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Induced and Coinduced Modules over Cluster-Tilted Algebras

Khrystyna Serhiyenko
University of Connecticut - Storrs, khrystyna.nechyporenko@uconn.edu

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University of Connecticut, 2015

ABSTRACT

We propose a new approach to study the relation between the module categories of a tilted algebra C and the corresponding cluster-tilted algebra $B = C \ltimes E$. This new approach consists of using the induction functor $- \otimes_C B$ as well as the coinduction functor $D(B \otimes_C D-)$. We give an explicit construction of injective resolutions of projective B-modules, and as a consequence, we obtain a new proof of the 1-Gorenstein property for cluster-tilted algebras. We show that DE is a partial tilting and a τ-rigid C-module and that the induced module $DE \otimes_C B$ is a partial tilting and a τ-rigid B-module. Furthermore, if $C = \text{End}_A T$ for a tilting module T over a hereditary algebra A, we compare the induction and coinduction functors to the Buan-Marsh-Reiten functor $\text{Hom}_{\mathcal{C}_A}(T, -)$ from the cluster-category of A to the module category of B. We also study the question which B-modules are actually induced or coinduced from a module over a tilted algebra.

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Khrystyna Serhiyenko

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Presented by

Khrystyna Serhiyenko, B.S. Applied Math., M.S. Math.

Major Advisor _____
Ralf Schiffler

Associate Advisor _____
Jerzy Weyman

Associate Advisor _____
Thomas Roby

University of Connecticut

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Contents

Chapter 1

Introduction

Representation theory is an area of mathematics that studies abstract algebraic objects such as groups, algebras, or Lie algebras by representing their elements as matrices and the operations between these elements as multiplication of matrices. This enables us to translate questions from an abstract algebraic setting to a more concrete linear algebra setting. Then one can use well-developed techniques of linear algebra such as Gaussian elimination, eigenvalue theory, and vector space bases to solve these questions. Representation theory was first introduced by Ferdinand Georg Frobenius about 100 years ago as a tool to analyze groups in an abstract way. Since then the theory vastly expanded to the study of other mathematical structures. Moreover, many connections to other fields have been developed, such as geometry, number theory, and particle physics. Ties to the last one have been noticed as early as 1930's by Eugene Wigner. It is known that a quantum state of an elementary particle corresponds to an irreducible representation of the Poincaré group.

More precisely, we are interested in studying the representation theory of cluster-

tilted algebras which are finite dimensional associative algebras that were introduced in [18] and, independently, in [22] for the type $\mathbb{A}$.

One motivation for introducing these algebras came from Fomin and Zelevinsky's cluster algebras [27]. Cluster algebras were developed as a tool to study dual canonical bases and total positivity in semisimple Lie groups, and cluster-tilted algebras were constructed as a categorification of these algebras. To every cluster in an acyclic cluster algebra one can associate a cluster-tilted algebra, such that the indecomposable rigid modules over the cluster-tilted algebra correspond bijectively to the cluster variables outside the chosen cluster. Generalizations of cluster-tilted algebras, the Jacobian algebras of quivers with potentials, were introduced in [26], extending this correspondence to the non-acyclic types. Many people have studied cluster-tilted algebras in this context, see for example [14, 18, 19, 20, 21, 23, 24, 29].

The second motivation came from classical tilting theory. Tilted algebras are the endomorphism algebras of tilting modules over hereditary algebras, whereas cluster-tilted algebras are the endomorphism algebras of cluster-tilting objects over cluster categories of hereditary algebras. This similarity in the two definitions lead to the following precise relation between tilted and cluster-tilted algebras, which was established in [3].

There is a surjective map

$$\{\text{tilted algebras}\} \longrightarrow\!\!\!\!\!\rightarrow \{\text{cluster-tilted algebras}\}$$

$$C \longmapsto B = C \ltimes E$$

where E denotes the C-C-bimodule $E = \text{Ext}^2_C(DC, C)$ and $C \ltimes E$ is the trivial extension.

This result allows one to define cluster-tilted algebras without using the cluster category. It is natural to ask how the module categories of C and B are related, and several results in this direction have been obtained, see for example [4, 5, 6, 13, 15, 25]. The Hochschild cohomology of the algebras C and B has been compared in [7, 9, 10, 31].

In this work, we use a new approach to study the relation between the module categories of a tilted algebra C and its cluster-tilted algebra $B = C \ltimes E$, namely *induction* and *coinduction*.

The induction functor $- \otimes_C B$ and the coinduction functor $\text{Hom}_C(B, -)$ from $\text{mod} \, C$ to $\text{mod} \, B$ are defined whenever C is a subring of B which has the same identity. If we are dealing with algebras over a field k, we can, and usually do, write the coinduction functor as $D(B \otimes_C D-)$, where $D = \text{Hom}(-, k)$ is the standard duality.

Induction and coinduction are important tools in the classical representation theory of finite groups. In this case, B would be the group algebra of a finite group G and C the group algebra of a subgroup of G (over a field whose characteristic is not dividing the group orders). In this situation, the algebras are semi-simple, induction and coinduction are the same functor, and this functor is exact. For arbitrary rings, and even for finite dimensional algebras, the situation is not that simple. In general, induction and coinduction are not the same functor and, since the C-module B is not projective (and not flat), induction and coinduction are not exact functors. However, the connection between tilted algebras and cluster-tilted algebras is close enough so that induction and coinduction are interesting tools for the study of the relation between the module categories.

Induction and coinduction have been studied for split extension algebras in [8, 12],

and we apply some of their results to our situation.

Our first main result is on the C-C-bimodule $E = \mathrm{Ext}^2_C(DC, C)$, considering its right C-module structure E_C as well as its left C-module structure $_C E$, but the latter is viewed a right C-module $D(_C E)$. In Chapter 3 we show the following.

Theorem 1.0.1. *If C is a tilted algebra and B is the corresponding cluster-tilted algebra, then*

(a) *DE is a partial tilting and τ_C-rigid C-module, and its corresponding induced module $DE \otimes B$ is a partial tilting and τ_B-rigid B-module.*

(b) *E is a partial cotilting and τ_C-corigid C-module, and its corresponding coinduced module $D(B \otimes DE)$ is a partial cotilting and τ_B-corigid B-module.*

We think it would be an interesting problem to study the possible completions of these partial (co)-tilting modules and their endomorphism algebras.

Our second main result presented in Chapter 4 is on injective and projective resolutions. The induction functor sends projective C-modules P_C to projective B-modules $P_B = P_C \otimes_C B$, and the coinduction functor sends injective C-modules I_C to injective B-modules $I_B = D(B \otimes_C DI_C)$. Hence, by considering injective resolutions in $\mathrm{mod}\, C$ we construct an explicit injective resolution in $\mathrm{mod}\, B$ for each projective B-module. Here ν denotes the Nakayama functor and Ω^{-1} the cosyzygy.

Theorem 1.0.2. *Let C be a tilted algebra, B the corresponding cluster-tilted algebra, P_C a projective C-module and $P_B = P_C \otimes_C B$ the corresponding projective B-module. Let*

$$0 \longrightarrow P_C \longrightarrow I^0_C \longrightarrow I^1_C \qquad and \qquad 0 \longrightarrow P_C \otimes E \longrightarrow \bar{I}^0_C \longrightarrow \bar{I}^1_C$$

be minimal injective presentations in $\mathrm{mod}\,C$, *and let* $\tilde{I}_C$ *be the injective* C-*module* $\tilde{I}_C = \nu\nu^{-1}\Omega^{-1}P_C$. *Then*

$$0 \longrightarrow P_B \longrightarrow I_B^0 \oplus \bar{I}_B^0 \longrightarrow \tilde{I}_B \oplus \bar{I}_B^1 \longrightarrow 0$$

is an injective resolution of P_B *in* $\mathrm{mod}\,B$.

The proof of the theorem uses both induction and coinduction, and it relies greatly on the particular structure of the bimodule E. As an immediate consequence, we obtain a new proof, which does not use cluster categories, of a result by Keller and Reiten [29].

Corollary 1.0.3. *Cluster-tilted algebras are 1-Gorenstein.*

In Chapter 5 we compare induction and coinduction with the well-known equivalence of categories $\mathrm{Hom}_{\mathcal{C}_A}(T,-) : \mathcal{C}_A/\mathrm{add}\,\tau T \to \mathrm{mod}\,B$ of Buan, Marsh and Reiten [18]. Here $\mathcal{C}_A$ denotes the cluster category. We show that induction commutes with this equivalence on the torsion subcategory $\mathcal{T}(T)$ and coinduction commutes with a variation of this equivalence on the torsion free subcategory $\mathcal{F}(T)$. We obtain the following result.

Theorem 1.0.4. *Let* C *be a tilted algebra and* B *the corresponding cluster-tilted algebra. Then*

$$\mathrm{Hom}_A(T,M) \otimes_C B \;\cong\; \mathrm{Hom}_{\mathcal{C}_A}(T,M), \text{ for every } M \in \mathcal{T}(T),$$
$$D(B \otimes_C D\mathrm{Ext}_A^1(T,M)) \;\cong\; \mathrm{Ext}_{\mathcal{C}_A}^1(T,M), \text{ for every } M \in \mathcal{F}(T).$$

Let us point out that these formulas implicitly use the tilting theorem of Brenner and Butler [16].

Finally, in Chapter 6 we address the question which B-modules can be obtained via induction or coinduction of modules over a tilted algebra.

Theorem 1.0.5. *Let B be a cluster-tilted algebra.*

(a) *If B is of finite representation type, then* every *B-module is induced and coinduced from some tilted algebra.*

(b) *If B is of arbitrary representation type, then every* transjective *B-module is induced or coinduced from some tilted algebra.*

(c) *If B is cluster concealed, then* every *B-module is induced or coinduced from some tilted algebra.*

(d) *If B is of tame representation type and there are no morphisms between indecomposable projective non-transjective modules, then* every *indecomposable B-module is induced or coinduced from some tilted algebra.*

We remark that for general cluster-tilted algebras, there are indecomposable modules that are not induced and not coinduced. We think it would be interesting to study the structure of these modules.

1.1 Modules over path algebras

The algebras discussed in this work can be realized using quivers. A *quiver $Q = (Q_0, Q_1)$* is an oriented graph, where Q_0 denotes a finite set of vertices, and Q_1 denotes a finite set of oriented edges, or arrows, in the quiver. A quiver is said to be *acyclic* if it does not contain a non constant oriented path starting and ending at the same vertex. We will only study quivers without loops ($\bullet\,\circlearrowright$) and oriented 2-cycles ($\bullet\rightleftarrows\bullet$). Let k be an algebraically closed field, and we define the associated path algebra kQ

below.

Definition 1.1.1. Given a quiver Q, the corresponding *path algebra kQ* is an asso-
ciative k-algebra with basis the set of all paths in Q. That is, the elements of this
algebra are k-linear combinations of paths in the quiver Q. Addition is the formal
addition of elements, and multiplication is given by composition of paths.

Example 1.1.2. Consider the following quiver which will be our running example.

$$\widetilde{Q}: \quad 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

The corresponding path algebra $k\widetilde{Q}$ is generated by the set of all possible paths
$\{\alpha, \beta, \alpha\beta, e_1, e_2, e_3\}$. Here e_1 corresponds to a constant path that starts at vertex 1
and ends at vertex 1, and the other e_i's are defined similarly. Moreover, if we multiply
$\alpha \cdot \beta$ we get a path $\alpha\beta$, and similarly $e_1 \cdot \alpha = \alpha$, $\beta \cdot e_3 = \beta$. However, $\beta \cdot \alpha$ and $\beta \cdot e_2$
are both zero because composition of these paths does not make sense when looking
at $\widetilde{Q}$.

As mentioned in the introduction, we do not want to study elements of the algebra
kQ, but rather use linear algebra and consider modules over this algebra. We always
take kQ to be finite dimensional, meaning Q contains only finitely many possible
paths or equivalently Q is acyclic. All (right) kQ-modules have a concrete realization
in terms of the quiver.

Definition 1.1.3. A (finite dimensional) *right kQ-module M* is a collection of finite
dimensional k-vector spaces $M_i = Me_i$, so that $M = \bigoplus_{i \in Q_0} M_i$, together with the
following kQ action. The module structure on M is equivalent to the set $M = (M_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$, where φ_α is a k-linear map, such that if α is an arrow in Q from

vertex i to vertex j, then φ_α is a map from M_i to M_j. Moreover, the map φ_α corresponds to the right action of the arrow α on M, sending $M_i = Me_i$ to $Me_i\alpha = Me_j = M_j$. Then extending this definition by linearity and multiplication yields a well-defined kQ action on M. (See Example 1.1.7.)

In other words any kQ-module M is obtained by placing a finite number of copies of k at each vertex in the quiver Q and a matrix between these spaces at each arrow in Q.

Definition 1.1.4. Given any two modules $M = (M_i, \varphi_\alpha)$ and $M' = (M'_i, \varphi'_\alpha)$ of an algebra kQ we can form a new module $M \oplus M'$ called a *direct sum* as follows.

$$M \oplus M' = (M_i \oplus M'_i, \begin{bmatrix} \varphi_\alpha & 0 \\ 0 & \varphi'_\alpha \end{bmatrix})$$

A nonzero module M is said to be *indecomposable* if it cannot be written as a direct sum of two other nonzero modules.

The following is a classical result known as the Krull-Schmidt theorem. It shows that in order to understand modules it suffices to study only the indecomposable ones.

Theorem 1.1.5. *Let M be a kQ-module. Then*

$$M \cong M_1 \oplus M_2 \oplus \cdots \oplus M_n$$

where M_i are indecomposable kQ-modules which are unique up to reordering.

We also want to consider morphisms or maps between modules.

Definition 1.1.6. Given two modules M and M' of kQ as before, a *morphism* $f :$ $M \to M'$ is a family of linear maps $f_i : M_i \to M'_i$ for all vertices i in Q such that for all arrows α from i to j the following diagram commutes.

$$
\begin{array}{ccc}
M_i & \xrightarrow{\varphi_\alpha} & M_j \\
f_i \downarrow & & \downarrow f_j \\
M'_i & \xrightarrow[\varphi'_\alpha]{} & M'_j
\end{array}
$$

Moreover, f is said to be an *isomorphism* if each f_i is an isomorphism.

Example 1.1.7. Let $\widetilde{M} = k \xrightarrow{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} k^2 \xrightarrow{\left[\begin{smallmatrix}1&0\\0&1\\0&0\end{smallmatrix}\right]} k^3$. Then $\widetilde{M}$ is a module of $k\widetilde{Q}$, where $\widetilde{Q}$ is defined in Example 1.1.2. The module $\widetilde{M}$ decomposes as

$$
(0 \xrightarrow{0} k \xrightarrow{1} k) \oplus (k \xrightarrow{1} k \xrightarrow{1} k) \oplus (0 \xrightarrow{0} 0 \xrightarrow{0} k).
$$

Denote the three summands of $\widetilde{M}$ by $\widetilde{M_1}$, $\widetilde{M_2}$, and $\widetilde{M_3}$ respectively. Let us find all maps $f \in \mathrm{Hom}_{k\widetilde{Q}}(\widetilde{M_1}, \widetilde{M_2})$, which is the space of all $k\widetilde{Q}$ morphisms mapping $\widetilde{M_1}$ to $\widetilde{M_2}$.

$$
\begin{array}{ccc}
\widetilde{M_1} & \qquad & 0 \xrightarrow{0} k \xrightarrow{1} k \\
\downarrow f & & \downarrow f_1 \quad\ \downarrow f_2 \quad\ \downarrow f_3 \\
\widetilde{M_2} & & k \xrightarrow{1} k \xrightarrow{1} k
\end{array}
$$

Note that according to the definition of a morphism each square in the diagram above must commute. Thus, we obtain two equations $f_1 = 0$ and $f_2 = f_3$. This shows that f must be of the form $f = (0, a, a)$ where $a \in k$, so $\mathrm{Hom}_{k\widetilde{Q}}(\widetilde{M_1}, \widetilde{M_2}) \cong k$. On the other hand similar computations yield $\mathrm{Hom}_{k\widetilde{Q}}(\widetilde{M_1}, \widetilde{M_3}) = 0$.

In the example above we can denote the representation $\widetilde{M_1}$ by $\genfrac{}{}{0pt}{}{2}{3}$, meaning there are one dimensional vector spaces at vertices 2 and 3, and the arrow is going down from 2 to 3 carrying the identity map. With this notation

$$\widetilde{M} \cong \genfrac{}{}{0pt}{}{2}{3} \oplus \genfrac{}{}{0pt}{}{1}{\genfrac{}{}{0pt}{}{2}{3}} \oplus 3\,.$$

From now on, we will only use the above notation to depict various modules.

It might seem that the algebras discussed in this section are very particular. However, path algebras make up a broad and well-studied class of algebras. In fact, any basic finite dimensional k-algebra is isomorphic to a quotient of a quiver path algebra kQ. See for example [11, Theorem II.3.7].

1.2   Projective and injective resolutions

Given a finite dimensional k-algebra Λ, let $\operatorname{mod}\Lambda$ denote the abelian category of finitely generated right Λ-modules. The set of all Λ-morphisms from M to M' is denoted by $\operatorname{Hom}_\Lambda(M, M')$, where $M, M' \in \operatorname{mod}\Lambda$. By $\operatorname{add}M$ we understand the full subcategory of $\operatorname{mod}\Lambda$ whose objects are the direct sums of direct summands of the module M. Next we recall the definitions of some important modules associated to a given Λ-module.

Definition 1.2.1. Given a finite dimensional k-algebra Λ, let $M \in \operatorname{mod}\Lambda$.

(a) The *right annihilator* of M is the module $\operatorname{Ann}M = \{a \in \Lambda \,|\, Ma = 0\}$.

(b) The (Jacobson) *radical* $\operatorname{rad}M$ of M is the intersection of all maximal submodules of M.

(c) The *top* of M is the quotient module $\operatorname{top}M = M/\operatorname{rad}M$.

(d) The *socle* submodule soc M of M is generated by all simple submodules of M.

The projective and injective modules make up another important class of modules. In fact, any Λ-module M can be approximated on the left by projective Λ-modules and on the right by injective Λ-modules. This means that there exist exact sequences of the form

$$\ldots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow \ldots$$

with P_i projective and I_i injective modules. Such sequences are called *projective* and *injective resolutions* of M. The *projective dimension* of M in $\mod \Lambda$, denoted by $\mathrm{pd}_\Lambda M$, is the smallest integer n such that there exists a projective resolution as above with $P_{n+1} = 0$. If no such n exists then $\mathrm{pd}_\Lambda M = \infty$. The injective dimension of M, denoted by $\mathrm{id}_\Lambda M$, is defined in a similar manner. These terms give rise to the following characterisation of algebras.

Definition 1.2.2. The *global dimension* gl.dim Λ of the algebra Λ is the supremum over all projective dimensions $\mathrm{pd}_\Lambda M$ of all $M \in \mod \Lambda$.

Next we define the concept of syzygies. Given a Λ-module M there exists a unique (up to isomorphism) projective Λ-module P_0, called a *projective cover* together with a surjective map $g_0 : P_0 \to M$, such that this map does not factor through another projective Λ-module. Then the *syzygy* ΩM of M is defined as the kernel of g_0. In particular, there exists the following short exact sequence of Λ-modules.

$$0 \longrightarrow \Omega M \longrightarrow P_0 \xrightarrow{g_0} M \longrightarrow 0$$

Similarly, we define the *cosyzygy* $\Omega^{-1}M$ of M. Let I_0 be a unique (up to isomorphisms) injective Λ-module, called an *injective envelope*, together with an injective map $f_0 : M \to I_0$, such that this map does not factor through another injective Λ-module. Then $\Omega^{-1}M$ is defined as the cokernel of f_0, and there exists the following short exact sequence.

$$0 \longrightarrow M \xrightarrow{f_0} I_0 \longrightarrow \Omega^{-1}M \longrightarrow 0$$

Moreover, there is a nice relationship between the functor Ext and (co)-syzygy operation, which will be used extensively later on.

Theorem 1.2.3. *For any Λ-modules M and M' and for any $n \geq 2$*

$$\operatorname{Ext}_\Lambda^n(M, M') \cong \operatorname{Ext}_\Lambda^{n-1}(\Omega M, M') \cong \operatorname{Ext}_\Lambda^{n-1}(M, \Omega^{-1}M').$$

1.3 Auslander-Reiten theory

We recall a number of well-known functors used in the study of the representation theory of algebras. The standard duality functor $\operatorname{Hom}_k(-, k)$ will always be denoted by D. Let $\nu = D\operatorname{Hom}_\Lambda(-, \Lambda)$ be the *Nakayama functor* and $\nu^{-1} = \operatorname{Hom}_\Lambda(D\Lambda, -)$ be the *inverse Nakayama functor*. These functors induce an equivalence between projective and injective Λ-modules.

$$\nu : \operatorname{proj}\Lambda \to \operatorname{inj}\Lambda \qquad\qquad \nu^{-1} : \operatorname{inj}\Lambda \to \operatorname{proj}\Lambda$$

Finally, given a Λ-module M we define its *Auslander-Reiten translate* τM as follows. Let

$$P_1 \xrightarrow{g_1} P_0 \xrightarrow{g_0} M \longrightarrow 0$$

be a minimal projective presentation of M. This means, P_0 is a projective cover of M, and P_1 is a projective cover of ΩM. Applying ν to this sequence one obtains τM as the kernel of νg_1. In particular, there exists an exact sequence

$$0 \longrightarrow \tau M \longrightarrow \nu P_1 \xrightarrow{\nu g_1} \nu P_0.$$

Dually, let

$$0 \longrightarrow M \xrightarrow{f_0} I_0 \xrightarrow{f_1} I_1$$

be a minimal injective presentation of M. This means, I_0 is an injective envelope of M, and I_1 is an inject envelope of $\Omega^{-1} M$. Applying ν^{-1} to this sequence one obtains the *inverse Auslander-Reiten translate* $\tau^{-1} M$ of M as the cockerel of $\nu^{-1} f_1$. In particular, there exists an exact sequence

$$\nu^{-1} I_0 \xrightarrow{\nu^{-1} f_1} \nu^{-1} I_1 \longrightarrow \tau^{-1} M \longrightarrow 0.$$

The Auslander-Reiten formula is an important result that describes a relationship between the spaces Hom and Ext via the Auslander-Reiten translation. Let $\underline{\mathrm{Hom}}_\Lambda(M, M')$ (respectively $\overline{\mathrm{Hom}}_\Lambda(M, M')$) denote the space of all Λ-morphisms from M to M' that do not factor through projective (respectively injective) Λ-modules. With this notation consider the following result.

