots ensures that the limit of the solution as $\gamma \to 1$ equals the well-known solution in the special case of log utility ($\gamma = \psi = 1$), for which $A = B = 0$, and the value function is simply $\log(X_t)$ (Merton, 1969, 1971, 1973).

Convergence to the known solution is obtained by selecting the root associated with the positive root of the discriminant of the quadratic equation (15) when $\gamma > 1$, and the negative root of the discriminant when $\gamma < 1$. This selection implies that $A < 0$ when $\gamma > 1$, and $A > 0$ when $\gamma < 1$; or, equivalently, that $A/(1-\gamma) > 0$. The alternative selection leads to a solution that diverges from the known solution in this special case.[6]

## 3.2 Optimal policies

We now state the complete solution, and discuss some of its most important properties:

**Proposition 1** *When $\psi = 1$, there is an exact analytical solution to problem (5)-(8) with value function given by*

$$J\left(X_t, y_t\right) = \exp\left\{Ay_t + B\right\} \frac{X_t^{1-\gamma}}{1-\gamma}. \tag{21}$$

[6] We would like to show that the alternative selection also implies that the unconditional expectation of the value function is not bounded, hence violating the standard transversality condition. However, we have not been able to show this analytically, because there is no closed form expression for the expectation of a exponential function of a square root process.

*This value function implies the following optimal consumption and portfolio rules:*

$$\frac{C_t}{X_t} = \beta, \tag{22}$$

*and*

$$\pi_t = \frac{1}{\gamma}(\mu - r)\, y_t + \left(1 - \frac{1}{\gamma}\right)(-\rho)\, \sigma \mathcal{A} y_t, \tag{23}$$

*where* $\mathcal{A} \equiv A/(1-\gamma) > 0$*, and* $A$ *and* $B$ *are given by the solution to the system of equations (15)-(16).*

**Proof**. The value function and its coefficients follow immediately from (**??**) and the ensuing discussion. The optimal policies follow immediately from direct substitution of the value function (21) and its derivatives into the first order conditions (10) and (11).■

Proposition 1 shows that for investors with unit elasticity of intertemporal substitution, the optimal log consumption-wealth ratio is invariant to changes in volatility and it is equal to their rate of time preference. For these investors, the income and substitution effects on consumption produced by a change in the investment opportunity set exactly cancel out, and it is optimal for them to consume a fixed fraction of her wealth each period. For this reason this consumption policy is usually termed "myopic."

Equation (23) shows the optimal portfolio rule. This rule has two components. The first component is myopic (or mean-variance) portfolio demand. The second component is Merton's intertemporal hedging demand. Both components are linear functions of precision, which implies that their ratio is independent of the current level of precision or volatility. This is the result of returns being instantaneously correlated with proportional changes in volatility rather than with absolute changes in volatility.

Inspection of equation (23) shows that intertemporal hedging demand is always zero—and myopic demand optimal—in three cases: when investment opportunities are constant ($\sigma = 0$); when they are time-varying, but investors cannot use the risky asset to hedge changes in investment opportunities ($\rho = 0$); when investors have unit coefficients of relative risk aversion ($\gamma = 1$). In those cases multiperiod investors behave as if they were facing a series of identical one-period problems (Merton 1969, 1971, 1973, Giovannini and Weil 1989). This is why the first component of the optimal portfolio rule is usually termed "myopic demand."

In all other cases, intertemporal hedging demand is not necessarily zero. It depends on all the parameters that characterize investor preferences and the investment opportunity set. In particular, its sign is a function of the sign of the correlation between unexpected returns and changes in volatility ($-\rho$) and the sign of $(1 - 1/\gamma)$. When this correlation is negative ($-\rho < 0$), intertemporal hedging demand is negative for investors with $\gamma > 1$, and positive for investors with $\gamma < 1$. Investors who are more risk averse than logarithmic investors have a negative hedging demand for the risky asset because it tends to do worse when there is an increase in risk. On the other hand, investors who are more aggressive than logarithmic investors have a positive intertemporal hedging demand for the risky asset; they are willing to trade off worse performance when volatility is high for extra performance when volatility is low.

We have noted that intertemporal hedging demand is zero when $\rho = 0$. It is also zero when investors are infinitely risk averse. This follows from the fact that

$\lim_{\gamma \to \infty} \mathcal{A} = 0$. For these investors, the optimal overall allocation to the risky asset is zero, since the myopic component of portfolio demand is also zero when $\gamma \to \infty$.

Finally, it is worth noting here that we can use the explicit solution for the optimal policies given in Proposition 1 to examine the effect on intertemporal hedging demand of changes in the parameters that determine the process for precision, particularly $\sigma$, $\kappa$ and $\rho$. We perform these comparative statics exercises in Section 6.

# 4 An Approximate Solution for the General Case

We now address the general case, where the investor's elasticity of intertemporal substitution of consumption can take any value. The general case is interesting for two reasons. First, it is empirically relevant, since estimates of $\psi$ available from both aggregate data and disaggregate data on individual investors suggest that $\psi$ is below one (Hall 1988, Campbell and Mankiw 1989, Campbell 1999, Vissin-Jorgensen 2001). Second, it nests as a special case the time-additive power utility case standard in the literature. Since $\gamma = 1/\psi$ with power utility, the $\psi = 1$ case does not nest power utility unless we restrict ourselves to the special case of log utility—where $\gamma = 1/\psi = 1$.

Unfortunately, there is no exact analytical solution to the model in the general case. However, we show in this section that we can still find an approximate analytical solution to the problem. This solution provides strong economic intuition about the nature of optimal portfolio choice with time-varying risk, and converges to an exact solution in those special cases such a solution is known. We argue in section 7 that, for all other cases, it is reasonably accurate.

## 4.1 Bellman equation and approximation

When $\psi$ is not restricted to one, the Bellman equation for the problem is still given by equation (10). The first order condition for portfolio choice is still given by (11), but the first order condition for consumption resulting from the envelope condition $\mathrm{f}_C = J_X$ is different, because the aggregator takes a different form, given in (6). The first order condition for consumption is now given by:

$$C_t = J_X^{-\psi} \left[ (1-\gamma) J \right]^{\frac{1-\gamma\psi}{1-\gamma}} \beta^{\psi}. \tag{24}$$

After plugging (11) and (24) into the Bellman equation (12), guessing that $J(X_t, y_t) = I(y_t) X_t^{1-\gamma}/(1-\gamma)$, and making the transformation $I = H^{-\frac{1-\gamma}{1-\psi}}$, we obtain the following *non-homogeneous* ODE:

$$\begin{aligned} 0 \;=\; & -\beta^{\psi} H^{-1} + \psi\beta + \frac{(1-\psi)(\mu - r)^2}{2\gamma} y_t - \frac{\rho\sigma(\mu - r)(1-\gamma)}{\gamma} \frac{H_y}{H} y_t \\ & + r(1-\psi) + \frac{\rho^2\sigma^2(1-\gamma)^2}{2\gamma(1-\psi)} \left( \frac{H_y}{H} \right)^2 y_t - \frac{H_y}{H} \kappa(\theta - y_t) \\ & + \frac{\sigma^2}{2} \left( \frac{1-\gamma}{1-\psi} + 1 \right) \left( \frac{H_y}{H} \right)^2 y_t - \frac{\sigma^2}{2} \frac{H_{yy}}{H} y_t. \end{aligned} \tag{25}$$

Unfortunately, equation (25) is a non-linear ODE in $H$ whose analytical solution is unknown except in three special cases. The first two cases are well-known from Merton's (1969, 1971, 1973) work, and correspond to log utility ($\gamma = \psi \equiv 1$) and constant investment opportunities ($\kappa, \sigma = 0$).

The third case corresponds to power utility and perfect instantaneous correlation of the state variable with the risky asset return—so that markets are complete.[7] This

[7]With $|\rho| = 1$, equation (25) becomes a non-homogeneous version of the Gauss' hypergeomet-

case has also been explored by Wachter (1998) and Liu (1998). Unfortunately, the assumption of perfect correlation between changes in volatility and asset returns is not empirically plausible. For example, in Section 6 we estimate that for the US market this correlation is large, but still far from perfect. This suggests that we should consider the general case.

In the general case, the nonlinear ODE (25) has no exact analytical solution. Nevertheless, it is still possible to find an approximate analytic solution based on a log-linear expansion of the consumption-wealth ratio around its unconditional mean. Campbell (1993), Campbell and Viceira (1999, 2001), and Campbell, Chan, and Viceira (2002) have used an identical approximation to solve for optimal intertemporal portfolio and consumption problems. However, while they work in discrete-time and use the approximation to linearize the log budget constraint, we work here in continuous-time and use it to linearize the Bellman equation. We can view this approach as a particular class of the perturbation methods of approximation described in Judd (1998), where the approximation takes place around a particular point in the state space—the unconditional mean of the log consumption-wealth ratio.

We start by noting that the envelope condition (10) implies

$$\beta^{\psi} H^{-1} = \exp\left\{c_t - x_t\right\},$$

where $c_t - x_t = \log\left(C_t / X_t\right)$. Therefore, using a first-order Taylor expansion of $\exp\{c_t - x_t\}$ around $\mathrm{E}[c_t - x_t] \equiv (\overline{c - x})$ we can write

$$\beta^{\psi} H^{-1} \approx h_0 + h_1\left(c_t - x_t\right), \tag{26}$$

ric ODE, which has a closed-form solution in terms of the confluent hypergeometric function (see Polyanin and Zaitsev 1995, p.143). Unfortunately this solution has a rather abstruse mathematical form, from which it is very dificult to obtain any useful economic insights.

where $h_1 = \exp\{\overline{c - x}\}$, and $h_0 = h_1(1 - \log h_1)$.

Substituting (26) for $\beta^{\psi} H^{-1}$ in the first term of (25), it is easy to see that the resulting ODE has a solution of the form $H = \exp\{A_1 y_t + B_1\}$. This solution implies a value function of the form

$$J(X_t, y_t) = \exp\left\{-\left(\frac{1-\gamma}{1-\psi}\right)(A_1 y_t + B_1)\right\} \frac{X_t^{1-\gamma}}{1-\gamma}. \tag{27}$$

The approximate ODE leads to two algebraic equations for $A_1$ and $B_1$:

$$a_1 A_1^2 + b_1 A_1 + c_1 = 0, \tag{28}$$

$$h_0 - h_1 [B_1 - \psi \log \beta] - \psi\beta - r(1-\psi) + \kappa\theta A_1 = 0, \tag{29}$$

where

$$a_1 = \frac{\sigma^2}{2\gamma}\left(\frac{1-\gamma}{1-\psi}\right)\left[\gamma\left(1-\rho^2\right)+\rho^2\right], \tag{30}$$

$$b_1 = (h_1 + \kappa) - \frac{(1-\gamma)\rho\sigma(\mu - r)}{\gamma}, \tag{31}$$

$$c_1 = \frac{(1-\psi)(\mu - r)^2}{2\gamma}. \tag{32}$$

The analysis of the quadratic equation (28) for $A_1$ is parallel to the analysis of the quadratic equation (15) for $A$ in the $\psi = 1$ case, so that we simply state here the properties of $A_1$ derived from this analysis. First, $A_1/(1-\psi)$ is independent of $\psi$ given $h_1$. Second, comparison of equations (30)-(32) with equations (17)-(19) shows that $-A_1/(1-\psi)$ and $A/(1-\gamma)$ are non-negative identical functions of $h_1$ and $\beta$, respectively.[8] Third, when $\gamma > 1$, the discriminant of equation (28) is always

[8] This equivalence is also apparent from comparing the value function (21) in the $\psi = 1$ case and the value function (27) in the general case.

positive and the roots of the equation are real and have opposite sign; when $\gamma < 1$, the discriminant can have either sign but, if it is positive, the roots of the equation are real and have the same sign. However, only the positive square root of the discriminant ensures in both cases that the approximate solution approaches the exact solution when $\psi = 1$.

