On the Application of Intervening Variables for Stochastic Finite Element Analysis

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Abstract

This paper investigates the application of first order Taylor series expansion considering intervening variables for estimating second order statistics for problems of stochastic finite element analysis. In particular, two types of intervening variables are considered. The focus is on studying the response of linear elastic static structures whose properties are characterized by means of log-normal random fields. Results obtained indicate that the use of intervening variables can lead to improved estimation of second order statistics when compared with traditional approaches based on Taylor series while numerical efforts remain similar.

Keywords: Stochastic Finite Element Method, Log-normal random field, Second order statistics, Intervening variables, Taylor series

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1. Introduction

Within the field of computational mechanics, several different methods have been developed for predicting structural response (see, e.g. [1, 2, 3, 4]). Among these methods, the Finite Element Method (FEM) is a well established technique in the engineering community [5, 6, 7]. The application of FEM demands that characteristics, properties and operation conditions of the structure are identified adequately in order to ensure an accurate prediction of structural response. Hence, on one hand, it becomes necessary to model the behavior of structural materials by means of appropriate constitutive laws. Similarly, the dimensions of structural members should be modeled properly. On the other hand, loading conditions that the structure will experience during its lifetime should be identified and quantified. However, it should be noted that in situations of practical interest, it will be quite challenging (if not impossible) to characterize all these parameters precisely due to the inherent uncertainty associated with future conditions of a structure, its properties and the external loading [8]. The inherent uncertainty associated with relevant parameters of the structural model implies that uncertainty is also present in the prediction of the structural response when applying FEM, i.e. the response can no longer be characterized as a

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deterministic, single magnitude. In this scenario, probability theory offers an appropriate means for characterizing the uncertainty in the structural response [9]. The incorporation of methods for assessing the effects of uncertainty using FEM leads to the Stochastic Finite Element Method (SFEM) (see, e.g. [10]). Within this context, the aim of this contribution is presenting an approach for uncertainty quantification for SFEM based on first order Taylor series expansion and intervening variables.

Approaches for SFEM have been classified in three main groups [11]. The first one refers to methods that employ *Taylor series expansions* (see, e.g [12, 13, 14]). Techniques belonging to this class have been often referred as *perturbation methods*. The key idea behind this type of strategies is generating an approximate expression for representing the structural response as an explicit function of the uncertain parameters. Perturbation methods are often simple to implement and usually numerically inexpensive, although numerical costs may grow rapidly when the number of uncertain parameters is large and a second order Taylor series is considered. Furthermore, methods based on expansions are usually not very accurate, particularly when the variability in the input parameters is large. A technique related conceptually to perturbation methods is the so-called *high dimensional model representation* [15], where the structural response is approximated by means of low order polynomials. The second class of approaches for SFEM involves the so called *spectral methods*, where uncertainty in structural response is captured by means of polynomials whose coefficients are determined using a Galerkin approach [16, 17, 18, 19, 20, 21]. Although this class of methods leads to results which may be more accurate than those produced with expansion methods [22], their main drawback is that the size of the matrices associated with the problem increases significantly according to the number of uncertain parameters involved and also the degree of the polynomials that capture the variability of the response [21]. However, reduction techniques may allow applying spectral approaches in problems involving a higher number of uncertain parameters [23, 24, 25]. The third group of approaches refers to methods based on *simulation*, including Monte Carlo Simulation (MCS) and its more advanced variants such as Importance Sampling, Line Sampling or Subset Simulation [26, 27, 28, 29, 30, 31]. Simulation methods are robust to both the number of uncertain input parameters and also the variability of these input parameters. However, numerical efforts involved with simulation methods are not negligible as a number of samples need to be computed.

From the discussion above, it is evident that one of the major challenges faced by all approaches for SFEM is the high numerical cost associated with the necessity of either constructing approximations of the structural response (as in the case of *Taylor expansions* and *spectral methods*) or repeated computation of the structural response (as in *simulation methods*). Interestingly, the field of structural optimization (see, e.g. [32, 33]) faces a similar problem: the identification of a structural design that is optimal (according to some prescribed criterion) is a numerically demanding task as it involves the repeated computation of the structural response for different sets of trial designs. In view of this issue, several different specialized techniques have been developed for coping with high numerical costs associated with optimization. Among the different techniques developed for performing structural optimization, Taylor series expansions have been used widely in a similar way as used for stochastic finite element analysis. Nonetheless, a major difference between the application of Taylor series for optimization and stochastic analysis is that in the former, the concept of *intervening variables* is applied extensively (see, e.g. [34, 35, 36, 37, 38, 39, 40, 41]). The so-called intervening variables allow constructing high quality approximations of the structural response with numerical costs similar to those of a regular Taylor series expansion. Although the concept of intervening variables may bring substantial advantages, its use within the context of SFEM has remained almost unexplored, except by efforts reported in [42], where the application of reciprocal intervening variables is investigated for predicting the second-order statistics of the response of truss structures for which the Young's moduli of the bars are modeled as uncorrelated random variables. Thus, this contribution (which is an expanded and revised version of [43]) studies the application of intervening variables and Taylor series expansion for estimating second order statistics of the structural response. In particular, two types of intervening variables are investigated. The focus is on linear, statical problems involving structures whose Young's modulus is modeled as a log-normal random field. The objective is determining whether or not the use of intervening variables and Taylor series expansions can lead to improved estimations of second order statistics when compared to approaches based solely on Taylor series (considering no intervening variables). In this context, it should be noted the scope of this contribution is restricted to estimation of second order statistics only. Estimation of higher order statistics of a certain response of interest (such as skewness and kurtosis) can be certainly relevant in several applications (as already studied in [44, 45]). Nonetheless, estimation of these higher order statistics is beyond the scope of this work.

The structure of this contribution is as follows. In Section 2, the problem under study is formulated. Section 3 describes estimation of second order statistics of the structural response using first and second order Taylor expansions as performed in the literature of SFEM. Novel contributions of this paper are contained in Section 4, where the concept of intervening variables and its application for quantifying the uncertainty in the structural response is presented. In Section 5, the application of methods presented in Sections 3 and 4 is illustrated by means of numerical examples involving different types of structures. The contribution closes with some conclusions and outlook for possible extensions of the work reported herein.

2. Formulation of the Problem

2.1. Characterization of Structural Response and Uncertainty Quantification

Consider a structure which is modeled as linear static. It is assumed uncertainties are present in the Young's modulus E. These uncertainties are modeled by means of an homogeneous log-normal random field $E(\boldsymbol{x})$, where $\boldsymbol{x}$ is a vector denoting position. The structure is modeled by means of the FEM, comprising N_e elements and N_d degrees-of-freedom (DOFs). Applying the mid-point method [46], the random field $E(\boldsymbol{x})$ is represented approximately by its discrete version $\hat{E}(\boldsymbol{x})$. Note that $\hat{E}(\boldsymbol{x})$ can be interpreted as a collection of N_e log-normal correlated random variables (recall N_e is the number of finite elements). This collection of random variables is expressed in terms of a vector of M independent, identically distributed standard normal random variables $\boldsymbol{\xi} = \langle \xi_1, \xi_2, \ldots, \xi_M \rangle^T$, $\boldsymbol{\xi} \in \mathbb{R}^M$, and its associated joint probability density function $f_\xi(\boldsymbol{\xi})$ (details on this representation are discussed in Section 2.2). As the structural system is modeled considering a total of N_d DOFs, the equilibrium condition is characterized by means of a system of N_d linear equations.

$$\boldsymbol{K}(\boldsymbol{\xi})\boldsymbol{u}(\boldsymbol{\xi}) = \boldsymbol{f} \tag{1}$$

In the above equation, $\boldsymbol{K}(\boldsymbol{\xi})$ is the $N_d \times N_d$ stiffness matrix of the structural system, $\boldsymbol{u}(\boldsymbol{\xi})$ is the $N_d \times 1$ vector of displacements and $\boldsymbol{f}$ is the $N_d \times 1$ vector of (equivalent) nodal forces. From eq. (1), it is noted that in view of the assumption that the Young's modulus is modeled as a random field, the stiffness matrix depends on the vector of uncertain parameters $\boldsymbol{\xi}$. However, it is assumed that the vector of nodal forces is deterministic. Clearly, in a more general case, the force vector may also depend on the random variable vector.

According to eq. (1), uncertainties associated with the Young's modulus (the *input* parameter of the model) propagate to the vector of displacements (the *output*). Hence, the vector $\boldsymbol{u}(\boldsymbol{\xi})$ is actually a vector of random variables that can be fully characterized by its probability density function $f_u(\boldsymbol{u})$. Although such characterization would be most informative, the estimation of $f_u(\boldsymbol{u})$ may be extremely involved, particularly for cases where the dimension M of the vector $\boldsymbol{\xi}$ is considerable. Hence, in this contribution, the uncertainty associated with the displacement vector $\boldsymbol{u}$ will be characterized by means of second-order statistics, namely expected value and covariance. For example, the expected value of the displacement of the n-th degree of freedom is:

$$\mathrm{E}[u_n] = \int_{\boldsymbol{\xi} \in \boldsymbol{\Xi}} u_n(\boldsymbol{\xi}) f_\xi(\boldsymbol{\xi}) d\boldsymbol{\xi}, \; n = 1, \ldots, N_d \tag{2}$$

where u_n is the n-th component of the displacement vector $\boldsymbol{u}$ and $\mathrm{E}[\cdot]$ denotes expected value. Similarly, the displacement covariance between the n-th and m-th degree of freedom is:

$$\mathrm{Cov}[u_n, u_m] = \int_{\boldsymbol{\xi}\in\boldsymbol{\Xi}} u_n(\boldsymbol{\xi}) u_m(\boldsymbol{\xi}) f_\xi(\boldsymbol{\xi}) d\boldsymbol{\xi} - \mathrm{E}[u_n]\mathrm{E}[u_m], \quad n, m = 1, \ldots, N_d \tag{3}$$

where $\mathrm{Cov}[\cdot,\cdot]$ denotes covariance.