Theorem 1.3.1 (Auslander-Reiten formula)**.** *Let M and M' be Λ-modules. Then*

$$\text{Ext}^1_\Lambda(M, M') \cong D\underline{\text{Hom}}_\Lambda(\tau^{-1}M', M) \cong D\overline{\text{Hom}}_\Lambda(M', \tau M).$$

Moreover, if $\text{pd}_\Lambda M \leq 1$ then $\text{Ext}^1_\Lambda(M, M') \cong D\text{Hom}_\Lambda(M', \tau M)$, and similarly if $\text{id}_\Lambda M' \leq 1$ then $\text{Ext}^1_\Lambda(M, M') \cong D\text{Hom}_\Lambda(\tau^{-1}M', M)$.

While studying the representation theory of algebras it is useful to consider $\text{ind}\,\Lambda$, which is a set of representatives of each isoclass of indecomposable right Λ-modules. If $\text{ind}\,\Lambda$ is a finite set then Λ is said to be a *representation-finite* algebra. Otherwise, it is said to be a *representation-infinite* algebra.

An important tool in studying the module category of Λ is the *Auslander-Reiten quiver* denoted by $\Gamma(\text{mod}\,\Lambda)$. The vertices of this quiver are the elements of $\text{ind}\,\Lambda$, and the arrows are the so-called *irreducible* Λ-morphisms. By an irreducible morphism we understand a morphism between indecomposable modules that does not factor through another module. Below we identify some key components of the Auslander-Reiten quiver.

Definition 1.3.2. Let $\Gamma(\text{mod}\,\Lambda)$ be the Auslander-Reiten quiver of a k-algebra Λ.

(a) A connected component $\mathcal{P}$ of $\Gamma(\text{mod}\,\Lambda)$ is called *preprojective* if $\mathcal{P}$ is acyclic, and for any indecomposable module $M \in \mathcal{P}$, there exists $t \geq 0$ such that $M \cong \tau^{-t}P$ and P is an indecomposable projective Λ-module. A Λ-module is called *preprojective* if it is a direct sum of indecomposable modules each belonging to a preprojective component.

(b) A connected component $\mathcal{Q}$ of $\Gamma(\text{mod}\,\Lambda)$ is called *preinjective* if $\mathcal{Q}$ is acyclic, and for any indecomposable module $N \in \mathcal{Q}$, there exists $s \geq 0$ such that $N \cong \tau^s I$ and I is an indecomposable injective Λ-module. A Λ-module is called *preinjective* if it is

a direct sum of indecomposable modules each belonging to a preinjective component.

(c) A connected component $\mathcal{R}$ of $\Gamma(\operatorname{mod}\Lambda)$ is called *regular* if it contains neither projective nor injective modules. A Λ-module is called *regular* if it is a direct sum of indecomposable modules each belonging to a regular component.

Moreover, if a representation infinite algebra Λ is isomorphic to some path algebra kQ, then its Auslander-Reiten quiver is precisely a disjoint union of preprojective, preinjective, and regular components.

For further details on representation theory we refer to [11, 35].

Example 1.3.3. Consider the path algebra $k\widetilde{Q}$ as defined in Example 1.1.2. This algebra is representation finite, so the Auslander-Reiten quiver $\Gamma(\operatorname{mod}k\widetilde{Q})$ shown below is also finite. The Auslander-Reiten quiver consists of a single connected component, which means every $k\widetilde{Q}$ module is both preinjective and preprojective.
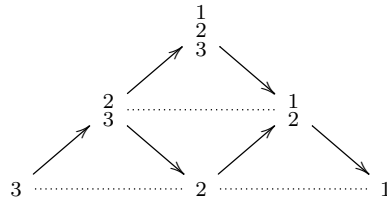
Given the Auslander-Reiten quiver it is very easy to compute τ and τ^{-1}, which corresponds to shifting a module left and right, respectively, along the dotted lines. For example,

$$\tau\,{\textstyle\frac{1}{2}}=\,{\textstyle\frac{2}{3}} \qquad \tau\,{\textstyle\frac{2}{3}}=0 \qquad \tau^{-1}2=1 \qquad \tau\,{\textstyle\frac{1}{\frac{2}{3}}}=\tau^{-1}{\textstyle\frac{1}{\frac{2}{3}}}=0.$$

1.4   Tilted algebras

Tilting theory is one of the main themes in the study of the representation theory of algebras. Given a k-algebra Λ one can construct a new algebra from Λ in such way that the corresponding module categories are closely related. The key idea is the notion of a tilting module, which we define below.

Definition 1.4.1. A module $T \in \operatorname{mod} \Lambda$ is called *tilting* if all of the following are true:

    (i) T is *rigid*, meaning $\operatorname{Ext}_\Lambda^1(T, T) = 0$.

    (ii) $\operatorname{pd}_\Lambda T \leq 1$.

    (iii) the number of indecomposable summands of T is the number of isoclasses of indecomposable simple Λ-modules.

Moreover T is called a *partial tilting module* if conditions (i) and (ii) hold.

An algebra A is said to be *hereditary* if every submodule of a projective A-module is also projective. It is known, that any finite-dimensional hereditary algebra over an algebraically closed field k is Morita equivalent to a path algebra of a uniquely determined finite quiver Q without oriented cycles.

Given a tilting module T as defined above over a hereditary algebra A we construct a new algebra $C = \operatorname{End}_A T$ called a *tilted algebra*. Observe that C is indeed an algebra. Given two maps f, g going from T to T, we can define two operations, addition and multiplication, by considering two new maps, $f + g$ and $f \circ g$ (composition), which again are morphisms from T to T.

Example 1.4.2. Let $k\widetilde{Q}$ be the algebra defined in Example 1.1.2. Then

$$\widetilde{T} = {}_1 \oplus {\textstyle\frac{1}{2}}_3 \oplus {}_3$$

is a tilting $k\widetilde{Q}$ module consisting of three distinct indecomposable direct summands. Observe that there is a nonzero map from the third summand to the second, and from the second summand to the first, but there is no such map from the third summand to the first. Hence, the corresponding tilted algebra $\widetilde{C} = \text{Hom}_{k\widetilde{Q}}(\widetilde{T}, \widetilde{T})$ can be realized as the path algebra of the following quiver

$$Q_{\widetilde{C}} = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

and requiring that $\alpha\beta = 0$. In other words $\widetilde{C} = kQ_{\widetilde{C}}/\langle\alpha\beta\rangle$, where $\langle\alpha\beta\rangle$ denotes the ideal generated by $\alpha\beta$.

Definition 1.4.3. A pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of mod Λ is called a *torsion pair* if the following conditions are satisfied:

(a) $\text{Hom}_\Lambda(M, N) = 0$ for all $M \in \mathcal{T}$, $N \in \mathcal{F}$.

(b) $\text{Hom}_\Lambda(M, -)|_{\mathcal{F}} = 0$ implies $M \in \mathcal{T}$.

(c) $\text{Hom}_\Lambda(-, N)|_{\mathcal{T}} = 0$ implies $N \in \mathcal{F}$.

Given a tilting module $T \in \text{mod } A$ and a tilted algebra $C = \text{End}_A T$, consider the following full subcategories of mod A.

$$\mathcal{T}(T) = \{M \in \text{mod } A \mid \text{Ext}_A^1(T, M) = 0\}$$

$$\mathcal{F}(T) = \{M \in \text{mod } A \mid \text{Hom}_A(T, M) = 0\}$$

Then $(\mathcal{T}(T), \mathcal{F}(T))$ is a torsion pair in mod A that determines another torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ in mod C, where

$$\mathcal{X}(T) = \{M \in \text{mod } C \mid M \otimes_C T = 0\}$$

$$\mathcal{Y}(T) = \{M \in \operatorname{mod} C \mid \operatorname{Tor}_1^C(M, T) = 0\}.$$

For more details refer to [11, chapters VI and VIII]. With this notation consider the following theorem due to Brenner and Butler.

Theorem 1.4.4. *Let A be a hereditary algebra, T a tilting A-module, and $C = \operatorname{End}_A T$. Then*

(a) *the functors $\operatorname{Hom}_A(T, -)$ and $- \otimes_C T$ induce quasi-inverse equivalences between $\mathcal{T}(T)$ and $\mathcal{Y}(T)$.*

(b) *the functors $\operatorname{Ext}_A^1(T, -)$ and $\operatorname{Tor}_1^C(-, T)$ induce quasi-inverse equivalences between $\mathcal{F}(T)$ and $\mathcal{X}(T)$.*

Moreover, $\operatorname{Tor}_1^C(\operatorname{Hom}_A(T, M), T) = 0$ and $\operatorname{Ext}_A^1(T, M) \otimes_C T = 0$ for any $M \in \operatorname{mod} A$, and $\operatorname{Hom}_A(T, \operatorname{Tor}_1^C(N, T)) = 0$ and $\operatorname{Ext}_A^1(T, N \otimes_C T) = 0$ for any $N \in \operatorname{mod} C$.

There is a close relationship between the Auslander-Reiten quivers of a hereditary algebra and its corresponding tilted algebra. We shall need the following result.

Theorem 1.4.5. *[11, p.330] Let A be a representation-infinite hereditary algebra, T be a preprojective tilting A-module, and $C = \operatorname{End}_A T$. Then*

(a) *$\mathcal{T}(T)$ contains all but finitely many nonisomorphic indecomposable A-modules, and any indecomposable A-module not in $\mathcal{T}(T)$ is preprojective.*

(b) *the image under the functor $\operatorname{Hom}_A(T, -)$ of regular A-modules yields all regular C-modules.*

(c) *the injective and projective dimension of all regular C-modules is at most one.*

The following proposition describes several facts about tilted algebras, which we will use throughout the thesis.

Proposition 1.4.6. *Let A be a hereditary algebra, T a tilting A-module, and $C =$* $\operatorname{End}_A T$ *the corresponding tilted algebra. Then*

(a) $\operatorname{gl.dim} C \leq 2$.

(b) *For all $M \in \operatorname{ind} C$ $\operatorname{id}_C M \leq 1$ or $\operatorname{pd}_C M \leq 1$.*

(c) *For all $M \in \mathcal{X}(T)$ $\operatorname{id}_C M \leq 1$.*

(d) *For all $M \in \mathcal{Y}(T)$ $\operatorname{pd}_C M \leq 1$.*

(e) *$(\mathcal{X}(T), \mathcal{Y}(T))$ is splitting, which means that every indecomposable C-module belongs either to $\mathcal{X}(T)$ or $\mathcal{Y}(T)$.*

(f) *$\mathcal{Y}(T)$ is closed under predecessors and $\mathcal{X}(T)$ is closed under successors.*

There is also a precise description of injective modules in a tilted algebra in terms of the corresponding hereditary algebra.

Proposition 1.4.7. *Let A be a hereditary algebra, T a tilting A-module, and $C =$* $\operatorname{End}_A T$. *Let $T_1, \ldots, T_n$ be a complete set of pairwise nonisomorphic indecomposable direct summands of T. Assume that the modules $T_1, \ldots, T_m$ are projective, the remaining modules $T_{m+1}, \ldots, T_n$ are not projective, and $I_1, \ldots, I_m$ are indecomposable injective A-modules with $\operatorname{soc} I_j \cong T_j / \operatorname{rad} T_j$, for $j = 1, \ldots, m$. Then the C-modules*

$$\operatorname{Hom}_A(T, I_1), \ldots, \operatorname{Hom}_A(T, I_m), \operatorname{Ext}^1_A(T, \tau T_{m+1}), \ldots, \operatorname{Ext}^1_A(T, \tau T_n)$$

form a complete set of pairwise nonisomorphic indecomposable injective modules.

1.5   Cluster-tilted algebras

Let $A = kQ$ and let $\mathcal{D}^b(\operatorname{mod} A)$ denote the derived category of bounded complexes of A-modules. The *cluster category* $\mathcal{C}_A$ is defined as the orbit category of the derived category with respect to the functor $\tau_{\mathcal{D}}^{-1}[1]$, where $\tau_{\mathcal{D}}$ is the Auslander-Reiten translation in the derived category and $[1]$ is the shift. Cluster categories were introduced in [17], and in [22] for type $\mathbb{A}$, and were further studied in [2, 28, 29, 32]. They are triangulated categories [28], that are 2-Calabi-Yau and have Serre duality [17].

An object T in $\mathcal{C}_A$ is called *cluster-tilting* if $\operatorname{Ext}^1_{\mathcal{C}_A}(T,T) = 0$ and T has $|Q_0|$ non-isomorphic indecomposable direct summands. The endomorphism algebra $\operatorname{End}_{\mathcal{C}_A} T$ of a cluster-tilting object is called a *cluster-tilted algebra* [18].

The following theorem will be used later.

Theorem 1.5.1. *[18] If T is a cluster-tilting object in $\mathcal{C}_A$, then $\operatorname{Hom}_{\mathcal{C}_A}(T,-)$ induces an equivalence of categories $\mathcal{C}_A/\operatorname{add}(\tau T) \to \operatorname{mod} \operatorname{End}_{\mathcal{C}_A} T$.*

1.6   Relation extensions

Let C be an algebra of global dimension at most two and let E be the C-C-bimodule $E = \operatorname{Ext}^2_C(DC, C)$. The *relation extension* of C is the trivial extension algebra $B = C \ltimes E$, whose underlying C-module is $C \oplus E$, and multiplication is given by $(c, e)(c', e') = (cc', ce' + ec')$. Relation extensions where introduced in [3]. In the special case where C is a tilted algebra, we have the following result.

Theorem 1.6.1. *[3] Let C be a tilted algebra. Then $B = C \ltimes \operatorname{Ext}^2_C(DC, C)$ is a cluster-tilted algebra. Moreover all cluster-tilted algebras are of this form.*

This shows that a tilted algebra C is both a subalgebra and a quotient of the associated cluster-tilted algebra B. In particular there exists a short exact sequence of right B-modules

$$0 \longrightarrow E \xrightarrow{\ i\ } B_B \xrightarrow{\ \pi\ } C_C \longrightarrow 0 . \tag{1.6.1}$$

In general the quiver of the corresponding cluster-tilted algebra can be obtained from the quiver of its tilted algebra by adding new arrows in the opposite direction for each zero relation.

Example 1.6.2. Recall that $\widetilde{C} = k\widetilde{Q}/\langle \alpha\beta \rangle$ is a tilted algebra constructed in Example 1.4.2, and there is only one zero relation $\alpha\beta = 0$. Hence, we obtain the following quiver

$$Q_{\widetilde{B}} = \ 1 \underset{\gamma}{\overset{\alpha}{\rightleftarrows}} 2 \xrightarrow{\ \beta\ } 3$$

and the associated cluster-tilted algebra $\widetilde{B} = kQ_{\widetilde{B}}/\langle \alpha\beta, \beta\gamma, \gamma\alpha \rangle$. Note that the quiver is not acyclic, however the resulting algebra $\widetilde{B}$ is still finite dimensional. The theory discussed in Section 1.1 still makes sense in this setting.

1.7 Slices and local slices

Let Λ be a k-algebra.

Definition 1.7.1. A *path* in $\operatorname{mod}\Lambda$ with source X and target Y is a sequence of non-zero morphisms $X = X_0 \to X_1 \to \cdots \to X_s = Y$ where $X_i \in \operatorname{mod}\Lambda$ for all i, and $s \geq 1$. A *path* in $\Gamma(\operatorname{mod}\Lambda)$ with source X and target Y is a sequence of arrows $X = X_0 \to X_1 \to \cdots \to X_s = Y$ in the Auslander-Reiten quiver. In addition, such path is called *sectional* if for each i with $0 < i < s$, we have $\tau_\Lambda X_{i+1} \neq X_{i-1}$.

Definition 1.7.2. A *slice* Σ in $\Gamma(\text{mod}\,\Lambda)$ is a set of indecomposable Λ-modules such that

(i) Σ is sincere, meaning $\text{Hom}_\Lambda(P, \Sigma) \neq 0$ for any projective Λ-module P.

(ii) Any path in $\text{mod}\,\Lambda$ with source and target in Σ consists entirely of modules in Σ.

(iii) If M is an indecomposable non-projective Λ-module then at most one of M, $\tau_\Lambda M$ belongs to Σ.

(iv) If $M \to S$ is an irreducible morphism with $M, S \in \text{ind}\,\Lambda$ and $S \in \Sigma$, then either M belongs to Σ or M is non-injective and $\tau_\Lambda^{-1}M$ belongs to Σ.

Given an indecomposable Λ-module Y in a connected component Γ of $\Gamma(\text{mod}\,\Lambda)$ define two full subquivers of Γ induced by the following set of vertices.

$$\Sigma(\to Y) = \{X \in \text{ind}\,\Lambda \mid \exists\, X \to \cdots \to Y \in \Gamma \text{ and every path from } X \text{ to } Y \text{ in } \Gamma \text{ is sectional}\}$$

$$\Sigma(Y \to) = \{X \in \text{ind}\,\Lambda \mid \exists\, Y \to \cdots \to X \in \Gamma \text{ and every path from } Y \text{ to } X \text{ in } \Gamma \text{ is sectional}\}$$

With this notation consider the next result due to Ringel.

Proposition 1.7.3. *[33] Let Y be an indecomposable sincere module in a standard, convex, and directed component. Then both $\Sigma(\to Y)$ and $\Sigma(Y \to)$ are slices. Moreover, a component that is preprojective or preinjective is standard, convex, and directed.*

The existence of slices is used to characterize tilted algebras in the following way.

Theorem 1.7.4. *[33] Let $C = \text{End}_A T$ be a tilted algebra. Then the class of C-modules $\text{Hom}_A(T, DA)$ forms a slice in $\text{mod}\,C$. Conversely, any slice in any module category is obtained in this way.*

The following notion of local slices has been introduced in [4] in the context of cluster-tilted algebras.

Definition 1.7.5. A *local slice* Σ in $\Gamma(\operatorname{mod}\Lambda)$ is a set of indecomposable Λ-modules inducing a connected full subquiver of $\Gamma(\operatorname{mod}\Lambda)$ such that

(i) If $X \in \Sigma$ and $X \to Y$ is an arrow in $\Gamma(\operatorname{mod}\Lambda)$ then either $Y \in \Sigma$ or $\tau_\Lambda Y \in \Sigma$.

(ii) If $Y \in \Sigma$ and $X \to Y$ is an arrow in $\Gamma(\operatorname{mod}\Lambda)$ then either $X \in \Sigma$ or $\tau_\Lambda^{-1} X \in \Sigma$.

(iii) For every sectional path $X = X_0 \to X_1 \to \cdots \to X_s = Y$ in $\Gamma(\operatorname{mod}\Lambda)$ with $X, Y \in \Sigma$ we have $X_i \in \Sigma$, for $i = 0, 1, \ldots, s$.

(iv) The number of indecomposables in Σ equals the number of nonisomorphic summands of T, where T is a tilting Λ-module.

Remark 1.7.6. The definition of a local slice makes sense if we replace the algebra Λ by a cluster category $\mathcal{C}_A$, and then consider objects of $\mathcal{C}_A$ instead of Λ-modules.

There is a relationship between tilted and cluster-tilted algebras given in terms of slices and local slices.

Theorem 1.7.7. *[4] Let C be a tilted algebra and B be the corresponding cluster-tilted algebra. Then any slice in $\operatorname{mod} C$ embeds as a local slice in $\operatorname{mod} B$ and any local slice Σ in $\operatorname{mod} B$ arises in this way. Moreover, $C = B/\operatorname{Ann}_B \Sigma$.*

The existence of local slices in a cluster-tilted algebra gives rise to the following definition. The unique connected component of $\Gamma(\operatorname{mod} B)$ that contains local slices is called the *transjective* component. In particular if B is of finite representation type then the transjective component is the entire Auslander-Reiten quiver of $\operatorname{mod} B$. In this case there is the following statement.

Theorem 1.7.8. *[4] Let B be a representation-finite cluster-tilted algebra. Then any indecomposable B-module lies on a local slice.*

We end this section with a lemma that we will need later.

Lemma 1.7.9. *Let $\widetilde{\Sigma}$ be a local slice in the cluster category $\mathcal{C}_A$. Then $\Sigma = \mathrm{Hom}_{\mathcal{C}_A}(T, \widetilde{\Sigma})$ is a local slice in $\mathrm{mod}\, B$ if and only if $\widetilde{\Sigma}$ contains no summand of $\tau_{\mathcal{C}_A} T$.*

Proof. $\mathrm{Hom}_{\mathcal{C}_A}(T, -) : \widetilde{\Sigma} \to \Sigma$ is a bijection if and only if $\widetilde{\Sigma}$ contains no summand of $\tau_{\mathcal{C}_A} T$. Now the statement follows from [4, Lemma 17]. $\qquad\square$

Chapter 2

Induction and coinduction functors

In this chapter we define two functors called induction and coinduction and describe some general results about them. Suppose there are two k-algebras C and B with the property that C is a subalgebra of B and they share the same identity. Then there is a general construction via the tensor product, also known as *extension of scalars*, that sends a C-module to a particular B-module. In general very little can be said about the B-modules obtained in this way. However, many interesting results can be deduced about the structure of induced and coinduced B-modules in the special case when B is a split extension of C.