## 4.2 Optimal policies

We now state the approximate solution in the following proposition:

**Proposition 2** *When $\psi \neq 1$, there is an approximate analytical solution to problem (5)-(8) with value function given by (27). The optimal consumption and portfolio rules implied by this value function are*

$$\frac{C_t}{X_t} = \beta^{\psi} \exp\left\{-A_1 y_t - B_1\right\}, \tag{33}$$

*and*

$$\pi_t = \frac{1}{\gamma}(\mu - r)\, y_t + \left(1 - \frac{1}{\gamma}\right)(-\rho)\, \sigma \mathcal{A}_1 y_t, \tag{34}$$

*where $\mathcal{A}_1 \equiv -A_1/(1-\psi) > 0$, and $A_1$ and $B_1$ are given by the solution to the system of equations (28)-(29). $\mathcal{A}_1$ does not depend on $\psi$ except through the loglinearization coefficient $h_1$, and it reduces to $\mathcal{A}$ in Proposition 1 when $h_1 = \beta$.*

**Proof**. The value function follows immediately from (25), (26) and (28)-(29). The optimal policies follow from (27) and the first order conditions (11) and (24). We have already shown above that $\mathcal{A}_1 \equiv -A_1/(1-\psi)$ and $\mathcal{A} \equiv A/(1-\gamma)$ are non-negative identical functions of $h_1$ and $\beta$, respectively. ■

The approximate solution depends on the loglinearization coefficient $h_1$, which is itself endogenous. However, Proposition 2 shows that we can still derive a number of properties of the solution without solving explicitly for $h_1$, using the fact that it lies between zero and one. We now comment on some of these properties, and leave for Section 6 the description of a simple procedure to compute numerical values for $h_1$ and the optimal policies.

Proposition 2 shows that the optimal log consumption-wealth ratio is an affine function of instantaneous precision. Since $A_1/(1-\psi) < 0$, the consumption-wealth ratio is a decreasing monotonic function of volatility for investors whose intertemporal elasticity of consumption $\psi$ is smaller than one, while it is an increasing function of volatility for investors whose elasticity is larger than one.

This property reflects the comparative importance of intertemporal income and substitution effects of volatility on consumption. To understand this, consider the effect on consumption of an unexpected increase in volatility. This increase implies a deterioration in investment opportunities, because returns on the risky asset are now more volatile, while its expected return is the same.

A deterioration in investment opportunities creates a positive intertemporal substitution effect on consumption—because the investment opportunities available are not as good as they are at other times—but also a negative income effect—because increased uncertainty increases the marginal utility of consumption. For investors with $\psi < 1$, the income effect dominates the substitution effect and they reduce their current consumption relative to wealth. For investors with $\psi > 1$, the substitution effect dominates, and they increase their current consumption relative to wealth.

Proposition 2 also characterizes optimal portfolio demand in the general case. This proposition implies that optimal portfolio demand in the $\psi \neq 1$ case is qualitatively analogous to optimal portfolio demand in the $\psi = 1$ case. This follows immediately from direct comparison of equations (34) and (23). These equations are identical, except for the positive coefficients $\mathcal{A}_1$ and $\mathcal{A}$. Section 6 shows that, for empirically plausible characterizations of the process for precision, these coefficients are very close, which implies that the effect of $\psi$ on optimal portfolio choice is quantitatively small. Campbell and Viceira (1999, 2001) and Campbell, Chan and Viceira (2002) show a similar result in models with time variation in risk premia and interest rates.

Finally, we want to note that an important feature of the approximate solution is that it delivers the exact expression for the optimal policies in the special cases of log utility ($\gamma = \psi \equiv 1$), unit elasticity of intertemporal substitution, and constant investment opportunities ($\kappa$, $\sigma = 0$ and $v_t \equiv v$). Appendix B shows this convergence result.

# 5 Consumption and Portfolio Choice When Expected Excess Returns Covary with Volatility

The analysis of optimal consumption and portfolio choice with time-varying risk in Sections 3 and 4 assumes that expected excess returns are constant. A natural extension of this analysis is to replace the assumption of constant expected excess returns with one that allows expected excess returns to vary with volatility:

$$\mathrm{E}_t\left[\frac{dS_t}{S_t} - r dt\right] = (\alpha_1 + \alpha_2 v_t)\, dt = \left(\alpha_1 + \alpha_2 y_t^{-1}\right) dt. \tag{35}$$

When $\alpha_2 > 0$, equation (35) implies increases in risk are rewarded with increases in expected excess returns. This model also nests the model in Section 2, which obtains when $\alpha_2 = 0$, and $\alpha_1 = \mu - r$.

To derive the optimal policies under this new assumption we follow the same method as in Section 4. We describe here the main steps of the derivation, and provide a detailed analysis in Appendix D. Guessing the same functional forms for $J(X_t, y_t)$ and $I(y_t)$ as in Section 4, the Bellman equation for this problem simplifies to an ODE in $H(y_t)$. This ODE has a closed form solution, provided that we make the approximation $\beta^{\psi} H^{-1} \approx h_0 + h_1(c_t - x_t)$.[9]

The solution takes the form $H = \exp\{A_1 y_t + A_2 \log y_t + B_2\}$, which implies a value function of the form

$$J(X_t, y_t) = \exp\left\{-\left(\frac{1-\gamma}{1-\psi}\right)(A_1 y_t + A_2 \log y_t + B_2)\right\}\frac{X_t^{1-\gamma}}{1-\gamma},$$

where $A_1$ and $A_2$ solve two independent quadratic equations and $B_2$ solves an equation which is linear, given $A_1$ and $A_2$.

Proposition 3 shows the optimal consumption and portfolio rules implied by this value function:

**Proposition 3** *The optimal consumption and portfolio rules when* $\mathrm{E}_t[(dS_t/S_t) - rdt] = (\alpha_1 + \alpha_2 v_t)dt = (\alpha_1 + \alpha_2/y_t)dt$ *are*

$$\frac{C_t}{X_t} = \beta^{\psi}\exp\{-A_1 y_t - A_2 \log y_t - B_2\}, \tag{36}$$

[9] When we substitute $h_0 + h_1(c_t - x_t)$ for $\beta^{\psi} H_t^{-1}$ in the Bellman equation, we still need to do a further approximation of $-\log y_t = \log v_t$ around its conditional mean.

*and*

$$\pi_t = \frac{1}{\gamma}(\alpha_1 y_t + \alpha_2) + \left(1 - \frac{1}{\gamma}\right)(-\rho)\,\sigma\,(\mathcal{A}_1 y_t + \mathcal{A}_2)\,, \tag{37}$$

*where* $\mathcal{A}_1 \equiv -A_1/(1-\psi) > 0$*, and* $\mathcal{A}_2 \equiv -A_2/(1-\psi) < 0$*.* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *do not depend on* $\psi$*, except through the loglinearization constant* $h_1$*. Moreover,* $\mathcal{A}_1$ *is mathematically identical to* $\mathcal{A}_1$ *in Proposition 2, with* $\alpha_1 = \mu - r$*. Thus it does not depend on* $\alpha_2$*, except through* $h_1$*.* $\mathcal{A}_2$ *does not depend on* $\alpha_1$*.*

**Proof**. See appendix C.■

Proposition 3 shows that the myopic component and the intertemporal hedging component of portfolio demand are both affine functions of precision—not simply linear functions of precision, as in the case with constant expected returns. Thus total portfolio demand is itself an affine function of precision. The slope of total portfolio demand is mathematically identical to the optimal portfolio rule in the case with constant expected returns—with $\alpha_1$ replacing $\mu - r$. It captures essentially the effect on portfolio choice of changes in volatility that are not rewarded by corresponding changes in expected excess returns.

The intercept of the optimal portfolio rule captures the additional effects caused by the fact that now a unit shift in volatility changes stock expected excess returns by $\alpha_2$ units. It is interesting to note that the magnitude of the intercept depends on $\alpha_2$, but its sign is independent of the sign of $\alpha_2$. To gain some intuition on why the sign of $\alpha_2$ is irrelevant for intertemporal hedging, consider myopic portfolio demand when $\alpha_1 = 0$. In that case, the myopic portfolio is long in stocks when $\alpha_2 > 0$, and short when $\alpha_2 < 0$, and it has a expected excess return equal to $\alpha_2^2 v_t/\gamma$: Shocks to volatility always drive the expected excess return on the myopic portfolio in the same direction, regardless of their impact on stock expected excess returns. Thus negative shocks to volatility always represent a worsening in investment opportunities. Equation (37) with $\alpha_1 = 0$ shows that whether this leads to a positive or a negative intertemporal hedging demand for stocks depends on the sign of $-\rho$ and $(1 - 1/\gamma)$. In particular, when $-\rho < 0$, an investor with $\gamma > 1$ will have a positive intertemporal hedging demand for the risky asset, because it tends to pay when investment opportunities worsen and the marginal utility of consumption is high.

# 6 Optimal Consumption and Portfolio Choice with Stochastic Volatility: The U.S. Experience

## 6.1 Parameter values

This section examines the implications for optimal portfolio choice and consumption of the patterns in volatility observed in the U.S. stock market. We start with an estimation of the continuous-time process (1)-(2) using the Spectral Generalized Method of Moments of Chacko and Viceira (2003), Jiang and Knight (2000) and Singleton (2000), which we describe in Appendix D.

We provide two sets of parameter estimates. The first set is based on monthly excess stock returns on the CRSP value-weighted portfolio over the T-bill rate from January 1926 through December 2000. The second set is based on annual excess equity returns on the Standard and Poor Composite Stock Price Index over the prime commercial paper rate from 1871 through 2000.[10] In both datasets, stock returns are inclusive of dividends. In our calibration exercises we set the riskless rate at 1.5% per year.

Table I reports Spectral GMM parameter estimates and their standard errors. Standard errors are bootstrapped, and parameter estimates are annualized to facilitate their interpretation. The estimates of both the unconditional mean of excess returns and precision have low standard errors in both samples. However, the estimates of the rest of the parameters—particularly the reversion parameter—are less precise.

These estimates imply a mean excess return around 8% per year in both samples and, using the approximate expression of the unconditional variance of stock returns given in (4), an unconditional standard deviation of returns of almost 20% per year in the monthly sample, and about 25% per year in the annual sample. The instantaneous correlation between shocks to volatility and stock returns ($-\rho$) is negative and relatively large—about $-53\%$ in the monthly sample and about $-37\%$ in the annual sample.

The estimate of the reversion parameter $\kappa$ in the precision equation implies a half-life of a shock to precision of about 2 years in the monthly sample. The rate of mean reversion is slower in the annual sample, where the estimate of the half-life of a shock to precision is about 16 years. French, Schwert and Stambaugh (1987), Schwert (1989), and Campbell and Hentschel (1990) have also found a relatively slow speed of adjustment of shocks to stock volatility in low frequency data. This slow reversion to

[10] This is an updated version of Shiller (1989) lont-term stock return data, which is publicly available at Robert Shiller's web home page at [http://www.econ.yale.edu/~shiller/].

the mean in low frequency data contrasts with the fast speed of adjustment detected in high frequency data by Andersen, Benzoni and Lund (1998).

These results suggest that there might be high frequency and low frequency (or long-memory) components in stock market volatility.[11] By construction, the single component model (1)-(2) cannot capture these components simultaneously. On the other hand, it is very difficult to find analytical solutions for a model with multiple components in volatility. We hope that by focusing on estimates of the single component model derived from low frequency data, we can capture the persistence and variability characteristics of the volatility process that are most relevant to long-term investors. Accordingly, in our calibration exercise we focus on the monthly and annual estimates of the single component model.

## 6.2 Calibration results

The optimal portfolio choice and consumption rules given in equations (33) and (34) depend on the loglinearization coefficient $h_1$. Since $h_1$is itself endogenous—it is the exponentiated mean log consumption-wealth ratio—, evaluating these expressions requires solving for $h_1$ simultaneously. To this end we use a simple recursive procedure. We take an initial value of $h_1$, solve for the corresponding optimal consumption-wealth ratio (33), and use this consumption-wealth ratio to calculate a new value for $h_1$. We repeat this procedure until convergence. In practice, convergence is extremely fast.

[11] In recent work, Chacko and Viceira (2001) offer some support for this conjecture. They generate data from a model of stock return volatility with multiple additive components, each one with a different persistence parameter $\kappa$, and show that fitting a single component model to returns sampled at different frequencies generates a pattern in the estimates of $\kappa$ similar to that observed in US data.

Table II explores the implications for portfolio choice of the monthly estimates, while Table III explores the implications of the annual estimates. We consider investors with coefficients of relative risk aversion ($\gamma$) in the interval [0.75, 40], elasticities of intertemporal substitution ($\psi$) in the interval [1/0.75, 1/40], and a rate of time preference ($\beta$) equal to 6% annually.[12]

Panel A of each table reports mean optimal percentage allocations to stocks. It shows that the mean optimal portfolio allocation to stocks varies widely across investors with different coefficients of relative risk aversion but similar elasticities of intertemporal substitution of consumption. By contrast, there is very little variation in the mean optimal portfolio allocations of investors with different elasticities of intertemporal substitution of consumption but similar coefficients of relative risk aversion. Campbell and Viceira (1999, 2001) and Campbell, Chan, and Viceira (2002) find similar results in models with time-varying expected returns and interest rates.