2.2. Modeling Spatial Variability of Young's Modulus by means of Log-normal Random Fields

Let $E(\boldsymbol{x})$ be the homogeneous log-normal random field model for the Young's modulus, characterized by its expected value $\bar{E}(\boldsymbol{x})$ and covariance between points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ equal to $C^{EE}(\boldsymbol{x_1}, \boldsymbol{x_2})$, where $\boldsymbol{x}_j,\ j = 1, 2$ is a vector describing the position of point j. Applying the mid-point method [46], the random field $E(\boldsymbol{x})$ is represented approximately by its discrete version $\hat{E}(\boldsymbol{x})$ comprising N_e random variables. This discrete log-normal random field is formulated by exponentiating an associated discrete Gaussian field (see, e.g. [47, 48]). The latter random field is characterized by means of an expected value vector $\boldsymbol{\mu}^G$ of dimension $N_e \times 1$ and a covariance matrix $\boldsymbol{C}^{EE,G}$ of dimension $N_e \times N_e$ which are defined as shown in eqs. (5) and (4), respectively.

$$C^{EE,G}_{pq} = \ln\left(\frac{C^{EE}(\bar{\boldsymbol{x}}_p, \bar{\boldsymbol{x}}_q)}{E(\bar{\boldsymbol{x}}_p)E(\bar{\boldsymbol{x}}_q)} + 1\right),\ p, q = 1, \ldots, N_e \tag{4}$$

$$\mu^G_p = \ln\left(\bar{E}(\bar{\boldsymbol{x}}_p)\right) - \frac{1}{2}C^{EE,G}_{pp},\ p = 1, \ldots, N_e \tag{5}$$

In the above equations, $C^{EE,G}_{pq}$ is the (p, q)-th term of the covariance matrix of the underlying Gaussian random field, μ^G_p is p-th component of vector $\boldsymbol{\mu}^G$ and $\bar{\boldsymbol{x}}_j$ is the vector describing the midpoint coordinates of the j-th finite element.

In case the underlying discrete Gaussian field is modeled using the Karhunen-Loève (K-L) expansion (see, e.g. [49, 50, 51]), the Young's modulus associated with the p-th finite element $\hat{E}_p$ is represented as:

$$\hat{E}_p = e^{\mu^G_p + \sum_{i=1}^{M} \zeta_{pi}\xi_i},\ p = 1, \ldots, N_e \tag{6}$$

where $\xi_i,\ i = 1, \ldots, M$ are independent standard Gaussian random variables, M is the number of terms retained for representing the aforementioned underlying Gaussian field when applying the K-L expansion (note that $M \leq N_e$), μ^G_p is p-th component of vector $\boldsymbol{\mu}^G$ (see eq. (5)) and ζ_{pi} is defined as:

$$\zeta_{pi} = \sqrt{\lambda_i}\phi_{pi},\ i = 1, \ldots, M,\ p = 1, \ldots, N_e \tag{7}$$

where λ_i denotes the i-th eigenvalue and ϕ_{pi} is the p-th component of the eigenvector $\boldsymbol{\phi}_i$ associated with following eigenproblem.

$$\boldsymbol{C}^{EE,G}\boldsymbol{\phi}_i = \lambda_i \boldsymbol{\phi}_i, \; i = 1, \ldots, M \tag{8}$$

In eq. (8), it is implicitly assumed that the M largest eigenvalues of $\boldsymbol{C}^{EE,G}$ are sorted such that $\lambda_1 \geq \lambda_2 \ldots \geq \lambda_M$.

3. Taylor Series Expansion

3.1. Approximate Representation of Structural Response

The estimation of second order statistics as stated in eqs. (2) and (3) is quite challenging, as it comprises multidimensional integrals where the displacement vector $\boldsymbol{u}(\boldsymbol{\xi})$ is known pointwise only. A possible means to overcome this issue is approximating the displacement vector by means of an expression that is an explicit function of the random variable vector $\boldsymbol{\xi}$ by means of a Taylor series (see, e.g. [12, 13, 14, 52]). In order to formulate the approximation for the displacement vector, assume the uncertain stiffness matrix $\boldsymbol{K}(\boldsymbol{\xi})$ involved in eq. (1) is expressed in terms of its corresponding Taylor series expansion about $\boldsymbol{\xi}^0 = \langle 0, \ldots, 0 \rangle^T$:

$$\boldsymbol{K}(\boldsymbol{\xi}) = \boldsymbol{K}\left(\boldsymbol{\xi}^0\right) + \sum_{i=1}^{M} \boldsymbol{K}_{,i}\xi_i + \frac{1}{2}\sum_{i=1}^{M}\sum_{j=1}^{M} \boldsymbol{K}_{,ij}\xi_i\xi_j + \ldots \tag{9}$$

where the matrices $\boldsymbol{K}_{,i}$ and $\boldsymbol{K}_{,ij}$ (each of dimension $N_d \times N_d$) are defined as shown below.

$$\boldsymbol{K}_{,i} = \left.\frac{\partial \boldsymbol{K}(\boldsymbol{\xi})}{\partial \xi_i}\right|_{\boldsymbol{\xi}=\boldsymbol{\xi}^0}, \; i = 1, \ldots, M \tag{10}$$

$$\boldsymbol{K}_{,ij} = \left.\frac{\partial^2 \boldsymbol{K}(\boldsymbol{\xi})}{\partial \xi_i \partial \xi_j}\right|_{\boldsymbol{\xi}=\boldsymbol{\xi}^0}, \; i, j = 1, \ldots, M \tag{11}$$

Then, the Taylor series associated with the displacement vector $\boldsymbol{u}(\boldsymbol{\xi})$ about $\boldsymbol{\xi}^0 = \langle 0, \ldots, 0 \rangle^T$ is:

$$\boldsymbol{u}(\boldsymbol{\xi}) = \boldsymbol{u}\left(\boldsymbol{\xi}^0\right) + \sum_{i=1}^{M} \boldsymbol{u}_{,i}\xi_i + \frac{1}{2}\sum_{i=1}^{M}\sum_{j=1}^{M} \boldsymbol{u}_{,ij}\xi_i\xi_j + \ldots \tag{12}$$

where the vectors $\boldsymbol{u}\left(\boldsymbol{\xi}^0\right)$, $\boldsymbol{u}_{,i}$ and $\boldsymbol{u}_{,ij}$ (each of dimension $N_d \times 1$) are the nominal displacement, first and second order derivative of the displacement, respectively. The latter three vectors are defined as (see, e.g. [10, 13]):

$$\boldsymbol{u}\left(\boldsymbol{\xi}^0\right) = \boldsymbol{K}\left(\boldsymbol{\xi}^0\right)^{-1}\boldsymbol{f} \tag{13}$$

$$\boldsymbol{u}_{,i} = -\boldsymbol{K}\left(\boldsymbol{\xi}^0\right)^{-1}\boldsymbol{K}_{,i}\boldsymbol{u}\left(\boldsymbol{\xi}^0\right),\ i = 1,\ldots,M \tag{14}$$

$$\begin{aligned}\boldsymbol{u}_{,ij} = -\boldsymbol{K}\left(\boldsymbol{\xi}^0\right)^{-1}\left(\boldsymbol{K}_{,i}\boldsymbol{u}_{,j} + \boldsymbol{K}_{,j}\boldsymbol{u}_{,i} + \boldsymbol{K}_{,ij}\boldsymbol{u}\left(\boldsymbol{\xi}^0\right)\right), \\ i,j = 1,\ldots,M\end{aligned} \tag{15}$$

Note that for approximating the displacement vector by means of a Taylor series, a single matrix factorization is required, as shown in eqs. (13), (14) and (15). In addition, the derivatives of the stiffness matrix are required as well, as shown in eqs. (10) and (11). These derivatives can be calculated using any suitable technique, e.g. finite differences, automatic differentiation, semi analytical methods, etc. (see [53, 54]). For the particular case of this contribution, the required derivatives are calculated by assembling the stiffness matrix considering the corresponding derivatives of the Young's modulus.

3.2. Estimation of Second Order Statistics

Let $\boldsymbol{u}^L(\boldsymbol{\xi})$ be the first order approximation of the displacement vector $\boldsymbol{u}(\boldsymbol{\xi})$, i.e. only the constant and linear term of eq. (12) are retained. Recalling that $\xi_i,\ i = 1,\ldots,M$, are independent standard Gaussian variables, it can be shown straightforwardly that the second order statistics of $\boldsymbol{u}^L(\boldsymbol{\xi})$ are the following [10, 13].

$$\mathrm{E}\left[u_n^L\right] = u_n\left(\boldsymbol{\xi}^0\right),\ n = 1,\ldots,N_d \tag{16}$$

$$\mathrm{Cov}\left[u_n^L, u_m^L\right] = \sum_{i=1}^{M}\sum_{j=1}^{M} u_{n,i}u_{m,j}\delta_{ij},\ n,m = 1,\ldots,N_d \tag{17}$$

In the above equations, u_n^L is the n-th component of the linear approximation of the displacement vector $\boldsymbol{u}^L(\boldsymbol{\xi})$, $u_n\left(\boldsymbol{\xi}^0\right)$ is the n-th component of the vector of nominal displacements $\boldsymbol{u}\left(\boldsymbol{\xi}^0\right)$, $u_{n,i}$ is the n-th component of the first order derivative vector of the displacement $\boldsymbol{u}_{,i}$ and δ_{ij} is the Kronecker delta, which is equal to 1 in case $i = j$ and zero, otherwise.