2.1 Generic properties

Definition 2.1.1. Let C be a subalgebra of B, such that $1_C = 1_B$, then

$$- \otimes_C B : \mod C \to \mod B$$

is called the *induction functor*, and dually

$$D(B \otimes_C D-) : \ \text{mod}\, C \to \text{mod}\, B$$

is called the *coinduction functor*. Moreover, given $M \in \text{mod}\, C$ the corresponding *induced module* is defined to be $M \otimes_C B$, and the *coinduced module* is defined to be $D(B \otimes_C DM)$.

First observe that both functors are covariant. The induction functor is right exact, while the coinduction functor is left exact. Now consider the following lemma.

Lemma 2.1.2. *Let C and B be two k-algebras and N a C-B-bimodule, then*

$$M \otimes_C N \cong D\text{Hom}_C(M, DN)$$

as B-modules for all $M \in \text{mod}\, C$.

Proof.

$$M \otimes_C N \cong D\text{Hom}_k(M \otimes_C N, k) \cong D\text{Hom}_C(M, \text{Hom}_k(N, k)) \cong D\text{Hom}_C(M, DN).$$

$\square$

The next proposition describes an alternative definition of these functors, and we will use these two descriptions interchangeably.

Proposition 2.1.3. *Let C be a subalgebra of B such that $1_C = 1_B$, then for every $M \in \text{mod}\, C$*

(a) $M \otimes_C B \cong D\text{Hom}_C(M, DB)$.

(b) $D(B \otimes_C DM) \cong \text{Hom}_C(B, M)$.

Proof. Both parts follow from Lemma 2.1.2. ∎

Next we show some basic properties of these functors.

Proposition 2.1.4. *Let C be a subalgebra of B such that $1_C = 1_B$. If e is an idempotent then*

(a) $(eC) \otimes_C B \cong eB$.

(b) $D(B \otimes_C Ce) \cong DBe$.

In particular, if $P(i)$ and $I(i)$ are indecomposable projective and injective C-modules at vertex i, then $P(i) \otimes_C B$ and $D(B \otimes_C DI(i))$ are respectively indecomposable projective and injective B-modules at vertex i.

Proof. Observe that $(eC) \otimes_C B = e(C \otimes_C B) \cong eB$, and similarly we compute $D(B \otimes_C (Ce)) = D((B \otimes_C C)e) \cong DBe$. The rest of the proposition follows from above if we let $e = e_i$ be the primitive idempotent given by the constant path at vertex i. ∎

2.2 Split extension algebras

We can say more about induction and coinduction functors in the situation when B is a split extension of C.

Definition 2.2.1. Let B and C be two algebras. We say B *is a split extension of C by a nilpotent bimodule E* if there exists a short exact sequence of B-modules

$$0 \longrightarrow E \overset{i}{\longrightarrow} B \underset{\sigma}{\overset{\pi}{\rightleftarrows}} C \longrightarrow 0 \tag{2.2.1}$$

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where π and σ are algebra homomorphisms, such that $\pi\sigma = 1_C$, and $E = \ker\pi$ is nilpotent.

For example, relation extensions described in Section 1.6 are split extensions.

If B is a split extension of C then σ is injective, which means C is a subalgebra of B. Also, E is a C-C-bimodule, and we require E to be nilpotent so that $1_B = 1_C$. Observe that $B \cong C \oplus E$ as C-modules, and there is an isomorphism of C-modules $M \otimes_C B \cong (M \otimes_C C) \oplus (M \otimes_C E) \cong M \oplus M \otimes_C E$. Similarly, $D(B \otimes_C DM) \cong M \oplus D(E \otimes_C DM)$ as C-modules. This shows that induction and coinduction of a module M yields the same module M plus possibly something else. The next proposition shows a precise relationship between a given C-module and its image under the induction and coinduction functors.

Proposition 2.2.2. *Suppose B is a split extension of C by a nilpotent bimodule E, then for every $M \in \operatorname{mod} C$ there exist two short exact sequences of B-modules*

(a) $0 \longrightarrow M \otimes_C E \longrightarrow M \otimes_C B \longrightarrow M \longrightarrow 0$.

(b) $0 \longrightarrow M \longrightarrow D(B \otimes_C DM) \longrightarrow D(E \otimes_C DM) \longrightarrow 0$.

Proof. To show part (a) we apply $M \otimes_C -$ to the short exact sequence (2.2.1) and obtain the following long exact sequence

$$\operatorname{Tor}_1^C(M, C) \longrightarrow M \otimes_C E \longrightarrow M \otimes_C B \longrightarrow M \otimes_C C \longrightarrow 0 .$$

However, $\operatorname{Tor}_1^C(M, C) = 0$ since C is a projective C-module, so part (a) follows. Similarly, to show part (b) we apply $D(- \otimes_C DM)$ to sequence (2.2.1). This yields a long exact sequence

$$0 \longrightarrow D(C \otimes_C DM) \longrightarrow D(B \otimes_C DM) \longrightarrow D(E \otimes_C DM) \longrightarrow D\operatorname{Tor}_1^C(C, DM) .$$

The last term in the sequence is again zero, which shows part (b). □

Thus, in this situation each module is a quotient of its induced module and a submodule of its coinduced module.

In general it is not the case that induction or coinduction of an indecomposable C-module produces an indecomposable B-module. However, in the particular situation discussed in this section indecomposable modules induce and coinduce indecomposable B-modules.

Proposition 2.2.3. *Suppose B is a split extension of C by a nilpotent bimodule E, then*

(a) $M \otimes_C B \in \text{ind } B$ *if and only if* $M \in \text{ind } C$.

(b) $D(B \otimes_C DM) \in \text{ind } B$ *if and only if* $M \in \text{ind } C$.

Proof. Part (a). Suppose $M \in \text{mod } C$ is not indecomposable. Then $M \cong M_1 \oplus M_2$ such that neither M_1 nor M_2 is the zero module. Consider $M \otimes_C B \cong (M_1 \otimes_C B) \oplus (M_2 \otimes_C B)$. Since $M_1 \neq 0$, Proposition 2.2.2 (a) implies that $M_1 \otimes_C B \neq 0$. The same reasoning shows that $M_2 \otimes_C B \neq 0$. Therefore, we conclude that $M \otimes_C B$ is a decomposable B-module. This shows the forward direction of the statement in part (a).

Now, suppose $M \in \text{ind } C$, but $M \otimes_C B \cong M_1 \oplus M_2$ where neither M_1 nor M_2 is the zero module. Because B is a split extension of C, it follows that C has a left B-module structure. Thus, $M \otimes_C B \otimes_B C_C \cong (M_1 \oplus M_2) \otimes_B C_C$, and we conclude that $M \cong (M_1 \otimes_B C_C) \oplus (M_2 \otimes_B C_C)$. By assumption M is indecomposable, so without loss of generality let $M_1 \otimes_B C_C = 0$. But $0 = M_1 \otimes_B C \otimes_C B \cong M_1 \otimes_B B \cong M_1$, which is a contradiction. Therefore, $M \otimes_C B$ is indecomposable.

Part (b) can be shown in a similar manner as above. □

Proposition 2.2.4. *Suppose B is a split extension of C by a nilpotent bimodule E, then for every $M, N \in \operatorname{mod} C$*

(a) $M \otimes_C B \cong N \otimes_C B$ *if and only if $M \cong N$.*

(b) $D(B \otimes_C DM) \cong D(B \otimes_C DN)$ *if and only if $M \cong N$.*

Proof. Part (a). Suppose $M \otimes_C B \cong N \otimes_C B$. Because B is a split extension of C, it follows that C has a left B-module structure. Thus, we have that $M \otimes_C B \otimes_B C_C \cong N \otimes_C B \otimes_B C_C$, which means $M \cong M \otimes_C C_C \cong N \otimes_C C_C \cong N$. The proof of part (b) is similar to that of part (a) and we omit it. $\square$

The next lemma describes a relationship between the Auslander-Reiten translations in $\operatorname{mod} C$ and $\operatorname{mod} B$, and induction and coinduction functors. This lemma together with the following theorem were shown in [8].

Lemma 2.2.5. *Suppose B is a split extension of C by a nilpotent bimodule, then for every $M \in \operatorname{mod} C$*

(a) $\tau_B(M \otimes_C B) \cong D(B \otimes_C D(\tau_C M))$.

(b) $\tau_B^{-1} D(B \otimes_C DM) \cong (\tau_C^{-1} M) \otimes_C B$.

Theorem 2.2.6. *Suppose B is a split extension of C by a nilpotent bimodule E and $T \in \operatorname{mod} C$, then*

(a) $T \otimes_C B$ *is a (partial) tilting B-module if and only if T is a (partial) tilting C-module, $\operatorname{Hom}_C(T \otimes_C E, \tau_C T) = 0$ and $\operatorname{Hom}_C(DE, \tau_C T) = 0$.*

(b) $D(B \otimes_C DT)$ *is a (partial) cotilting B-module if and only if T is a (partial) cotilting C-module, $\operatorname{Hom}_C(\tau_C^{-1} T, D(E \otimes_C DT)) = 0$ and $\operatorname{Hom}_C(\tau_C^{-1} T, E) = 0$.*

Chapter 3

Induced and coinduced modules over cluster-tilted algebras

In this chapter we develop properties of the induction and coinduction functors particularly when C is an algebra of global dimension at most two and $B = C \ltimes E$ is the trivial extension of C by the C-C-bimodule $E = \operatorname{Ext}_C^2(DC, C)$. In the specific case when C is also a tilted algebra, then B is the corresponding cluster-tilted algebra. Some of the results in this chapter only hold when C is tilted, but many hold in a more general situation when gl.dim $C \leq 2$, and we make that distinction clear in the assumptions of each statement. However, throughout this chapter we always assume $E = \operatorname{Ext}_C^2(DC, C)$ and a tensor product $\otimes$ is a tensor product over C.

The main result of this chapter holds when C is a tilted algebra. It says that DE is a partial tilting and τ_C-rigid C-module, and the corresponding induced module $DE \otimes B$ is a partial tilting and τ_B-rigid B-module.

3.1 Basic properties

We begin by presenting a number of preliminary results that lead to the main theorem. Many of these statements will also be used in the subsequent chapters.

Proposition 3.1.1. *Let C be an algebra of global dimension at most 2. Then*

(a) $E \cong \tau^{-1}\Omega^{-1}C$.

(b) $DE \cong \tau\Omega DC$.

(c) $M \otimes E \cong \tau^{-1}\Omega^{-1}M$.

(d) $D(E \otimes DM) \cong \tau\Omega M$.

Proof. Since the global dimension of C is at most 2, we have $\mathrm{id}_C\Omega^{-1}M \leq 1$ (respectively $\mathrm{pd}_C\Omega M \leq 1$) for all C-modules M. Therefore, when applying the Auslander-Reiten formula below, we obtain the full Hom-space and not its quotient by the space of morphisms factoring through projectives (respectively injectives).

Part (a). $E = \mathrm{Ext}^2_C(DC, C) \cong \mathrm{Ext}^1_C(DC, \Omega^{-1}C) \cong D\mathrm{Hom}_C(\tau^{-1}\Omega^{-1}C, DC) \cong \tau^{-1}\Omega^{-1}C$.

Part (b). $DE = D\mathrm{Ext}^2_C(DC, C) \cong D\mathrm{Ext}^1_C(\Omega DC, C) \cong \mathrm{Hom}_C(C, \tau\Omega DC) \cong \tau\Omega DC$.

Part (c). Using Lemma 2.1.2 we have $M \otimes E \cong D\mathrm{Hom}_C(M, DE)$, which in turn by part (b) is isomorphic to $D\mathrm{Hom}_C(M, \tau\Omega DC)$. Then, we have the following chain of isomorphisms $M \otimes E \cong D\mathrm{Hom}_C(M, \tau\Omega DC) \cong \mathrm{Ext}^1_C(\Omega DC, M) \cong \mathrm{Ext}^2_C(DC, M) \cong \mathrm{Ext}^1_C(DC, \Omega^{-1}M) \cong D\mathrm{Hom}_C(\tau^{-1}\Omega^{-1}M, DC) \cong \tau^{-1}\Omega^{-1}M$. Part (d) can be shown in a similar manner as above. $\square$

Proposition 3.1.2. *Let C be an algebra of global dimension at most 2, and let $B = C \ltimes E$. Suppose $M \in \mathrm{mod}\,C$, then*

(a) $\mathrm{id}_C M \leq 1$ *if and only if* $M \otimes B \cong M$.

(b) $\mathrm{pd}_C M \leq 1$ *if and only if* $D(B \otimes DM) \cong M$.

Proof. Part (a). Recall that $M \otimes B \cong M \oplus M \otimes E$ as C-modules. Therefore, it suffices to show that $M \otimes E = 0$ if and only if $\mathrm{id}_C M \leq 1$. Proposition 3.1.1(c) implies that $M \otimes E \cong \tau^{-1} \Omega^{-1} M$, which is zero if and only if $\mathrm{id}_C M \leq 1$.

Part (b). Similarly it suffices to show that $D(E \otimes DM) = 0$ if and only if $\mathrm{pd}_C M \leq 1$. However, Proposition 3.1.1(d) implies that $D(E \otimes DM) \cong \tau \Omega M$, which again is zero if and only if $\mathrm{pd}_C M \leq 1$. $\square$

As a consequence of this proposition and Lemma 2.2.5 we obtain the following corollary, which says that a slice in a tilted algebra together with its τ and τ^{-1}-translates fully embeds in the cluster-tilted algebra. This result was already shown in [ABS4] relying on the main theorem of [AZ]. Here we present a new proof using induction and coinduction functors.

Corollary 3.1.3. *Let C be a tilted algebra and B the corresponding cluster-tilted algebra. Let Σ be a slice in $\mathrm{mod}\, C$ and M a module in Σ. Then*

    (a) $\tau_C M \cong \tau_B M$.

    (b) $\tau_C^{-1} M \cong \tau_B^{-1} M$.

Proof. We show part (a) and the proof of part (b) is similar. Since the module M lies on a slice in $\mathrm{mod}\, C$ then $\mathrm{pd}_C M \leq 1$ and $\mathrm{id}_C M \leq 1$. By Lemma 2.2.5(a) we have the following isomorphism

$$\tau_B(M \otimes_C B) \cong D(B \otimes_C D(\tau_C M)).$$

It follows from Proposition 3.1.2(a) that the left hand side is isomorphic to $\tau_B M$. It remains to show that the right hand side is isomorphic to $\tau_C M$. We may suppose without loss of generality that $C = \mathrm{End}_A T$ and $\Sigma = \mathrm{Hom}_A(T, DA)$ lies in $\mathcal{Y}(T)$.

Hence, in particular $M \in \mathcal{Y}(T)$. Proposition 1.4.6(f) implies that $\tau_C M \in \mathcal{Y}(T)$, so part (d) of the same proposition shows that $\mathrm{pd}_C \tau_C M \leq 1$. Therefore, we have $D(B \otimes_C D(\tau_C M)) \cong \tau_C M$ by Proposition 3.1.2(b), and this completes the proof. $\square$

Lemma 3.1.4. *Let C be an algebra of global dimension 2. Then for all $M \in \mathrm{mod}\, C$*

(a) $\mathrm{pd}_C N = 2$ *for all nonzero $N \in \mathrm{add}\, M \otimes E$.*

(b) $\mathrm{id}_C N = 2$ *for all nonzero $N \in \mathrm{add}\, D(E \otimes DM)$.*

Proof. Part (a). By Proposition 3.1.1 (c), $M \otimes E \cong \tau^{-1}\Omega^{-1}M$, which is nonzero if and only if $\mathrm{id}_C M = 2$. Now, consider a minimal injective resolution of M and $\Omega^{-1}M$

$$0 \longrightarrow M \longrightarrow I^0 \xrightarrow{\quad\pi\quad} \Omega^{-1}M \longrightarrow I^1 \longrightarrow I^2 \longrightarrow 0.$$

Apply ν^{-1} to find a minimal projective resolution of $\tau^{-1}\Omega^{-1}M$.

$$0 \longrightarrow \nu^{-1}\Omega^{-1}M \longrightarrow \nu^{-1}I^1 \longrightarrow \nu^{-1}I^2 \longrightarrow \tau^{-1}\Omega^{-1}M \longrightarrow 0.$$

Since C has global dimension two, $\nu^{-1}\Omega^{-1}M$ is a projective C-module. It remains to show that it is nonzero. By definition $\nu^{-1}\Omega^{-1}M = \mathrm{Hom}_C(DC, \Omega^{-1}M)$, which is nonzero since we have a nonzero map π. Finally, observe that the argument above holds if we replace $\tau^{-1}\Omega^{-1}M$ by a nonzero direct summand of $\tau^{-1}\Omega^{-1}M$. This completes the proof. Part (b) can be shown in a similar manner as above. $\square$

Lemma 3.1.5. *Let C be a tilted algebra. Then for all $M \in \mathrm{mod}\, C$*

(a) $\mathrm{id}_C M \otimes E \leq 1$.

(b) $\mathrm{pd}_C D(E \otimes DM) \leq 1$.

Proof. Part (a) follows from Lemma 3.1.4(a) and Proposition 1.4.6(b). Similarly, part (b) follows from Lemma 3.1.4(b) and Proposition 1.4.6(b). □

**Proposition 3.1.6.** *Let C be a tilted algebra. Then*

(a) $E \otimes E = 0$.

(b) $D(E \otimes D(DE)) = 0$.

Proof. Part (a). Proposition 3.1.1(c) implies that $E \otimes E \cong \tau^{-1}\Omega^{-1}E$, but this is zero since $\mathrm{id}_C E \leq 1$, by Lemma 3.1.5(a) with $M = C$. Part (b) follows directly from part (a). □

Remark 3.1.7. The above proposition does not hold if C has global dimension 2, but is not tilted. For example, consider an algebra C given by the following quiver with relations.

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4 \xrightarrow{\delta} 5 \qquad\qquad \alpha\beta = \gamma\delta = 0.$$

The Auslander-Reiten quiver of C is the following.
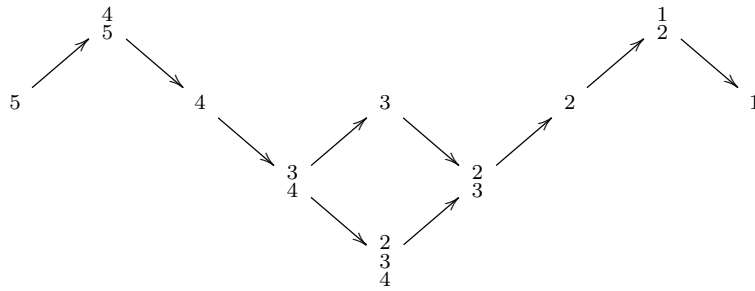
Here $E = S(1) \oplus S(3)$, where $S(i)$ is the simple module at vertex i. Hence, we see

$E \otimes E \cong \tau^{-1}\Omega^{-1}(S(1) \oplus S(3)) = S(1)$.

The following lemma is used throughout the paper.

Lemma 3.1.8. *Let C be an algebra of global dimension at most 2, then*

(a) $\text{Ext}^1_C(E, C) = 0$.

(b) $\text{Ext}^1_C(DC, DE) = 0$.

(c) $\text{Ext}^1_C(E, E) = 0$.

(d) $\text{Ext}^1_C(DE, DE) = 0$.

Proof. We will show parts (a) and (c), and the rest of the lemma can be proven similarly.

Part (a). By Proposition 3.1.1(a), we see that $\text{Ext}^1_C(E, C) \cong \text{Ext}^1_C(\tau^{-1}\Omega^{-1}C, C)$, which in turn by the Auslander-Reiten formula is isomorphic to $D\overline{\text{Hom}}_C(C, \Omega^{-1}C)$. Let $i : C \to I$ be an injective envelope of C, thus we have the following short exact sequence

$$0 \longrightarrow C \overset{i}{\longrightarrow} I \overset{\pi}{\longrightarrow} \Omega^{-1}C \longrightarrow 0 . \tag{3.1.1}$$

Applying $\text{Hom}_C(C, -)$ to this sequence we obtain an exact sequence

$$0 \longrightarrow \text{Hom}_C(C, C) \longrightarrow \text{Hom}_C(C, I) \overset{\pi_*}{\longrightarrow} \text{Hom}_C(C, \Omega^{-1}C) \longrightarrow \text{Ext}^1_C(C, C) .$$

However, $\text{Ext}^1_C(C, C) = 0$ shows that π_* is surjective. This implies that every morphism from C to $\Omega^{-1}C$ factors through the injective I. Thus, $\overline{\text{Hom}}_C(C, \Omega^{-1}C) = 0$, and this shows part (a).

Part (c). As above observe that $\text{Ext}^1_C(E, E) \cong D\overline{\text{Hom}}_C(E, \Omega^{-1}C)$. Applying the functor $\text{Hom}_C(E, -)$ to the sequence (3.1.1) we get an exact sequence

$$0 \longrightarrow \text{Hom}_C(E, C) \longrightarrow \text{Hom}_C(E, I) \overset{\pi_{**}}{\longrightarrow} \text{Hom}_C(E, \Omega^{-1}C) \longrightarrow \text{Ext}^1_C(E, C) .$$

But then by part (a) we have $\text{Ext}^1_C(E, C) = 0$, which shows that π_{**} is surjective.