Panel B evaluates the empirical importance of intertemporal hedging demands resulting from volatility risk. It reports the percentage ratio of hedging portfolio demand over myopic portfolio demand. Equations (23) and (34) show that this ratio is independent of the level of precision or volatility. Consistent with the results in Propositions 1 and 2, the estimated negative instantaneous correlation of volatility with stock returns implies a positive intertemporal hedging demand for investors with $\gamma < 1$, and a negative demand for investors with $\gamma > 1$.

[12] Our solution procedure fails to converge for investors with $\gamma = 0.75$, $\psi = 1/0.75$ and $\beta = 6\%$ when we use parameter values based on the annual estimates. For these investors, the loglinearization parameter $h_1$ converges to zero. That is, this set of parameter values results in extremely low levels of optimal consumption relative to optimally invested wealth. For investors with $\gamma = 0.8$, $\psi = 1/0.8$ and $\beta = 6\%$ the procedure converges, so we use these preference parameteres instead.

More importantly, Panel B shows that our estimates of volatility risk imply intertemporal hedging demands that are typically small. By contrast, Brandt (1999), Campbell and Viceira (1999, 2001, 2002), Campbell, Chan and Viceira (2003), and others have shown that the time variation in risk premia or in interest rates estimated from U.S. data imply large intertemporal hedging demands for investors with similar preferences.

There are, however, striking differences across both samples. The monthly estimates generate very small intertemporal hedging demands: Even for highly risk averse investors ($\gamma = 40$), hedging demand reduces myopic demand by less than 4%. By contrast, the annual estimates generate much larger intertemporal hedging demands: Hedging demand reduces myopic demand by 4.7% for investors with $\gamma = 1.5$, and by almost 16% for investors with $\gamma = 40$.

Figures 1 through 4 report the results of comparative statics exercises that evaluate the sensitivity of intertemporal hedging demand to changes in the persistence, mean and variance of precision, and in its correlation with stock returns. These are the main dimensions along which the monthly estimates differ from the annual estimates. These figures plot the ratio of hedging demand to myopic demand for investors with $\psi = 1/2$ and $\gamma = \{2, 4, 20\}$ as we consider changes in the parameters of interest, and keep the rest of the parameters at the values implied by the monthly estimates. We show in Appendix E that qualitatively similar results hold for general parameter values in the case $\psi = 1$—the case for which our analytical solution is exact and we can evaluate analytically the derivative of the intertemporal hedging demand function.

First, we examine in Figure 1 the effect on intertemporal hedging of changes in the persistence of shocks to precision, holding the first and second unconditional moments of stock returns and precision constant at the values implied by the monthly estimates. We achieve this by varying $\sigma$ appropriately as we change the persistence parameter $\kappa$.[13] We consider values $\kappa$ implying half-lives of a shock to precision between 6 months and 30 years. Figure 1 shows that a compensated increase in persistence leads to an increase in the size of intertemporal hedging demand. However, this increase is small.[14]

Interestingly, Figure 1 shows that the absolute magnitude of intertemporal hedging demand does not increase monotonically with compensated increases in persistence. The exact analytical solution in the case $\psi = 1$, for which we can evaluate this effect for general parameter values, provides some intuition about this result. Appendix E shows that a compensated increase in persistence increases the size of intertemporal hedging demand only when $\beta < \kappa$, i.e., only when the rate at which investors discount future utility of consumption is smaller than the rate at which shocks to precision die out.

Second, we consider the effect of correlation. Figure 2 repeats the experiment of Figure 1, except that it considers changes in the correlation coefficient $\rho$. The effect of changes in correlation is somewhat larger than the effect of compensated changes in persistence, especially when we consider correlations close to perfect, but it is still modest. Figure 2 also shows that intertemporal hedging demand increases monotonically with compensated increases in persistence.

[13]Recall from Section 2 that $\text{Var}(y_t) = \sigma^2\theta/2\kappa$ and $\text{Var}(dS_t/S_t) \approx 1/\theta + \text{Var}(y_t)/\theta^3$. Thus, setting $\sigma^2 = 2\kappa\,\text{Var}(y_t)/\theta$ as we vary $\kappa$ leaves these second moments constant. Note that the unconditional means of precision ($\theta$) and stock returns ($\mu$) do not change with either $\kappa$ or $\sigma$.

[14]Results are qualitatively similar when we consider compensated changes in $\sigma$. To save space we do not report these results here, but they are readily available upon request.

Third, we explore the effect on intertemporal hedging demand of changes in the unconditional variance of precision, while keeping its mean constant. Since $\mathrm{Var}(y) = \sigma^2\theta/2\kappa$, we can implement this exercise by considering uncompensated changes in $\sigma$ or $\kappa$; we report results based on varying $\sigma$.[15] To determine a reasonable range of variation for $\sigma$, we use the fact that the unconditional variance of stock returns also changes with $\sigma$ (see equation [4]), and consider values of $\sigma$ implying stock return volatilities between 18% and 30%.

Figure 3 reports the result of this experiment. To facilitate interpretation, we plot stock return volatility instead of $\sigma$ or $\mathrm{Var}(y_t)$ on the horizontal axis. Figure 3 shows that intertemporal hedging demand is highly responsive to changes in the variance of the precision, especially when investors are highly risk averse. However, this could be the result of changes in the unconditional variance of stock returns rather than the result of changes in the unconditional variance of precision—both of them increase as we increase $\sigma$.

To isolate one effect from the other, Figure 4 evaluates the effect on intertemporal hedging demand of changes in the unconditional variance of stock returns that leave the unconditional variance of precision constant. Equation (4) shows that we achieve this by considering compensated changes in $\theta$.[16] Once again, we consider values of $\theta$ that imply stock return volatilities between 18% and 30% and plot stock return volatility instead of $\theta$ on the horizontal axis. Figure 4 shows that this effect is relatively small. Thus this analysis suggests that intertemporal hedging demand is comparatively more responsive to changes in the unconditional variance of the state

[15] Varying $\kappa$ instead of $\sigma$ produces similar results. See Appendix E.

[16] Since $\mathrm{Var}(y) = \sigma^2\theta/2\kappa$, we can achieve this by varying $\sigma$ or $\kappa$ as we vary $\theta$. We choose to vary $\sigma$. Varying $\kappa$ instead of $\sigma$ does not change the conclusions.

variable than to changes in the persistence of shocks to this variable, its mean, or its correlation with stock returns.

Table IV explores the implications for consumption and savings of time variation in volatility. Panel A in the table reports the exponentiated optimal mean log consumption-wealth ratio and Panel B reports the long-term expected return on wealth. The numbers in the table are based on the monthly sample. Panel A shows that optimal consumption depends on both $\gamma$ and $\psi$. It is a positive monotonic function of $\gamma$ when $\psi > 1$, while it is a negative monotonic function of $\gamma$ when $\psi < 1$. It is independent of $\gamma$ and equal to the rate of time preference $\beta$ (6%) when $\psi = 1$—as shown in Section 3.

These patterns are identical to those found by Campbell and Viceira (1999) in the context of a model with time-varying expected returns. We simply summarize here their explanation for those patterns. Investors in the bottom of the panel are highly risk averse, so they are almost fully invested in the riskless asset. If they are also very reluctant to substitute consumption intertemporally ($\psi$ close to zero), they optimally choose to consume the long-term yield on wealth—allowing for some precautionary savings—; this is why the optimal consumption-wealth ratios of investors with high $\gamma$ and low $\psi$ are very close to the long-term expected return on their wealth portfolios. For investors with $\psi$ still close to zero but less risk averse, it is optimal to invest a larger fraction of their wealth in stocks; hence they choose a larger consumption-wealth ratio—though precautionary savings also increase, as they take on more risk: This explains the pattern we observe as we move upwards in Panel A from the southeast corner.

When the expected return on wealth is smaller than the rate of time preference, investors who are willing to substitute consumption intertemporally value current consumption more than future consumption; hence the increasing pattern in consumption-wealth ratios we observe as we move to the left in Panel A. As we move upwards in the panel this pattern becomes less pronounced—or even reverses—because the long-term expected return on wealth increases and eventually is larger than the rate of time preference.

Finally, Table V investigates the effect on optimal portfolio choice when expected returns change with volatility, using the model of Section 5. This model assumes that

$$\mathrm{E}_t\left[\frac{dS_t}{S_t} - r dt\right] = \alpha_1 + \alpha_2 v_t. \tag{38}$$

Table V explores how the allocations of Table II, which are based on $\alpha_2 = 0$, change as $\alpha_2$ moves away from zero, holding the unconditional mean and variance of stock returns constant. To hold the unconditional mean excess return at the same value as in the in the benchmark case $\alpha_2 = 0$, we recompute $\alpha_1$ for each value of $\alpha_2$ as $\alpha_1 = \mu - r - \alpha_2\,\mathrm{E}[v_t]$, where $\mathrm{E}[v_t]$ is given in (4). To hold the unconditional variance of stock returns constant, we recompute $\theta$ for each value of $\alpha_2$. Since we do not have an analytical expression for the variance of stock returns when expected returns are time varying, we use Monte Carlo simulation to determine the value of $\theta$ that leaves the variance unchanged as we move $\alpha_2$ away from zero.[17]

We consider values of $\alpha_2$ equal to {-2.00, -0.75, -0.25, 0.00, 0.25, 0.75, 2.00}.[18] Each row of Table V reports allocations corresponding to this set of values of $\alpha_2$,

[17]For each run we have generated 10,000 time series of the process, each 30 years in length, with a time step $dt = 0.01$ (or about 3 days).

[18]This choice is based on the fact that an estimation of the model with time-varying expected excess returns gives a point estimate of 0.75 for $\alpha_2$, with a standard error of 0.41.

given a particular value of $\gamma$. All entries in the table assume $\psi = 1/2$. Panel A reports mean optimal allocations to stocks based on equation (37) in Proposition 3. Panel B reports the percentage value of the intercept of the intertemporal hedging component, and Panel C reports the percentage value of the slope of the intertemporal hedging demand times $\theta$, the unconditional mean of precision. Of course, the mean intertemporal hedging demands obtain by adding the numbers in Panel B and C.

Panel A shows that compensated changes in $\alpha_1$ have a large impact on mean optimal portfolio demands. However, this effect operates mainly through the myopic component of portfolio demand: Panel B and C show that the intercept and the slope (times the mean of precision) of intertemporal hedging demands are too small in absolute value to have any significant impact on total portfolio demand when using parameter estimates based on the monthly dataset.

The negative sign of the instantaneous correlation between volatility and stock returns implies that the intercept is negative for $\gamma < 1$ and positive for $\gamma > 1$, while the slope is positive for $\gamma < 1$ and negative for $\gamma > 1$. We have noted in Section 5 that the slope captures intertemporal hedging effects of uncompensated changes in volatility, while the intercept captures intertemporal hedging effects of compensated changes in volatility—it is zero when expected excess returns are constant, and it increases as $\alpha_2$ becomes larger in absolute value. Table V shows that the effect of the slope is relatively more important than the effect of the intercept, at least for values of $\alpha_2$ close to zero.

### 6.3 The Accuracy of the Approximate Solution

The portfolio allocations and consumption-wealth ratios shown in Section 6 are based on an analytical solution for the optimal rules that is exact only in the case of $\psi = 1$. In all other cases this solution is approximate, based on an expansion of the optimal log consumption-wealth ratio around its unconditional mean $\mathrm{E}[c_t - x_t]$. Campbell (1993) and Campbell and Viceira (2002) note that this solution method is accurate provided that the log consumption-wealth ratio is not too variable around its unconditional mean.[19] Table VI reports annualized, percentage values of the unconditional standard deviation of the optimal log consumption-wealth ratio.[20] These results suggest that this solution is likely to be accurate for values of $\psi$ far from one, in line with the results of Campbell (1993), Campbell and Koo (1997) and Campbell, Cocco, Gomes, Maenhout, and Viceira (2002) for models with time variation in interest rates and expected excess returns. The log consumption-wealth ratio exhibits low volatility in most cases, both in absolute terms and relative to its mean. The exception are investors with very low elasticities of intertemporal substitution of consumption.

# 7 Conclusion

We have explored in this paper dynamic optimal consumption and portfolio choice when asset return volatility is time-varying. We have considered a model where long-horizon investors with Duffie-Epstein (1992) recursive preferences over intermediate

[19] In the $\psi = 1$ case, this ratio is constant, and the solution is exact.

[20] From equation (33) in Proposition 2, this standard deviation is equal to $|A_1|\sqrt{\mathrm{Var}(y_t)} = |A_1|\sigma\sqrt{\theta/2\kappa}$.

consumption have two assets available for investment, a riskless bond and a risky asset ("stocks"). Stock return precision—the reciprocal of volatility—follows a mean-reverting process which is instantaneously correlated with stock returns.