Now, let $\boldsymbol{u}^Q(\boldsymbol{\xi})$ be the second order approximation of the displacement vector $\boldsymbol{u}(\boldsymbol{\xi})$, i.e. the constant, linear and quadratic terms of eq. (12) are retained. Then, the second order statistics of $\boldsymbol{u}^Q(\boldsymbol{\xi})$ are the following [10, 13].

$$\mathrm{E}\left[u_n^Q\right] = u_n\left(\boldsymbol{\xi}^0\right) + \frac{1}{2}\sum_{i=1}^{M}\sum_{j=1}^{M} u_{n,ij}\delta_{ij},\ n = 1,\ldots,N_d \tag{18}$$

$$\begin{aligned}\mathrm{Cov}\left[u_n^Q, u_m^Q\right] = &\sum_{i=1}^{M}\sum_{j=1}^{M} u_{n,i}u_{m,j}\delta_{ij} + \\ &\frac{1}{4}\sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{k=1}^{M}\sum_{l=1}^{M} u_{n,ij}u_{m,kl}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right), \\ &n, m = 1,\ldots,N_d\end{aligned} \tag{19}$$

In the above equations, u_n^Q is the n-th component of the quadratic approximation of the displacement vector $\boldsymbol{u}^Q(\boldsymbol{\xi})$ and $u_{n,ij}$ is the n-th component of the second order derivative vector of the displacement $\boldsymbol{u}_{,ij}$.

From a numerical viewpoint, it should be noted that the application of a linear or a quadratic approximation involves different costs. As described in the preceding sections of this contribution, a linear approximation requires an appropriate representation of the random field (this implies in turn an eigenvalue/eigenvector analysis), the factorization of the nominal stiffness matrix $\boldsymbol{K}(\boldsymbol{\xi}^{\mathbf{0}})$ and the calculation of first order derivatives of the displacement $\boldsymbol{u}_{,i}$. A quadratic approximation requires the exact same information as in the case of the linear approximation plus the calculation of second order derivatives of the displacement $\boldsymbol{u}_{,ij}$. For the case where the number of random variables M involved in the problem is high, a quadratic approximation can be considerably more involved than a linear one as the number of second order derivatives to be calculated grows exponentially with the number of random variables M. This poses a challenge not only from the time required to calculate these second order derivatives but also from the point of view of storage of the associated matrices and vectors.

4. Application of Intervening Variables

4.1. Description

As already mentioned in Section 1, stochastic finite element analysis faces challenges similar to those found in structural optimization. That is, in both fields it is required to evaluate the structural response for different realizations of uncertain parameters and sets of design variables, respectively. Hence, the application of Taylor series expansion allows generating approximate models for representing the structural response as an explicit function of the variables of interest that can be evaluated with almost negligible numerical efforts. In this way, for the linear static case, repeated factorization of the stiffness matrix is avoided. However, the application of Taylor series within the context of stochastic finite element analysis (as shown in Section 3) or structural optimization is often limited to first order expansions, as the implementation of higher order expansions would demand evaluating higher order derivatives as well as increased computational resources for storing those derivatives. However, the accuracy of first order expansions may be quite low, even in the neighborhood of the expansion point. One of the strategies developed within the context of structural optimization for overcoming the lack of accuracy of first order Taylor expansions is the use of the so called *intervening variables* [33]. This strategy is employed because the response that is being approximated behaves more linearly with respect to these intervening variables. Thus, the accuracy of the first order expansion might be improved.

4.2. Basic Idea

In order to illustrate the application of intervening variables within the context of structural optimization, consider a linear static optimization problem involving a set of design variables grouped in the vector $\boldsymbol{x}$ of dimension n_x. Moreover, consider a set of intervening variables $y_i,\ i = 1, \ldots, n_x$ such that $y_i = y_i(x_i)$. Then, the first order Taylor series expansion of the displacement vector in terms of the intermediate design variables (denoted as $\boldsymbol{u}^I$) about the point $\boldsymbol{x}^0$ is [33]:

$$\boldsymbol{u}(\boldsymbol{x}) \approx \boldsymbol{u}^I\left(\boldsymbol{y}\left(\boldsymbol{x}\right)\right) = \boldsymbol{u}\left(\boldsymbol{y}\left(\boldsymbol{x}^0\right)\right) + \sum_{i=1}^{n_x} \left.\frac{\partial \boldsymbol{u}}{\partial y_i}\right|_{\boldsymbol{y}=\boldsymbol{y}(\boldsymbol{x}^0)} \left(y_i(x_i) - y_i\left(x_i^0\right)\right) \tag{20}$$

or alternatively:

$$\boldsymbol{u}^I\left(\boldsymbol{x}\right) = \boldsymbol{u}\left(\boldsymbol{x}^0\right) + \sum_{i=1}^{n_x} \left.\left(\frac{\partial \boldsymbol{u}}{\partial x_i}\frac{\partial x_i}{\partial y_i}\right)\right|_{\boldsymbol{x}=\boldsymbol{x}^0} \left(y_i(x_i) - y_i\left(x_i^0\right)\right) \tag{21}$$

The accuracy of the approximation shown in eq. (21) will depend upon the selection of an appropriate intervening variable y_i. For example, for the optimal design of statically determinate trusses where the design variables are the cross section areas of the bars, the displacement vector can be represented exactly by a first order Taylor expansion in case the intermediate variable is selected as the *reciprocal* [34, 35]. That is:

$$y_i = \frac{1}{x_i},\ i = 1, \ldots, n_x \tag{22}$$

For situations involving more general types of structures (not necessarily statically determinate trusses), the application of reciprocal intervening variables for constructing a first order Taylor expansion can still lead to improved results when compared to performing the expansion with respect to the original variables [36]. Another type of intervening variable that offers more flexibility than the reciprocal is the so-called *exponential* intervening variable [37, 38], that is defined as:

$$y_i = x_i^{m_i},\ i = 1, \ldots, n_x \tag{23}$$

where m_i is a real constant. Clearly, in case $m_i = -1$, the exponential intervening variable reduces to the reciprocal one. Hence, the so-called exponential intervening variable could be interpreted as a *generalization* of the reciprocal variable as an additional degree of freedom is introduced in the model. In order to determine the coefficient m_i, an approach based on a two-point approximation has been proposed [38], i.e. the coefficient is calculated based on information of the function value and its gradient at two different points. In order to ensure the exponential approximation is robust, different criteria have been proposed for limiting the values the coefficient m_i can assume, e.g. in [38], it is proposed that $-1 \leq m_i \leq 1$ while in [55] it is proposed $-3 \leq m_i \leq 0$. It has been observed that the expansion considering exponential intervening variables may improve considerably the quality of the approximation of the structural response [39].

Although the use of intervening variables for constructing approximations using first order Taylor series may bring substantial advantages, its use for stochastic finite element analysis remains almost unexplored. To the authors' best knowledge, the application of intervening variables of the reciprocal type has been investigated in [42]. In the latter contribution, second order statistics of displacement and internal forces of trusses with varying degree of static determinacy were calculated. The uncertainties involve the Young's moduli of the bars, which were modeled as uncorrelated uniform random variables. Hence, in the following, the application of first order Taylor expansion considering intervening variables is investigated. More specifically, two types of intervening variables are proposed that are based on reciprocal and exponential variables described above, respectively.

4.3. Application of Intervening Variable Type I

In order to approximate the displacement vector $\boldsymbol{u}(\boldsymbol{\xi})$ by means of a first order Taylor expansion considering intervening variables, a particular case is first analyzed. Assume a statically determinate truss of N_d DOF's and N_e bars such that the Young's moduli of each bar are modeled by uncorrelated log-normal random variables. This implies the covariance $\boldsymbol{C}^{EE}$ is a diagonal matrix. Hence, in eq. (6), the number of independent standard Gaussian random variables is $M = N_e$. That is:

$$\hat{E}_p = e^{\mu_p^G + \zeta_{pp}\xi_p}, \; p = 1, \ldots, N_e \tag{24}$$

For this particular case, the *exact* displacement of the n-th degree of freedom is:

$$u_n(\boldsymbol{\xi}) = \sum_{p=1}^{N_e} \frac{C_{pn}}{\hat{E}_p} = \sum_{p=1}^{N_e} \left(\frac{C_{pn}}{e^{\mu_p^G}} \right) \frac{1}{e^{\zeta_{pp}\xi_p}} \tag{25}$$

where C_{pn} is a constant depending on the particular DOF under analysis, the dimensions of the truss and the loading [56]. Examining eq. (25), it becomes evident that the following intervening variable would allow representing the displacement exactly for the particular case under analysis:

$$y_i(\xi_i) = \frac{1}{z_i^*(\xi_i)}, \; i = 1, \ldots, M \tag{26}$$

where $z_i(\xi_i)$ is an auxiliary variable defined as:

$$z_i^*(\xi_i) = e^{\zeta_{ii}\xi_i}, \; i = 1, \ldots, M \tag{27}$$

Clearly, the intervening variable introduced in eq. (26) is of the reciprocal type. Note the auxiliary variable $z_i^*(\xi_i)$ is introduced in view of the exponential function involved in eq. (25) due to the assumption of a log-normal random variable for modeling the Young's moduli. Now, for a general case where the structure being analyzed is neither a statically determinate truss nor are the Young's moduli modeled using uncorrelated random variables, the following intervening variable is proposed, based on eq. (26). This variable is termed in the following as intervening variable of type I.

$$y_i(\xi_i) = \frac{1}{z_i(\xi_i)} \text{ where } z_i(\xi_i) = e^{\alpha_i\xi_i}, \; i = 1, \ldots, M \tag{28}$$

In the above equation, $z_i(\xi_i)$ is an auxiliary variable while α_i is a real, constant coefficient defined such that:

$$\alpha_i = \frac{\sum_{p=1}^{N_e} |\zeta_{pi}|}{\sum_{p=1}^{N_e} I\left[|\zeta_{pi}| > 0\right]}, \; i = 1, \ldots, M \tag{29}$$

where $|\cdot|$ denotes absolute value and $I[\cdot]$ is an indicator function that is equal to 1 in case its argument holds and 0, otherwise. The coefficient α_i could be interpreted as the *mean* variability associated with the i-th term of the K-L expansion of the underlying Gaussian random field. That is, it takes into account how much the i-th random variable ξ_i contributes to the variability of the Young's modulus for each of the N_e elements of the model in terms of an average. It can be easily verified that the proposed intervening variable of type I reduces to eq. (26) for the particular case of a statically determinate truss whose Young's moduli are modeled as uncorrelated log-normal random variables. Note the coefficient α_i introduced in eq. (29) is always larger than zero as long as $\lambda_M > 0$.