Thus $\overline{\mathrm{Hom}}_C(E, \Omega^{-1}C) = 0$, and this completes the proof of part (c). $\qquad\square$

Corollary 3.1.9. *If the global dimension of C is at most two, then both $E \oplus C$ and $DE \oplus DC$ are rigid modules.*

Proof. Observe that $\mathrm{Ext}_C^1(E \oplus C, E \oplus C) \cong \mathrm{Ext}_C^1(E, E \oplus C) \oplus \mathrm{Ext}_C^1(C, E \oplus C)$. The first summand is zero because of Lemma 3.1.8, and the second is zero because C is projective. The proof of the rigidity of $DE \oplus DC$ is similar. $\qquad\square$

3.2 On the relation-extension bimodule

The following theorem is the main result of this section. Following [1] we say that a Λ-module M is τ_Λ-*rigid* if $\mathrm{Hom}_\Lambda(M, \tau_\Lambda M) = 0$.

Theorem 3.2.1. *If C is a tilted algebra and B is the corresponding cluster-tilted algebra, then*

(a) *DE is a partial tilting and τ_C-rigid C-module, and its corresponding induced module $DE \otimes B$ is a partial tilting and τ_B-rigid B-module.*

(b) *E is a partial cotilting and τ_C-corigid C-module, and its corresponding coinduced module $D(B \otimes DE)$ is a partial cotilting and τ_B-corigid B-module.*

Proof. We show part (a), and the proof of part (b) is similar. First let us show that DE is a partial tilting and τ_C-rigid C-module. Because C is tilted, Lemma 3.1.5(b) implies that $\mathrm{pd}_C DE \leq 1$, but also DE is rigid by Lemma 3.1.8(d). This shows that DE is a partial tilting C-module. On the other hand, by Lemma 3.1.4(b), all nonzero indecomposable summands of DE have injective dimension 2. Thus, $\mathrm{pd}_C DE \leq 1$, by Proposition 1.4.6(b). So, applying the Auslander-Reiten formula we

have $\mathrm{Hom}_C(DE, \tau_C DE) \cong D\mathrm{Ext}^1_C(DE, DE) = 0$, where the last step follows from Lemma 3.1.8(d). This shows that DE is τ_C-rigid.

Now, it remains to shows that $DE \otimes B$ is a partial tilting and τ_B-rigid B-module. First we show that $DE \otimes B$ is partial tilting. From above we know that DE is a partial tilting C-module, thus by Theorem 2.2.6 it suffices to show the two conditions $\mathrm{Hom}_C(DE \otimes_C E, \tau_C DE) = 0$ and $\mathrm{Hom}_C(DE, \tau_C DE) = 0$. The second identity follows from the work above, so we need to show the first identity. Observe that by Lemma 3.1.4(a), $\mathrm{pd}_C DE \otimes E = 2$. Then Proposition 1.4.6 implies that $DE \otimes E \in \mathcal{X}(T)$. Similarly, by Lemma 3.1.4(b), $\mathrm{id}_C DE = 2$, so Proposition 1.4.6 implies that $DE \in \mathcal{Y}(T)$. However, by Proposition 1.4.6(f), $\mathcal{Y}(T)$ is closed under predecessors, which means $\tau_C DE \in \mathcal{Y}(T)$. By definition of a torsion pair there are no nonzero morphisms from $\mathcal{X}(T)$ to $\mathcal{Y}(T)$, so $\mathrm{Hom}_C(DE \otimes E, \tau_C DE) = 0$. This shows that $DE \otimes B$ is a partial tilting B-module.

Now we show that $DE \otimes B$ is τ_B-rigid, that is $\mathrm{Hom}_B(DE \otimes B, \tau_B(DE \otimes B)) = 0$. First observe that Lemma 2.2.5(a) yields $\tau_B(DE \otimes B) \cong D(B \otimes D\tau_C DE)$, which in turn by Proposition 3.1.2(b) is isomorphic to $\tau_C DE$. Let $f \in \mathrm{Hom}_B(DE \otimes B, \tau_C DE)$. Then we have the following diagram, whose top row is the short exact sequence of Proposition 2.2.2(a).

$$0 \longrightarrow DE \otimes E \overset{i_*}{\longrightarrow} DE \otimes B \overset{\pi_*}{\longrightarrow} DE \longrightarrow 0 \ .$$

with f and g morphisms into $\tau_C DE$.

Observe that $fi_* \in \mathrm{Hom}_C(DE \otimes E, \tau_C DE)$, which is zero by our calculation above. Next, the universal property of $\mathrm{coker}\, i_*$ implies that there exists $g \in \mathrm{Hom}_C(DE, \tau_C DE)$ such that $g\pi_* = f$. However, we know that DE is τ_C-rigid, which implies that $g = 0$.

This means $f = 0$. Thus, we conclude that $DE \otimes B$ is τ_B-rigid. $\qquad\square$

Remark 3.2.2. Unlike DE, the C-module E is not τ_C-rigid, that is $\mathrm{Hom}_C(E, \tau_C E) \neq 0$. Consider, for example, the tilted algebra C given by the following quiver with relations.

$$\delta\alpha = \gamma\beta = 0.$$
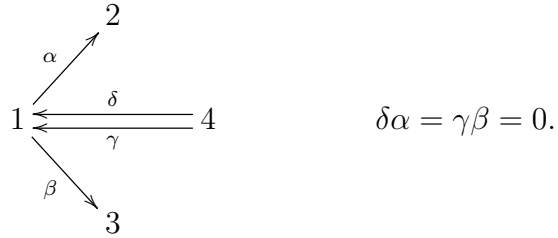
Here $E = \begin{smallmatrix} 4 \\ 1 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 1 \\ 3 \end{smallmatrix}$ and $\tau_C E = \begin{smallmatrix} 4 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 1 \end{smallmatrix}$.

Chapter 4

Injective resolutions

In this chapter we construct an explicit injective resolution of an arbitrary projective B-module in a cluster-tilted algebra B using only induction and coinduction functors applied to modules over a tilted algebra C. This resolution is described completely in terms of C-modules and has length at most one.

We begin by deriving a number of commutative diagrams. These enable us to define modules and morphisms involved in the construction of an injective resolution. Most of the statements are also true when the global dimension of C is at most 2 and we make that distinction clear.

4.1 Preliminary constructions

Lemma 4.1.1. *Let* $\mathrm{gl.dim}\, C = 2$ *and* $M \in \mathrm{mod}\, C$. *Suppose that* $\mathrm{id}_C M = 2$, *and that*

$$0 \longrightarrow M \xrightarrow{\ i_0\ } I_C^0 \xrightarrow{\ i_2\ } I_C^1 \xrightarrow{\ i_3\ } I_C^2 \longrightarrow 0$$

is a minimal injective resolution of M. Then there is a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{\ i_0\ } & I_C^0 & \xrightarrow{\ i_1\ } & \Omega^{-1}M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_0} & & \| & & \\
0 & \longrightarrow & \tau\Omega\tau^{-1}\Omega^{-1}M & \xrightarrow{\ \pi_0\ } & \tilde{I}_C & \xrightarrow{\ \pi_1\ } & \Omega^{-1}M & \longrightarrow & 0
\end{array}
$$

where $\tilde{I}_C = \nu\nu^{-1}\Omega^{-1}M$ is an injective C-module.

Proof. Consider the diagram below. The injective resolution of M also gives a minimal injective resolution of $\Omega^{-1}M$, which is shown in the top row of the diagram. Then we apply the inverse Nakayama functor $\nu^{-1} = \mathrm{Hom}_C(DC, -)$ to this resolution and obtain a projective presentation of $\tau^{-1}\Omega^{-1}M$ in the second row. First, note that $\nu^{-1}\Omega^{-1}M = \mathrm{Hom}_C(DC, \Omega^{-1}M)$ is nonzero, because it contains the nonzero map i_1. Also, the global dimension of C is two, which means that $\nu^{-1}\Omega^{-1}M$ is projective and we actually have a projective resolution of $\tau^{-1}\Omega^{-1}M$.

Next we apply the Nakayama functor $\nu = D\mathrm{Hom}_C(-, C)$ to the projective resolution of $\Omega\tau^{-1}\Omega^{-1}M$, and obtain an injective presentation of $\tau\Omega\tau^{-1}\Omega^{-1}M$ in the third row of the diagram. Observe that $\nu\Omega\tau^{-1}\Omega^{-1}M = \nu\mathrm{Im}(\nu^{-1}i_3) = \mathrm{Im}(\nu\nu^{-1}i_3) = \mathrm{Im}\, i_3 = I_C^2$.

Let $g_0 = \nu\nu^{-1}i_1$. By commutativity in the diagram below, we see that $i_4\pi_1 g_0 = i_2 = i_4 i_1$. Since i_4 is injective, we conclude $\pi_1 g_0 = i_1$. This shows that we have two short exact sequences as in the statement of the lemma and that the second square in the diagram is commutative. Then by the universal property of $\ker \pi_1$ there exists $g_1 \in \mathrm{Hom}_C(M, \tau\Omega\tau^{-1}\Omega^{-1}M)$ that makes the first square commute.

$$\begin{array}{c}
\Omega^{\text{-}1}M
\end{array}$$

The diagram shows:

Top row: $0 \xrightarrow{} M \xrightarrow{i_0} I_C^0 \xrightarrow{i_2} I_C^1 \xrightarrow{i_3} I_C^2 \xrightarrow{} 0$ with $\Omega^{\text{-}1}M$ above connected by i_1 and i_4.

Middle row: $0 \xrightarrow{} \nu^{\text{-}1}\Omega^{\text{-}1}M \xrightarrow{\nu^{\text{-}1}i_4} \nu^{\text{-}1}I_C^1 \xrightarrow{\nu^{\text{-}1}i_3} \nu^{\text{-}1}I_C^2 \to \tau^{\text{-}1}\Omega^{\text{-}1}M \to 0$ with $\nu^{\text{-}1}I_C^0$ above connected by $\nu^{\text{-}1}i_1$ and $\nu^{\text{-}1}i_2$, and $\Omega\tau^{\text{-}1}\Omega^{\text{-}1}M$ below. Vertical maps labeled $\nu^{\text{-}1}$.

Bottom row: $0 \to \tau\Omega\tau^{\text{-}1}\Omega^{\text{-}1}M \xrightarrow{\pi_0} \nu\nu^{\text{-}1}\Omega^{\text{-}1}M \xrightarrow{\nu\nu^{\text{-}1}i_4} I_C^1 \xrightarrow{i_3} I_C^2 \xrightarrow{} 0$ with I_C^0 above connected by $\nu\nu^{\text{-}1}i_1$ and i_2, and $\Omega^{\text{-}1}M$ below connected by π_1 and i_4. Vertical maps labeled ν.

$\square$

Proposition 4.1.2. *Let* $\mathrm{gl.dim}\,C \leq 2$ *and* $E = \mathrm{Ext}_C^2(DC, C)$, *then for every* $M \in \mathrm{mod}\,C$

$$M \otimes E \cong \mathrm{Hom}_C(E, M \otimes E) \otimes E.$$

Proof. Observe that by Proposition 3.1.1(c) the left hand side of the statement above is isomorphic to $\tau^{-1}\Omega^{-1}M$. Now consider the right hand side

$$\begin{aligned}
\mathrm{Hom}_C(E, M \otimes E) \otimes E &\cong D(E \otimes D(M \otimes E)) \otimes E && \text{by Lemma 2.1.2}\\
&\cong \tau\Omega(M \otimes E) \otimes E && \text{by Proposition 3.1.1(d)}\\
&\cong \tau\Omega\tau^{-1}\Omega^{-1}M \otimes E && \text{by Proposition 3.1.1(c)}\\
&\cong \tau^{-1}\Omega^{-1}\tau\Omega\tau^{-1}\Omega^{-1}M && \text{by Proposition 3.1.1(c).}
\end{aligned}$$

Therefore, it suffices to show that $\tau^{-1}\Omega^{-1}M \cong \tau^{-1}\Omega^{-1}\tau\Omega\tau^{-1}\Omega^{-1}M$. If $\mathrm{id}_C M \leq 1$ then both sides are zero and the statement follows. If $\mathrm{id}_C M = 2$, then by Lemma 4.1.1 we have a short exact sequence

$$0 \longrightarrow \tau\Omega\tau^{-1}\Omega^{-1}M \xrightarrow{\pi_0} \tilde{I}_C \longrightarrow \Omega^{-1}M \longrightarrow 0 \ .$$

Next we construct the following commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \tau\Omega\tau^{-1}\Omega^{-1}M & \longrightarrow & I & \longrightarrow & \Omega^{-1}\tau\Omega\tau^{-1}\Omega^{-1}M & \longrightarrow & 0 \\
& & \Big\| & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} & & \\
0 & \longrightarrow & \tau\Omega\tau^{-1}\Omega^{-1}M & \xrightarrow{\pi_0} & \tilde{I}_C & \longrightarrow & \Omega^{-1}M & \longrightarrow & 0
\end{array}
$$

where I is an injective envelope of $\tau\Omega\tau^{-1}\Omega^{-1}M$. The map i exists and is a monomorphism by the properties of an injective envelope; j exists by the universal property of the cokernel. Moreover, $\tilde{I}_C \cong i(I) \oplus I'$, where I' is some injective C-module. By commutativity $\mathrm{Im}\,\pi_0 \subset i(I)$, which implies that $\Omega^{-1}M \cong I' \oplus \Omega^{-1}\tau\Omega\tau^{-1}\Omega^{-1}M$. Hence, $\tau^{-1}\Omega^{-1}M \cong \tau^{-1}\Omega^{-1}\tau\Omega\tau^{-1}\Omega^{-1}M$. $\qquad\square$

To simplify the notation we will write I_C for an injective C-module, and I_B for the corresponding injective B-module, that is $I_B = D(B \otimes DI_C)$. Also, the morphisms in the statements of the proceeding lemmas will be used in the proof of the main theorem, so we keep the notation consistent throughout this chapter.

Lemma 4.1.3. *Suppose P_C is a projective module over a tilted algebra C such that $\mathrm{id}_C P_C = 2$ with an injective envelope $0 \to P_C \xrightarrow{i_0} I_C^0$. Then there is a commutative*

diagram with exact rows

$$0 \longrightarrow P_C \xrightarrow{\left(\begin{smallmatrix} j_0 i_0 \\ g_1 \end{smallmatrix}\right)} I_B^0 \oplus \tau\Omega\tau^{-1}\Omega^{-1}P_C \xrightarrow{\left(\alpha,\, -l_1\pi_0\right)} \tilde{I}_B \longrightarrow 0$$

with vertical maps i_0, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \pi_0 \end{smallmatrix}\right)$, and equality

$$0 \longrightarrow I_C^0 \xrightarrow{\left(\begin{smallmatrix} j_0 \\ g_0 \end{smallmatrix}\right)} I_B^0 \oplus \tilde{I}_C \xrightarrow{\left(\alpha,\, -l_1\right)} \tilde{I}_B \longrightarrow 0$$

where $\tilde{I}_C = \nu\nu^{-1}\Omega^{-1}P_C$ is the injective C-module of Lemma 4.1.1.

Proof. By Lemma 4.1.1 there is a short exact sequence

$$0 \longrightarrow P_C \xrightarrow{\left(\begin{smallmatrix} i_0 \\ g_1 \end{smallmatrix}\right)} I_C^0 \oplus \tau\Omega\tau^{-1}\Omega^{-1}P_C \xrightarrow{\left(g_0,\, -\pi_0\right)} \tilde{I}_C \longrightarrow 0.$$

We will apply the coinduction functor $\mathrm{Hom}_C(B, -)$ to this sequence, but first observe that using Proposition 3.1.1(d) we have $\tau\Omega\tau^{-1}\Omega^{-1}P_C \cong D(E \otimes D\tau^{-1}\Omega^{-1}P_C)$. Since C is tilted, Lemma 3.1.5(b) shows that the projective dimension of $\tau\Omega\tau^{-1}\Omega^{-1}P_C$ is strictly less then 2, and Proposition 3.1.2(b) implies that both P_C and $\tau\Omega\tau^{-1}\Omega^{-1}P_C$ do not change under coinduction. Also, by Corollary 3.1.9 there is a C-module isomorphism $\mathrm{Ext}^1_C(B, P_C) \cong \mathrm{Ext}^1_C(C \oplus E, P_C) = 0$.

Hence, coinducing the sequence above we obtain the following short exact sequence.

$$0 \longrightarrow P_C \xrightarrow{\left(\begin{smallmatrix} i_0 j_0 \\ g_1 \end{smallmatrix}\right)} I_B^0 \oplus \tau\Omega\tau^{-1}\Omega^{-1}P_C \xrightarrow{\left(\alpha,\, -l_1\pi_0\right)} \tilde{I}_B \longrightarrow 0$$

Here, α is the coinduced morphism from g_0, and l_1, j_0 are the inclusions as in Propo-

sition 2.2.2(b), such that there is a commutative diagram

$$
\begin{array}{ccc}
0 \longrightarrow I_C^0 & \xrightarrow{j_0} & I_B^0 \\
\downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle \alpha} \\
0 \longrightarrow \tilde{I}_C & \xrightarrow{l_1} & \tilde{I}_B.
\end{array}
\tag{4.1.1}
$$

So far we have constructed the top short exact sequence as in the statement of the lemma. It suffices to show that the bottom row is exact and that the diagram is commutative. First observe that the second square commutes trivially and the first one commutes because $\pi_0 g_1 = g_0 i_0$ by Lemma 4.1.1. Next, note that the bottom row is exact at I_C^0, because j_0 is injective, and it is exact at $\tilde{I}_B$, because the diagram is commutative.

Thus it remains to show that $\mathrm{Im}\left(\begin{smallmatrix} j_0 \\ g_0 \end{smallmatrix}\right) = \ker\left(\alpha, -l_1\right)$. To show the forward inclusion consider $\left(\alpha, -l_1\right)\left(\begin{smallmatrix} j_0 \\ g_0 \end{smallmatrix}\right) = \alpha j_0 - l_1 g_0$ which is zero by diagram (4.1.1). Now suppose $\alpha(a) - l_1(b) = 0$ for some $a \in I_B^0$ and $b \in \tilde{I}_C$. By Lemma 4.1.1, there exists $d \in I_C^0$ such that $i_1(d) = \pi_1(b)$. Then $\pi_1 g_0(d) = i_1(d) = \pi_1(b)$ or equivalently $\pi_1(b - g_0(d)) = 0$. Again Lemma 4.1.1 implies that there exists $c \in \tau \Omega \tau^{-1} \Omega^{-1} P_C$ such that $\pi_0(c) = b - g_0(d)$. Now we have $\alpha(a) = l_1(b) = l_1(\pi_0(c) + g_0(d)) = l_1 \pi_0(c) + \alpha j_0(d)$, where the last identity follows from diagram (4.1.1), so $\alpha(a - j_0(d)) - l_1 \pi_0(c) = 0$. Using the fact that the top row in our diagram is exact we can find $p \in P_C$ such that $j_0 i_0(p) = a - j_0(d)$ and $g_1(p) = c$. Finally, $i_0(p) + d \in I_C^0$ and $\left(\begin{smallmatrix} j_0 \\ g_0 \end{smallmatrix}\right)(i_0(p) + d) = \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$ since $g_0 i_0 = \pi_0 g_1$. This shows the reverse inclusion and completes the proof of the lemma. $\qquad \square$

Lemma 4.1.4. *Let* $\mathrm{gl.dim}\, C = 2$. *Suppose* P_C *is a projective* C-*module with an injective envelope* $0 \to P \xrightarrow{i_0} I_C^0$ *and* $\mathrm{id}_C P = 2$. *Then there exists a commutative*

diagram with exact rows

$$0 \longrightarrow P_B \xrightarrow{\binom{u_1}{g_2}} P_C \oplus \tau\Omega\tau^{-1}\Omega^{-1}P_C \otimes B \xrightarrow{\left(g_1,\ -\delta_1\right)} \tau\Omega\tau^{-1}\Omega^{-1}P_C \longrightarrow 0$$

$$\Big\| \qquad \downarrow \left(\begin{smallmatrix} i_0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \qquad \qquad \downarrow \pi_0$$

$$0 \longrightarrow P_B \xrightarrow{\binom{i_0 u_1}{g_2}} I_C^0 \oplus \tau\Omega\tau^{-1}\Omega^{-1}P_C \otimes B \xrightarrow{\left(g_0,\ -\pi_0\delta_1\right)} \tilde{I}_C \longrightarrow 0$$

where $\tilde{I}_C = \nu\nu^{-1}\Omega^{-1}P_C$ is the injective C-module of Lemma 4.1.1 and $P_B = P_C \otimes B$ is the corresponding projective B-module.

