

ON HOMOMORPHISMS FROM RINGEL-HALL ALGEBRAS TO QUANTUM CLUSTER ALGEBRAS

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ABSTRACT. In [1], the authors defined algebra homomorphisms from the dual Ringel-Hall algebra of certain hereditary abelian category \mathcal{A} to an appropriate q -polynomial algebra. In the case that \mathcal{A} is the representation category of an acyclic quiver, we give an alternative proof by using the cluster multiplication formulas in [7]. Moreover, if the underlying graph of Q is bipartite and the matrix B associated to the quiver Q is of full rank, we show that the image of the algebra homomorphisms is in the corresponding quantum cluster algebra.

1. BACKGROUND

The Ringel-Hall algebra $\mathcal{H}(\mathcal{A})$ of a (small) finitary abelian category \mathcal{A} was introduced by Ringel ([13]). When \mathcal{A} is the category $\text{Rep}_{\mathbb{F}_q} Q$ of finite dimensional representations for a simply-laced quiver Q over a finite field \mathbb{F}_q , the Ringel-Hall algebra $\mathcal{H}(\mathcal{A})$ is isomorphic to the positive part $U_q(\mathfrak{n})$ of the corresponding quantum group $U_q(\mathfrak{g})$ ([13]). Lusztig ([11]) constructed the canonical basis of the quantum group $U_q(\mathfrak{n})$ under the context of Ringel-Hall algebras. In order to study the canonical basis algebraically and combinatorially, Berenstein and Zelevinsky ([2]) defined quantum cluster algebras as a noncommutative analogue of cluster algebras (see [8][9]). A quantum cluster algebra is a subalgebra of a skew field of rational functions in q -commuting variable and generated by a set of generators called the *cluster variables*.

A natural question is to study the relations between Ringel-Hall algebras and quantum cluster algebras. Geiss, Leclerc and Schröer ([10]) showed that quantum groups of type A , D and E have quantum cluster structures. Recently, Berenstein and Rupel [1] constructed algebra homomorphisms from Ringel-Hall algebras to quantum cluster algebras. We recall their main result. Let \mathcal{A} be a finitary hereditary abelian category and $\mathbf{i} = (i_1, \dots, i_m)$ be a sequence of simple objects in \mathcal{A} . They showed that, under certain co-finiteness conditions, the assignment $[V]^* \rightarrow X_{V;\mathbf{i}}$ defines a

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homomorphism of algebras

$$\Psi_{\mathbf{i}} : \mathcal{H}^*(\mathcal{A}) \rightarrow P_{\mathbf{i}}$$

where $\mathcal{H}^*(\mathcal{A})$ is dual Ringel-Hall algebra and $X_{V;\mathbf{i}}$ is the quantum cluster \mathbf{i} -character of V in an appropriate q -polynomial algebra $P_{\mathbf{i}}$. Moreover, for an appropriate \mathbf{i} , the image restricting to the composition algebra of $\mathcal{H}^*(\mathcal{A})$ is in the corresponding upper cluster algebra.

The aim of this note is to give an alternative proof of the above result when \mathcal{A} is the representation category of an acyclic quiver. Different from [1], a key ingredient of our proof is to apply the cluster multiplication formulas proved in [7] (see also Theorem 3.3). We show that if the underlying graph of Q is bipartite (i.e, we can associate this graph an orientation such that every vertex is a sink or a source) and the matrix B associated to the quiver Q is of full rank, then the algebra $\mathcal{AH}_{|k|}(Q)$ generated by all quantum cluster characters is exactly the quantum cluster algebra $\mathcal{A}_{|k|}(Q)$ (see Theorem 4.5). As a corollary, the image of the algebra homomorphism is in the quantum cluster algebra $\mathcal{A}_{|k|}(Q)$ (see Corollary 4.6). We expect that the approach in this note can be extended to construct algebra homomorphisms from derived Hall algebras to quantum cluster algebras.