We have shown that this model has an analytical solution which is exact when investors have unit elasticity of intertemporal substitution of consumption—but not necessarily unit coefficient of relative risk aversion—, and approximate in all other cases. Optimal portfolio demand for stocks is a linear combination of two components: a myopic (or mean-variance) component, and an intertemporal hedging component, both of which change linearly with precision. We have used this solution to analytically characterize intertemporal hedging demand in the presence of volatility risk, and to assess its quantitative importance. To this end, we have conducted a comprehensive calibration exercise based on estimates of the joint process for stock market returns and volatility using monthly U.S. stock market returns from 1926 to 2000, and annual returns from 1871 to 2000.

Our estimates of the instantaneous correlation of precision with stock returns are large and positive, implying a large negative correlation of volatility with stock returns. Shocks to precision exhibit low persistence and variance, especially in the monthly sample. These estimates generate small, negative intertemporal hedging demands for investors with coefficients of relative risk aversion larger than one. By contrast, Brandt (1999), Campbell and Viceira (1999, 2001, 2002), Campbell, Chan and Viceira (2003) and others have shown that the estimated time variation in risk premia or in real interest rates in the U.S. results in much lager intertemporal hedging demands for investors with similar preferences. A comprehensive comparative statics exercise suggests that the unconditional variance of precision has to be much larger to generate intertemporal hedging demands of comparable size.

An important caveat of our empirical analysis is that we have counterfactually assumed that investors observe volatility (or precision), and that they take as true parameters our empirical estimates of the joint process for returns and volatility. In practice, however, investors do not observe volatility, and they do not know the parameters of the process for volatility or even the process itself. They must infer all of that from observed returns, and they must account for this uncertainty when they make their portfolio decisions. The large standard errors of some of our point estimates, and the significant differences in the estimates from the monthly sample and the annual sample, suggest that these may be important issues. Barberis (2000), Brennan (1998), Xia (2001), and others have shown that parameter uncertainty and learning can have a large effect on optimal long-term investment strategies. Integrating all of these effects into a one single empirically implementable framework is beyond the scope of this paper, and a challenging task for future research.

We have also considered a model where expected stock excess returns are an affine function of volatility. In this case, optimal portfolio demand and its hedging component are both affine functions of precision. A possible extension of this model could allow for both expected stock returns and risk to vary over time as a function of a vector of state variables. Intertemporal hedging demand would then depend on the resulting process for the Sharpe ratio of stocks, and how it correlates with the vector of state variables. However, Campbell (1987), Harvey (1989, 1991), Glosten, Jagannathan, and Runkle (1993), Ait-Sahalia and Brandt (2001) and others have modelled time-varying returns and volatility jointly, and found that the effects of state variables on expected returns are stronger than their effects on volatility. This suggests that the negative hedging demand associated with volatility risk will be modest even in a framework that combines time-varying volatility with time-varying expected returns.

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# A Appendix A: Analysis of the roots of the quadratic equation (15) for $A$ in the $\psi = 1$ case

The roots of the quadratic equation (15) are given by

$$\begin{aligned} A &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad (39) \\ &= \frac{\frac{\beta+\kappa}{1-\gamma} - \frac{\rho\sigma(\mu-r)}{\gamma} \pm \sqrt{\left(\frac{\beta+\kappa}{1-\gamma} - \frac{\rho\sigma(\mu-r)}{\gamma}\right)^2 - \frac{\sigma^2(\mu-r)^2[\gamma(1-\rho^2)+\rho^2]}{\gamma^2(1-\gamma)}}}{\frac{\sigma^2[\gamma(1-\rho^2)+\rho^2]}{\gamma(1-\gamma)}}, \end{aligned}$$

where note that $[\gamma(1-\rho^2)+\rho^2] > 0$.

We start by discussing when the roots of (15) are real. These roots are real if the discriminant of the equation, $\Delta = b^2 - 4ac$, is non-negative. When $\gamma > 1$, it is immediate to see by simple inspection of the $\Delta$ in (39) that the discriminant is always non-negative. When $\gamma < 1$, the discriminant is positive provided that

$$\Delta = \left(\frac{\beta+\kappa}{1-\gamma}\right) - 2\frac{\rho\sigma(\mu-r)}{\gamma} - \frac{\sigma^2(\mu-r)^2}{\gamma(\beta+\kappa)} > 0. \qquad (40)$$

This expression results from some simple algebraic manipulation of the expression for $\Delta$ in (39). Reordering terms in this expression we obtain inequality (20) in text.

To determine the sign of the roots of (15), we use a standard result on quadratic equations. This result is that the product of the roots of the equation is equal to

$$\frac{c}{a} = \frac{(1-\gamma)(\mu-r)^2}{\sigma^2[\gamma(1-\rho^2)+\rho^2]}.$$

It is immediate to see that this ratio is always negative when $\gamma > 1$, and positive when $\gamma < 1$. Therefore, the roots of the equation have opposite signs when $\gamma > 1$, and they have the same sign when $\gamma < 1$.

We can further show that when $\gamma < 1$, the condition for the roots to be real also implies that they are positive. This requires showing that the numerator of (39) is positive, because the denominator $2a$ is always positive in the $\gamma < 1$ case. Now, the condition for real roots (40) also implies that $b < 0$. Since $ac > 0$, we have that $-b > |\sqrt{\Delta}|$ so that $-b \pm \sqrt{\Delta} > 0$. Therefore, (40) implies that $(-b \pm \sqrt{\Delta})/2a > 0$.

Equation (39) has two roots. To determine which one we choose when $\gamma > 1$ and when $\gamma < 1$, we compute the limit of equation (39) as $\gamma \to 1$ in each case. This limit is given by

$$\lim_{\gamma \to 1} A = \frac{(\beta + \kappa) \mp \sqrt{(\beta + \kappa)^2}}{\sigma^2},$$

where the negative sign holds in the $\gamma > 1$ case, and the positive sign holds in the $\gamma < 1$ case. For general parameter values this expression is zero only if we pick the positive root of the discriminant in (39) when $\gamma > 1$, and the negative root when $\gamma < 1$. Note that $A = 0$ and $\gamma = 1$ implies immediately that $B = 0$.

# B Appendix B: Convergence of approximate solution to exact solution in special cases

An important feature of this approximate solution is that it delivers the exact solution in those cases where this solution is known: log utility, unit elasticity of intertemporal substitution, and constant investment opportunities.

For convenience, we note here that the solution to the quadratic equation (28) for

$A_1$ is

$$A_1 = \frac{\frac{(1-\gamma)\rho\sigma(\mu-r)}{\gamma} - (h_1+\kappa) + \sqrt{\left((h_1+\kappa) - \frac{(1-\gamma)\rho\sigma(\mu-r)}{\gamma}\right)^2 - \frac{\sigma^2(\mu-r)^2(1-\gamma)[\gamma(1-\rho^2)+\rho^2]}{\gamma^2}}}{\frac{\sigma^2}{\gamma}\left(\frac{1-\gamma}{1-\psi}\right)\left[\gamma\left(1-\rho^2\right)+\rho^2\right]}. \tag{41}$$

In the case of log utility ($\gamma = \psi \equiv 1$), we find by direct substitution of $\psi = 1$ into (41) that $A_1 = 0$. Further substitution of $\psi = 1$ into equation (29) for $B_1$ shows that $B_1 = 0$ when $h_1 = \beta$. This implies that $C_t/X_t = \beta$, which is in turn consistent with $h_1 \equiv \exp\{\mathrm{E}[\log(C_t/X_t)]\} = \beta$. Now, using (41) it is straightforward to check that

$$\lim_{\gamma\to 1} \frac{1-\gamma}{(1-\psi)\,\gamma} A_1 = \lim_{\gamma\to 1}\left(1-\frac{1}{\gamma}\right)\mathcal{A}_1 = 0, \tag{42}$$

so that $\pi_t = (\mu - r)y_t$.This is the exact solution to the problem with log utility reported in Merton (1969).

When $\psi = 1$ but $\gamma \neq 1$, we have that $C_t/X_t = \beta$ and $h_1 = \beta$ using the same arguments as in the log utility case. However, the limit (42) is not necessarily zero and the hedging component of $\pi_t$ does not vanish. That is, the optimal consumption rule is myopic, while the optimal portfolio rule is not. This is the case discussed in section 3. Note that, when $\gamma = 1$ but $\psi \neq 1$, the result is reversed: The hedging component of $\pi_t$ vanishes and $\pi_t = (\mu - r)y_t$, but consumption relative to wealth is not constant.

Finally, when investment opportunities are constant, implying $\kappa, \sigma = 0$ and $v_t \equiv v$, both policies are myopic. Substitution of these parameter values into the expressions for $a_1$, $b_1$ and $c_1$ in (30)-(32), shows that equation (28) reduces to a linear equation for $A_1$ with solution

$$A_1 = -\frac{(\mu-r)^2\,(1-\psi)}{2\gamma h_1}. \tag{43}$$

so that $\mathcal{A}_1 = (\mu - r)^2/2\gamma h_1$. Note, however, that $\sigma = 0$ implies that the optimal portfolio rule is myopic, even though $\mathcal{A}_1$ is not necessarily zero: $\pi_t = (\mu - r)/\gamma v$.

Further substitution of $\kappa$, $\sigma = 0$ and $v_t \equiv v$ into equation (28) shows that

$$B_1 = \frac{h_1 (1 - \log h_1) + \psi \log \beta - \psi\beta - r (1 - \psi)}{h_1}. \tag{44}$$

Substitution of the solutions for $A_1$ and $B_1$ given in (43) and (44) into equation (33) gives

$$\begin{aligned} c_t - x_t &= \frac{-h_1 (1 - \log h_1) + \psi\beta + r (1 - \psi)}{h_1} + \frac{(\mu - r)^2 (1 - \psi)}{2\gamma h_1} \frac{1}{v} && (45) \\ &= \log h_1, && (46) \end{aligned}$$

where the second equality follows from the fact that $c_t - x_t$ is constant. After cancelling the terms in $\log h_1$, we find that

$$h_1 = \frac{C_t}{X_t} = \psi\beta + \frac{1}{2} (1 - \psi) \frac{(\mu - r)^2}{\gamma v} + (1 - \psi) r.$$

This is a generalized version of the exact solution given in Merton (1969) for the power utility case ($\psi = 1/\gamma$) when investment opportunities are constant.

# C Appendix C: Derivation of Optimal Policies Under Time-Varying Expected Returns

The simplified ODE for this problem is given by:

$$
\begin{aligned}
0 &= -\beta^{\psi} H^{-1} + \psi\beta + \frac{(1-\psi)}{2\gamma}\left(2\alpha_1\alpha_2 + \alpha_1^2 y_t + \frac{\alpha_1^2}{y_t}\right) - \frac{\rho\sigma(1-\gamma)}{\gamma}(\alpha_1 y_t + \alpha_2)\frac{H_y}{H} y_t \\
&\quad + r(1-\psi) + \frac{\rho^2\sigma^2(1-\gamma)^2}{2\gamma(1-\psi)}\left(\frac{H_y}{H}\right)^2 y_t - \frac{H_y}{H}\kappa(\theta - y_t) \\
&\quad + \frac{\sigma^2}{2}\left(\frac{1-\gamma}{1-\psi} + 1\right)\left(\frac{H_y}{H}\right)^2 y_t - \frac{\sigma_y^2}{2}\frac{H_{yy}}{H_t} y_t. \qquad (47)
\end{aligned}
$$

We now guess that $H = \exp\{A_1 y_t + A_2 \log y_t + B\}$ and we make the substitution

$$
\beta^{\psi} H^{-1} \approx h_0 + h_1(c_t - x_t),
$$

and

$$
\begin{aligned}
-\log y_t &= \log v_t \\
&\approx \log \theta_v + \frac{1}{\theta_v}(v - \theta_v) \\
&= \log \kappa - \log\left(\kappa\theta - \sigma^2\right) - 1 + \frac{\kappa\theta - \sigma^2}{\kappa}\frac{1}{y_t}.
\end{aligned}
$$

After collecting terms in $1/y_t$, $y_t$ and 1 we find:

$$
\begin{aligned}
0 &= \frac{(1-\psi)\alpha_2^2}{2\gamma} - \left[\theta(h_1 + \kappa) - \left(\frac{h_1}{\kappa} + \frac{1}{2}\right)\sigma^2 + \frac{(1-\gamma)\rho\sigma\alpha_2}{\gamma}\right] A_2 + a_2 A_2^2, \qquad (48) \\
0 &= \frac{(1-\psi)\alpha_1^2}{2\gamma} + \left[(h_1 + \kappa) - \frac{(1-\gamma)\rho\sigma\alpha_1}{\gamma}\right] A_1 + a_1 A_1^2, \qquad (49) \\
0 &= -h_0 - h_1\psi\log\beta - h_1\left(\log\kappa - \log\left(\kappa\theta - \sigma^2\right) - \frac{\kappa}{\kappa\theta - \sigma^2}\right) A_2 + h_1 B + \psi\beta \\
&\quad + \frac{2(1-\psi)\alpha_1\alpha_2}{2\gamma} - \frac{\rho\sigma(1-\gamma)}{\gamma}(\alpha_1 A_2 + \alpha_2 A_1) + r(1-\psi) \\
&\quad + 2aA_1A_2 + \kappa(A_2 - \theta A_1),
\end{aligned}
$$

where $a_2 = a_1$ and $a_1$ is given in (30). The optimal policies obtain immediately from substitution of the value function into the first order conditions (10) and (11).