Proof. By Lemma 4.1.1 there is a short exact sequence

$$0 \longrightarrow P_C \xrightarrow{\binom{i_0}{g_1}} I_C^0 \oplus \tau\Omega\tau^{-1}\Omega^{-1}P_C \xrightarrow{\left(g_0,\ -\pi_0\right)} \tilde{I}_C \longrightarrow 0.$$

We will apply the induction functor $D\mathrm{Hom}_C(-, DB)$ to this sequence. Since I_C^0 and $\tilde{I}_C$ are injective, Proposition 3.1.2(a) implies that these modules do not change under induction. Also, there is a C-module isomorphism $\mathrm{Ext}_C^1(\tilde{I}_C, DB) \cong \mathrm{Ext}_C^1(\tilde{I}_C, DC \oplus DE) = 0$, by Corollary 3.1.9. Hence, inducing the sequence above we obtain the following short exact sequence.

$$0 \longrightarrow P_B \xrightarrow{\binom{i_0 u_1}{g_2}} I_C^0 \oplus \tau\Omega\tau^{-1}\Omega^{-1}P_C \otimes B \xrightarrow{\left(g_0,\ -\pi_0\delta_1\right)} \tilde{I}_C \longrightarrow 0$$

Here, g_2 is the image of g_1 under the induction functor, and δ_1, u_1 are projections as in Proposition 2.2.2(a), such that there is the following commutative diagram

$$\begin{array}{ccc} P_B & \xrightarrow{\ u_1\ } & P_C \longrightarrow 0 \\ \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_1} \\ \tau\Omega\tau^{-1}\Omega^{-1}P_C \otimes B & \xrightarrow{\ \delta_1\ } & \tau\Omega\tau^{-1}\Omega^{-1}P_C \longrightarrow 0. \end{array} \qquad (4.1.2)$$

Thus we have constructed the bottom row as in the conclusion of the lemma. It suffices to show that the top row is exact and that the diagram is commutative. First observe that the first square commutes trivially and the second one commutes because $\pi_0 g_1 = g_0 i_0$, by Lemma 4.1.1. Next, note that the top row is exact at P_B by commutativity of the diagram and it is exact at $\tau\Omega\tau^{-1}\Omega^{-1}P_C$ because δ_1 is surjective.

Therefore, it remains to show that $\operatorname{Im}\left(\begin{smallmatrix} u_1 \\ g_2 \end{smallmatrix}\right) = \ker\left(\begin{smallmatrix} g_1, & -\delta_1 \end{smallmatrix}\right)$. The forward inclusion holds, since $\left(\begin{smallmatrix} g_1, & -\delta_1 \end{smallmatrix}\right)\left(\begin{smallmatrix} u_1 \\ g_2 \end{smallmatrix}\right) = g_1 u_1 - \delta_1 g_2 = 0$, by diagram (4.1.2). Now suppose that $g_1(a) - \delta_1(b) = 0$ for some $a \in P_C$ and $b \in \tau\Omega\tau^{-1}\Omega^{-1}P_C \otimes B$. Applying π_0 to both sides we obtain $\pi_0 g_1(a) - \pi_0 \delta_1(b) = 0$ or equivalently $g_0 i_0(a) - \pi_0 \delta_1(b) = 0$, by commutativity. Since the bottom row is exact we can find $p \in P_B$ such that $i_0 u_1(p) = i_0(a)$ and $g_2(p) = b$. Note that i_0 is injective so $u_1(p) = a$. Finally, $\left(\begin{smallmatrix} u_1 \\ g_2 \end{smallmatrix}\right)(p) = \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$. This shows the reverse inclusion and completes the proof of the lemma. $\square$

The next proposition describes an isomorphism between induction and coinduction for a particular type of modules.

Proposition 4.1.5. *Suppose* $\operatorname{gl.dim} C = 2$ *and* $M = \tau\Omega\tau^{-1}\Omega^{-1}N$ *for some* $N \in \operatorname{mod} C$. *Then the induction of* M *is isomorphic to the coinduction of* $M \otimes E$ *and there is a commutative diagram with exact rows*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M \otimes E & \xrightarrow{1 \otimes i} & M \otimes B & \xrightarrow{1 \otimes \pi} & M \otimes C & \longrightarrow & 0 \\
& & \downarrow{\phi} & & \downarrow{\theta} & & \downarrow{\psi} & & \\
0 & \longrightarrow & \operatorname{Hom}_C(C, M \otimes E) & \xrightarrow{\pi^*} & \operatorname{Hom}_C(B, M \otimes E) & \xrightarrow{i^*} & \operatorname{Hom}_C(E, M \otimes E) & \longrightarrow & 0
\end{array}
$$

where $\phi, \theta, \text{and } \psi$ *are isomorphisms of* B-*modules.*

Proof. First observe that the top row is a short exact sequence by Proposition 2.2.2(a), and the bottom row is a short exact sequence by Propositions 2.2.2(b) and 2.1.3(b).

The maps i and π are as in sequence (1.6.1). Next we explicitly describe the rest of the maps that appear in the diagram above.

Let us begin with the definition of ψ. First, consider

$$M \cong \tau\Omega\tau^{-1}\Omega^{-1}N$$

$$\cong D(E \otimes D\tau^{-1}\Omega^{-1}N) \qquad \text{by Proposition 3.1.1(d)}$$

$$\cong \operatorname{Hom}_C(E, \tau^{-1}\Omega^{-1}N) \qquad \text{by Lemma 2.1.2}$$

$$\cong \operatorname{Hom}_C(E, N \otimes E) \qquad \text{by Proposition 3.1.1(c).}$$

Then Proposition 4.1.2 implies that

$$M \otimes E \cong N \otimes E$$

and in turn

$$\operatorname{Hom}_C(E, M \otimes E) \cong \operatorname{Hom}_C(E, N \otimes E) \cong M.$$

Now, there exists a unique C-module homomorphism ψ such that

$$\psi : M \otimes C \to \operatorname{Hom}_C(E, M \otimes E)$$

$$m \otimes c \longmapsto (e \mapsto mc \otimes e).$$

Next, we want to show that ψ is injective. Since $M \otimes_C C \cong M$, it suffices to show that $\psi(m \otimes 1) = 0$ if and only if $m = 0$. Now suppose that $\psi(m \otimes 1) = 0$ for some $m \otimes 1 \in M \otimes C$, which means $m \otimes e = 0$, for all $e \in E$, and we need to show that $m = 0$. For this we use the universal property of the tensor product. Consider the

diagram

$$
\begin{array}{ccc}
M \times E & \longrightarrow & M \otimes_C E \\
& \overline{\psi} \searrow & \downarrow \Psi \\
& & N \otimes E
\end{array}
$$

where $\overline{\psi}(m, e) = m(e)$. Recall that we can think of M as $\mathrm{Hom}_C(E, N \otimes E)$, so here by $m(e)$ we understand a map m evaluated at an element $e \in E$. One can check that $\overline{\psi}$ is a C-balanced map, and the universal property of the tensor product implies that there exists a unique C-module homomorphism Ψ such that $\Psi(m \otimes e) = m(e)$. Now suppose $m \otimes e = 0$ for all $e \in E$. Then $\Psi(m \otimes e) = m(e) = 0$ for all $e \in E$, which means that m is the zero map, thus $m = 0$. This shows that ψ is an injective C-module homomorphism, but since $M \cong \mathrm{Hom}_C(E, M \otimes E)$ are finite dimensional, this shows that ψ is a C-module isomorphism. Because every C-module is also a B-module by defining the action of E to be trivial, then ψ is also a B-module isomorphism.

Now we define θ, and again we use the universal property of the tensor product. Consider the diagram

$$
\begin{array}{ccc}
M \times B & \longrightarrow & M \otimes_C B \\
& \varphi \searrow & \downarrow \theta \\
& & \mathrm{Hom}_C(B, M \otimes E)
\end{array}
$$

where

$$
\varphi : (m, (c, e)) \longmapsto ((c', e') \mapsto m \otimes (ce' + ec')).
$$

Again one can easily check that this map is C-balanced, so the universal property of the tensor product implies that there exists a unique C-module homomorphism θ

such that

$$\theta(m \otimes (c,e)) = ((c',e') \mapsto m \otimes (ce' + ec')).$$

Now we want to show that θ is a B-module homomorphism. Observe that for all $(\tilde{c}, \tilde{e}) \in B$ and $m \otimes (c,e) \in M \otimes B$ we have

$$\theta(m \otimes (c,e)) \cdot (\tilde{c}, \tilde{e}) = ((c',e') \mapsto m \otimes (ce' + ec')) \cdot (\tilde{c}, \tilde{e})$$
$$= ((c',e') \mapsto m \otimes (c(\tilde{c}e' + \tilde{e}c') + e\tilde{c}c')).$$

On the other hand

$$\theta(m \otimes (c,e) \cdot (\tilde{c}, \tilde{e})) = \theta(m \otimes (c\tilde{c}, c\tilde{e} + e\tilde{c}))$$
$$= ((c',e') \mapsto m \otimes (c\tilde{c}e' + (c\tilde{e} + e\tilde{c})c')),$$

and the two expressions are the same. This shows that θ is a B-module homomorphism.

Finally, we define the morphism ϕ. Let

$$\phi : M \otimes E \to \mathrm{Hom}_C(C, M \otimes E)$$

$$\phi : m \otimes e \longmapsto (c \mapsto (m \otimes e) \cdot c)$$

which is a standard isomorphism of C-modules. By the same reasoning as above it is also an isomorphism of B-modules. Thus we defined all morphisms appearing in the proposition, so it remains to show that the corresponding diagram is commutative.

Let $m \otimes e \in M \otimes E$, then consider

$$\pi^*\phi(m \otimes e) = \pi^*(c \mapsto (m \otimes e) \cdot c)$$
$$= (c \mapsto (m \otimes e) \cdot c) \circ \pi$$
$$= ((c', e') \mapsto m \otimes e \cdot \pi(c', e'))$$
$$= ((c', e') \mapsto m \otimes e \cdot (c', 0))$$
$$= ((c', e') \mapsto m \otimes ec').$$

On the other hand

$$\theta(1 \otimes i)(m \otimes e) = \theta(m \otimes i(e))$$
$$= \theta(m \otimes (0, e))$$
$$= ((c', e') \mapsto m \otimes ec').$$

This shows that the first square commutes. Now let $m \otimes (c, e) \in M \otimes B$ and consider

$$i^*\theta(m \otimes (c, e)) = i^*((c', e') \mapsto m \otimes (ce' + ec'))$$
$$= (e' \mapsto m \otimes ce').$$

Also,

$$\psi(1 \otimes \pi)(m \otimes (c, e)) = \psi(m \otimes c)$$
$$= (e' \mapsto mc \otimes e')$$
$$= (e' \mapsto m \otimes ce').$$

This shows that the second square commutes.

Next the Five Lemma implies that θ is a B-module isomorphism. This completes the proof of the proposition. □

The following corollary is a reformulation of Proposition 4.1.5 and will be used in the proof of Theorem 4.2.1.

Corollary 4.1.6. *Suppose* $\mathrm{gl.dim}\, C = 2$ *and* $M = \tau\Omega\tau^{-1}\Omega^{-1}N$ *for some* $N \in \mathrm{mod}\, C$. *Then there is a commutative diagram with exact rows*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & N \otimes E & \xrightarrow{\delta_0} & M \otimes B & \xrightarrow{\delta_1} & M & \longrightarrow & 0 \\
& & \downarrow{\phi'} & & \downarrow{\theta'} & & \downarrow{\psi'} & & \\
0 & \longrightarrow & N \otimes E & \xrightarrow{\beta_0} & \mathrm{Hom}_C(B, N \otimes E) & \xrightarrow{\beta_1} & M & \longrightarrow & 0
\end{array}
$$

where ϕ', θ', *and* ψ' *are isomorphisms of* B-*modules.*

Proof. First observe that the top row of this diagram is equivalent to the top row of the diagram in Proposition 4.1.5. Also, in the proof of this proposition we showed that $N \otimes E \cong M \otimes E$ and $M \cong \mathrm{Hom}_C(E, M \otimes E)$. This implies that the bottom rows of the two diagrams are also equivalent. Thus, we conclude that there exist ϕ', θ', and ψ' that make the diagram above commute. □

4.2 The main theorem

The following theorem is the main theorem of this chapter. It explicitly describes an injective resolution of length at most one for each projective module in a cluster-tilted algebra.

Theorem 4.2.1. *Let C be a tilted algebra, B the corresponding cluster-tilted algebra, P_C a projective C-module and P_B the corresponding projective B-module. Let*

$$0 \longrightarrow P_C \longrightarrow I_C^0 \longrightarrow I_C^1 \qquad and \qquad 0 \longrightarrow P_C \otimes E \longrightarrow \bar{I}_C^0 \longrightarrow \bar{I}_C^1$$

be minimal injective presentations in $\operatorname{mod} C$, and let $\tilde{I}_C$ be the injective C-module $\tilde{I}_C = \nu\nu^{-1}\Omega^{-1}P_C$. Then

$$0 \longrightarrow P_B \longrightarrow I_B^0 \oplus \bar{I}_B^0 \longrightarrow \tilde{I}_B \oplus \bar{I}_B^1 \longrightarrow 0$$

is an injective resolution of P_B in $\operatorname{mod} B$.

Remark 4.2.2. If the injective dimension of P_C is at most one, then $P_C \otimes E = 0$, by Proposition 3.1.2(a), and $\tilde{I}_C = \nu\nu^{-1}\Omega^{-1}P_C = \nu\nu^{-1}I_C^1 = I_C^1$. Thus in this case the injective resolution is

$$0 \longrightarrow P_B \longrightarrow I_B^0 \longrightarrow I_B^1 \longrightarrow 0.$$

Moreover, this resolution is minimal.

Proof. If $\operatorname{id}_C P_C \leq 1$, then consider a minimal injective resolution of P_C in $\operatorname{mod} C$

$$0 \longrightarrow P_C \longrightarrow I_C^0 \longrightarrow I_C^1 \longrightarrow 0$$

of length at most one. We apply the coinduction functor $\operatorname{Hom}_C(B, -)$ to the injective resolution of P_C and obtain

$$0 \longrightarrow P_C \longrightarrow I_B^0 \longrightarrow I_B^1 \longrightarrow \operatorname{Ext}_C^1(B, P_C).$$

Indeed the injectives in $\operatorname{mod} C$ will map to the corresponding injectives in $\operatorname{mod} B$, and P_C will not change, by Proposition 3.1.2(b). Also, there is a C-module isomorphism $\operatorname{Ext}^1_C(B, P_C) \cong \operatorname{Ext}^1_C(C \oplus E, P_C) \cong \operatorname{Ext}^1_C(E, P_C)$ which is zero, by Lemma 3.1.8(a). Finally, observe that since $\operatorname{id}_C P_C \leq 1$ then $P_B \cong P_C \otimes B \cong P_C$ by Propositions 2.1.4(a) and 3.1.2(a). It also shows that $P_C \otimes E = 0$, which means that both injectives $\bar{I}^0_B$ and $\bar{I}^1_B$ are zero. Finally, the remark above implies that $\tilde{I}_B = I^1_B$. Thus, if $\operatorname{id}_C P_C \leq 1$ we obtain the injective resolution of P_B as in the statement of the theorem.

If $\operatorname{id}_C P_C = 2$. We start by defining the morphisms in the injective resolution. In order to do so, consider the following commutative diagram

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \qquad (4.2.1)\\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow & P_C \otimes E & \xrightarrow{\beta_0} & \operatorname{Hom}_C(B, P_C \otimes E) & \xrightarrow{\beta_1} & \tau \Omega \tau^{-1} \Omega^{-1} P_C & \longrightarrow 0 \\
& \downarrow{\epsilon} & & \downarrow{\gamma_0} & & \downarrow{\gamma_1} & & \\
0 \longrightarrow & \bar{I}^0_C & \xrightarrow{v_0} & \bar{I}^0_B & \xrightarrow{v_1} & \tau \Omega \bar{I}^0_C & \longrightarrow 0 \\
& \downarrow{\kappa_1} & & \downarrow{\pi} & & \downarrow{\gamma_2} & & \\
0 \longrightarrow & \bar{I}^1_C & \xrightarrow{l_2} & \bar{I}^1_B & \xrightarrow{\kappa_2} & \tau \Omega \bar{I}^1_C & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

where every row and column is exact. To construct this diagram, we begin with the injective resolution in $\operatorname{mod} C$ of $P_C \otimes E$ as in the statement of the theorem. Note that, by Lemma 3.1.5(a), $\operatorname{id}_C P_C \otimes E \leq 1$. This sequence appears in the left most column of the diagram above. Then we apply the coinduction functor $\operatorname{Hom}_C(B, -)$ to this sequence and recall that $\operatorname{Ext}^1_C(E, P_C \otimes E) = 0$, by Corollary 3.1.9. This gives

us the short exact sequence to the right of the one we started with, which is the middle column of the diagram. We also obtain inclusions β_0, v_0, l_2 from the given C-modules to the corresponding coinduced B-modules and projections β_1, v_1, k_2 to the corresponding cokernels as in Proposition 2.2.2(b). By the same proposition, $\mathrm{cok}\, \beta_0 = D(E \otimes D(P_C \otimes E)) \cong \tau\Omega\tau^{-1}\Omega^{-1}P_C$, where the last identity follows from Proposition 3.1.1. Similarly, $\mathrm{cok}\, v_0 \cong \tau\Omega\bar{I}_C^0$, and $\mathrm{cok}\, l_2 \cong \tau\Omega\bar{I}_C^1$. Thus we obtain the commutative diagram (4.2.1).

Now we construct the commutative diagram (4.2.2) with exact rows, which appears below. We obtain the bottom two rows from Corollary 4.1.6 by letting $N = P_C$. Next, we draw a commutative diagram (4.1.2) in the top right corner, where the maps g_1 and g_2 are as in Lemma 4.1.4. We complete the top row by $P_C \otimes E$, the kernel of u_1, as in Proposition 2.2.2(a). Finally, by the universal property of $\ker \delta_1$ there exists a morphism ϵ_0 that completes the diagram and makes the upper left square commute. Note that the bottom row of diagram (4.2.2) is the same as the top row of diagram (4.2.1).

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & P_C \otimes E & \xrightarrow{u_0} & P_B & \xrightarrow{u_1} & P_C & \longrightarrow & 0 \\
& & \downarrow{\epsilon_0} & & \downarrow{g_2} & & \downarrow{g_1} & & \\
0 & \longrightarrow & P_C \otimes E & \xrightarrow{\delta_0} & \tau\Omega\tau^{-1}\Omega^{-1}P_C \otimes B & \xrightarrow{\delta_1} & \tau\Omega\tau^{-1}\Omega^{-1}P_C & \longrightarrow & 0 \\
& & \downarrow{\phi'} & & \downarrow{\theta'} & & \downarrow{\psi'} & & \\
0 & \longrightarrow & P_C \otimes E & \xrightarrow{\beta_0} & \mathrm{Hom}_C(B, P_C \otimes E) & \xrightarrow{\beta_1} & \tau\Omega\tau^{-1}\Omega^{-1}P_C & \longrightarrow & 0.
\end{array}
\tag{4.2.2}
$$

Next we want to show that ϵ_0 is an isomorphism. Observe that, since $P_C \otimes E$ is finite dimensional, it suffices to show that ϵ_0 is injective. Suppose $\epsilon_0(a) = 0$, for some $a \in P_C \otimes E$. Let $u_0(a) = b$. Because u_0 is injective, it is enough to show that $b = 0$.

By commutativity $g_2(b) = 0$, but looking at the top row of Lemma 4.1.4, we conclude that either $b = 0$ or $u_1(b) \neq 0$. In the first case we are done, so suppose $u_1(b) \neq 0$. Then $u_1(b) = u_1 u_0(a) \neq 0$, which is a contradiction since the top row of the diagram (4.2.2) is exact. This shows that $a = b = 0$ and that ϵ_0 is an isomorphism.

Now we show that the following sequence as in the statement of the theorem is exact

$$0 \xrightarrow{\hspace{2cm}} P_B \xrightarrow{\left(\begin{smallmatrix} j_0 i_0 u_1 \\ \gamma_0 \theta' g_2 \end{smallmatrix} \right)} I_B^0 \oplus \bar{I}_B^0 \xrightarrow{\left(\begin{smallmatrix} \alpha & -l_1 \gamma v_1 \\ 0 & \pi \end{smallmatrix} \right)} \tilde{I}_B \oplus \bar{I}_B^1 \xrightarrow{\hspace{2cm}} 0$$

where the maps are v_1, π, γ_0 are given in diagram (4.2.1), the maps θ', u_1, g_2 in diagram (4.2.2), the maps i_0, j_0, α, l_1 in Lemma 4.1.3, and the map γ in diagram (4.2.3) below.

$$\begin{array}{ccc} \tau \Omega \tau^{-1} \Omega^{-1} P_C & \xrightarrow{\pi_0} & \tilde{I}_C \\ {\scriptstyle \gamma_1 \psi'} \downarrow & \nearrow & \\ & \scriptstyle \gamma & \\ \tau \Omega \bar{I}_C^0 & & \end{array} \qquad (4.2.3)$$

Here π_0 is the map of Lemma 4.1.1. Observe that γ exists because $\tilde{I}_C$ is injective and $\gamma_1 \psi'$ is an injective map. Moreover, the map γ makes the diagram commute, that is $\gamma \gamma_1 \psi' = \pi_0$.

First we show that the sequence we defined above is exact at P_B. Suppose $\left(\begin{smallmatrix} j_0 i_0 u_1 \\ \gamma_0 \theta' g_2 \end{smallmatrix} \right)(p) = 0$ for some $p \in P_B$. So on the one hand $j_0 i_0 u_1(p) = 0$, but j_0 and i_0 are injective, which means $u_1(p) = 0$. By diagram (4.2.2) there exists $b \in P_C \otimes E$ such that $u_0(b) = p$. On the other hand, $\gamma_0 \theta' g_2 u_0(b) = 0$, but by commutativity in diagram (4.2.2) this is equivalent to $\gamma_0 \beta_0 \phi' \epsilon_0(b) = 0$. Because, all of these maps are injective it follows that $b = 0$, which implies $p = 0$. This shows that $\left(\begin{smallmatrix} j_0 i_0 u_1 \\ \gamma_0 \theta' g_2 \end{smallmatrix} \right)$ is an injective map.