2. QUANTUM CLUSTER ALGEBRAS AND CALDERO-CHAPOTON MAPS

2.1. Quantum cluster algebras. We briefly recall the definition of quantum cluster algebras. Let L be a lattice of rank m and $\Lambda : L \times L \rightarrow \mathbb{Z}$ a skew-symmetric bilinear form. We will need a formal variable q and consider the ring of integer Laurent polynomials $\mathbb{Z}[q^{\pm 1/2}]$. Define the *based quantum torus* associated to the pair (L, Λ) to be the $\mathbb{Z}[q^{\pm 1/2}]$ -algebra \mathcal{T} with a distinguished $\mathbb{Z}[q^{\pm 1/2}]$ -basis $\{X^e : e \in L\}$ and the multiplication given by

$$X^e X^f = q^{\Lambda(e,f)/2} X^{e+f}.$$

It is easy to see that \mathcal{T} is associative and the basis elements satisfy the following relations:

$$X^e X^f = q^{\Lambda(e,f)} X^f X^e, \quad X^0 = 1, \quad (X^e)^{-1} = X^{-e}.$$

It is known that \mathcal{T} is an Ore domain, i.e., is contained in its skew-field of fractions \mathcal{F} . The quantum cluster algebra will be defined as a $\mathbb{Z}[q^{\pm 1/2}]$ -subalgebra of \mathcal{F} .

A *toric frame* in \mathcal{F} is a map $M : \mathbb{Z}^m \rightarrow \mathcal{F} \setminus \{0\}$ of the form

$$M(\mathbf{c}) = \varphi(X^{\eta(\mathbf{c})})$$

where φ is an automorphism of \mathcal{F} and $\eta : \mathbb{Z}^m \rightarrow L$ is an isomorphism of lattices. By the definition, the elements $M(\mathbf{c})$ form a $\mathbb{Z}[q^{\pm 1/2}]$ -basis of the based quantum torus $\mathcal{T}_M := \varphi(\mathcal{T})$ and satisfy the following relations:

$$M(\mathbf{c})M(\mathbf{d}) = q^{\Lambda_M(\mathbf{c},\mathbf{d})/2} M(\mathbf{c} + \mathbf{d}), \quad M(\mathbf{c})M(\mathbf{d}) = q^{\Lambda_M(\mathbf{c},\mathbf{d})} M(\mathbf{d})M(\mathbf{c}),$$

$$M(\mathbf{0}) = 1, \quad M(\mathbf{c})^{-1} = M(-\mathbf{c}),$$

where Λ_M is the skew-symmetric bilinear form on \mathbb{Z}^m obtained from the lattice isomorphism η . Let Λ_M also denote the skew-symmetric $m \times m$ matrix defined by $\lambda_{ij} = \Lambda_M(e_i, e_j)$ where $\{e_1, \dots, e_m\}$ is the standard basis of \mathbb{Z}^m . Given a toric frame M , let $X_i = M(e_i)$. Then we have

$$\mathcal{T}_M = \mathbb{Z}[q^{\pm 1/2}] \langle X_1^{\pm 1}, \dots, X_m^{\pm 1} : X_i X_j = q^{\lambda_{ij}} X_j X_i \rangle.$$

Let Λ be an $m \times m$ skew-symmetric matrix and let \tilde{B} be an $m \times n$ ($m > n$) matrix, whose principal part denoted by B . We call the pair (Λ, \tilde{B}) *compatible* if $\tilde{B}^T \Lambda = (D|0)$ is an $n \times m$ matrix with $D = \text{diag}(d_1, \dots, d_n)$ where $d_i \in \mathbb{N}$ for $1 \leq i \leq n$. The pair (M, \tilde{B}) is called a *quantum seed* if the pair (Λ_M, \tilde{B}) is compatible. Now we define the mutation of the quantum seed (Λ_M, \tilde{B}) in direction k for $1 \leq k \leq n$.

Define the $m \times m$ matrix $E = (e_{ij})$ by

$$e_{ij} = \begin{cases} \delta_{ij} & \text{if } j \neq k; \\ -1 & \text{if } i = j = k; \\ \max(0, -b_{ik}) & \text{if } i \neq j = k. \end{cases}$$

For $n, k \in \mathbb{Z}$, $k \geq 0$, denote $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n - q^{-n}) \cdots (q^{n-r+1} - q^{-n+r-1})}{(q^r - q^{-r}) \cdots (q - q^{-1})}$. Let $\mathbf{c} = (c_1, \dots, c_m) \in \mathbb{Z}^m$ with $c_k \geq 0$. Define the toric frame $M' : \mathbb{Z}^m \rightarrow \mathcal{F} \setminus \{0\}$ as follows:

$$(2.1) \quad M'(\mathbf{c}) = \sum_{p=0}^{c_k} \begin{bmatrix} c_k \\ p \end{bmatrix}_{q^{d_k/2}} M(E\mathbf{c} + p\mathbf{b}^k), \quad M'(-\mathbf{c}) = M'(\mathbf{c})^{-1}.$$

where the vector $\mathbf{b}^k \in \mathbb{Z}^m$ is the k -th column of \tilde{B} .