Coefficient $A_2$ obtains as the solution to the quadratic equation (48). Note that $A_2$ does not depend on $\alpha_1$. This equation has two roots. However, only the root associated with the negative root of the discriminant ensures that $A_2 = 0$ when $\alpha_2 = 0$—i.e., it ensures the mutual consistency between the solution given in Proposition 2 and this solution. When $\gamma > 1$, the roots of the equation are real and have opposite signs. The root associated with the negative root of the discriminant implies that $A_2/(1-\psi) > 0$. When $\gamma < 1$, the roots may be real or complex conjugate. The condition that ensures that the discriminant of the equation is nonnegative, so that the roots are real, also implies that $A_2/(1-\psi) > 0$.

Coefficient $A_1$ obtains as the solution to the quadratic equation (49). Simple inspection of this equation and equation (28) shows that they are identical except that $\alpha_1$ replaces $(\mu - r)$ in (49). Hence the analysis of $A_1$ presented in Section 4 is also valid here, and we have that $A_2/(1-\psi) < 0$, and $A_1 = 0$ when $\alpha_1 = 0$. Note that $A_1$ does not depend on $\alpha_2$.

# D Appendix D: Model Estimation: Spectral GMM

This section examines the implications for optimal portfolio choice and consumption of the patterns in volatility observed in the U.S. stock market. This requires estimating first the parameters of the continuous-time process (1)-(2). Unfortunately, the likelihood function of this process is unknown in analytical form, which makes direct estimation of the parameters of the process via maximum likelihood unfeasible. However, the characteristic function of the process is known, which makes direct estimation feasible via the Spectral Generalized Method of Moments of Chacko and

Viceira (2003), Jiang and Knight (2000) and Singleton (2000).[21] This estimation method is essentially GMM using the complex moments generated by the characteristic function of the process and it does not require the discretization of the stochastic process.

## D.1 Derivation of the conditional characteristic function

The conditional characteristic function of the log stock price is defined as

$$\phi\left(\log S_t, y_t, \tau; \boldsymbol{\theta}, \omega\right) = \mathrm{E}\left[\exp\left\{i\omega \log S_{t+\tau}\right\} \mid \log S_t, y_t\right],$$

where $\omega \in \mathbb{R}$, $\tau \geq 1$, and $\boldsymbol{\theta} = (\mu,\, \kappa,\, \theta,\, \sigma,\, \rho)'$.

The conditional characteristic function $\phi\left(\log S_t, y_t, \tau; \boldsymbol{\theta}, \omega\right)$ for the (log) stock price process is given by the solution to the following Kolmogorov backward equation (KBE):

$$\begin{aligned} 0 \;=\; & \frac{1}{2y_t}\frac{\partial^2 \phi}{\partial(\log S_t)^2} + (\mu - \frac{1}{2y_t})\frac{\partial \phi}{\partial \log S_t} + \kappa(\theta - y_t)\frac{\partial \phi}{\partial y_t} + \frac{1}{2}\sigma^2 y_t \frac{\partial^2 \phi}{\partial y_t^2} \\ & + \rho\sigma \frac{\partial^2 \phi}{\partial(\log S_t)\partial y_t} - \frac{\partial \phi}{\partial \tau}, \end{aligned} \tag{50}$$

where $\phi \equiv \phi\left(\log S_t, y_t, \tau; \boldsymbol{\theta}, \omega\right)$. The boundary condition for this partial differential equation is simply

$$\phi\left(\log S_{t+\tau}, y_{t+\tau}, \tau; \boldsymbol{\theta}, \omega\right) = \exp[i\omega \log S_{t+\tau}].$$

[21] Other techniques used to estimate stochastic volatility models include simulated method of moments, Kalman filtering, simulated maximum likelihood, and Bayesian estimation. See Melino and Turnbull (1990), Gallant, Hsieh, and Tauchen (1994), Harvey, Ruiz, and Shephard (1994), Danielsson (1994), and Jacquier, Polson, and Rossi (1994).

Due to the $1/2y_t$ terms in (50), this equation clearly does not fit into the affine specification required in Chacko and Viceira (2003). However, we will use a Taylor-series expansion to linearize this term. We use the approximation

$$\frac{1}{y_t} \approx \frac{2}{\theta} - \frac{1}{\theta^2} y_t.$$

With this linearization, the KBE becomes

$$\begin{aligned} 0 &= \frac{1}{2}\left(\frac{2}{\theta} - \frac{1}{\theta^2} y_t\right) \frac{\partial^2 \phi}{\partial (\log S_t)^2} + \left[\mu - \frac{1}{2}\left(\frac{2}{\theta} - \frac{1}{\theta^2} y_t\right)\right] \frac{\partial \phi}{\partial \log S_t} + \kappa(\theta - y_t) \frac{\partial \phi}{\partial y_t} \\ &\quad + \frac{1}{2}\sigma^2 y_t \frac{\partial^2 \phi}{\partial y_t^2} + \rho\sigma \frac{\partial^2 \phi}{\partial (\log S_t) \partial y_t} - \frac{\partial \phi}{\partial \tau}. \end{aligned} \tag{51}$$

Since the KBE is now a linear equation, we know from Chacko and Viceira (2001) that the characteristic function is exponential affine. The solution to (51) is given by

$$\phi\left(\log S_t, y_t, \tau; \boldsymbol{\theta}, \omega\right) = \exp[A(\tau) \log S_t + B(\tau) y_t + C(\tau)], \tag{52}$$

where

$$\begin{aligned} A(\tau) &= i\omega, \\ B(\tau) &= \frac{2r_1 r_2}{\sigma^2}\left[\frac{\exp(r_1\tau) - \exp(r_2\tau)}{r_1 \exp(r_2\tau) - r_1 \exp(r_1\tau)}\right], \\ C(\tau) &= \left(\mu - \frac{1}{\theta}\right) i\omega\tau + \frac{1}{\theta}(i\omega)^2\tau + (\kappa\theta + \rho\sigma i\omega)\frac{2}{\sigma^2}\log\left[\frac{r_2 - r_1}{r_2 \exp(r_1\tau) - r_1 \exp(r_2\tau)}\right], \\ r_1 &= -\frac{1}{2}\kappa + \frac{1}{2}\sqrt{\kappa^2 + \frac{\sigma^2}{\theta^2} i\omega(i\omega - 1)}, \\ r_2 &= -\frac{1}{2}\kappa - \frac{1}{2}\sqrt{\kappa^2 + \frac{\sigma^2}{\theta^2} i\omega(i\omega - 1)}. \end{aligned}$$

Since $y_t$ is unobservable, we choose to integrate this variable out of the characteristic function (52) before we proceed to estimate the model. This is easily accomplished as follows. Let $f(y_t)$ denote the unconditional density of volatility. Then,

$$\begin{aligned}
\int_0^\infty \phi\left(\log S_t, y_t, \tau; \boldsymbol{\theta}, \omega\right) f(y_t)\, dy_t &= \int_0^\infty \exp[A(\tau)\log S_t + B(\tau)y_t + C(\tau)] f(y_t)\, dy_t \\
&= \exp[A(\tau)\log S_t + C(\tau)] \int_0^\infty \exp[B(\tau)y_t] f(y_t)\, dy_t \\
&= \phi\left(\log S_t, \tau; \boldsymbol{\theta}, \omega\right).
\end{aligned}$$

Of course, $\int_0^\infty \exp[B(\tau)y_t] f(y_t)\, dy_t$ is simply the moment-generating function of the square-root process (2). This moment generating function is well known. Therefore, we have the result that the characteristic function of $\log S_T$ conditional only on the prior observation of the stock price, $\log S_t$, is given by

$$\phi\left(\log S_t, \tau; \boldsymbol{\theta}, \omega\right) = \exp[A(\tau)\log S_t + C(\tau)] \left[\frac{2\kappa}{2\kappa - \sigma^2 B(\tau)}\right]^{\frac{2\kappa\theta}{\sigma_y^2}}. \tag{53}$$

Thus, we have effectively integrated out the unobservable state variable, $y_t$, from the characteristic function and, therefore, from the estimation process.

## D.2 Estimation

Chacko and Viceira (2003), Jiang and Knight (2001), and Singleton (2000) estimate stochastic volatility models similar to that used in this paper, using the Spectral GMM procedure. We provide here a brief description of this estimation procedure and refer the reader to these papers for full details and further examples.

The characteristic function (53) defines a set of moments of the complex stochastic variable $\exp(i \log S_{t+\tau})$, since

$$\phi\left(\log S_t, \tau; \boldsymbol{\theta}, \omega\right) = \mathrm{E}\left[\exp\left\{i\omega \log S_{t+\tau}\right\} \mid \log S_t\right]. \tag{54}$$

For example, when $\omega = 1$, the characteristic function is simply the first non-central moment of $\exp(i \log S_{t+\tau})$. For $\omega = 2$, the characteristic function is simply the second non-central moment of $\exp(i \log S_{t+\tau})$. This procedure can be repeated to obtain any desired number of moments. Furthermore, obtaining these population moments is trivial since it involves only the evaluation of the characteristic function (53).

Equation (54) implicitly defines a set of moment conditions based on the stock price, which is not a stationary process. For GMM estimation we need to use moment conditions for a stationary process. We can easily generate moment conditions for a stationary process by noting that, for stock returns, we have that

$$\mathrm{E}\left[\exp\left\{i\omega(\log S_{t+\tau} - \log S_t)\right\} \mid \log S_t\right] = \frac{\phi(\log S_t, \tau; \theta, \omega)}{\exp(i\omega \log S_t)} \equiv \lambda(\log S_t, \tau; \theta, \omega). \quad (55)$$

The moment conditions

$$\mathrm{E}\left[\mathbf{h}(\mathbf{X}, t) \otimes \left(\lambda\left(\log S_t, \tau; \boldsymbol{\theta}, \omega\right) - \exp\left\{i\omega\left(\log S_{t+\tau} - \log S_t\right)\right\}\right)\right] = 0, \quad (56)$$

where $\omega \in \mathbb{R}$ and $\mathbf{h}(\mathbf{X}, t)$ is any vector of (real-valued or complex-valued) instruments, are now based on stock returns rather than stock prices.

We use (56) for GMM estimation. Since there are five parameters to be estimated, we choose $\omega = 1, \ldots, 5$. We also set $\mathbf{h}(\mathbf{X}, t) = 1$ and $\tau = 1$ for simplicity. If we let $g(\boldsymbol{\theta})$ represent the sample analog of the moment conditions in (56), we choose parameter estimates such that

$$\widehat{\boldsymbol{\theta}} = \arg \min_{\{\boldsymbol{\theta}\}} g(\boldsymbol{\theta})' W(\boldsymbol{\theta}) g(\boldsymbol{\theta})$$

where $W(\boldsymbol{\theta})$ is a positive-definite, symmetric weighting matrix. Because the moment conditions exactly match the number of parameters we have in the model, the parameters are exactly identified, and $g(\boldsymbol{\theta})' W(\boldsymbol{\theta}) g(\boldsymbol{\theta})$ attains zero for all choices $W(\boldsymbol{\theta})$. Consequently, we use the identity matrix for $W(\boldsymbol{\theta})$.

# E Appendix E: Properties of intertemporal hedging demand when $\psi = 1$.

For reference throughout this section, we rewrite here the expression for intertemporal component of total portfolio demand when $\psi = 1$:

$$\pi_t^h = \left(1 - \frac{1}{\gamma}\right)(-\rho)\,\sigma\mathcal{A}y_t, \tag{57}$$

where $\mathcal{A} \equiv A/(1-\gamma) > 0$, and $A$ is given by the solution to equation (15).

## E.1 The effect on intertemporal hedging demand of uncompensated changes in $\sigma$ and $\kappa$.

It is useful to start this exercise by considering the effect on intertemporal hedging demand of uncompensated changes in $\sigma$ and $\kappa$. First, we consider the effect on $\pi_t^h$ of changes in $\sigma$. An examination of equation (57) shows that $\partial\pi_t^h/\partial\sigma$ is proportional to $\partial(\sigma\mathcal{A})/\partial\sigma$, and that $\partial(\sigma\mathcal{A})/\partial\sigma > 0$ implies that the magnitude of $\pi_t^h$ increases with $\sigma$—i.e., that $|\pi_t^h|$ increases with $\sigma$.