Next we show that $\mathrm{im}\left(\begin{smallmatrix} j_0 i_0 u_1 \\ \gamma_0 \theta' g_2 \end{smallmatrix}\right) = \mathrm{ker}\left(\begin{smallmatrix} \alpha & -l_1 \gamma v_1 \\ 0 & \pi \end{smallmatrix}\right)$, meaning that the sequence is exact at $I_B^0 \oplus \bar{I}_B^0$. To show the forward inclusion consider $\left(\begin{smallmatrix} \alpha & -l_1 \gamma v_1 \\ 0 & \pi \end{smallmatrix}\right)\left(\begin{smallmatrix} j_0 i_0 u_1 \\ \gamma_0 \theta' g_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \alpha j_0 i_0 u_1 - l_1 \gamma v_1 \gamma_0 \theta' g_2 \\ \pi \gamma_0 \theta' g_2 \end{smallmatrix}\right)$.

Observe that $\pi \gamma_0 = 0$ by diagram (4.2.1), which means that the bottom entry is zero. The top entry is also zero, because

$$\alpha j_0 i_0 u_1 - l_1 \gamma v_1 \gamma_0 \theta' g_2 = l_1(\pi_0 g_1 u_1 - \gamma v_1 \gamma_0 \theta' g_2) \qquad \text{by first row in Lemma 4.1.3}$$

$$= l_1(\gamma \gamma_1 \psi' g_1 u_1 - \gamma v_1 \gamma_0 \theta' g_2) \quad \text{by diagram (4.2.3)}$$

$$= l_1 \gamma(\gamma_1 \psi' g_1 u_1 - v_1 \gamma_0 \theta' g_2)$$

$$= l_1 \gamma(\gamma_1 \beta_1 \theta' g_2 - v_1 \gamma_0 \theta' g_2) \qquad \text{by diagram (4.2.2),}$$

which is zero by commutativity of diagram (4.2.1). To show the reverse inclusion suppose $\left(\begin{smallmatrix} \alpha & -l_1 \gamma v_1 \\ 0 & \pi \end{smallmatrix}\right)\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$ for some $a \in I_B^0, b \in \bar{I}_B^0$. Since $\alpha(a) - l_1 \gamma v_1(b) = 0$ and the bottom row in the diagram of Lemma 4.1.3 is exact, there exists $c \in I_C^0$ such that $j_0(c) = a$ and $g_0(c) = \gamma v_1(b)$. Also, we have $\pi(b) = 0$, so by diagram (4.2.1) there exists $d \in \mathrm{Hom}_C(B, P_C \otimes E)$, such that $\gamma_0(d) = b$. Since θ' is an isomorphism we can find $e \in \tau \Omega \tau^{-1} \Omega^{-1} P_C \otimes B$ such that $\theta'(e) = d$. Now we have

$$g_0(c) = \gamma v_1 \gamma_0 \theta'(e)$$

$$= \gamma \gamma_1 \beta_1 \theta'(e) \qquad \qquad \text{by diagram (4.2.1)}$$

$$= \gamma \gamma_1 \psi' \delta_1(e) \qquad \qquad \text{by diagram (4.2.2)}$$

$$= \pi_0 \delta_1(e) \qquad \qquad \text{by diagram (4.2.3).}$$

Equivalently we can write $g_0(c) - \pi_0 \delta_1(e) = 0$, and since the bottom row in the diagram of Lemma 4.1.4 is exact, there exists $p \in P_B$ such that $i_0 u_i(p) = c$ and $g_2(p) = e$.

Now consider $\left(\begin{smallmatrix} j_0 i_0 u_1 \\ \gamma_0 \theta' g_2 \end{smallmatrix}\right)(p) = \left(\begin{smallmatrix} j_0(c) \\ \gamma_0(d) \end{smallmatrix}\right) = \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$. This shows exactness at $I_B^0 \oplus \bar{I}_B^0$.

It remains to show that $\left(\begin{smallmatrix} \alpha & -l_1 \gamma v_1 \\ 0 & \pi \end{smallmatrix}\right)$ is surjective. By definition, π is surjective, thus it suffices to show that $\left(\alpha \ -l_1 \gamma v_1\right)$ is surjective. That is, given $a \in \tilde{I}_B$ find $c \in I_B^0$, $d \in \bar{I}_B^0$ such that $\alpha(c) - l_1 \gamma v_1(d) = a$. By Lemma 4.1.3, there exist $c' \in I_B^0$ and $e \in \tilde{I}_C$ such that $\alpha(c') - l_1(e) = a$. Then by Lemma 4.1.4, there exist $m \in I_C^0$ and $n \in \tau \Omega \tau^{-1} \Omega^{-1} P_C \otimes B$ such that $g_0(m) - \pi_0 \delta_1(n) = e$. Finally, let $d = \gamma_0 \theta'(n) \in \bar{I}_B^0$ and $c = j_0(m) + c' \in I_B^0$ and observe that

$$\alpha(j_0(m) + c') - l_1 \gamma v_1(\gamma_0 \theta'(n)) =$$

$$= l_1 g_0(m) + \alpha(c') - l_1 \gamma v_1 \gamma_0 \theta'(n) \quad \text{by diagram (4.1.1)}$$

$$= \alpha(c') + l_1(g_0(m) - \gamma v_1 \gamma_0 \theta'(n))$$

$$= \alpha(c') - l_1(g_0(m) - \gamma \gamma_1 \psi' \delta_1(n)) \quad \text{by diagrams (4.2.1), (4.2.2)}$$

$$= \alpha(c') - l_1(g_0(m) - \pi_0 \delta_1(n)) \quad \text{by diagram (4.2.3)}$$

$$= a.$$

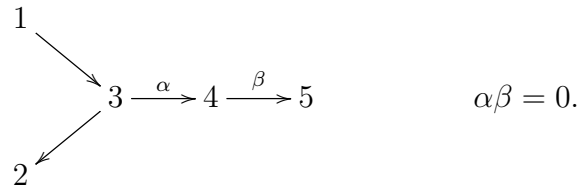This shows surjectivity and finishes the proof of the theorem. $\qquad \square$

Remark 4.2.3. The theorem above can be dualized. That is, in a similar manner one can construct a projective resolution of an injective B-module of length at most one.

We obtain therefore a new proof of the following result which was first proved by Keller and Reiten using cluster categories [29].
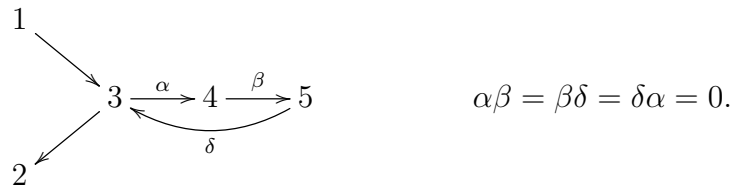
Corollary 4.2.4. *If C is a tilted algebra and B is the corresponding cluster-tilted algebra, then B is 1-Gorenstein.*

4.3 Examples

Example 4.3.1. Let C be the tilted algebra given by the following quiver with relations.

$$
\begin{array}{c}
1 \\
\quad \searrow \\
\qquad 3 \xrightarrow{\ \alpha\ } 4 \xrightarrow{\ \beta\ } 5 \qquad\qquad \alpha\beta = 0. \\
\quad \nearrow \\
2
\end{array}
$$

The corresponding cluster-tilted algebra B is given by the quiver with relations below.

$$
\begin{array}{c}
1 \\
\quad \searrow \\
\qquad 3 \underset{}{\overset{\alpha}{\rightleftarrows}} 4 \xrightarrow{\ \beta\ } 5 \qquad\qquad \alpha\beta = \beta\delta = \delta\alpha = 0. \\
\quad \nearrow \\
2
\end{array}
$$

We want to construct the injective resolution of $P_B(1) = \begin{smallmatrix} 1 \\ 3 \\ 2\ 4 \end{smallmatrix}$ as described in the theorem above. Observe that $P_B(1) = P_C(1)$, hence $P_C(1) \otimes E = 0$ and $\mathrm{id}_C P_C(1) \leq 1$. In this case we only need to consider the minimal injective resolution of $P_C(1)$ given below.

$$ 0 \longrightarrow P_C(1) \longrightarrow I_C(2) \oplus I_C(4) \longrightarrow I_C(3) \longrightarrow 0 $$
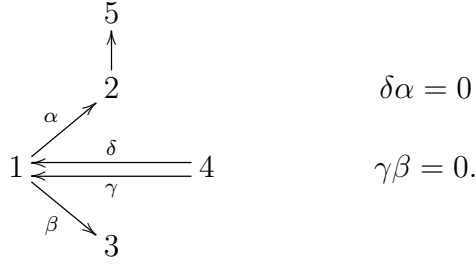
Now according to the remark following the statement of Theorem 4.2.1 we obtain the minimal injective resolution of $P_B(1)$ as shown below.

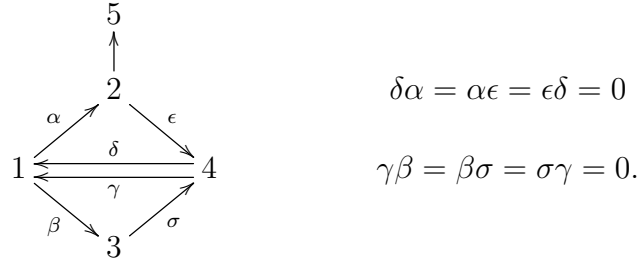$$ 0 \longrightarrow P_B(1) \longrightarrow I_B(2) \oplus I_B(4) \longrightarrow I_B(3) \longrightarrow 0 $$

Note that here $I_B(2) \neq I_C(2)$ and $I_B(3) \neq I_C(3)$.

Example 4.3.2. Let C be the tilted algebra given by the following quiver with

relations.

$$5 \uparrow 2$$

(diagram with vertices)

$$\delta\alpha = 0$$

$$\gamma\beta = 0.$$

The corresponding cluster-tilted algebra B is of type $\tilde{\mathbb{A}}_{(3,2)}$ and it is given by the quiver with relations below.

$$\delta\alpha = \alpha\epsilon = \epsilon\delta = 0$$

$$\gamma\beta = \beta\sigma = \sigma\gamma = 0.$$

We want to construct the injective resolution of $P_B(2) = \begin{smallmatrix} & 2 & \\ 5 & & 4 \\ & 1 & \\ & 2 & \\ & 5 & \end{smallmatrix}$, the projective B-module at vertex 2, as in Theorem 4.2.1. First, we find minimal injective presentations of $P_C(2) = \begin{smallmatrix} 2 \\ 5 \end{smallmatrix}$ and $P_C(2) \otimes E = \begin{smallmatrix} 4 \\ 1 \\ 2 \\ 5 \end{smallmatrix}$ in mod C, which are given below.

$$0 \longrightarrow P_C(2) \longrightarrow I_C(5) \longrightarrow I_C(1) \qquad\qquad 0 \longrightarrow P_C(2) \otimes E \longrightarrow I_C(5) \longrightarrow 0$$

We write out the explicit representations involved in the sequences above.

$$0 \longrightarrow \begin{smallmatrix} 2 \\ 5 \end{smallmatrix} \longrightarrow \begin{smallmatrix} 4 \\ 1 \\ 2 \\ 5 \end{smallmatrix} \longrightarrow \begin{smallmatrix} 4 & 4 \\ & 1 \end{smallmatrix} \qquad\qquad 0 \longrightarrow \begin{smallmatrix} 4 \\ 1 \\ 2 \\ 5 \end{smallmatrix} \longrightarrow \begin{smallmatrix} 4 \\ 1 \\ 2 \\ 5 \end{smallmatrix} \longrightarrow 0$$

Here $I_C(i)$ denotes the injective C-module at vertex i, while $I_B(i)$ will denote the

corresponding injective B-module. Next, we calculate $\nu\nu^{-1}\Omega^{-1}P_C(2)$. Observe that $\Omega^{-1}P_C(2)$ is the module with dimension vector $(1,0,0,1,0)$ such that $\gamma = 1$ and $\delta = 0$. Then $\nu^{-1}\Omega^{-1}P_C(2) = \text{Hom}_C(DC, \Omega^{-1}P_C(2))$, and the only injectives that have a nonzero morphism into $\Omega^{-1}P_C(2)$ are $I_C(2)$ and $I_C(5)$. Hence, $\nu^{-1}\Omega^{-1}P_C(2) = P_C(2)$, so $\nu\nu^{-1}\Omega^{-1}P_C(2) = I_C(2)$. Then according to Theorem 4.2.1 we construct the injective resolution of $P_B(2)$

$$0 \longrightarrow P_B(2) \longrightarrow I_B(5) \oplus I_B(5) \longrightarrow I_B(2) \oplus 0 \longrightarrow 0$$

or equivalently substituting the representations we have

$$0 \longrightarrow \begin{smallmatrix} & 2 & \\ 5 & & 4 \\ & 1 & \\ & 2 & \\ & 5 & \end{smallmatrix} \longrightarrow \begin{smallmatrix} 2 \\ 4 \\ 1 \\ 2 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 4 \\ 1 \\ 2 \\ 5 \end{smallmatrix} \longrightarrow \begin{smallmatrix} 2 \\ 4 \\ 1 \\ 2 \end{smallmatrix} \longrightarrow 0.$$

Chapter 5

Relation to cluster categories

In this chapter we study the relationship between induction and coinduction functors and the cluster category. Here, we assume that A is a hereditary algebra and $T \in \text{mod } A$ is a basic tilting module. Let $C = \text{End}_A T$ be the corresponding tilted algebra, and B be the associated cluster-tilted algebra. Finally, let $\mathcal{C}_A$ denote the cluster category of A. We know that $\text{mod } A$ naturally embeds in $\mathcal{C}_A$, which in turn maps surjectively onto $\text{mod } B$ via the functor $\text{Hom}_{\mathcal{C}_A}(T, -)$. Recall that the induction functor $- \otimes_C B$ and the coinduction functor $D(B \otimes_C D-)$ both map $\text{mod } C$ to $\text{mod } B$, while the module categories of A and C are closely related via the Tilting Theorem 1.4.4. Now, we want to study how the induction and the coinduction functors fit into this larger picture.

$$
\begin{array}{ccc}
\text{mod } A & \xleftarrow{\quad \text{Tilting Theorem} \quad} & \text{mod } C \\
\downarrow & & \downarrow {\scriptstyle \begin{array}{c} -\otimes_C B \\ D(B\otimes_C D-) \end{array}} \\
\mathcal{C}_A & \xrightarrow{\quad \text{Hom}_{\mathcal{C}_A}(T,-) \quad} & \text{mod } B
\end{array}
$$

5.1 Induction functor

The following theorem describes the relationship between the induction functor and the cluster category.

Theorem 5.1.1. *Let A be a hereditary algebra and $T \in \mathrm{mod}\, A$ a basic tilting module. Let $\mathcal{C}_A$ be the cluster category of A, $C = \mathrm{End}_A T$ be the corresponding tilted algebra, and $B = C \ltimes E$ the corresponding cluster-tilted algebra. Recall the definitions of $\mathcal{T}(T)$ and $\mathcal{Y}(T)$, the associated torsion class of $\mathrm{mod}\, A$ and the torsion free class of $\mathrm{mod}\, C$ below.*

$$\mathcal{T}(T) = \{M \in \mathrm{mod}\, A \mid \mathrm{Ext}_A^1(T, M) = 0\} \qquad \mathcal{Y}(T) = \{N \in \mathrm{mod}\, C \mid \mathrm{Tor}_1^C(N, T) = 0\}$$

Then the following diagram commutes.

$$
\begin{array}{ccc}
\mathcal{T}(T) & \xrightarrow{\ \mathrm{Hom}_A(T,-)\ } & \mathcal{Y}(T) \\
\downarrow & & \downarrow{\scriptstyle -\otimes_C B} \\
\mathcal{C}_A & \xrightarrow{\ \mathrm{Hom}_{\mathcal{C}_A}(T,-)\ } & \mathrm{mod}\, B
\end{array}
$$

That is, $\mathrm{Hom}_A(T, M) \otimes_C B \cong \mathrm{Hom}_{\mathcal{C}_A}(T, M)$ for every $M \in \mathcal{T}(T)$.

Proof. Let M be an indecomposable module belonging to $\mathcal{T}(T)$. If $M \in \mathrm{add}\, T$, then $\mathrm{Hom}_A(T, M)$ is a projective C-module, and $\mathrm{Hom}_A(T, M) \otimes_C B$ is the corresponding projective B-module. On the other hand, $\mathrm{Hom}_{\mathcal{C}_A}(T, M)$ is also the same projective B-module. Hence, in this case the theorem above holds. If $M \notin \mathrm{add}\, T$, then Proposition 1.4.6(d) implies that the projective dimension of $\mathrm{Hom}_A(T, M) \in \mathcal{Y}(T)$ is one. Let

$$0 \longrightarrow P_C^1 \xrightarrow{\ f\ } P_C^0 \longrightarrow \mathrm{Hom}_A(T, M) \longrightarrow 0 \qquad\qquad (5.1.1)$$

be its minimal projective resolution in $\mod C$. By Theorem 1.4.4(a), there exist $T^0, T^1 \in \mathrm{add}\, T$, and $g \in \mathrm{Hom}_A(T^1, T^0)$ such that

$$f = \mathrm{Hom}_A(T, g), \qquad P_C^1 = \mathrm{Hom}_A(T, T^1), \qquad P_C^0 = \mathrm{Hom}_A(T, T^0).$$

Moreover, since $M \in \mathcal{T}(T)$, we have that $M = \mathrm{coker}\, g$. Applying $- \otimes_C T$ to the projective resolution (5.1.1) and using Theorem 1.4.4(a), we obtain an exact sequence

$$\mathrm{Tor}_1^C(\mathrm{Hom}_A(T, M), T) \longrightarrow T^1 \xrightarrow{\ g\ } T^0 \longrightarrow M \longrightarrow 0.$$

Furthermore, $\mathrm{Tor}_1^C(\mathrm{Hom}_A(T, M), T) = 0$ by Tilting Theorem 1.4.4, which means that the sequence above is a short exact sequence in $\mod A$. Observe that g is also a morphism in the cluster category, and the short exact sequence above corresponds to a triangle

$$T^1 \xrightarrow{\ g\ } T^0 \longrightarrow M \longrightarrow T^1[1]$$

in $\mathcal{C}_A$. Applying $\mathrm{Hom}_{\mathcal{C}_A}(T, -)$ to this triangle yields an exact sequence in $\mod B$

$$\mathrm{Hom}_{\mathcal{C}_A}(T, T^1) \xrightarrow{\ g_*\ } \mathrm{Hom}_{\mathcal{C}_A}(T, T^0) \longrightarrow \mathrm{Hom}_{\mathcal{C}_A}(T, M) \longrightarrow 0.$$

Note that there is zero on the right of the sequence as $\mathrm{Hom}_{\mathcal{C}_A}(T, T^1[1]) = 0$, because T is a tilting object in $\mathcal{C}_A$. Also, $\mathrm{Hom}_{\mathcal{C}_A}(T, T^1) = P_B^1$ and $\mathrm{Hom}_{\mathcal{C}_A}(T, T^0) = P_B^0$, thus

$$\mathrm{Hom}_{\mathcal{C}_A}(T, M) = \mathrm{coker}\, \mathrm{Hom}_{\mathcal{C}_A}(T, g) : P_B^1 \to P_B^0.$$

On the other hand, applying the induction functor $- \otimes_C B$ to the sequence (5.1.1)

we obtain

$$P_B^1 \xrightarrow{f \otimes 1} P_B^0 \longrightarrow \operatorname{Hom}_A(T, M) \otimes_C B \longrightarrow 0.$$

Hence, we see that

$$\operatorname{Hom}_A(T, M) \otimes_C B = \operatorname{coker} \operatorname{Hom}_A(T, g) \otimes 1 : P_B^1 \to P_B^0.$$

But since $\operatorname{top} P_B^1 = \operatorname{top} P_C^1$, we have

$$\operatorname{Hom}_{\mathcal{C}_A}(T, g)(\operatorname{top} P_B^1) = \operatorname{Hom}_A(T, g)(\operatorname{top} P_C^1) = \operatorname{Hom}_A(T, g) \otimes 1 (\operatorname{top} P_B^1).$$

Because P_B^1 is projective both maps $\operatorname{Hom}_{\mathcal{C}_A}(T, g)$ and $\operatorname{Hom}_A(T, g) \otimes 1$ are determined by their restrictions to $\operatorname{top} P_B^1$. Therefore, we construct a commutative diagram

$$
\begin{array}{ccccccc}
P_B^1 & \xrightarrow{\operatorname{Hom}_{\mathcal{C}_A}(T,g)} & P_B^0 & \longrightarrow & \operatorname{Hom}_{\mathcal{C}_A}(T, M) & \longrightarrow & 0 \\
\| & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle h} & & \\
P_B^1 & \xrightarrow{\operatorname{Hom}_A(T,g) \otimes 1} & P_B^0 & \longrightarrow & \operatorname{Hom}_A(T, M) \otimes_C B & \longrightarrow & 0
\end{array}
$$

where h is an isomorphism by the Five Lemma. This shows that for every $M \in \mathcal{T}(T)$ we have $\operatorname{Hom}_{\mathcal{C}_A}(T, M) \cong \operatorname{Hom}_A(T, M) \otimes_C B$, and finishes the proof of the theorem. $\qquad\square$

This result shows that the induction functor can be reformulated in terms of other well-studied functors. In particular consider the following corollary.