Define $\tilde{B}' = (b'_{ij})$ by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k; \\ b_{ij} + \text{sgn}(b_{ik}[b_{ik}b_{kj}]_+) & \text{otherwise.} \end{cases}$$

where $[b]_+ = \max(b, 0)$.

Then the quantum seed (M', \tilde{B}') is called to be the mutation of (M, \tilde{B}) in direction k . Quantum seeds are mutation-equivalent if they can be obtained from each other by a sequence of mutations. Let $\mathcal{C} = \{M'(e_i) : i \in [1, n]\}$ where (M', \tilde{B}') is mutation-equivalent to (M, \tilde{B}) . Let $\mathbb{Z}\mathbb{P}$ be the ring of integral Laurent polynomials in the (quasi-commuting) variables in $\{q^{1/2}, X_{n+1}, \dots, X_m\}$. The *quantum cluster algebra* $\mathcal{A}_q(\Lambda_M, \tilde{B})$ is the $\mathbb{Z}\mathbb{P}$ -subalgebra of \mathcal{F} generated by \mathcal{C} .

Proposition 2.1. (*Mutation of cluster variables*)[2] *The toric frame X' is determined by*

$$\begin{aligned} X'_k &= X \left(\sum_{1 \leq i \leq m} [b_{ik}]_+ e_i - e_k \right) + X \left(\sum_{1 \leq j \leq m} [-b_{jk}]_+ e_j - e_k \right), \\ X'_k &= X_i, \quad 1 \leq i \leq m, \quad i \neq k. \end{aligned}$$

Theorem 2.2. (*Quantum Laurent phenomenon*)[2] *The quantum cluster algebra $\mathcal{A}_q(\Lambda_M, \tilde{B})$ is a subalgebra of \mathcal{T}_M .*

Set $\mathbf{X} = \{X_1, \dots, X_n\}$ and $\mathbf{X}_k = \mathbf{X} - \{X_k\} \cup \{X'_k\}$ for any $k \in [1, n]$. Denote by $\mathcal{U}(\Lambda_M, \tilde{B})$ is the $\mathbb{Z}\mathbb{P}$ -subalgebra of \mathcal{F} given by

$$\mathcal{U}(\Lambda_M, \tilde{B}) = \mathbb{Z}\mathbb{P}[\mathbf{X}^{\pm 1}] \cap \mathbb{Z}\mathbb{P}[\mathbf{X}_1^{\pm 1}] \cap \dots \cap \mathbb{Z}\mathbb{P}[\mathbf{X}_n^{\pm 1}].$$

We call $\mathcal{U}(\Lambda_M, \tilde{B})$ the *quantum upper cluster algebra*. The following result shows that the acyclicity condition closes the gap between the upper bounds and the corresponding quantum cluster algebras.

Theorem 2.3. [2] *If the principal matrix B is acyclic, then $\mathcal{U}(\Lambda_M, \tilde{B}) = \mathcal{A}_q(\Lambda_M, \tilde{B})$.*

2.2. Quantum Caldero-Chapoton maps. Let k be a finite field with cardinality $|k| = q$ and $m \geq n$ be two positive integers and \tilde{Q} an acyclic quiver with vertex set $\{1, \dots, m\}$. Denote the subset $\{n+1, \dots, m\}$ by C . The full subquiver Q on the vertices $1, \dots, n$ is called the *principal part* of \tilde{Q} . For $1 \leq i \leq m$, let S_i be the i th simple module for $k\tilde{Q}$.