Following eq. (21), the first order Taylor expansion of the n-th DOF in terms of the intervening variable of type I introduced eq. (28) (which is denoted as $u_n^{TI}(\boldsymbol{\xi})$) is:

$$u_n^{\mathrm{TI}}(\boldsymbol{\xi}) = u_n\left(\boldsymbol{\xi}^0\right) + \sum_{i=1}^{M} u_{n,i}\left(\frac{1 - e^{-\alpha_i\xi_i}}{\alpha_i}\right) \tag{30}$$

The second order statistics of $u_n^{\mathrm{TI}}(\boldsymbol{\xi})$ are given by the expressions shown below.

$$\mathrm{E}\left[u_n^{\mathrm{TI}}\right] = u_n\left(\boldsymbol{\xi}^0\right) + \sum_{i=1}^{M} u_{n,i}\left(\frac{1-e^{\alpha_i^2/2}}{\alpha_i}\right), \quad n = 1,\ldots,N_d \tag{31}$$

$$\mathrm{Cov}\left[u_n^{\mathrm{TI}}, u_m^{\mathrm{TI}}\right] = \sum_{i=1}^{M} u_{n,i}u_{m,i}\left(\frac{e^{2\alpha_i^2}-e^{\alpha_i^2}}{\alpha_i^2}\right), \quad n,m = 1,\ldots,N_d \tag{32}$$

Within the scope of this contribution, it should be noted that numerical costs associated with the implementation of a first order Taylor expansion comprising the intervening variable of type I should be slightly larger than those associated with a first order Taylor expansion considering no intervening variables. This is due to the fact although the same information is required by both approaches (i.e. appropriate representation of random field, factorization of the nominal stiffness matrix and calculation of first order derivatives), the estimation of the expected value and covariance using the intervening variables of type I comprises somewhat more involved expressions (the difference is negligible).

4.4. Application of Intervening Variable Type II

This section introduces a first order Taylor expansion of the displacement vector considering an intervening variable that resembles the exponential variable for structural optimization (see eq. (23)). Recall that in eq. (28), an intervening variable of type I is introduced considering an auxiliary variable $z_i\left(\xi_i\right)$. Following this idea, the intervening variable of type II is defined as:

$$y_i(\xi_i) = \left(z_i\left(\xi_i\right)\right)^{m_{ni}} = e^{\alpha_i m_{ni}\xi_i},\ n = 1,\ldots,N_d,\ i = 1,\ldots,M \tag{33}$$

where y_i is the intervening variable for approximating the n-th DOF associated with the i-th random variable and where $m_{ni},\ i = 1,\ldots,M$, are real, constant coefficients. Note the coefficient m_{ni} fulfills a role similar to the coefficient m_i in eq.(23), i.e. it introduces an additional degree of freedom when compared to the intervening variable of type I. Of course, for the case $m_{ni} = -1,\ i = 1,\ldots,M$, the intervening variable of type II reduces to type I.

In order to determine the numerical value of the coefficients m_{ni} involved in the intervening variable of type II, an appropriate criterion should be considered. Within the context of structural optimization considering exponential intervening variables, it is proposed in [38] to evaluate these coefficients using information on the gradients of the displacement function at two different realizations of the vector of design variables. The application of such criterion implies constructing a *two-point* approximation of the displacement function. That is, the expression approximating the quantity of interest is generated based on the function value and gradient at one point (the expansion point) plus the information on gradients collected at an additional point. Although such criterion could be applied for determining the coefficients m_{ni} of the intervening variable type II, this idea is not pursued further in this contribution as it departs considerably from traditional perturbation methods that consider a single expansion point. Nonetheless, the use of *two-point* (or eventually *multipoint*) approximations may bring advantages for SFEM analysis, particularly in view of the fact it has already been used successfully in the field of structural optimization, see e.g. [38, 57, 58, 59]. The criterion used in this contribution for calculating m_{ni} consists in imposing the condition that some of the second order partial derivatives of the exact displacement vector match the second order derivatives of the approximate displacement vector considering intervening variables of type II (note the latter displacement vector is denoted as $\boldsymbol{u}^{\mathrm{TII}}(\boldsymbol{\xi})$) [43]. This last condition reads as:

$$\left.\frac{\partial^2 u_n}{\partial \xi_i^2}\right|_{\boldsymbol{\xi}=\boldsymbol{\xi}^0} = \left.\frac{\partial^2 u_n^{\mathrm{TII}}}{\partial \xi_i^2}\right|_{\boldsymbol{\xi}=\boldsymbol{\xi}^0} \tag{34}$$

Imposing the latter condition, the coefficients m_{ni} are calculated considering the following formula.

$$m_{ni} = \frac{u_{n,ii}}{\alpha_i u_{n,i}},\ n = 1, \ldots, N_d,\ i = 1, \ldots, M \tag{35}$$

In the above equation, $u_{n,i}$ and $u_{n,ii}$ represent the first- and second-order derivative of the displacement of n-th DOF at $\boldsymbol{\xi}^0$, respectively. Thus, the coefficient m_{ni} associated with the intervening variable of type II introduces information contained on second-order derivatives. However, note that the criterion for defining m_{ni} introduced in eq. (34) does not depend on all second-order derivatives but only on the diagonal terms of the associated Hessian matrix. It should be noted the definition for the coefficient m_{ni} proposed in eq. (35) possesses two disadvantages. First, it is undefined in case the associated first-order derivative is equal to zero. Second, in case the first-order derivative tends to be small, the values of m_{ni} may grow rapidly. These issues are overcome by considering the following formula for defining m_{ni}:

$$m_{ni} = \begin{cases} 1 & \text{if } u_{n,i} = 0 \\ m_{ni}^{\star} & \text{if } u_{n,i} \neq 0 \end{cases} \tag{36}$$

where $m_{ni}^{\star}$ is defined as:

$$m_{ni}^{\star} = \begin{cases} -3 & \text{if } \frac{u_{n,ii}}{\alpha_i u_{n,i}} \leq -3 \\ \frac{u_{n,ii}}{\alpha_i u_{n,i}} & \text{if } -3 \leq \frac{u_{n,ii}}{\alpha_i u_{n,i}} \leq 3 \\ 3 & \text{if } \frac{u_{n,ii}}{\alpha_i u_{n,i}} \geq 3 \end{cases} \tag{37}$$

According to eq. (36), whenever the associated first-order derivative is equal to zero, the coefficient m_{ni} is set arbitrarily as one, thus preventing undefined values; note that this arbitrary value is irrelevant as the first-order derivative that is equal to zero does not affect the first order Taylor expansion. Moreover, the criterion in eq. (37) prevents the coefficient m_{ni} from growing unboundedly (see, e.g. [38]). In particular, the bound for the coefficient m_{ni} is chosen as 3 following the criterion suggested in [55]. Numerical validations carried out by the authors indicate this criterion is appropriate within the scope of this contribution. Therefore, considering the definitions in eqs. (21), (33) and (36), the first order Taylor expansion of the n-th DOF in terms of type II intervening variables (denoted as $u_n^{\mathrm{TII}}(\boldsymbol{\xi})$) is:

$$u_n^{\mathrm{TII}}(\boldsymbol{\xi}) = u_n\left(\boldsymbol{\xi}^0\right) + \sum_{i=1}^{M} u_{n,i}\left(\frac{e^{\alpha_i m_{ni}\xi_i} - 1}{\alpha_i m_{ni}}\right) \tag{38}$$

while the second order statistics of $u_n^{\mathrm{TII}}(\boldsymbol{\xi})$ are given by the expressions shown below.

$$\mathrm{E}\left[u_n^{\mathrm{TII}}\right] = u_n\left(\boldsymbol{\xi}^0\right) + \sum_{i=1}^{M} u_{n,i}\left(\frac{e^{(\alpha_i m_{ni})^2/2} - 1}{\alpha_i m_{ni}}\right), \quad n = 1, \ldots, N_d \tag{39}$$

$$\mathrm{Cov}\left[u_n^{\mathrm{TII}}, u_m^{\mathrm{TII}}\right] = \sum_{i=1}^{M} u_{n,i} u_{m,i}\left(\frac{e^{\left(\frac{(\alpha_i m_{ni})^2 + (\alpha_i m_{mi})^2}{2}\right)}\left(e^{\alpha_i^2 m_{ni} m_{mi}} - 1\right)}{\alpha_i^2 m_{ni} m_{mi}}\right), \quad n, m = 1, \ldots, N_d \tag{40}$$

The application of a first order Taylor expansion considering intervening variables of type II involves numerical efforts which are larger than those associated with the application of an intervening variable of type I. These efforts stem from the calculation of the coefficients $m_{n,i}$ which require evaluating second order derivatives $\boldsymbol{u}_{,ii}$. Hence, the numerical efforts for applying the intervening variable of type II grow linearly with the number of random variables M. Additionally, it should be noted the approximation involving the intervening variable of type II is comparable to the quadratic Taylor approximation considering no intervening variables in the sense they both consider second order information. However, the numerical costs associated with the approximation involving the intervening variable of type II are smaller than the costs associated with a quadratic Taylor approximation as mixed partial derivatives of the displacement vector $\boldsymbol{u}_{,ij}$ are not required in the former approach.