Since $\mathcal{A} \equiv A/(1-\gamma)$ we have that

$$\frac{\partial(\sigma\mathcal{A})}{\partial\sigma} = \frac{1}{1-\gamma}\frac{\partial(\sigma A)}{\partial\sigma}.$$

We can obtain an expression for $\sigma A$ by premultiplying equation (39) by $\sigma$, after taking the appropriate root for the discriminant—positive when $\gamma > 1$, and negative when $\gamma < 1$.

Simple but cumbersome algebraic manipulation of the resulting expression gives

$$\frac{\partial(\sigma A)}{\partial\sigma} = -\frac{2\gamma(\beta+\kappa)}{\sigma^2\left[\gamma\left(1-\rho^2\right)+\rho^2\right]}\left[\frac{\sqrt{b^2-4ac}\mp b}{\sqrt{b^2-4ac}}\right], \tag{58}$$

where the negative sign in the expression in brackets corresponds to the $\gamma > 1$ case, and the positive sign to the $\gamma < 1$ case.

To determine the sign of the derivative (58) we need to examine the sign of the expression in brackets, because the term multiplying this expression is always negative. When $\gamma > 1$, we have $ac < 0$, so that $\sqrt{b^2-4ac} - b > 0$ and $\partial(\sigma A)/\partial\sigma < 0$. When $\gamma < 1$, we have $ac > 0$; since $b < 0$ when the roots are real (see Appendix A), we have that $\sqrt{b^2-4ac} + b < 0$ and $\partial(\sigma A)/\partial\sigma > 0$.

These results for $\partial(\sigma A)/\partial\sigma$ imply in turn that

$$\frac{\partial(\sigma\mathcal{A})}{\partial\sigma} > 0$$

for all $\gamma$. This in turn implies that an increase in $\sigma$ leads to an increase in the magnitude of $\pi_t^h$.

Second, we consider the effect on $\pi_t^h$ of changes in $\kappa$. Again, an examination of equation (57) shows that $\partial\pi_t^h/\partial\kappa$ is proportional to $\partial\mathcal{A}/\partial\kappa$, and that $\partial\mathcal{A}/\partial\kappa < 0$ implies that the magnitude of $\pi_t^h$ decreases as $\kappa$ increases—i.e., that $|\pi_t^h|$ increases with persistence. From the definition of $\mathcal{A}$ we have

$$\frac{\partial\mathcal{A}}{\partial\kappa} = \frac{1}{1-\gamma}\frac{\partial A}{\partial\kappa}.$$

Computing $\partial A/\partial\kappa$ from (39) is straightforward, after noting from equations (17)-(19) that the only coefficient that depends on $\kappa$ is $b$. Thus

$$\frac{\partial A}{\partial\kappa} = \mp\frac{A}{\sqrt{b^2-4ac}}\frac{\partial b}{\partial\kappa},$$

where the negative sign holds for $\gamma > 1$, and the positive sign holds for $\gamma < 1$.

From equation (18) for $b$ we have that

$$\frac{\partial b}{\partial \kappa} = -\frac{1}{1-\gamma},$$

which leads to

$$\frac{\partial A}{\partial \kappa} = \pm \frac{\mathcal{A}}{\sqrt{b^2 - 4ac}}.$$

This result implies that

$$\frac{\partial \mathcal{A}}{\partial \kappa} < 0$$

for all $\gamma$.

## E.2 The effect on intertemporal hedging demand of compensated changes in $\kappa$.

We want to examine the effect on $\pi_t^h$ of changes in $\kappa$ while keeping constant the first and second moments of stock returns and precision constant. Since $E[y_t]/dt = \theta$, $E[dS_t/S_t]/dt = \mu dt$, $\text{Var}(y_t)/dt = \sigma^2\theta/2\kappa \equiv \mathcal{V}_y$, and $\text{Var}(dS_t/S_t) \approx 1/\theta + \text{Var}(y_t)/\theta^3$, this implies that by setting

$$\sigma^2 = \frac{2\kappa \mathcal{V}_y}{\theta}, \tag{59}$$

these moments will stay constant as we vary $\kappa$.

Thus we can determine the effect on $\pi_t^h$ of compensated changes in $\kappa$ by making the substitution (59) everywhere in the expression for $\sigma\mathcal{A} = \sigma A/(1-\gamma)$, and computing again $\partial(\sigma\mathcal{A})/\partial\kappa$.

Cumbersome but straightforward algebraic manipulation of the derivative leads to

$$\left.\frac{\partial\left(\sigma A\right)}{\partial\kappa}\right|_{\sigma^2=2\kappa\mathcal{V}_y/\theta}=\mp\left(\beta-\kappa\right)\sqrt{\frac{\mathcal{V}_y}{2\kappa}}\left(\sigma\mathcal{A}\right).$$

where the negative sign holds for $\gamma > 1$, and the positive sign holds for $\gamma < 1$. Since $\mathcal{A} > 0$, we have that this derivative has the same sign as the difference $(\beta - \kappa)$ when $\gamma > 1$, and the same sign when $\gamma < 1$. This in turn implies that

$$\operatorname{sign}\left(\left.\frac{\partial\left(\sigma\mathcal{A}\right)}{\partial\kappa}\right|_{\sigma^2=2\kappa v_y/\theta}\right)=\operatorname{sign}\left(\beta-\kappa\right),$$

so that this derivative is always negative when $\beta < \kappa$. Since persistence increases as $\kappa$ approaches zero, this implies that $\sigma\mathcal{A}$ is increasing in persistence when $\beta < \kappa$, and decreasing otherwise.

## E.3 The effect on intertemporal hedging demand of compensated changes in $\rho$.

Equation (57) implies that $\partial\pi_t^h/\partial\rho$ is proportional to $\partial(\rho\mathcal{A})/\partial\rho$. Thus we want to analyze

$$\frac{\partial\left(\rho\mathcal{A}\right)}{\partial\rho}=\frac{1}{1-\gamma}\frac{\partial\left(\rho A\right)}{\partial\rho},$$

where we can easily obtain $\rho A$ by premultiplying equation (39) by $\rho$.

Once again, straightforward but cumbersome algebraic manipulation leads to the following expression for $\partial\left(\rho A\right)/\partial\rho$ when $\gamma > 1$:

$$\frac{\partial\left(\rho A\right)}{\partial\rho}=\frac{2\gamma\left(1-\gamma\right)\mathcal{A}}{\left[\gamma\left(1-\rho^2\right)+\rho^2\right]}+\frac{\beta+k}{\gamma-1}\frac{1}{\sqrt{b^2-4ac}}\left[\frac{\sqrt{b^2-4ac}+\left(2\rho\sigma\left(\mu-r\right)/\gamma\right)}{2a}\right]<0.$$

To see why this expression is negative, note that under the assumption that $\gamma > 1$, both elements of the sum are negative. The first element of the sum is negative because $(1-\gamma) < 0$ but both $\mathcal{A}$ and the term in brackets in the denominator are always positive. The second element is always negative because the expression in brackets is negative—the numerator of the expression in brackets is positive, while the denominator is negative—, and the other factors are all positive.

The expression for $\partial(\rho A)/\partial\rho$ when $\gamma < 1$ is slightly different:

$$\frac{\partial(\rho A)}{\partial \rho} = \frac{2\gamma(1-\gamma)\mathcal{A}}{[\gamma(1-\rho^2)+\rho^2]} + \frac{\beta+k}{\gamma-1}\frac{1}{\sqrt{b^2-4ac}}\left[\frac{\sqrt{b^2-4ac}-(2\rho\sigma(\mu-r)/\gamma)}{2a}\right].$$

Unfortunately, this expression is difficult to sign. The first element of the sum is always positive when $\gamma < 1$. However, the second element is difficult to sign, because the numerator of the term in brackets does not necessarily have the same sign for all parameter values.

Therefore, we have that

$$\gamma > 1 \Rightarrow \frac{\partial(\rho\mathcal{A})}{\partial\rho} > 0,$$

but when $\gamma < 1$ we cannot determine unequivocally the sign of this derivative. In the case $\gamma > 1$, simple examination of equation (57) shows that this result implies that the magnitude of intertemporal hedging demand increases with the magnitude of $\rho$—i.e. that $|\pi_t^h|$ increases with $|\rho|$.

## E.4 The effect on intertemporal hedging demand of changes in the unconditional moments of precision.

First, consider the effects on intertemporal hedging demand of an increase in the unconditional variance of precision, while keeping its mean constant. We have already noted in text that, since $E[y_t]/dt = \theta$, and $\text{Var}(y_t)/dt = \sigma^2\theta/2\kappa$, this is equivalent to examine the effect on $\pi_t^h$ of an uncompensated increase in $\sigma$, or an uncompensated decrease in $\kappa$—both of them increase $\text{Var}(y_t)$ and leave $E[y_t]$ unchanged—, which we have already analyzed.

Second, consider the effects on intertemporal hedging demand of an decrease in the unconditional mean of precision $\theta$, while keeping its variance constant—note from equation (4) that this is equivalent to an increase in the unconditional variance of stock returns. We have already noted in text that this is equivalent to examine the effect on $\pi_t^h$ of decreasing $\theta$ while, at the same time, increasing $\sigma$ or decreasing $\kappa$ as to keep $\text{Var}(y_t)$ unchanged. But simple inspection of equations (57) and (15) shows that $\pi_t^h$ does not depend directly on $\theta$. This implies that uncompensated changes in $\theta$ do not affect $\pi_t^h$, and that compensated changes in $\theta$ affect $\pi_t^h$ only through the companion changes in $\sigma$ or $\kappa$, which we have already analyzed.

**TABLE I**

**Estimates of the Stochastic Process for Returns and Volatility**

**Model:**

$$dS_t/S_t - dB_t/B_t = (\mu - r)dt + \sqrt{v_t}dW_s,$$

$$v_t = 1/y_t$$

$$dy_t = \kappa(\theta - y_t)dt + \sigma\sqrt{y_t}dW_y,$$

$$dW_s dW_y = \rho dt$$

**Parameter estimates (s.e.):**

| | **1926.01 - 2000.12** | **1871 - 2000** |
|---|---|---|
| $\boldsymbol{\mu - r}$ | .0811 | .0848 |
| | (.0235) | (.0369) |
| $\boldsymbol{\kappa}$ | .3374 | .0438 |
| | (.3025) | (.0443) |
| $\boldsymbol{\theta}$ | 27.9345 | 25.2109 |
| | (1.7961) | (12.5738) |
| $\boldsymbol{\sigma}$ | .6503 | 1.1703 |
| | (.4802) | (.6892) |
| $\boldsymbol{\rho}$ | .5241 | .3688 |
| | (.2274) | (.3665) |

**Note**:Table 1 reports Spectral GMM estimates of the stochastic process driving stock returns and volatility. Appendix D describes the estimation procedure in detail. The monthly estimates are based on excess stock returns on the CRSP value-weighted portfolio over the T-bill rate from January 1926 through December 2000. The annual estimates are based on excess equity returns on the Standard and Poor Composite Stock Price Index over the prime commercial paper rate from 1871 through 2000. The annual dataset is an updated version of Shiller's (1989) long run data, publicly available at his website [http://www.econ.yale.edu/ shiller/]. Standard errors are bootstrapped, and parameter estimates are annualized to facilitate their interpretation.