Let \tilde{B} be the $m \times n$ matrix associated to the quiver \tilde{Q} whose entry in position (i, j) given by

$$b_{ij} = |\{\text{arrows } i \longrightarrow j\}| - |\{\text{arrows } j \longrightarrow i\}|$$

for $1 \leq i \leq m$, $1 \leq j \leq n$. Denote by \tilde{I} the left $m \times n$ submatrix of the identity matrix of size $m \times m$. Assume that there exists some antisymmetric $m \times m$ integer matrix Λ such that

$$(2.2) \quad \Lambda(-\tilde{B}) = \begin{bmatrix} I_n \\ 0 \end{bmatrix},$$

where I_n is the identity matrix of size $n \times n$. Let $\tilde{R} = \tilde{R}_{\tilde{Q}}$ be the $m \times n$ matrix with its entry in position (i, j) given by

$$\tilde{r}_{ij} := \dim_k \text{Ext}_{k\tilde{Q}}^1(S_j, S_i) = |\{\text{arrows } j \longrightarrow i\}|.$$

for $1 \leq i \leq m$, $1 \leq j \leq n$. Set $\tilde{R}^{tr} = \tilde{R}_{\tilde{Q}^{op}}$. Denote the principal $n \times n$ submatrices of \tilde{B} and \tilde{R} by B and R respectively. Note that $\tilde{B} = \tilde{R}^{tr} - \tilde{R}$ and $B = R^{tr} - R$.

Let $\mathcal{C}_{\tilde{Q}}$ be the cluster category of $k\tilde{Q}$, i.e., the orbit category of the derived category $\mathcal{D}^b(\tilde{Q})$ under the action of the functor $F = \tau \circ [-1]$ (see [3]). Let I_i be the indecomposable injective $k\tilde{Q}$ module for $1 \leq i \leq m$. Then the indecomposable $k\tilde{Q}$ -modules and $I_i[-1]$ for $1 \leq i \leq m$ exhaust all indecomposable objects of the cluster category $\mathcal{C}_{\tilde{Q}}$. Each object M in $\mathcal{C}_{\tilde{Q}}$ can be uniquely decomposed as

$$M = M_0 \oplus I_M[-1]$$

where M_0 is a module and I_M is an injective module.

The Euler form on $k\tilde{Q}$ -modules M and N is given by

$$\langle M, N \rangle = \dim_k \text{Hom}(M, N) - \dim_k \text{Ext}^1(M, N).$$

Note that the Euler form only depends on the dimension vectors of M and N .

The quantum Caldero-Chapoton map of an acyclic quiver Q has been defined in [14][12][7][6]:

$$X_{\tau} : \text{obj}\mathcal{C}_{\tilde{Q}} \longrightarrow \mathcal{T}$$

by the following rules:

(1) If M is a kQ -module, then

$$X_M = \sum_{\underline{e}} |\text{Gr}_{\underline{e}}M| q^{-\frac{1}{2}\langle \underline{e}, \underline{m} - \underline{e} \rangle} X^{-\tilde{B}\underline{e} - (\tilde{I} - \tilde{R}^{tr})\underline{m}},$$

(2) If M is a kQ -module and I is an injective $k\tilde{Q}$ -module, then

$$X_{M \oplus I[-1]} = \sum_{\underline{e}} |\text{Gr}_{\underline{e}}M| q^{-\frac{1}{2}\langle \underline{e}, \underline{m} - \underline{e} - \underline{i} \rangle} X^{-\tilde{B}\underline{e} - (\tilde{I} - \tilde{R}^{tr})\underline{m} + \underline{\dim} \text{soc} I},$$

where $\underline{\dim} I = \underline{i}$, $\underline{\dim} M = \underline{m}$ and $\text{Gr}_{\underline{e}}M$ denotes the set of all submodules V of M with $\underline{\dim} V = \underline{e}$. We note that

$$X_{P[1]} = X_{\tau P} = X^{\underline{\dim} P / \text{rad} P} = X^{\underline{\dim} \text{soc} I} = X_{I[-1]} = X_{\tau^{-1}I}.$$

for any projective $k\tilde{Q}$ -module P and injective $k\tilde{Q}$ -module I with $\text{soc} I = P / \text{rad} P$.

In the following, we always use the underlined lower letter \underline{x} to denote the corresponding dimension vector of a kQ -module X and view \underline{x} as a column vector in \mathbb{Z}^n .