Corollary 5.1.2. *Let A be a hereditary algebra and $T \in \operatorname{mod} A$ a basic tilting module. Let $\mathcal{C}_A$ be the cluster category of A, $C = \operatorname{End}_A T$ be the corresponding tilted algebra,*

and $B = C \ltimes E$ the corresponding cluster-tilted algebra. Then

$$M \otimes_C B \cong \begin{cases} \mathrm{Hom}_{\mathcal{C}_A}(T, M \otimes_C T) & \text{if} \quad M \in \mathcal{Y}(T) \\ M & \text{if} \quad M \in \mathcal{X}(T) \end{cases}$$

for every $M \in \mathrm{ind}\, C$.

Proof. First, note that every $M \in \mathrm{ind}\, C$ belongs to either $\mathcal{X}(T)$ or $\mathcal{Y}(T)$ by Proposition 1.4.6(e). If $M \in \mathcal{X}(T)$, Proposition 1.4.6(c) implies that $\mathrm{id}_C M \leq 1$, and then Proposition 3.1.2(a) yields $M \otimes_C B \cong M$. If $M \in \mathcal{Y}(T)$, then Theorem 1.4.4(a) implies that $M \cong \mathrm{Hom}_A(T, N)$ for $N = M \otimes_C T \in \mathcal{T}(T)$. Hence $M \otimes_C B \cong \mathrm{Hom}_{\mathcal{C}_A}(T, N)$, by Theorem 5.1.1. $\square$

5.2   Coinduction functor

Similarly, there is a dual statement that provides an alternative description of the coinduction functor.

Theorem 5.2.1. *Let A be a hereditary algebra and $T \in \mathrm{mod}\, A$ a basic tilting module. Let $\mathcal{C}_A$ be the cluster category of A, $C = \mathrm{End}_A T$ be the corresponding tilted algebra, and $B = C \ltimes E$ the corresponding cluster-tilted algebra. Recall the definitions of $\mathcal{F}(T)$ and $\mathcal{X}(T)$, the associated torsion free class of $\mathrm{mod}\, A$ and the torsion class of $\mathrm{mod}\, C$ below.*

$$\mathcal{F}(T) = \{M \in \mathrm{mod}\, A \mid \mathrm{Hom}_A(T, M) = 0\} \qquad \mathcal{X}(T) = \{N \in \mathrm{mod}\, C \mid N \otimes_C T = 0\}$$

Then the following diagram commutes.

$$
\begin{array}{ccc}
\mathcal{F}(T) & \xrightarrow{\mathrm{Ext}^1_A(T,-)} & \mathcal{X}(T) \\
\downarrow & & \downarrow{\scriptstyle D(B\otimes_C D-)} \\
\mathcal{C}_A & \xrightarrow{\mathrm{Ext}^1_{\mathcal{C}_A}(T,-)} & \mathrm{mod}\, B
\end{array}
$$

That is, $D(B \otimes_C D\mathrm{Ext}^1_A(T, M)) \cong \mathrm{Ext}^1_{\mathcal{C}_A}(T, M)$, for every $M \in \mathcal{F}(T)$.

Proof. Let M be an indecomposable module belonging to $\mathcal{F}(T)$. Consider $\mathrm{Ext}^1_A(T, M)$, which belongs to $\mathcal{X}(T)$ by Theorem 1.4.4(b). If $\mathrm{Ext}^1_A(T, M) = I^i_C$ is an injective C-module, then $D(B \otimes_C DI^i_C) \cong I^i_B$ is the corresponding injective in $\mathrm{mod}\, B$. On the other hand, by Proposition 1.4.7 we have $M \cong \tau T^i$ for some $T^i \in \mathrm{add}\, T$. Therefore, by Serre Duality

$$
\mathrm{Ext}^1_{\mathcal{C}_A}(T, \tau T^i) \cong D\mathrm{Hom}_{\mathcal{C}_A}(T^i, T) \cong I^i_B. \tag{5.2.1}
$$

This shows that the theorem holds if $\mathrm{Ext}^1_A(T, M)$ is injective.

If $\mathrm{Ext}^1_A(T, M) \in \mathcal{X}(T)$ is not injective then according to Proposition 1.4.6(c) it has injective dimension one. Consider a minimal injective resolution of this module in $\mathrm{mod}\, C$ below.

$$
0 \longrightarrow \mathrm{Ext}^1_A(T, M) \longrightarrow I^0_C \xrightarrow{f} I^1_C \longrightarrow 0 \tag{5.2.2}
$$

Because I^0_C, I^1_C are successors of $\mathrm{Ext}^1_A(T, M)$, Proposition 1.4.6(f) implies that these injectives belong to $\mathcal{X}(T)$. By Proposition 1.4.7 and Theorem 1.4.4(b), there exist $T^1, T^0 \in \mathrm{add}\, T$, and $g \in \mathrm{Ext}^1_A(\tau T^0, \tau T^1)$ such that

$$
f = \mathrm{Ext}^1_A(T, g), \qquad I^1_C = \mathrm{Ext}^1_A(T, \tau T^1), \qquad I^0_C = \mathrm{Ext}^1_A(T, \tau T^0).
$$

Applying $\mathrm{Tor}_1^C(-, T)$ to sequence (5.2.2) and using Theorem 1.4.4, we obtain a long exact sequence

$$\mathrm{Tor}_2^C(I_C^1, T) \longrightarrow M \longrightarrow \tau T^0 \xrightarrow{g} \tau T^1 \longrightarrow \mathrm{Ext}_A^1(T, M) \otimes_C T.$$

Observe that by the same theorem the last term in the sequence above is zero, likewise $\mathrm{Tor}_2^C(I_C^1, T) = 0$, because T as a left C-module has projective dimension at most one. Thus we obtain a short exact sequence in $\mathrm{mod}\, A$

$$0 \longrightarrow M \longrightarrow \tau T^0 \xrightarrow{g} \tau T^1 \longrightarrow 0$$

which induces a triangle

$$\tau T^1[-1] \longrightarrow M \longrightarrow \tau T^0 \xrightarrow{g} \tau T^1$$

in the cluster category. Applying $\mathrm{Ext}_{\mathcal{C}_A}^1(T, -)$ to this triangle we obtain an exact sequence in $\mathrm{mod}\, B$

$$\mathrm{Ext}_{\mathcal{C}_A}^1(T, \tau T^1[-1]) \longrightarrow \mathrm{Ext}_{\mathcal{C}_A}^1(T, M) \longrightarrow \mathrm{Ext}_{\mathcal{C}_A}^1(T, \tau T^0) \xrightarrow{g_*} \mathrm{Ext}_{\mathcal{C}_A}^1(T, \tau T^1) \,.$$

Observe that because T is a tilting object in $\mathcal{C}_A$, we have $\mathrm{Ext}_{\mathcal{C}_A}^1(T, \tau T^1[-1]) = 0$. Moreover, equation (5.2.1) yields $\mathrm{Ext}_{\mathcal{C}_A}^1(T, \tau T^0) = I_B^0$ and $\mathrm{Ext}_{\mathcal{C}_A}^1(T, \tau T^1) = I_B^1$. Thus,

$$\mathrm{Ext}_{\mathcal{C}_A}^1(T, M) = \ker \mathrm{Ext}_{\mathcal{C}_A}^1(T, g) : I_B^0 \to I_B^1.$$

On the other hand, applying the coinduction functor to sequence (5.2.2) we obtain

$$0 \longrightarrow D(B \otimes_C D\mathrm{Ext}_A^1(T, M)) \longrightarrow I_B^0 \xrightarrow{D(1 \otimes Dg)} I_B^1.$$

Hence, we see that

$$D(B \otimes_C D\mathrm{Ext}_A^1(T, M)) = \ker D(1 \otimes Dg) : I_B^0 \to I_B^1.$$

Because I_B^1 is injective, both maps $\mathrm{Ext}_{\mathcal{C}_A}^1(T, g)$ and $D(1 \otimes Dg)$ are determined by their preimages of $\mathrm{soc}\, I_B^1$. But since $\mathrm{soc}\, I_B^1 = \mathrm{soc}\, I_C^1$, we have

$$\mathrm{Ext}_{\mathcal{C}_A}^1(T, g)^{-1}(\mathrm{soc}\, I_B^1) = \mathrm{Ext}_A^1(T, g)^{-1}(\mathrm{soc}\, I_C^1) = D(B \otimes_C D\mathrm{Ext}_A^1(T, g))^{-1}(\mathrm{soc}\, I_B^1).$$

Thus we have a commutative diagram

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Ext}_{\mathcal{C}_A}^1(T, M) & \longrightarrow & I_B^0 & \xrightarrow{\mathrm{Ext}_{\mathcal{C}_A}^1(T,g)} & I_B^1 \\
& & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \cong} & & \| \\
0 & \longrightarrow & D(B \otimes_C D\mathrm{Ext}_A^1(T, M)) & \longrightarrow & I_B^0 & \xrightarrow{D(B \otimes_C D\mathrm{Ext}_A^1(T,g))} & I_B^1
\end{array}
$$

and h is an isomorphism by the Five Lemma. This shows that for every $M \in \mathcal{F}(T)$ we have $\mathrm{Ext}_{\mathcal{C}_A}^1(T, M) \cong D(B \otimes_C D\mathrm{Ext}_A^1(T, M))$, and finishes the proof of the theorem.

$\square$

Corollary 5.2.2. *Let A be a hereditary algebra and $T \in \mathrm{mod}\, A$ a basic tilting module. Let $\mathcal{C}_A$ be the cluster category of A, $C = \mathrm{End}_A T$ be the corresponding tilted algebra,*

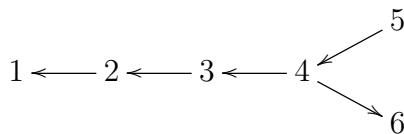and $B = C \ltimes E$ the corresponding cluster-tilted algebra. Then

$$D(B \otimes_C DM) \cong \begin{cases} \operatorname{Ext}^1_{\mathcal{C}_A}(T, \operatorname{Tor}^C_1(M, T)) & \text{if} \quad M \in \mathcal{X}(T) \\ M & \text{if} \quad M \in \mathcal{Y}(T) \end{cases}$$

for every $M \in \operatorname{ind} C$.

Proof. The proof is similar to that of Corollary 5.1.2 and we omit it. $\qquad\square$

5.3 Example

Example 5.3.1. Let A be the path algebra of the following quiver.

Let

$$T = \underset{1}{1} \oplus \begin{smallmatrix} 5 \\ 4 \\ 6\ 3 \\ 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 5 \\ 4 \\ 6\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 5 \\ 4 \\ 6 \end{smallmatrix} \oplus \underset{5}{5}$$

be the tilting A-module. The Auslander-Reiten quiver of $\operatorname{mod} A$ is shown at the top of Figure 5.3.1. The modules belonging to $\mathcal{T}(T)$ are in the dark shaded regions enclosed by a solid line, while modules belonging to $\mathcal{F}(T)$ are in the light shaded regions enclosed by a dotted line.

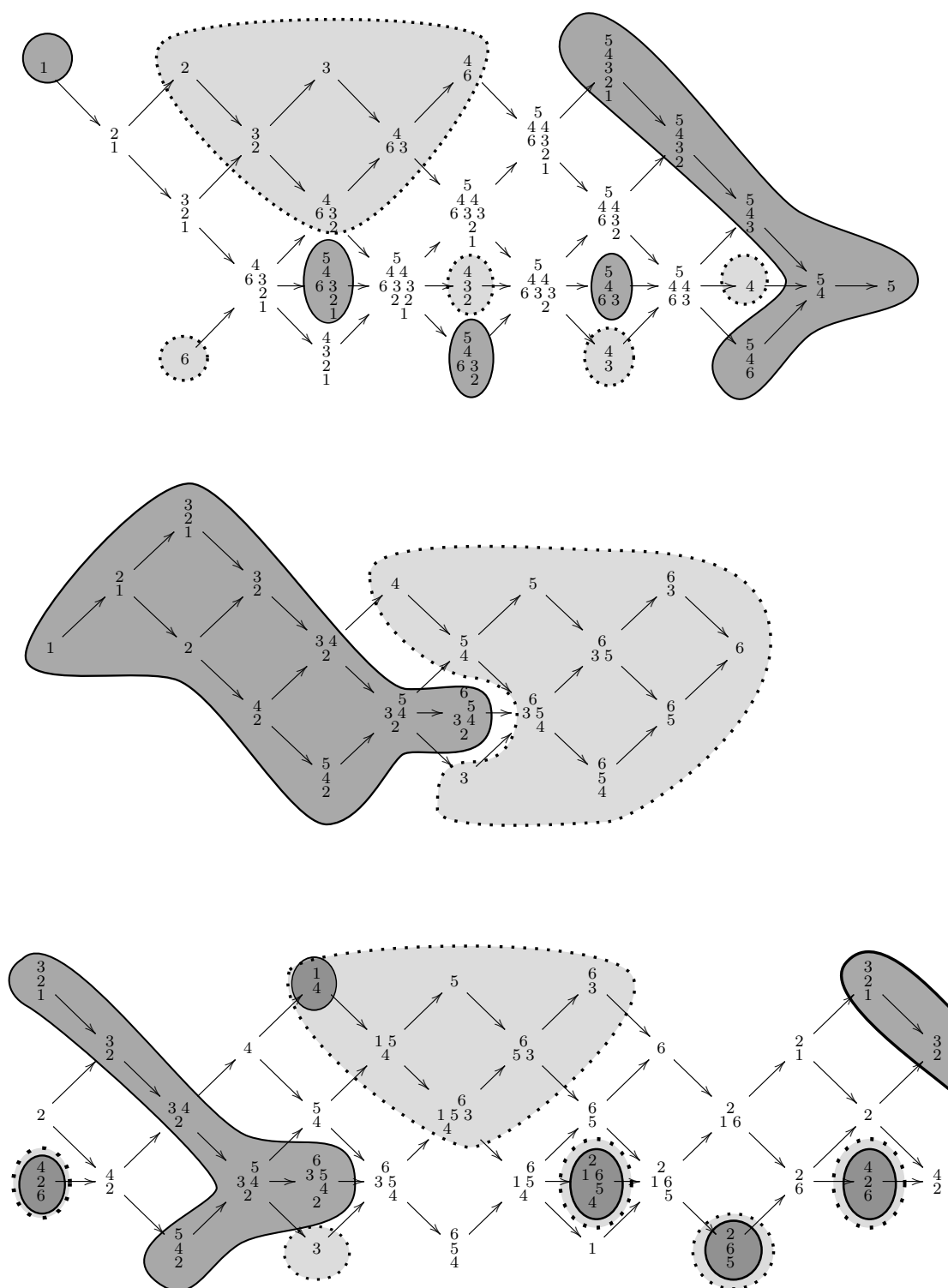Then the corresponding tilted algebra $C = \operatorname{End}_A T$ is the algebra of the following

FIGURE 5.3.1: Auslander-Reiten quivers of $\operatorname{mod} A$ (top), $\operatorname{mod} C$ (middle) and $\operatorname{mod} B$ (bottom).

quiver with relations.

$$1 \xleftarrow{\ \beta\ } 2 \xleftarrow{\ \alpha\ } 4 \xleftarrow{\ \gamma\ } 5 \xleftarrow{\ \delta\ } 6 \qquad\qquad \alpha\beta = 0$$

with σ going from 2 to 3 and ϵ from 6 to 3:

$$\delta\gamma\alpha = \epsilon\sigma.$$

The Auslander-Reiten quiver of $\operatorname{mod} C$ is shown in the middle of Figure 5.3.1. The modules belonging to $\mathcal{Y}(T)$ are in the dark shaded regions enclosed by a solid line, while the modules belonging to $\mathcal{X}(T)$ are in the light shaded regions enclosed by a dotted line.

The cluster-tilted algebra $B = C \ltimes E$ is represented by the following quiver with relations.

$$\alpha\beta = \omega\alpha = 0 \quad \gamma\alpha\rho = \alpha\rho\delta = 0 \quad \sigma\rho = \rho\epsilon = 0$$

$$\beta\omega = \rho\delta\gamma \quad \delta\gamma\alpha = \epsilon\sigma.$$

The Auslander-Reiten quiver of $\operatorname{mod} B$ is shown at the bottom of Figure 5.3.1, where we identify the modules which have the same labels. The modules in the dark shaded regions enclosed by a solid line correspond to the set $\{Y \otimes_C B \mid Y \in \mathcal{Y}(T)\}$, while those in the light shaded regions enclosed by a dotted line correspond to the set $\{D(B \otimes_C DX) \mid X \in \mathcal{X}(T)\}$. Note that there are four modules that lie in both sets. As stated in Theorems 5.1.1 and 5.2.1 the shape and relative position of $\mathcal{T}(T)$ and $\mathcal{F}(T)$ in $\operatorname{mod} A$ correspond exactly to the set of induced $\mathcal{Y}(T)$-modules and the set of coinduced $\mathcal{X}(T)$-modules respectively. Moreover, if we apply τ_B^{-1} to the set

of modules $\{D(B \otimes_C DX) \mid X \in \mathcal{X}(T)\}$, then their position relative to the induced $\mathcal{Y}(T)$ modules will correspond exactly to the position of $\mathcal{F}(T)$ relative to $\mathcal{T}(T)$ in $\bmod A$.

Chapter 6

Which modules are induced or coinduced

If we want to study the module category of a cluster-tilted algebra via the induction and coinduction functors, then it is natural to ask which modules in a cluster-tilted algebra are actually induced or coinduced from the modules over some tilted algebra. Recall from Theorem 1.7.7 that, given a cluster-tilted algebra B, every local slice Σ gives rise to a tilted algebra $C = B/\operatorname{Ann}\Sigma$, whose relation extension is B. Therefore, there are a number of different tilted algebras whose relation extension results in the same algebra B. In general the module category of B is much larger then the module category of the corresponding tilted algebra C. Thus, it is unrealistic to expect that every B-module belongs to the image of the induction or coinduction functor applied to the module category of a single tilted algebra C. However, it makes sense to ask the same question but consider all tilted algebras that give rise to the same B. In this chapter we explore this idea in more detail.

6.1 Transjective component

Definition 6.1.1. Let B be a cluster-tilted algebra and M a B-module.

1. M is said to be *induced from some tilted algebra* if there exists a tilted algebra C and a C-module X such that B is the relation extension of C and $M = X \otimes_C B$.

2. M is said to be *coinduced from some tilted algebra* if there exists a tilted algebra C and a C-module X such that B is the relation extension of C and $M = D(B \otimes_C DM)$.

We begin by looking at algebras of finite representation type.

Theorem 6.1.2. *If B is a cluster-tilted algebra of finite representation type then every indecomposable B-module M is both induced and coinduced from some tilted algebra C, depending on M.*

Proof. If B is of finite representation type and $M \in \operatorname{ind} B$ then by Theorem 1.7.8 there exists a local slice $\Sigma \in \Gamma(\operatorname{mod} B)$ containing M. Theorem 1.7.7 implies that there is a tilted algebra C such that Σ is a slice in $\Gamma(\operatorname{mod} C)$ and B is the relation extension of C. Hence, M is an indecomposable C-module lying on a slice, which means $\operatorname{id}_C M \leq 1$ and $\operatorname{pd}_C M \leq 1$. It follows from Proposition 3.1.2 that $M \cong M \otimes_C B \cong D(B \otimes_C DM)$. Therefore, $M \in \operatorname{ind} B$ is both induced and coinduced from the same module $M \in \operatorname{ind} C$, for some tilted algebra C. □

Let us make the following observation.

Proposition 6.1.3. *Suppose B is a cluster-tilted algebra and M is an indecomposable B-module. If M is also an indecomposable C-module, for some tilted algebra C, whose relation extension is B, then M is induced or coinduced.*

Proof. Observe that if M is a C-module, then Proposition 1.4.6(b) implies that $\mathrm{id}_C M \leq 1$ or $\mathrm{pd}_C M \leq 1$. In the first case $M \cong M \otimes_C B$ and in the second one $M \cong D(B \otimes_C B)$ by Proposition 3.1.2. Hence, in either case M is induced or coinduced from a C-module. $\qquad\square$

Therefore, it makes sense to ask which indecomposable modules over a cluster-tilted algebra are also indecomposable modules over a tilted algebra.

Theorem 6.1.4. *Let B be a cluster-tilted algebra. Then for every transjective indecomposable B-module M, there exists a tilted algebra C, such that B is the relation extension of C, and M is an indecomposable C-module. In particular, every transjective B-module is induced or coinduced from C.*

Proof. We will show that if M is a transjective indecomposable B-module then M or $\tau_B^{-1} M$ lies on a local slice in $\mathrm{mod}\, B$. According to Corollary 3.1.3 this will prove the theorem.

Let A be a hereditary algebra and $T \in \mathcal{C}_A$ a cluster-tilting object such that $B = \mathrm{End}_{\mathcal{C}_A} T$. Let M be an indecomposable B-module lying in the transjective component $\mathcal{T}$ of $\Gamma(\mathrm{mod}\, B)$, and let $\widetilde{M} \in \mathcal{C}_A$ be an indecomposable object such that $\mathrm{Hom}_{\mathcal{C}_A}(T, \widetilde{M}) = M$. Finally, let $\widetilde{\Sigma} = \Sigma(\to M)$ be the full subquiver of the Auslander-Reiten quiver of $\mathcal{C}_A$ defined in section 1.7.

Since $B \cong \mathrm{End}_{\mathcal{C}_A}(\tau_{\mathcal{C}_A}^\ell T)$, for all $\ell \in \mathbb{Z}$, we may assume without loss of generality that $\widetilde{\Sigma}$ lies in the preprojective component of $\mathrm{mod}\, A$. Furthermore, we may assume that every preprojective successor of $\widetilde{\Sigma}$ in $\mathrm{mod}\, A$ is sincere. Indeed, this follows from the fact that there are only finitely many (isoclasses of) indecomposable preprojective A-modules that are not sincere. For tame algebras (see section 6.3) this holds, because non-sincere modules are supported on a Dynkin quiver, and for wild algebras see [30].