5. Examples

5.1. Preliminary Remarks

This Section presents 3 examples where the performance of different strategies for estimating second order statistics of the displacement in problems of stochastic finite elements is compared. In this regard, Section 4 introduces two approximations for estimating second order statistics for problems of stochastic finite elements. These approximations correspond to first order Taylor expansions considering intervening variables of types I and II. In the remaining part of this contribution, these approximations are termed simply as *type I* and *type II*, respectively. It should be noted that type I approximation involves information on the first order derivatives of the displacement vector and the model for uncertainty. Hence, this approximation is comparable in terms of numerical costs to the first order Taylor approximation of the displacement considering no intervening variables (cf. eqs. (16) and (17)). The latter approximation is termed as *linear* approximation. In addition, it should be noted that type II approximation involves information on the first- and second-order derivatives of the displacement vector and the model for uncertainty. Thus, type II approximation is comparable to the *quadratic* approximation of the displacement vector. Therefore, in this Section, the performance of type I and II approximations is compared against the linear and quadratic approximations, respectively, as the numerical efforts associated with each of these pairs of approximations is similar.

As the estimates of second order statistics calculated with any of the four approaches mentioned above (i.e. linear, type I, quadratic and type II approximations) are not exact, these estimates are compared against the results obtained by means of Monte Carlo Simulation (MCS) [60]. In this regard, it is important to note that MCS does not produce *exact* values of the sought statistics as a finite number of samples is considered [61]. However, provided an appropriate number of samples is considered, it is possible to ensure the accuracy of the estimates calculated using MCS is sufficient (i.e. the error in the estimated statistics is negligible with respect to the value being estimated). Such an appropriate number of samples can be determined studying the convergence of the estimator (see e.g. [62]). In each of the examples of this contribution, results obtained by means of MCS comprise 10^6 samples. Such a large number ensures sufficiently accurate estimates of the second order statistics for the particular cases considered herein. Hence, in order to determine the accuracy of the estimates obtained using linear, type I, quadratic and type II approximations, the error of the estimates with respect to the estimates obtained via MCS is calculated. That is,

$$e_{s,X} = \frac{|s_X - s_{MCS}|}{|s_{MCS}|} \tag{41}$$

where s represents the statistic being estimated (note s can assume either the label μ for the mean or σ^2 for the variance), s_{MCS} is the statistic s calculated using MCS and s_X is the statistic s calculated using an approximation (where X can assume the label *linear*, *type I*, *quadratic* or *type II*) and $e_{s,X}$ is the error associated with the estimation of the statistic s using approximation X. Thus, for example, $e_{\sigma^2,\text{type I}}$ represents the error in estimating the variance using approximation type I.

5.2. Example 1: Truss Structure

In this example, the second order statistics of a truss structure vertical displacement are estimated. Uncertainty is considered in the Young's moduli of the bars of the structure. These Young's moduli are modeled as log-normal random variables. For the sake of simplicity, no correlations are considered between the Young's moduli of different bars. As already noted in Section 4, the application of intervening variables may lead to an exact representation of the structural response for statically determinate trusses. Hence, the applicability of the different approaches for estimating second order statistics is analyzed in view of the static determinacy.

A total of 6 trusses of varying degree of static determinacy are considered in the example. The layout and dimensions of the truss involving 23 bars are indicated in fig. 1. The other 5 trusses have a topology similar to the truss represented in fig. 1, except that they comprise a different number of bars. The specific bars comprised in each of the 6 trusses are listed in table 1. In addition, table 1 includes the degree of static determinacy of each of these 6 trusses, expressed in terms of the relative redundancy η [42]. The so called relative redundancy is calculated based on the number of bars N_b and the number of degrees-of-freedom N_d of the structure.

$$\eta = \frac{N_b - N_d}{N_d} \tag{42}$$

As shown in table 1, the selected trusses possess relative redundancies ranging from 0 to 0.917.

Figure 1 here

Table 1 here

Each of the 6 trusses under study withstands three nodal loads of magnitude 10 [kN], characterized as deterministic quantities. The cross section area of all bars is equal to 10^{-3} [m^2]. The Young's moduli of each bar are modeled by means of independent log-normal random variables with expected value equal to 2×10^{11} [Pa] and a coefficient of variation (CoV_E) varying between 5% and 30%. Thus, the dimension M of the vector of random variables $\boldsymbol{\xi}$ is equal to the number of bars N_b of a particular truss. The number of random variables involved in each truss considered is indicated in table 1. The objective of the example is determining the second-order statistics of the vertical displacement of node 8.

The results obtained for the estimates of the mean (μ) and variance (σ^2) of the displacement are presented in figs. 2, 3, 4 and 5. In each of these figures, the results are presented in terms of the relative error associated with mean ($e_{\mu,X}$) or variance ($e_{\sigma^2,X}$), where the relative error $e_{s,X}$ is calculated as described in eq. (41).

Figure 2 here

Figure 3 here

Figure 4 here

Figure 5 here

Figures 2, 3 and 4 illustrate the error in the estimation of second-order statistics of the vertical displacement of node 8 as a function of the coefficient of variation of the Young's modulus. The trusses presented in these figures correspond to the cases of 13, 19 and 23 bars, respectively. The analysis of the aforementioned figures indicate that the quadratic approximation is – for most cases – more accurate than the linear one. This was an expected result. However, the improvement observed between the linear and the quadratic approximation is relatively low, particularly for the case of the variance. This issue has already been noted in the literature, see e.g. [10, 52]. Figures 2, 3 and 4 also allow noting that the quality of all the approximations analyzed decreases as the variability of the input variables increases. Furthermore, there is a clear trend with respect to the linear and quadratic expansions: the error in the estimation of the statistics decreases as the relative redundancy increases. That is, these approximations seem to be more accurate for structures with a higher degree of redundancy. On the contrary, the error associated with type I approximation increases as the redundancy increases: for statically determinate structures, type I approximation produces exact results while for structures with a higher level of redundancy, type I approximation becomes worse than both linear and quadratic approximations. Finally, the behavior of type II approximation is a most interesting one. It is observed that for a statically determined truss (13-bar truss), the expansion is exact (see fig. 2). For the case of structures with a higher degree of redundancy (figs. 3 and 4), the property of structural determinacy does not hold. However, it can be seen that still type II approximation performs similar or even better than both linear and quadratic approximations.

Figure 5 presents the error in the estimation of the variance of the vertical displacement of node 8 as a function of the relative redundancy for a fixed value $\mathrm{CoV_E} = 30\%$. It is observed that for both linear and quadratic approximations, the error in the estimation of the variance decreases with an increase of the relative redundancy. On the contrary, the error associated with type I approximation increases as the relative redundancy increases. Finally, the error associated with type II approximation increases and then tends to decrease for an increase of the relative redundancy. Most remarkably, the error associated with the latter approximation is – for almost all cases – lower than the error associated with linear and quadratic approximations. The overall performance of type II approximation can be explained in terms of the associated coefficients m_{ni} (see eqs. (36) and (37)). These coefficients – which are calculated based on first and second order derivatives – provide sufficient flexibility to type II approximation to behave similar to type I approximation for statically determined structures while it can also resemble a quadratic approximation for redundant structures.

5.3. Example 2: Plate in Tension

In this example, a plate in tension is considered. The plate is assumed to be in a plain stress state and the Young's modulus is modeled as a log-normal random field. The model is taken from [17, 25] and is represented schematically in fig. 6. The objective is estimating the second order statistics of the vertical displacement at the corner B.

Figure 6 here

The dimensions of the plate are indicated in fig. 6 while the thickness is 0.01 [m]. A uniform loading of 1 [MN] is applied over the edge AB. The plate is modeled by considering quadrilateral finite elements of 8 DOF's. The model comprises a total of $N_d = 40$ DOF's and $N_e = 16$ elements. The Young's modulus of the plate is modeled as an homogeneous log-normal discrete random field using the midpoint method. For representing the underlying Gaussian random field, 16 terms are considered in the K-L expansion, i.e. the number of random variables involved in the problem is $M = 16$. The expected value of the Young's modulus is 200 [GPa] and the covariance matrix $\boldsymbol{C}^{EE}$ is defined as:

$$C_{pq}^{EE} = \sigma^2 e^{-\frac{||\bar{\boldsymbol{x}}_p - \bar{\boldsymbol{x}}_q||^2}{L^2}, \ p,q=1,\ldots,16} \tag{43}$$

where C_{pq}^{EE} is the (p, q)-th term of $\boldsymbol{C}^{EE}$, $\bar{\boldsymbol{x}}_p$ is the vector of midpoint coordinates of finite element p, $||\cdot||$ denotes Euclidean norm, σ^2 is the standard deviation of the Young's modulus and L is the correlation length. The values considered for σ^2 and L vary within predefined ranges. Thus, two cases are studied. In the first one, the correlation length of random field for the Young's modulus is fixed at the value $L = 1$ [m] while the parameter σ^2 assumes a value such that the coefficient of variation of the Young's modulus (CoV_E) is within the range [0.05,0.40]. The second order statistics are calculated considering type I and II, linear and quadratic approximations. As in the first example, the accuracy of the different approximations is compared in terms of their relative error with respect to MCS (see eq. (41)). The results obtained are shown in fig. 7(a) for the error in the estimation of expected value of the displacement ($e_{\mu,X}$) and in fig. 7(b) for the error in the estimation of the variance of the displacement ($e_{\sigma^2,X}$).