3. THE DUAL RINGEL-HALL ALGEBRAS AND THE CLUSTER MULTIPLICATION FORMULAS

Let \mathcal{A} be the representation category of an acyclic quiver Q . For an object $V \in \mathcal{A}$, we will write $[V]$ for the isomorphism class of V and write $|V|$ for the class of V in the Grothendieck group $K(\mathcal{A})$. Let $\mathcal{H}(\mathcal{A}) = \bigoplus k[V]$ be the free $K(\mathcal{A})$ -graded k -vector space spanned by the isomorphism classes of objects of \mathcal{A} with the natural grading via class in $K(\mathcal{A})$. For $U, V, W \in \mathcal{A}$ define

$$g_{UV}^V = |\{R \subset V \mid R \cong W, V/R \cong U\}|.$$

The assignment $[U][W] = \sum_{[V]} g_{UV}^V [V]$ defines an associative multiplication on $\mathcal{H}(\mathcal{A})$. The algebra $\mathcal{H}(\mathcal{A})$ is known as the Ringel-Hall algebra. Denote by $\mathcal{H}^*(\mathcal{A})$ the dual Ringel-Hall algebra, which is the space of linear functions $\mathcal{H}(\mathcal{A}) \rightarrow k$ with a basis of all delta-functions δ_V labeled by isomorphism classes $[V]$ of objects of \mathcal{A} .

Proposition 3.1. *Let M and N be kQ -modules, then the assignment*

$$\delta_M * \delta_N = q^{\frac{1}{2}\Lambda((\tilde{I} - \tilde{R}')\underline{m}, (\tilde{I} - \tilde{R}')\underline{n}) + \langle \underline{m}, \underline{n} \rangle} \sum_E h_E^{MN} \delta_E$$

defines an associative multiplication on $\mathcal{H}^(\mathcal{A})$, where $h_E^{MN} = \frac{|\text{Ext}_{kQ}^1(M, N)_E|}{|\text{Hom}_{kQ}(M, N)|}$.*

Proof. Note that $\sum_E g_{MN}^E g_{EL}^F = \sum_G g_{NL}^G g_{MG}^F$, and the relation of h_E^{MN} and g_{MN}^E is given by the Riedtmann-Peng's formula

$$h_E^{MN} = g_{MN}^E |Aut(M)| |Aut(N)| |Aut(E)|^{-1}.$$

Thus we have $\sum_E h_{MN}^E h_{EL}^F = \sum_G h_{NL}^G h_{MG}^F$. It is easy to see that $\phi(\underline{m}, \underline{n}) := \frac{1}{2} \Lambda((\tilde{I} - \tilde{R}') \underline{m}, (\tilde{I} - \tilde{R}') \underline{n}) + \langle \underline{m}, \underline{n} \rangle$ is a bilinear form on \mathbb{Z}^n . Hence the associativity can be deduced. \square

For any $k\tilde{Q}$ -modules M, N and E , denote by ε_{MN}^E the cardinality of the set $\text{Ext}_{k\tilde{Q}}^1(M, N)_E$ which is the subset of $\text{Ext}_{k\tilde{Q}}^1(M, N)$ consisting of those equivalence classes of short exact sequences with middle term isomorphic to E . Define

$$\text{Hom}_{k\tilde{Q}}(M, I)_{BI'} := \{f : M \rightarrow I \mid \ker f \cong B, \text{coker } f \cong I'\}.$$

Denote

$$\begin{aligned} [M, N] &= \dim_k \text{Hom}_{k\tilde{Q}}(M, N), \\ [M, N]^1 &= \dim_k \text{Ext}_{k\tilde{Q}}^1(M, N). \end{aligned}$$

We have the following cluster multiplication formulas.

Theorem 3.2. [7][6] *Let M and N be any kQ -modules, and I any injective $k\tilde{Q}$ -module, then*

$$(1) \quad q^{[M, N]^1} X_M X_N = q^{\frac{1}{2} \Lambda((\tilde{I} - \tilde{R}') \underline{m}, (\tilde{I} - \tilde{R}') \underline{n})} \sum_E \varepsilon_{MN}^E X_E,$$

$$(2) \quad q^{[M, I]} X_M X_{I[-1]} = q^{\frac{1}{2} \Lambda((\tilde{I} - \tilde{R}') \underline{m}, -\underline{\dim Soc I})} \sum_{B, I'} |\text{Hom}_{k\tilde{Q}}(M, I)_{BI'}| X_{B \oplus I'[-1]}.$$

Note that Theorem 3.2(1) implies the following result which has been proved by Berenstein-Rupel using generalities on bialgebras in braided monoidal categories.