## TABLE II

**Mean Optimal Percentage Allocation to Stocks and Percentage Hedging Demand Over Myopic Demand (Sample: 1926.01 - 2000.12)**

| R.R.A. | E.I.S. | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | **(A) Mean optimal allocation to stocks (%):** $\mathbf{E}\,[\pi_t(y_t)] = \pi(\theta) \times \mathbf{100}$ | | | | | | | |
| | 1/0.75 | 1.00 | 1/1.5 | 1/2 | 1/4 | 1/10 | 1/20 | 1/40 |
| 0.75 | 305.92 | 305.66 | 305.42 | 305.32 | 305.17 | 305.09 | 305.07 | 305.05 |
| 1.00 | 226.55 | 226.55 | 226.55 | 226.55 | 226.55 | 226.55 | 226.55 | 226.55 |
| 1.50 | 149.30 | 149.32 | 149.34 | 149.35 | 149.37 | 149.38 | 149.38 | 149.38 |
| 2.00 | 111.38 | 111.37 | 111.37 | 111.37 | 111.37 | 111.37 | 111.37 | 111.37 |
| 4.00 | 55.26 | 55.24 | 55.21 | 55.20 | 55.18 | 55.16 | 55.16 | 55.16 |
| 10.0 | 22.01 | 21.99 | 21.97 | 21.96 | 21.94 | 21.93 | 21.93 | 21.93 |
| 20.0 | 10.99 | 10.98 | 10.97 | 10.96 | 10.95 | 10.94 | 10.94 | 10.94 |
| 40.0 | 5.49 | 5.48 | 5.48 | 5.47 | 5.47 | 5.47 | 5.47 | 5.46 |
| | **(B) Ratio of hedging demand over myopic demand (%):** | | | | | | | |
| | 1/0.75 | 1.00 | 1/1.5 | 1/2 | 1/4 | 1/10 | 1/20 | 1/40 |
| 0.75 | 1.28 | 1.19 | 1.11 | 1.08 | 1.03 | 1.00 | 0.99 | 0.99 |
| 1.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 1.50 | -1.15 | -1.13 | -1.12 | -1.11 | -1.10 | -1.10 | -1.09 | -1.09 |
| 2.00 | -1.68 | -1.68 | -1.68 | -1.68 | -1.68 | -1.68 | -1.68 | -1.68 |
| 4.00 | -2.43 | -2.47 | -2.52 | -2.54 | -2.58 | -2.60 | -2.61 | -2.61 |
| 10.0 | -2.86 | -2.94 | -3.02 | -3.07 | -3.14 | -3.18 | -3.20 | -3.21 |
| 20.0 | -3.00 | -3.09 | -3.19 | -3.25 | -3.33 | -3.38 | -3.40 | -3.41 |
| 40.0 | -3.06 | -3.17 | -3.28 | -3.33 | -3.42 | -3.48 | -3.50 | -3.51 |

**Note**: Panel A reports mean optimal percentage allocations to stocks for different coefficients of relative risk aversion and elasticities of intertemporal subsitution of consumption. Panel B reports the percentage ratio of intertemporal hedging portfolio demand over myopic portfolio demand, which is independent of the level of precision or volatility. These numbers are based on the monthly parameter estimates of the joint process for return and volatility reported in Table I.

## TABLE III

### Mean Optimal Percentage Allocation to Stocks and Percentage Hedging Demand Over Myopic Demand (Sample: 1871 - 2000)

| R.R.A. | E.I.S. | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | **(A) Mean optimal allocation to stocks (%):** $\mathbf{E}\ [\pi_t(y_t)] = \pi(\theta) \times 100$ | | | | | | | |
| | 1/0.80 | 1.00 | 1/1.5 | 1/2 | 1/4 | 1/10 | 1/20 | 1/40 |
| 0.80 | 289.13 | 281.23 | 277.71 | 276.64 | 275.43 | 274.85 | 274.68 | 274.60 |
| 1.00 | 213.79 | 213.79 | 213.79 | 213.79 | 213.79 | 213.79 | 213.79 | 213.79 |
| 1.50 | 135.34 | 135.42 | 135.54 | 135.60 | 135.69 | 135.74 | 135.76 | 135.77 |
| 2.00 | 99.54 | 99.42 | 99.25 | 99.16 | 99.01 | 98.91 | 98.88 | 98.86 |
| 4.00 | 48.54 | 48.31 | 47.97 | 47.78 | 47.45 | 47.24 | 47.16 | 47.11 |
| 10.0 | 19.16 | 19.03 | 18.82 | 18.71 | 18.51 | 18.38 | 18.34 | 18.31 |
| 20.0 | 9.54 | 9.47 | 9.35 | 9.29 | 9.18 | 9.11 | 9.08 | 9.07 |
| 40.0 | 4.76 | 4.72 | 4.66 | 4.63 | 4.57 | 4.54 | 4.52 | 4.51 |
| | **(B) Ratio of hedging demand over myopic demand (%):** | | | | | | | |
| | 1/0.80 | 1.00 | 1/1.5 | 1/2 | 1/4 | 1/10 | 1/20 | 1/40 |
| 0.80 | 8.19 | 5.24 | 3.92 | 3.52 | 3.06 | 2.85 | 2.79 | 2.76 |
| 1.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 1.50 | -5.04 | -4.98 | -4.90 | -4.86 | -4.80 | -4.76 | -4.74 | -4.74 |
| 2.00 | -6.88 | -6.99 | -7.15 | -7.24 | -7.37 | -7.47 | -7.50 | -7.51 |
| 4.00 | -9.19 | -9.61 | -10.25 | -10.60 | -11.22 | -11.62 | -11.77 | -11.85 |
| 10.0 | -10.38 | -11.00 | -11.95 | -12.49 | -13.40 | -14.01 | -14.23 | -14.35 |
| 20.0 | -10.76 | -11.44 | -12.49 | -13.09 | -14.10 | -14.77 | -15.02 | -15.14 |
| 40.0 | -10.94 | -11.66 | -12.76 | -13.38 | -14.44 | -15.14 | -15.40 | -15.53 |

**Note**: Panel A reports mean optimal percentage allocations to stocks for different coefficients of relative risk aversion and elasticities of intertemporal subsitution of consumption. Panel B reports the percentage ratio of intertemporal hedging portfolio demand over myopic portfolio demand, which is independent of the level of precision or volatility. These numbers are based on the annual parameter estimates of the joint process for return and volatility reported in Table I.

**TABLE IV**

**Optimal Consumption-Wealth Ratio and Long-Term Expected Return on Wealth (Sample: 1926.01 - 2000.12)**

| R.R.A. | E.I.S. | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | **(A) Consumption-Wealth ratio (%):** $C_t/X_t = \exp\{E[c_t - x_t]\} \times 100$ | | | | | | | |
| | 1/0.75 | 1.00 | 1/1.5 | 1/2 | 1/4 | 1/10 | 1/20 | 1/40 |
| 0.75 | 3.30 | 6.00 | 8.68 | 10.01 | 12.01 | 13.21 | 13.61 | 13.81 |
| 1.00 | 4.44 | 6.00 | 7.56 | 8.34 | 9.51 | 10.22 | 10.45 | 10.57 |
| 1.50 | 5.51 | 6.00 | 6.49 | 6.74 | 7.11 | 7.33 | 7.41 | 7.45 |
| 2.00 | 6.02 | 6.00 | 5.98 | 5.97 | 5.95 | 5.94 | 5.93 | 5.93 |
| 4.00 | 6.77 | 6.00 | 5.23 | 4.84 | 4.25 | 3.90 | 3.79 | 3.73 |
| 10.0 | 7.21 | 6.00 | 4.79 | 4.18 | 3.27 | 2.72 | 2.54 | 2.45 |
| 20.0 | 7.36 | 6.00 | 4.64 | 3.96 | 2.95 | 2.33 | 2.13 | 2.03 |
| 40.0 | 7.43 | 6.00 | 4.57 | 3.86 | 2.78 | 2.14 | 1.93 | 1.82 |
| | **(B) Long-Term expected return on wealth (%):** $(\pi(\theta)(\mu - r) + r) \times 100$ | | | | | | | |
| | 1/0.75 | 1.00 | 1/1.5 | 1/2 | 1/4 | 1/10 | 1/20 | 1/40 |
| 0.75 | 26.31 | 26.29 | 26.27 | 26.26 | 26.25 | 26.24 | 26.24 | 26.24 |
| 1.00 | 19.87 | 19.87 | 19.87 | 19.87 | 19.87 | 19.87 | 19.87 | 19.87 |
| 1.50 | 13.61 | 13.61 | 13.61 | 13.61 | 13.61 | 13.61 | 13.61 | 13.61 |
| 2.00 | 10.53 | 10.53 | 10.53 | 10.53 | 10.53 | 10.53 | 10.53 | 10.53 |
| 4.00 | 5.98 | 5.98 | 5.98 | 5.98 | 5.97 | 5.97 | 5.97 | 5.97 |
| 10.0 | 3.28 | 3.28 | 3.28 | 3.28 | 3.28 | 3.28 | 3.28 | 3.28 |
| 20.0 | 2.39 | 2.39 | 2.39 | 2.39 | 2.39 | 2.39 | 2.39 | 2.39 |
| 40.0 | 1.95 | 1.94 | 1.94 | 1.94 | 1.94 | 1.94 | 1.94 | 1.94 |

**Note**: Panel A reports percentage exponentiated mean optimal log consumption-wealth ratios, i.e., 100 times the exponential of $E[log(C_t/W_t)]$, for different coefficients of relative risk aversion and elasticities of intertemporal substitution of consumption. Panel B reports the percentage unconditional mean of the log return on wealth. These numbers are based on the monthly parameter estimates of the joint process for return and volatility reported in Table I.

## TABLE V

**Mean Optimal Percentage Allocation to Stocks When Expected Stock Excess Returns Are an Affine Function of Volatility (Sample: 1926.01 - 2000.12)**

$$E_t[dS_t/S_t - rdt] = \alpha_1 + \alpha_2 v_t$$

| R.R.A. | $\alpha_2$ | | | | | | |
|---|---|---|---|---|---|---|---|
| **(A) Mean optimal allocation to stocks (%):** | | | | | | | |
| | -2.00 | -0.75 | -0.25 | 0.00 | 0.25 | 0.75 | 2.00 |
| 0.75 | 495.58 | 373.40 | 327.35 | 305.32 | 283.36 | 239.73 | 133.11 |
| 1.00 | 357.20 | 274.97 | 242.22 | 226.55 | 210.88 | 179.55 | 101.21 |
| 1.50 | 231.22 | 179.73 | 159.20 | 149.35 | 139.47 | 119.53 | 68.56 |
| 2.00 | 170.46 | 133.40 | 118.52 | 111.37 | 104.17 | 89.59 | 51.88 |
| 4.00 | 82.88 | 65.64 | 58.60 | 55.20 | 51.76 | 44.75 | 26.30 |
| 10.0 | 32.58 | 26.00 | 23.28 | 21.96 | 20.62 | 17.89 | 10.62 |
| 20.0 | 16.20 | 12.95 | 11.61 | 10.96 | 10.30 | 8.94 | 5.32 |
| 40.0 | 8.07 | 6.47 | 5.80 | 5.47 | 5.15 | 4.47 | 2.67 |
| **(B) Intercept of hedging demand (%):** | | | | | | | |
| | -2.00 | -0.75 | -0.25 | 0.00 | 0.25 | 0.75 | 2.00 |
| 0.75 | -2.84 | -0.33 | -0.04 | 0.00 | -0.04 | -0.37 | -2.40 |
| 1.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 1.50 | 1.12 | 0.19 | 0.02 | 0.00 | 0.02 | 0.20 | 1.42 |
| 2.00 | 1.34 | 0.21 | 0.02 | 0.00 | 0.03 | 0.23 | 1.65 |
| 4.00 | 1.09 | 0.17 | 0.02 | 0.00 | 0.02 | 0.18 | 1.30 |
| 10.0 | 0.54 | 0.08 | 0.01 | 0.00 | 0.01 | 0.09 | 0.64 |
| 20.0 | 0.29 | 0.04 | 0.00 | 0.00 | 0.01 | 0.05 | 0.34 |
| 40.0 | 0.15 | 0.02 | 0.00 | 0.00 | 0.00 | 0.02 | 0.18 |
| **(C) Slope of hedging demand times $\theta$ (%):** | | | | | | | |
| | -2.00 | -0.75 | -0.25 | 0.00 | 0.25 | 0.75 | 2.00 |
| 0.75 | 22.15 | 7.10 | 4.44 | 3.25 | 2.22 | 0.71 | 0.56 |
| 1.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 1.50 | -8.03 | -3.77 | -2.30 | -1.68 | -1.14 | -0.37 | -0.33 |
| 2.00 | -9.48 | -4.30 | -2.61 | -1.90 | -1.30 | -0.42 | -0.38 |
| 4.00 | -7.51 | -3.27 | -1.98 | -1.44 | -0.98 | -0.32 | -0.30 |
| 10.0 | -3.68 | -1.58 | -0.95 | -0.70 | -0.47 | -0.15 | -0.15 |
| 20.0 | -1.95 | -0.84 | -0.50 | -0.37 | -0.25 | -0.08 | -0.08 |
| 40.0 | -1.01 | -0.43 | -0.26 | -0.19 | -0.13 | -0.04 | -0.04 |

**Note**: Panel A reports mean optimal allocations to stocks based on equation (37) in Proposition 3, for different coefficients of relative risk aversion and values of $\alpha_2$. The elasticity of intertemporal subsitution of consumption is set to 1/2 throughout the table. Panel B reports the percentage value of the intercept of the intertemporal hedging component, and Panel C reports the percentage value of the slope of the intertemporal hedging demand times $\theta$, the unconditional mean of precision. Mean intertemporal hedging demands obtain by adding the numbers in Panel B and C. These numbers are based on the monthly parameter estimates of the joint process for return and volatility reported in Table I, except that we vary $\alpha_1$ and $\theta$ as we vary $\alpha_2$ as to hold the unconditional mean and variance of stock returns constant throughout the table. The benchmark values for the mean and variance are those implied by monthly estimates of the model with constant expected returns (i.e., with $\alpha_2 = 0$) shown in Table I. To hold the mean excess return constant, we set $\alpha_1 = \mu - r - \alpha_2 E[v_t]$, where $E[v_t]$ is given in equation (4). To keep the variance of stock returns constant, for each value of $\alpha_2$, we repeatedly simulate the joint process of stock returns and volatility with different values of $\theta$ until we find the value of $\theta$ for which the unconditional variance of stock returns equals the benchmark value. Each Monte Carlo simulation is based on 10,000 paths of stock returns and volatility, each one 30 years long, with a time step $dt = 0.01$ (about 3 days).