Figure 7 here

According to the results obtained in figs. 7(a) and 7(b), it is observed that the error associated with the estimation of second order statistics is similar for small values of the coefficient of variation of the Young's modulus (below 10%). However, this situation changes drastically as the coefficient of variation increases. For the particular case of fig. 7(a), it is seen that the most accurate estimate of the mean value is produced by the quadratic approximation. However, numerical costs associated with this approach largely exceed the numerical costs associated with the linear, type I and even type II approximation. For estimating the variance, both type I and II approximations are the most accurate ones. In this context, it should be noted that the numerical costs associated with type I approximation are similar to those of the linear approximation while the costs associated with type II approximation are smaller than those associated with the quadratic one. This fact highlights the potential of applying intervening variables for generating an approximation, i.e. with the same information, it is possible to generate improved estimates of second order statistics.

The second case of analysis consists in assuming that the coefficient of variation of the Young's modulus remains fixed at a value $\mathrm{CoV_E} = 40\%$ while the correlation length of the associated random field varies in the range [0.25,5] [m]. Again, the errors associated with the statistics $e_{\mu,X}$ and $e_{\sigma^2,X}$ are assessed. These results are presented in figs. 8(a) and 8(b), respectively. It can be noted that in all cases, the worst approximation is the linear one. For small values of correlation length, type I and II and quadratic approximations tend to have similar accuracy. However, as the correlation length increases, the accuracy of the quadratic approximation decreases while type I and II approximations improve. This is due to the fact for a larger correlation length (i.e. a highly correlated random field), the variability of the random field tends to depend mostly on the first terms of the K-L expansion. Thus, the displacement under analysis becomes a function of the reciprocal of the first terms of the K-L expansion. The relation between the displacement and the first terms can be captured better by both type I and II approximations.

Figure 8 here

Finally, Table 2 provides a summary on the relative execution time required by the linear, quadratic, type I and type II approximations for producing estimates of second order statistics for a fixed value of the coefficient of variation of the Young's modulus and correlation length. The linear approximation is assigned with an arbitrary execution time of 100%. It can be observed that type I approximation requires a slightly larger execution time (as already pointed out in Section 4.3). The execution time of approximation type II is larger than both linear and type I but it is substantially smaller than in the case of the quadratic approximation.

Table 2 here

5.4. Example 3: Foundation Over Elastic Soil

This example aims at estimating the second order statistics of the vertical displacement of an elastic soil layer due to the loading of a shallow foundation. The Young's modulus of the elastic soil layer is modeled as a Log-normal random field. This example is taken from [20, 48].

The elastic soil layer is of 30 [m] thickness and it rests over a rock bed assumed as infinitely rigid, as shown in fig. 9. A shallow foundation of 10 [m] width applies a load of 0.2 [MPa] over the elastic soil layer. In order to model the problem, a finite element (FE) mesh comprising a total of $N_e = 160$ quadrilateral elements of 8 DOF each are considered, thus involving a total of $N_d = 304$ DOF's. The finite elements are modeled considering a unit thickness. A plain strain state is assumed. The objective is estimating the mean and the variance of the vertical displacement of the central node of the FE mesh directly under the load (see fig. 9).

Figure 9 here

The Young's modulus of the elastic soil layer is modeled as an homogeneous log-normal discrete random field using the midpoint method. For representing the underlying Gaussian random field, all terms of the K-L expansion are considered, i.e. 160 terms. Therefore, the number of random variables involved in the problem is $M = 160$. The expected value of the Young's modulus is 50 [MPa] and the covariance matrix $\boldsymbol{C}^{EE}$ is defined as:

$$C_{pq}^{EE} = \sigma^2 e^{-\frac{||\bar{\boldsymbol{x}}_p - \bar{\boldsymbol{x}}_q||^2}{L^2}}, \ p, q = 1, \ldots, 160 \tag{44}$$

where C_{pq}^{EE} is the (p, q)-th term of $\boldsymbol{C}^{EE}$, $\bar{\boldsymbol{x}}_p$ is the vector of midpoint coordinates of finite element p, $|| \cdot ||$ denotes Euclidean norm, σ^2 is the standard deviation of the Young's modulus and L is the correlation length. The values considered for σ^2 and L vary within predefined ranges. For the first case of analysis, the correlation length is fixed at $L = 10$ [m] while the coefficient of variation of the Young's modulus ($\mathrm{CoV_E}$) takes values from 0.05 up to 0.5. As in the previous examples, four approximations are considered for estimating second order statistics. These approximations are compared in terms of their relative error with respect to MCS. The results obtained are shown in figs. 10(a) and 10(b) for the error in the mean ($e_{\mu,X}$) and variance ($e_{\sigma^2,X}$) of the displacement of the DOF of interest, respectively. The results obtained indicate that on one hand, type I approximation outperforms the linear one; on the other hand, the quadratic approximation is better than type II for estimating mean while the opposite holds for estimating the variance of the target displacement. For the second case of analysis, the standard deviation of the elastic soil layer is kept constant such that $\mathrm{CoV_E}$=50% while the correlation length L varies within [5,25] [m]. The results obtained for estimating the second order statistics of the displacement of interest are shown in figs. 11(a) and 11(b) for the mean ($e_{\mu,X}$) and variance ($e_{\sigma^2,X}$), respectively. The results obtained are similar to those obtained for the first case. However, it should be noted that as the correlation length increases, the error associated with the type I and II approximations decrease. This is due to the fact the variability of the random field associated with the Young's modulus tends to depend on the first terms of the K-L expansion. Then, the displacement becomes inversely proportional to these first terms and thus, this relation can be captured by both type I and II approximations.

Figure 10 here

Figure 11 here

6. Conclusions

In this contribution, the application of intervening variables for stochastic finite element analysis has been investigated. According to the results obtained, the two intervening variables introduced (denoted as type I and II) outperform in general expansions based on Taylor series of first and second order involving direct variables. More important, the approximation considering intervening variables of type I can be constructed using the very same information required for constructing a regular first order Taylor series expansion. In addition, the approximation based on intervening variables of type II can be constructed using less information than a second order Taylor expansion, i.e. instead of requiring the full Hessian matrix of the displacement (with respect to the random variables), only the diagonal terms of this matrix are required.

The results reported herein are promising, as they show the proposed approach can lead to better estimators of second order statistics than those produced by conventional perturbation methods. However, it should be kept in mind that the examples presented are of a limited scope. Therefore, it remains as an open issue exploring the application of intervening variables to different types of examples involving more challenging structures, in order to establish precisely the range of application of the approach reported herein for stochastic finite element analysis. In addition, the possibility of estimating second order statistics of the structural response by means of multipoint approximations is also another open issue. Similarly, the applicability of intervening variables for estimating not only second order statistics but also higher order moments needs to be explored. In fact, current research efforts are under way to address these issues.

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List of Tables

Truss	N_b	M	Bars	η
1$^{(*)}$	13	13	all bars shown in fig. (1) except bars 4-6, 10-14, 1-5, 9-11, 3-5, 9-13, 6-7, 7-10, 3-6, 10-13	0
2	15	15	all bars shown in fig. (1) except bars 4-6, 10-14, 1-5, 9-11, 3-5, 9-13, 6-7, 7-10	0.250
3	17	17	all bars shown in fig. (1) except bars 4-6, 10-14, 1-5, 9-11, 3-5, 9-13	0.417
4	19	19	all bars shown in fig. (1) except bars 4-6, 10-14, 1-5, 9-11	0.583
5	21	21	all bars shown in fig. (1) except bars 4-6, 10-14	0.750
6	23	23	all bars shown in fig. (1)	0.917

(*) for truss 1, the pin support in node 12 is replaced by a roller support (that rolls in the horizontal direction).