Theorem 3.3. [1] *The assignment $\delta_V \rightarrow X_V$ defines an algebra homomorphism $\Psi : \mathcal{H}^*(\mathcal{A}) \rightarrow \mathcal{T}$.*

An alternative proof: Note that the first cluster multiplication formula in Theorem 3.2 can be rewritten as

$$X_M X_N = q^{\frac{1}{2} \Lambda((\tilde{I} - \tilde{R}') \underline{m}, (\tilde{I} - \tilde{R}') \underline{n}) + \langle \underline{m}, \underline{n} \rangle} \sum_E h_E^{MN} X_E.$$

Thus we have

$$\begin{aligned} \Psi(\delta_M * \delta_N) &= \Psi(q^{\frac{1}{2} \Lambda((\tilde{I} - \tilde{R}') \underline{m}, (\tilde{I} - \tilde{R}') \underline{n}) + \langle \underline{m}, \underline{n} \rangle} \sum_E h_E^{MN} \delta_E) \\ &= q^{\frac{1}{2} \Lambda((\tilde{I} - \tilde{R}') \underline{m}, (\tilde{I} - \tilde{R}') \underline{n}) + \langle \underline{m}, \underline{n} \rangle} \sum_E h_E^{MN} X_E \\ &= X_M X_N = \Psi(\delta_M) \Psi(\delta_N). \end{aligned}$$

This completes the proof. \square

4. QUANTUM CLUSTER ALGEBRAS FOR BIPARTITE GRAPHS

In this section, we assume that Q is an acyclic quiver whose underlying graph is bipartite and the matrix B associated to the quiver Q is of full rank. Note that in this case the corresponding quantum cluster algebras are coefficient-free. We will show that the algebra $\mathcal{AH}_{|k|}(Q)$ generated by all quantum cluster characters is equal to the quantum cluster algebra $\mathcal{A}_{|k|}(Q)$.

Definition 4.1. X_L is called *the quantum cluster character* if $L \in \mathcal{C}_Q$.

Definition 4.2. For a quiver Q , denote by $\mathcal{AH}_{|k|}(Q)$ the \mathbb{Z} -subalgebra of \mathcal{F} generated by all the quantum cluster characters.

Let Q be an acyclic quiver and i be a sink or a source in Q . We define the reflected quiver $\sigma_i(Q)$ by reversing all the arrows ending at i . An *admissible sequence of sinks (resp. sources)* is a sequence (i_1, \dots, i_l) such that i_1 is a sink (resp. source) in Q and i_k is a sink (resp source) in $\sigma_{i_{k-1}} \cdots \sigma_{i_1}(Q)$ for any $k = 2, \dots, l$. A quiver Q' is called *reflection-equivalent* to Q if there exists an admissible sequence of sinks or sources (i_1, \dots, i_l) such that $Q' = \sigma_{i_l} \cdots \sigma_{i_1}(Q)$. Note that mutations can be viewed as generalizations of reflections, i.e, if i is a sink or a source in a quiver Q , then $\mu_i(Q) = \sigma_i(Q)$ where μ_i denotes the mutation in the direction i . Thus if Q' is a quiver mutation-equivalent to Q , there is a natural canonical isomorphism between $\mathcal{A}_q(Q)$ and $\mathcal{A}_q(Q')$, denoted by

$$\Phi_i : \mathcal{A}_q(Q) \rightarrow \mathcal{A}_q(Q')$$

Let $\Sigma_i^+ : \text{rep}(Q) \rightarrow \text{rep}(Q')$ be the standard BGP-reflection functor and $R_i^+ : \mathcal{C}_Q \rightarrow \mathcal{C}_{Q'}$ be the extended BGP-reflection functor defined in [16]:

$$R_i^+ : \begin{cases} X \mapsto \Sigma_i^+(X) & \text{if } X \not\cong S_i \text{ is a module} \\ S_i \mapsto P_i[1] \\ P_j[1] \mapsto P_j[1] & \text{if } j \neq i \\ P_i[1] \mapsto S_i \end{cases}$$

In [14], the author proved the following result.