**TABLE VI**

**Unconditional Standard Deviation of the Optimal Log Consumption-Wealth Ratio (%) (Sample: 1926.01 - 2000.12)**

$\sqrt{A_1^2\sigma^2\theta/2\kappa}\times 100$

| R.R.A. | E.I.S. | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 1/0.75 | 1.00 | 1/1.5 | 1/2 | 1/4 | 1/10 | 1/20 | 1/40 |
| 0.75 | 1.70 | 0.00 | 1.48 | 2.14 | 3.07 | 3.59 | 3.76 | 3.84 |
| 1.00 | 1.20 | 0.00 | 1.11 | 1.63 | 2.39 | 2.82 | 2.96 | 3.03 |
| 1.50 | 0.76 | 0.00 | 0.74 | 1.11 | 1.64 | 1.96 | 2.07 | 2.12 |
| 2.00 | 0.56 | 0.00 | 0.56 | 0.84 | 1.25 | 1.51 | 1.59 | 1.63 |
| 4.00 | 0.27 | 0.00 | 0.28 | 0.42 | 0.64 | 0.78 | 0.82 | 0.85 |
| 10.0 | 0.11 | 0.00 | 0.11 | 0.17 | 0.26 | 0.32 | 0.34 | 0.35 |
| 20.0 | 0.05 | 0.00 | 0.06 | 0.09 | 0.13 | 0.16 | 0.17 | 0.17 |
| 40.0 | 0.03 | 0.00 | 0.03 | 0.04 | 0.07 | 0.08 | 0.08 | 0.09 |

**Note**: This table reports annualized, percentage values of the unconditional standard deviation of the optimal log consumption-wealth ratio for different coefficients of relative risk aversion and elasticities of intertemporal substitution of consumption. These numbers are based on the monthly parameter estimates of the joint process for return and volatility reported in Table I. .

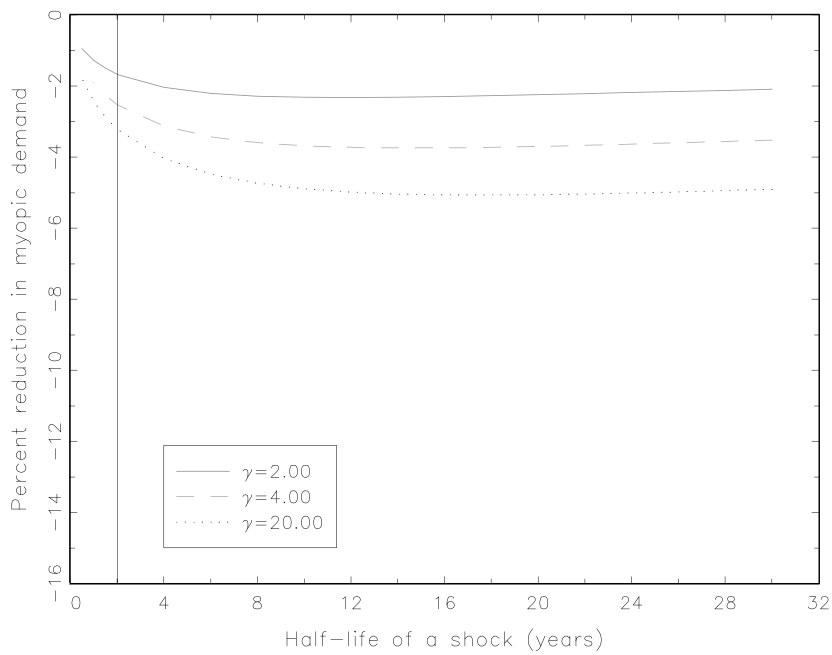


Figure 1: Effects on optimal portfolio demand of compensated changes in the persistence of shocks to precision.

This figure plots the ratio of intertemporal hedging demand to myopic demand for investors with $\psi = 1/2$ and $\gamma = \{2, 4, 20\}$ as we consider changes compensated changes in $\kappa$ that leave the first and second unconditional moments of stock returns and precision constant at the values implied by the monthly estimates shown in Table I. This figure considers values $\kappa$ implying half-lives of a shock to precision between 6 months and 30 years. The vertical line intersects the horizontal axis at the value implied by the monthly estimate of $\kappa$.

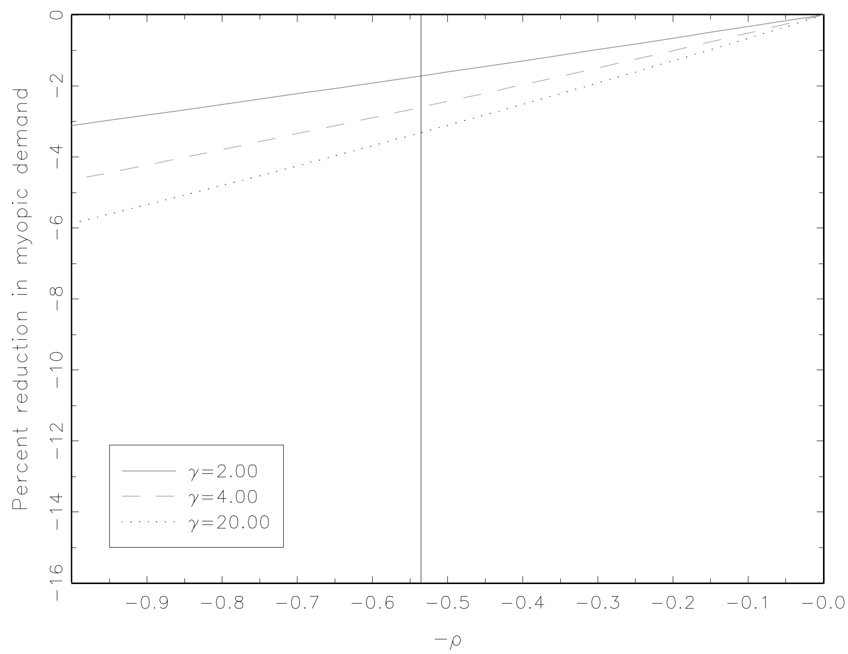


Figure 2: Effects on optimal portfolio demand of compensated changes in the instantaneous correlation of shocks to volatility and stock returns ($-\rho$).

This figure plots the ratio of intertemporal hedging demand to myopic demand for investors with $\psi = 1/2$ and $\gamma = \{2, 4, 20\}$ as we consider changes in the instantaneous correlation betwen shocks to volatility and stock returns, while holding the rest of the parameters constant at their monthly estimates shown in Table I. The vertical line intersects the horizontal axis at the value implied by the monthly estimate of $-\rho$.

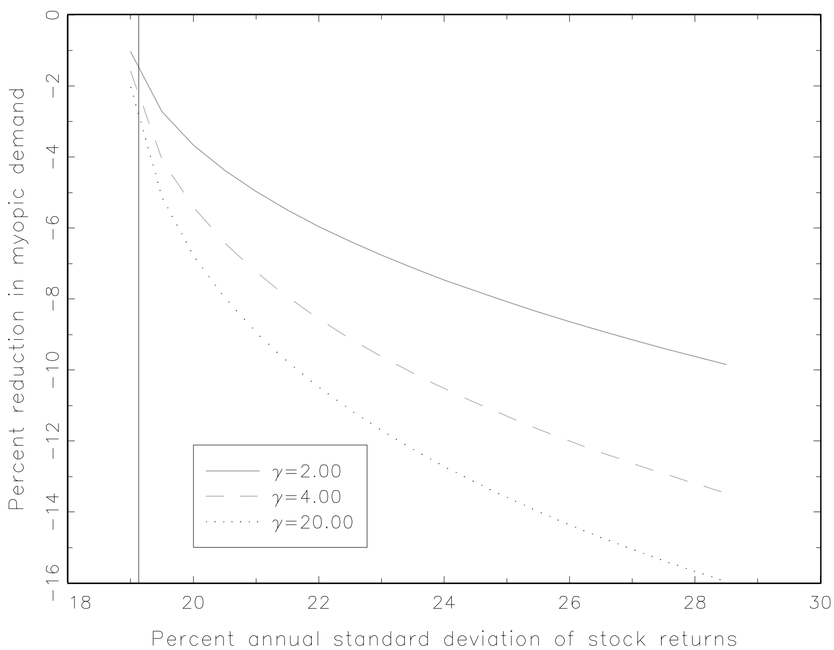


Figure 3: Effect on optimal portfolio demand of changes in the variance of precision.

This figure plots the ratio of intertemporal hedging demand to myopic demand for investors with $\psi = 1/2$ and $\gamma = \{2, 4, 20\}$ as we consider changes in the unconditional variance of precision, while keeping its mean constant. Since $E[y_t] = \theta$ and $\mathrm{Var}(y_t) = \sigma^2\theta/2\kappa$, we implement this exercise by changing $\sigma$, and holding the rest of the parameters constant at their monthly estimates shown in Table I. We consider values of $\sigma$ implying stock return volatilities between 18% and 30%. To facilitate interpretation, the horizontal axis plots stock return volatility nstead of $\sigma$ or $\mathrm{Var}(y_t)$. The vertical line intersects the horizontal axis at the value implied by the monthly estimate of the unconditional standard deviation of stock returns.

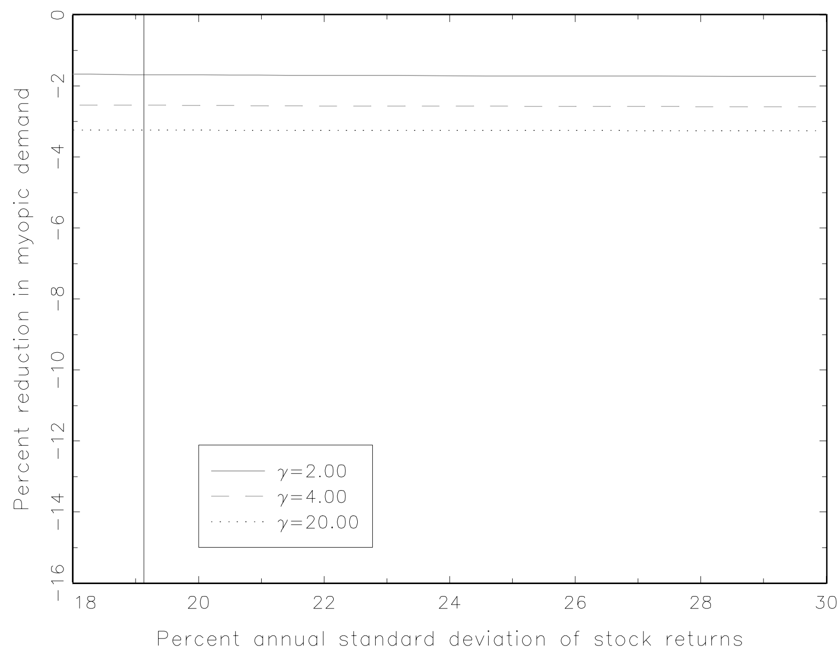


Figure 4: Effect on optimal portfolio demand of changes in the unconditional variance of stock returns, holding the unconditional variance of precision constant.

This figure plots the ratio of intertemporal hedging demand to myopic demand for investors with $\psi = 1/2$ and $\gamma = \{2, 4, 20\}$ as we consider changes in the unconditional variance of stock returns, holding the unconditional variance of precision constant. Since $\mathrm{Var}(dS_t/S_t) \approx 1/\theta + \mathrm{Var}(y_t)/\theta^3$ and $\mathrm{Var}(y_t) = \sigma^2\theta/2\kappa$, we implement the change in $\mathrm{Var}(dS_t/S_t)$ by varying $\theta$, and we hold $\mathrm{Var}(y_t)$ constant by varying $\sigma$ appropriately. We hold the rest of the parameters constant at their monthly estimates shown in Table I. We consider values of $\theta$ implying stock return volatilities between 18% and 30%. The vertical line intersects the horizontal axis at the value implied by the monthly estimate of the unconditional standard deviation of stock returns.