Table 1: Configuration of trusses considered in Example 1

Approach	Relative execution time [%]
Linear	100%
Quadratic	642%
Type I	102%
Type II	125%

Table 2: Relative execution time associated with the estimation of second order statistics for example 2

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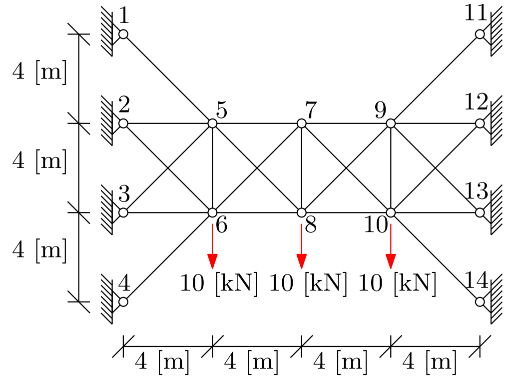
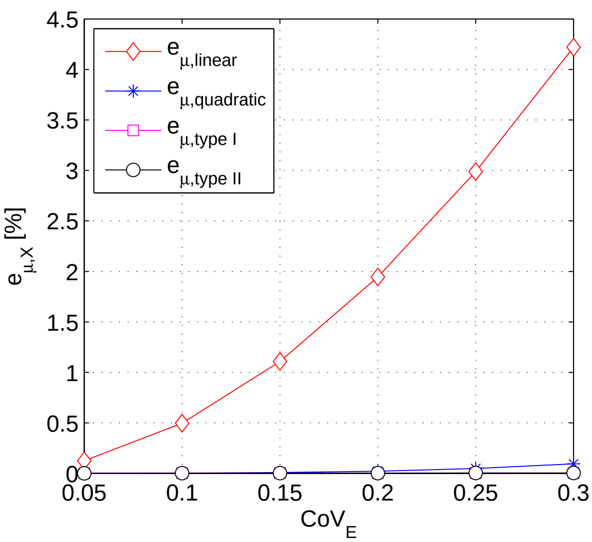
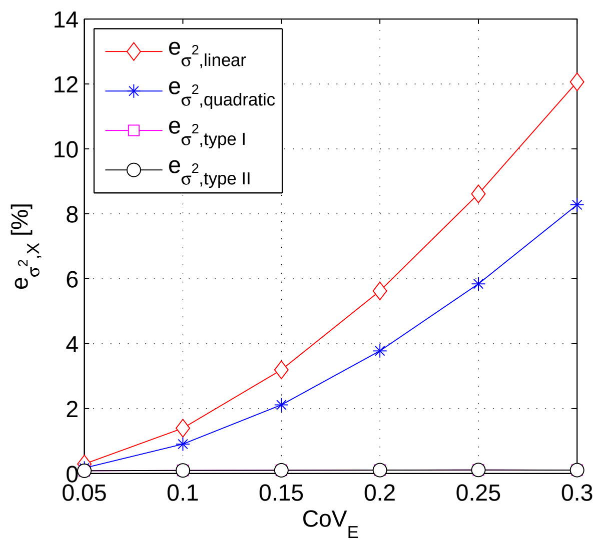


Figure 1: Schematic representation 23-bar truss structure

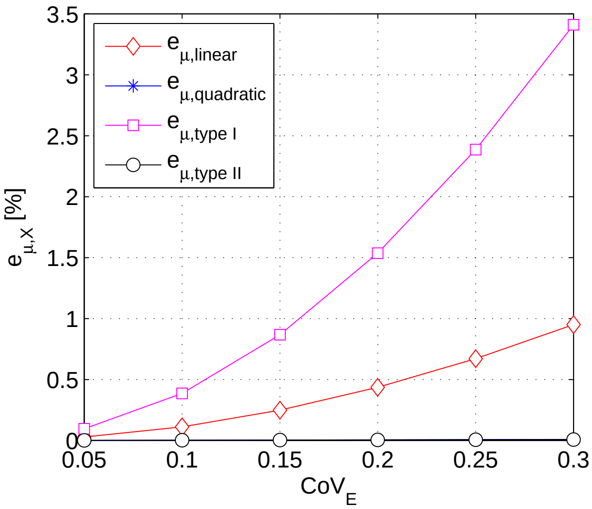


(a) Error in estimation of mean as a function of $\mathrm{CoV_E}$

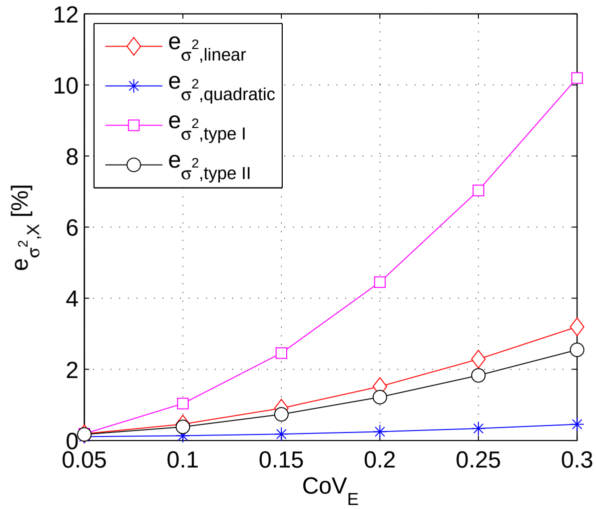


(b) Error in estimation of variance as a function of $\mathrm{CoV_E}$

Figure 2: Error in estimation of second-order statistics of vertical displacement of node 8 as a function of the CoV of the Young's modulus (13-bar truss, $\eta = 0$)

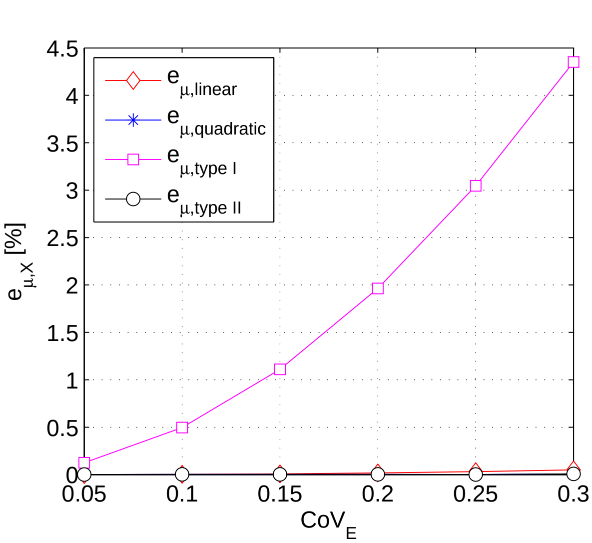


(a) Error in estimation of mean as a function of $\mathrm{CoV_E}$

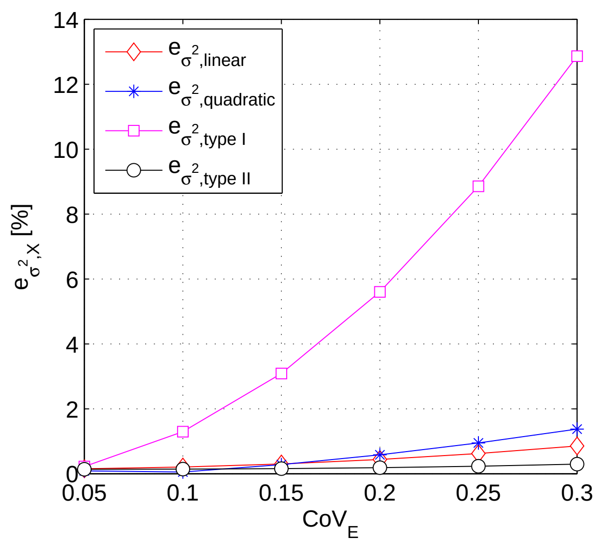


(b) Error in estimation of variance as a function of $\mathrm{CoV_E}$

Figure 3: Error in estimation of second-order statistics of vertical displacement of node 8 as a function of the CoV of the Young's modulus (19-bar truss, $\eta = 0.583$)



(a) Error in estimation of mean as a function of $\mathrm{CoV_E}$



(b) Error in estimation of variance as a function of $\mathrm{CoV_E}$

Figure 4: Error in estimation of second-order statistics of vertical displacement of node 8 as a function of the CoV of the Young's modulus (23-bar truss, $\eta = 0.917$)

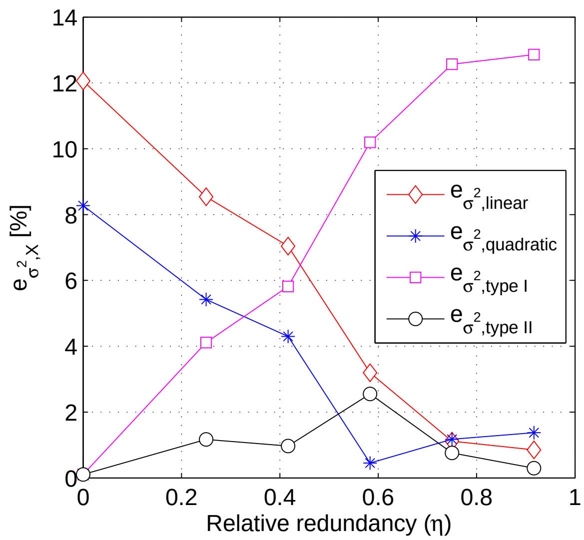


Figure 5: Error in estimation of variance of vertical displacement of node 8 as a function of the relative redundancy (η) for a fixed value of $\mathrm{CoV_E}$=30%