Theorem 4.3. [14] *For any indecomposable object M in \mathcal{C}_Q , we have $\Phi_i(X_M^Q) = X_{R_i^+ M}^{Q'}$.*

The following lemma is well-known.

Lemma 4.4. [5, Lemma 8(b)] *Let*

$$M \rightarrow E \rightarrow N \rightarrow M[1]$$

be a non-split triangle in \mathcal{C}_Q . Then

$$\dim_k \text{Ext}_{\mathcal{C}_Q}^1(E, E) < \dim_k \text{Ext}_{\mathcal{C}_Q}^1(M \oplus N, M \oplus N).$$

Theorem 4.5. *Assume that Q is an acyclic quiver whose underlying graph is bipartite and the matrix B associated to the quiver Q is of full rank, then $\mathcal{AH}_{|k|}(Q) = \mathcal{A}_{|k|}(Q)$.*

Proof. Firstly, we prove that for any indecomposable object $M \in \mathcal{C}_Q$, X_M is in the quantum cluster algebra $\mathcal{A}_{|k|}(Q)$.

Case 1: If Q is an alternating quiver (i.e., whose vertex is a sink or a source).

Denoted by

$$\Phi_i : \mathcal{A}_{|k|}(Q) \rightarrow \mathcal{A}_{|k|}(Q')$$

the canonical isomorphism of quantum cluster algebras associated to sink or source $1 \leq i \leq n$. It follows from Theorem 4.3, we obtain that $\Phi_i(X_M^Q) = X_{R_i^\pm M}^{Q'}$ for any indecomposable object $M \in \mathcal{C}_Q$. It is easy to see that Q' is again an acyclic quiver. Then we obtain that

$$X_M \in \mathbb{Z}[\mathbf{X}^{\pm 1}] \cap \mathbb{Z}[\mathbf{X}_1^{\pm 1}] \cap \cdots \cap \mathbb{Z}[\mathbf{X}_n^{\pm 1}].$$

Note that the quiver Q is acyclic, thus the corresponding quantum upper cluster algebra associated to Q coincides with the quantum cluster algebra $\mathcal{A}_{|k|}(Q)$ (see Theorem 2.3). Hence X_M is in the quantum cluster algebra $\mathcal{A}_{|k|}(Q)$.

Case 2: If Q is an acyclic quiver whose underlying graph is bipartite.

Note that Q is reflection equivalent to some alternating quiver Q' and in the *Case 1* we have showed that for any indecomposable object $M \in \mathcal{C}_{Q'}$, X_M is in the corresponding quantum cluster algebra $\mathcal{A}_{|k|}(Q')$. Thus the rest of the proof immediately follows from Theorem 4.3.

Now we need to prove that for any quantum cluster character $X_L \in \mathcal{AH}_{|k|}(Q)$, then $X_L \in \mathcal{A}_{|k|}(Q)$. Let $L \cong \bigoplus_{i=1}^l L_i^{\oplus n_i}$, $n_i \in \mathbb{N}$ where L_i ($1 \leq i \leq l$) are indecomposable objects in \mathcal{C}_Q . According to Theorem 3.2, we arrive at the following equality

$$X_{L_1}^{n_1} X_{L_2}^{n_2} \cdots X_{L_l}^{n_l} = q^{\frac{1}{2}n_L} X_L + \sum_{\dim_k \text{Ext}_{\mathcal{C}_Q}^1(E, E) < \dim_k \text{Ext}_{\mathcal{C}_Q}^1(L, L)} f_{n_E}(q^{\pm \frac{1}{2}}) X_E$$

where $n_L \in \mathbb{Z}$ and $f_{n_E}(q^{\pm \frac{1}{2}}) \in \mathbb{Z}[q^{\pm \frac{1}{2}}]$. Using Lemma 4.4 and proceeding by induction, it is straightforward to verify that $X_L \in \mathcal{A}_{|k|}(Q)$. \square

By Theorem 4.5, we can deduce the following corollary.

Corollary 4.6. *Assume that Q is an acyclic quiver whose underlying graph is bipartite, and the matrix B associated to the quiver Q is of full rank, then $\Psi(\mathcal{H}^*(\mathcal{A})) \subseteq \mathcal{A}_{|k|}(Q)$.*

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