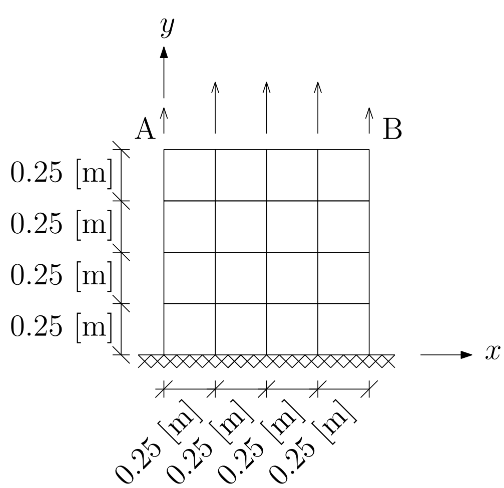
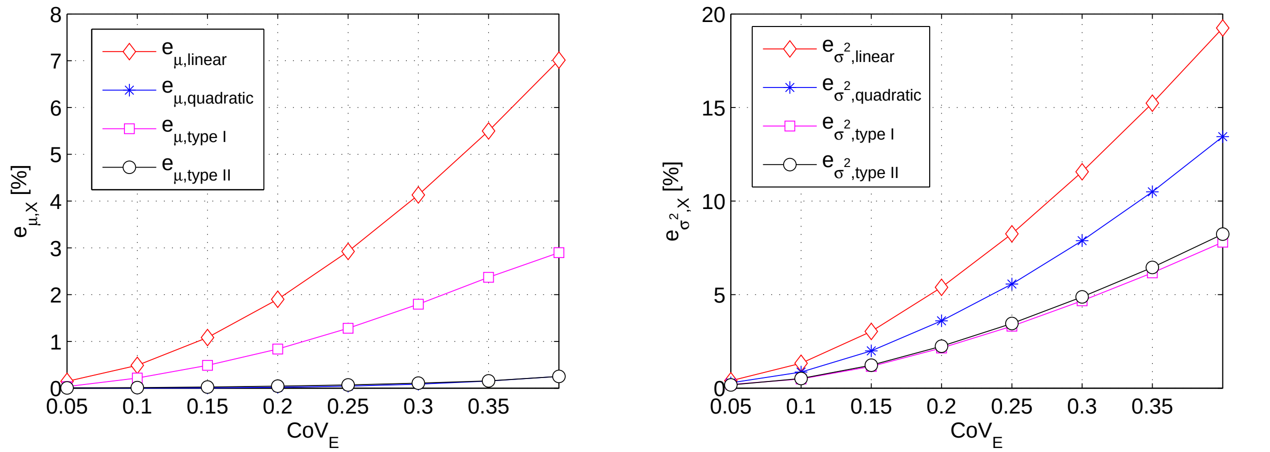
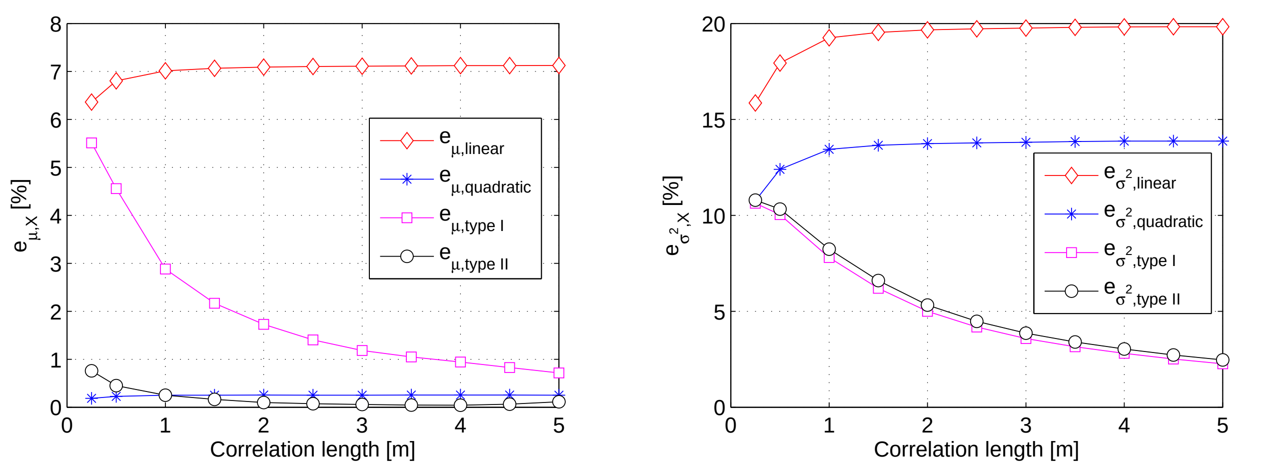


Figure 6: Schematic representation of plate in tension



(a) Error in estimation of mean as a function of $\mathrm{CoV_E}$ (b) Error in estimation of variance as a function of $\mathrm{CoV_E}$

Figure 7: Error in estimation of second-order statistics of vertical displacement at the corner B as a function of the coefficient of variation of the Young's modulus ($\mathrm{CoV_E}$) for a fixed value of the correlation length ($L = 1$ [m])



(a) Error in estimation of mean as a function of correlation length (b) Error in estimation of variance as a function of correlation length

Figure 8: Error in estimation of second-order statistics of vertical displacement at the corner B as a function of the correlation length (L) for a fixed value of the coefficient of variation of the Young's modulus ($\mathrm{CoV_E} = 40\%$)

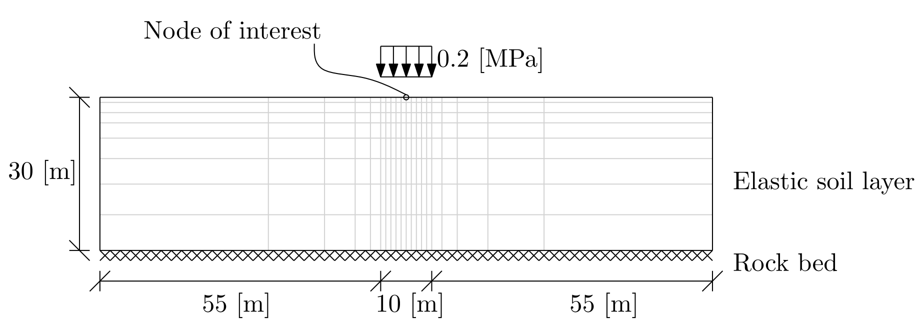
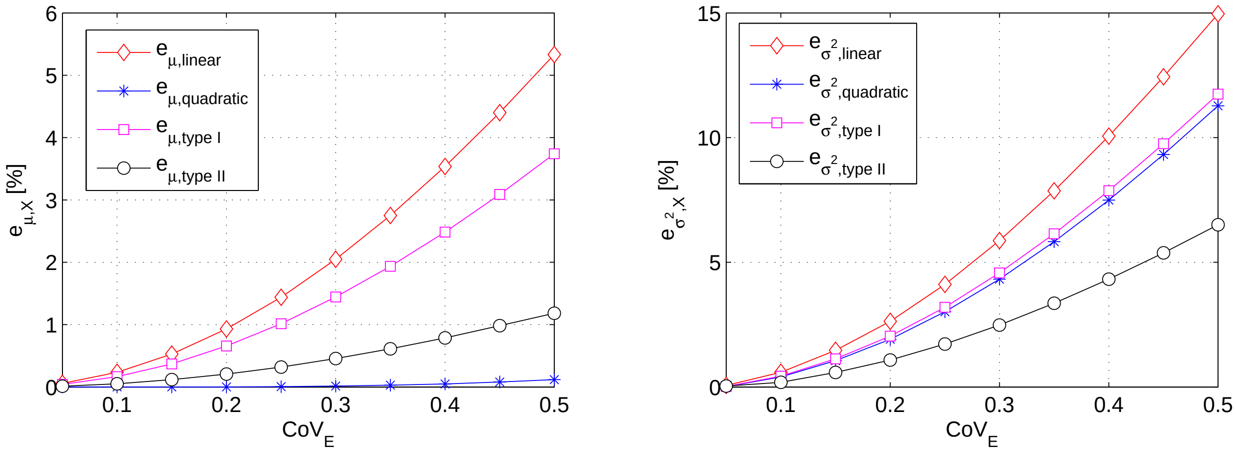


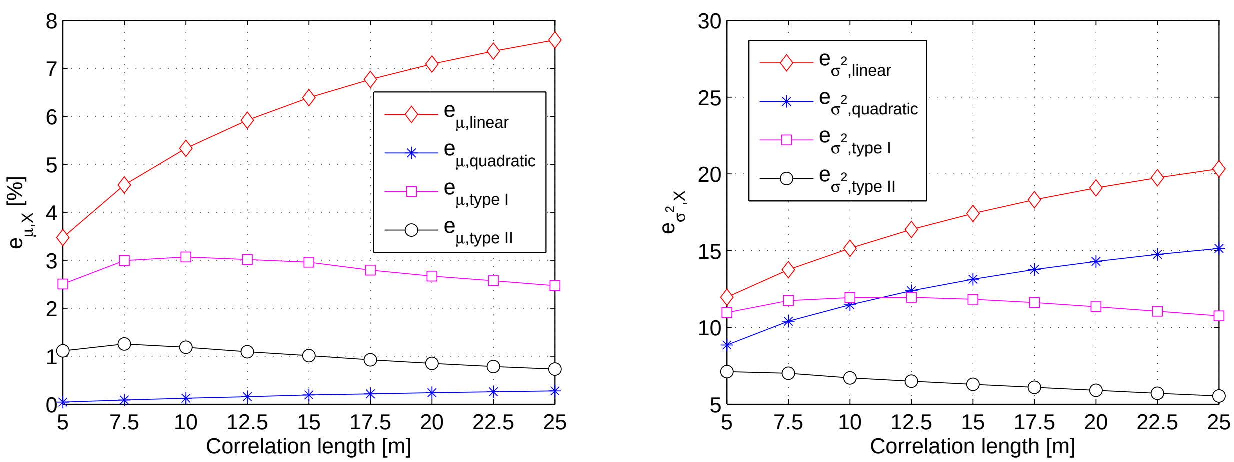
Figure 9: Schematic representation of foundation over elastic soil



(a) Error in estimation of mean as a function of $\mathrm{CoV_E}$

(b) Error in estimation of variance as a function of $\mathrm{CoV_E}$

Figure 10: Error in estimation of second-order statistics of vertical displacement of node under foundation as a function of the coefficient of variation of the Young's modulus ($\mathrm{CoV_E}$) for a fixed value of the correlation length ($L = 10$ [m])



(a) Error in estimation of mean as a function of correlation length

(b) Error in estimation of variance as a function of correlation length

Figure 11: Error in estimation of second order statistics of vertical displacement of node under foundation as a function of the correlation length (L) for a fixed value of the coefficient of variation of the Young's modulus ($\mathrm{CoV_E} = 50\%$)