Now since $\widetilde{M}$ is a sincere A-module, Proposition 1.7.3 implies that $\widetilde{\Sigma}$ is a slice in $\operatorname{mod} A$, and therefore $\widetilde{\Sigma}$ is a local slice in $\mathcal{C}_A$. Let $\Sigma = \operatorname{Hom}_{\mathcal{C}_A}(T, \widetilde{\Sigma})$. Then $M \in \Sigma$, and thus, if Σ is a local slice in $\operatorname{mod} B$, we are done.

Suppose to the contrary that Σ is not a local slice. Then Lemma 1.7.9 implies that $\widetilde{\Sigma}$ contains an indecomposable summand $\tau_{\mathcal{C}_A} T_i$ of $\tau_{\mathcal{C}_A} T$. Let $\widetilde{\Sigma}' = \Sigma(T_i \to)$ in $\mathcal{C}_A$. Then $\widetilde{\Sigma}'$ contains $\tau_{\mathcal{C}_A}^{-1} \widetilde{M}$. Since T_i is sincere, it again follows from Proposition 1.7.3 that $\widetilde{\Sigma}'$ is a local slice in $\mathcal{C}_A$.

Moreover $\widetilde{\Sigma}'$ cannot contain any summands of $\tau_{\mathcal{C}_A} T$, because if it did, there would be a sectional path from T_i to a summand of $\tau_{\mathcal{C}_A} T$, hence $\operatorname{Hom}_{\mathcal{C}_A}(T_i, \tau_{\mathcal{C}_A} T) \neq 0$, which is impossible, since T is a cluster-tilting object. Therefore Lemma 1.7.9 implies that $\operatorname{Hom}_{\mathcal{C}_A}(T, \widetilde{\Sigma}')$ is a local slice in $\operatorname{mod} B$ containing $\tau_B^{-1} M = \operatorname{Hom}_{\mathcal{C}_A}(T, \tau_{\mathcal{C}_A}^{-1} \widetilde{M})$. $\square$

6.2 Cluster-concealed algebras

Following [34], we say that a cluster-tilted algebra B is *cluster-concealed* if $B = \operatorname{End}_{\mathcal{C}_A}(T)$ where T is obtained from a preprojective A-module. This means that all projective B-modules lie in the transjective component of $\Gamma(\operatorname{mod} B)$. In this case, we show that the previous theorem holds not only for the transjective modules but for all B-modules.

Theorem 6.2.1. *Let B be a cluster-concealed algebra. Then for every indecomposable B-module M there exists a tilted algebra C whose relation extension is B, such that M is an indecomposable C-module. Moreover, for all non-transjective modules one can take the same C. In particular, every B-module is induced or coinduced from some tilted algebra.*

Proof. Observe that by Theorem 6.1.4 it suffices to consider the case when M does not belong to the transjective component of $\operatorname{mod} B$. In this case, B is of infinite representation type.

Because B is cluster-concealed there exists a hereditary algebra A, and a preprojective tilting A-module T such that $B \cong \operatorname{End}_{\mathcal{C}_A}(T)$. Observe that A is of infinite representation type. Let $C = \operatorname{End}_A(T)$ be the corresponding tilted algebra. Then B is the relation extension of C, thus $B = C \ltimes \mathrm{E}$, where $E = \operatorname{Ext}^2_C(DC, C)$. Let $\mathcal{R}_A$ be the set of all regular A-modules. It is nonempty because A is of infinite representation type. It follows from Theorem 1.4.5(a) that $\mathcal{R}_A \in \mathcal{T}(T)$. According to Theorem 1.4.5(b) the set of regular C-modules $\mathcal{R}_C$ is obtained from $\mathcal{R}_A$ by applying the functor $\operatorname{Hom}_A(T, -)$. Also, because of Theorem 1.5.1, the set of regular B-modules $\mathcal{R}_B$ is obtained from $\mathcal{R}_A$ by applying $\operatorname{Hom}_{\mathcal{C}_A}(T, -)$. Hence, given $M \in \mathcal{R}_B$ there exists $\widetilde{M} \in \mathcal{R}_A$ such that $\operatorname{Hom}_{\mathcal{C}_A}(T, \widetilde{M}) \cong M$. Because $\mathcal{R}_A \in \mathcal{T}(T)$, Theorem 5.1.1 implies that $M \cong \operatorname{Hom}_A(T, \widetilde{M}) \otimes_C B$. Hence, M is induced from the indecomposable C-module $\operatorname{Hom}_A(T, \widetilde{M})$, and now it remains to show that M is actually an indecomposable C-module itself, that is $M \cong \operatorname{Hom}_A(T, \widetilde{M})$. It follows from Theorem 1.4.5(c) that $\operatorname{id}_C \operatorname{Hom}_A(T, \widetilde{M}) \leq 1$, so Proposition 3.1.2(a) implies that

$$M \cong \operatorname{Hom}_A(T, \widetilde{M}) \otimes_C B \cong \operatorname{Hom}_A(T, \widetilde{M}).$$

This shows that M is an indecomposable C-module. □

6.3   Tame algebras

A particular class of hereditary algebras whose regular components are well-understood consists of tame hereditary algebras. These algebras have been classified in the following way. We refer to [33, Chapter 1] for further details.

Theorem 6.3.1. *A representation infinite path algebra kQ is tame if and only if the underlying graph of the acyclic quiver Q is a union of extended Dynkin diagrams.*

This means that if we forget about the orientation of the arrows in Q then we obtain a collection of extended Dynkin diagrams. It is known, that if A is a tame hereditary algebra then its regular components $\mathcal{R}_A$ are a collection of stable tubes. We give a precise definition below.

Consider the following infinite quiver $\mathbb{Z}\mathbb{A}_\infty$ with vertices labeled (i, j)
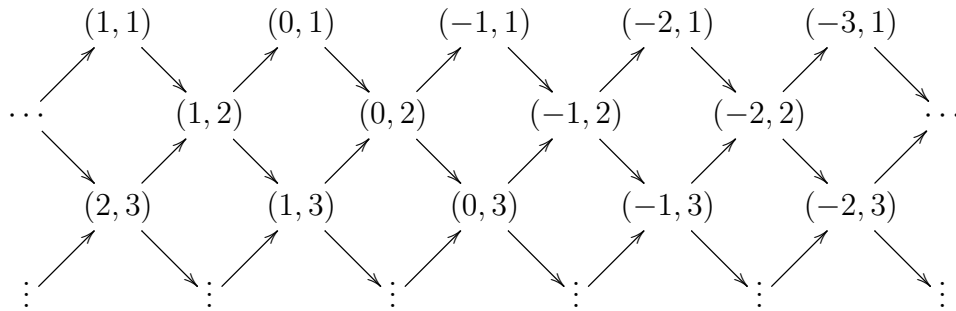
and $\tau(i, j) = (i + 1, j)$ for $i \in \mathbb{Z}$ and $j \geq 1$.

Definition 6.3.2. A *stable tube of rank r* is an infinite quiver isomorphic to the quotient $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$ obtained from $\mathbb{Z}\mathbb{A}_\infty$ by identifying each point (i, j) of $\mathbb{Z}\mathbb{A}_\infty$ with the point $\tau^r(i, j) = (i + r, j)$ and each arrow $\alpha : x \to y$ in $\mathbb{Z}\mathbb{A}_\infty$ with the arrow $\tau^r \alpha : \tau^r x \to \tau^r y$.

The following definitions are of importance to the theory.

Definition 6.3.3. Let $\mathcal{S}$ be a stable tube.

(a) The set of all points in $\mathcal{S}$ having exactly one immediate predecessor (or, equivalently, exactly one immediate successor) is called the *mouth* of $\mathcal{S}$.

(b) Given a point x lying on the mouth of $\mathcal{S}$, a *ray* starting at x is defined to be the unique infinite sectional path in $\mathcal{S}$ with source x.

(c) Given a point x lying on the mouth of $\mathcal{S}$, a *coray* ending at x is defined to be the unique infinite sectional path in $\mathcal{S}$ with target x.

For further details and examples of stable tubes see [36].

We say that a cluster-tiled algebra B is *tame* if $B \cong \operatorname{End}_{\mathcal{C}_A}(T)$, where T is a cluster-tilting object in a cluster category $\mathcal{C}_A$ and A is a tame hereditary algebra. The next result describes a situation when all modules over a cluster-tilted algebra of tame type are induced or coinduced from some tilted algebra.

Theorem 6.3.4. *Let B be a tame cluster-tilted algebra. Let $\mathcal{S}_B$ be a tube in $\operatorname{mod} B$ and let*

$$P_B(1), P_B(2), \ldots, P_B(\ell)$$

denote all distinct indecomposable projective B-modules belonging to $\mathcal{S}_B$. If

$$\operatorname{Hom}_B(P_B(i), P_B(j)) = 0 \text{ for all } i \neq j \text{ and } 1 \leq i, j \leq \ell$$

then every module in $\mathcal{S}_B$ is induced or coinduced from the same tilted algebra C.

Proof. Let A be a hereditary algebra and T a tilting A-module such that $B = \operatorname{End}_{\mathcal{C}_A} T$. Thus A is tame, and we can suppose without loss of generality that T has no preinjective summands. The regular components of $\operatorname{mod} A$ form a family of pairwise orthogonal stable tubes, see [36, Theorem XI.2.8]. The term orthogonal means that

there are no nonzero morphisms between indecomposable modules lying in different tubes.

Let $\mathcal{S}_A$ be the tube in $\operatorname{mod} A$ whose image under $\operatorname{Hom}_{\mathcal{C}_A}(T, -)$ is the tube $\mathcal{S}_B$ in the statement of the theorem. Let r denote the rank of the tube $\mathcal{S}_A$. Let $\mathcal{S}_C$ denote the skew tube in $\operatorname{mod} C$ defined as the image of $\mathcal{S}_A$ under the functor $\operatorname{Hom}_A(T, -)$. Thus $\mathcal{S}_C$ lies inside $\mathcal{Y}(T)$, which implies that each module in $\mathcal{S}_C$ has projective dimension at most one, by Proposition 1.4.6, and therefore the coinduction functor is the identity on $\mathcal{S}_C$, by Proposition 3.1.2.

Theorem 5.1.1 yields the following commutative diagram.

$$
\begin{array}{ccc}
\mathcal{S}_A \cap \mathcal{T}(T) & \xrightarrow{\operatorname{Hom}_A(T,-)} & \mathcal{S}_C \\
 & \searrow{\scriptstyle \operatorname{Hom}_{\mathcal{C}_A}(T,-)} & \downarrow{\scriptstyle -\otimes_C B} \\
 & & \mathcal{S}_C \otimes_C B \subset \mathcal{S}_B.
\end{array}
$$

Let $T_1, T_2, \ldots, T_\ell$ be the indecomposable summands of T that lie inside $\mathcal{S}_A$, such that $P_B(i) = \operatorname{Hom}_{\mathcal{C}_A}(T, T_i)$, for $i = 1, 2, \ldots \ell$, are the indecomposable projective modules in $\mathcal{S}_B$. Because of our assumption $\operatorname{Hom}_A(T_i, T_j) = 0$, if $i \neq j$, we have that each T_i lies on the mouth of $\mathcal{S}_A$, see for example [37, Proposition XII.2.1].

The local configuration in the Auslander-Reiten quiver of $\mathcal{S}_A$ is the following.
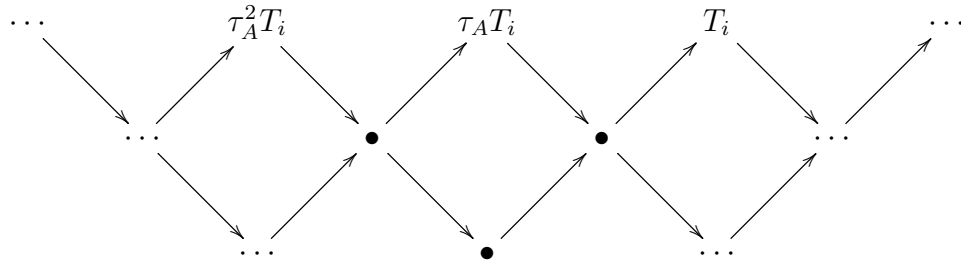
The corresponding local configuration in the Auslander-Reiten quiver of $\mathcal{S}_B$ is the following.
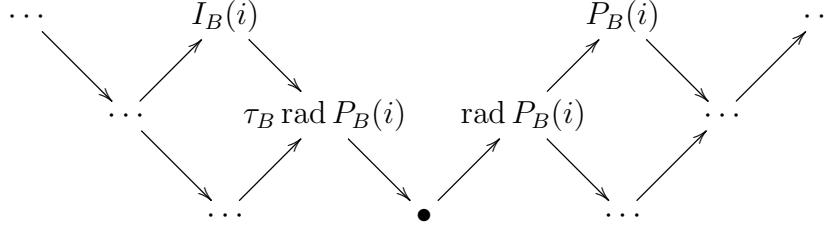
Since $\operatorname{Ext}_A^1(T_i, -) \cong D\operatorname{Hom}_A(-, \tau T_i)$, and τT_i lies on the mouth of $\mathcal{S}_A$, we see that $\mathcal{S}_A \cap \mathcal{T}(T)$ consists of the $r - \ell$ corays not ending in one of the τT_i, $i = 1, 2, \ldots, \ell$. Therefore $\mathcal{S}_C \otimes_C B$ has $r - \ell$ corays, because of our commutative diagram. So these $r - \ell$ corays are in the image of the induction functor $- \otimes_C B$.

Moreover, the ℓ corays in $\mathcal{S}_B$ that are not in the image of the induction functor are precisely those ending in $\tau_B \operatorname{rad} P_B(i)$, $i = 1, 2, \ldots, \ell$. We will show that these corays are equal to the corays in $\mathcal{S}_C$ ending in $\tau_C \operatorname{rad} P_C(i)$, which implies that these corays are coinduced.

Since $I_B(i) = \operatorname{Hom}_{\mathcal{C}_A}(T, \tau_A^2 T_i)$ we have

$$\tau_B \operatorname{rad} P_B(i) = I_B(i)/S(i). \tag{6.3.1}$$

By our assumption on T, we have $\operatorname{Hom}_A(T_i, T) = \operatorname{Hom}_A(T_i, T_i)$ and thus $I_C(i) = S(i)$ is simple. Moreover, by Proposition 2.2.2(b), there is a short exact sequence in mod B of the form

$$0 \longrightarrow I_C(i) \longrightarrow I_B(i) \longrightarrow \tau_C \Omega_C I_C(i) \longrightarrow 0,$$

and thus

$$I_B(i)/S(i) \cong \tau_C \Omega_C I_C(i). \tag{6.3.2}$$

Again using that $I_C(i)$ is simple, we see that

$$\Omega_C I_C(i) = \operatorname{rad} P_C(i). \tag{6.3.3}$$

Combining equations (6.3.1)-(6.3.3) yields

$$\tau_B \operatorname{rad} P_B(i) = \tau_C \operatorname{rad} P_C(i).$$

Finally, let M be any indecomposable C-module on the coray in $\mathcal{S}_C$ ending in $P_C(i)$, but $M \neq P_C(i)$. Then $M \otimes_C B$ lies on the coray in $\mathcal{S}_B$ ending in $P_B(i)$. Lemma 2.2.5 implies that

$$\tau_B(M \otimes_C B) = D(B \otimes_C D(\tau_C M)) = \tau_C M,$$

where the last identity holds because the coinduction functor is the identity on the tube $\mathcal{S}_C$. This shows that the τ_B-translate of the coray in $\mathcal{S}_B$ ending in $P_B(i)$ is equal to the coray in $\mathcal{S}_C$ ending in $\tau_C \operatorname{rad} P_C(i)$. In particular the modules on this coray are coinduced from C.

This finishes the proof of the theorem. $\qquad\square$

6.4   Example

There are cluster-tilted algebras B with indecomposable modules that are not induced and not coinduced from any tilted algebra. We provide an example of this situation below.

Example 6.4.1. Let B be a cluster-tilted algebra given by the following quiver with relations.

$$1 \rightrightarrows 5 \qquad\qquad \alpha\beta = \beta\epsilon = \epsilon\alpha = 0$$

$$\gamma\delta = \delta\sigma = \sigma\gamma = 0.$$
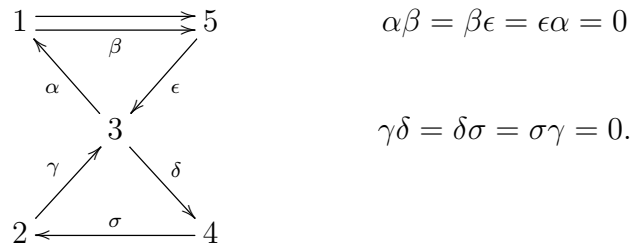
Then there are exactly two tilted algebras C and C', both of infinite representation type, whose relation extension is B. C and C' can be represented by the following quivers with relations.

$$C: \qquad \alpha\beta = 0 \qquad\qquad C': \qquad \beta\epsilon = 0$$

$$\gamma\delta = 0. \qquad\qquad\qquad \gamma\delta = 0.$$
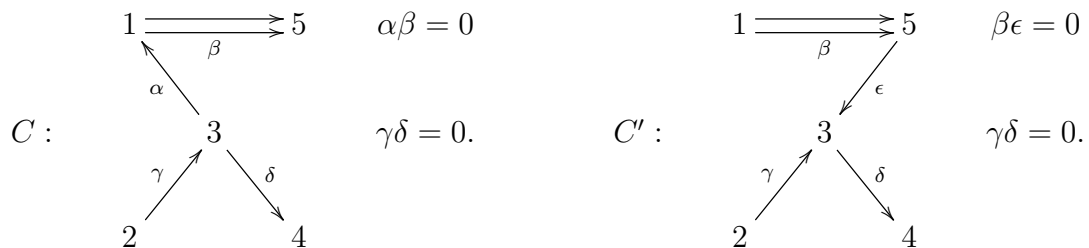
There is a tube in $\Gamma(\operatorname{mod} B)$ of rank four containing three distinct projective modules. We describe it below and identify the modules that have the same label. We will show that the four modules emphasized in bold are not induced and not coinduced.
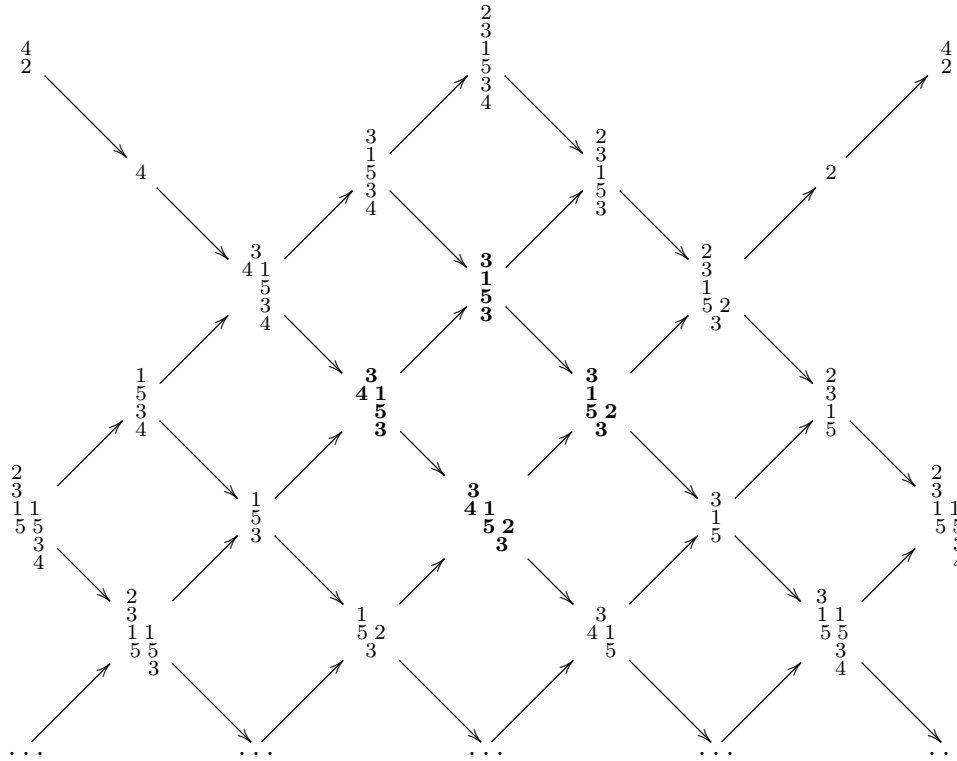
The corresponding skew tube in $\Gamma(\operatorname{mod} C)$ is given below. Observe that it consists of one coray ending in the projective at vertex 2, labeled $\begin{smallmatrix}2\\3\\1\\5\end{smallmatrix}$, and four rays starting in $\begin{smallmatrix}2\\3\\1\\5\end{smallmatrix}, \begin{smallmatrix}3\\1\\5\end{smallmatrix}, 4$ and $\frac{1}{5}$. Let $E = \tau^{-1}\Omega^{-1}C$ and $DE = \tau\Omega DC$, then we make the following computation.

$$E = \tfrac{3}{4} \oplus \tfrac{3}{4} \oplus \tfrac{3}{4} \oplus 2 \oplus \tfrac{3}{4} \qquad\qquad DE = 0 \oplus 4 \oplus \begin{smallmatrix}2\\3\\1\\5\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\\1\\5\end{smallmatrix} \oplus 0$$
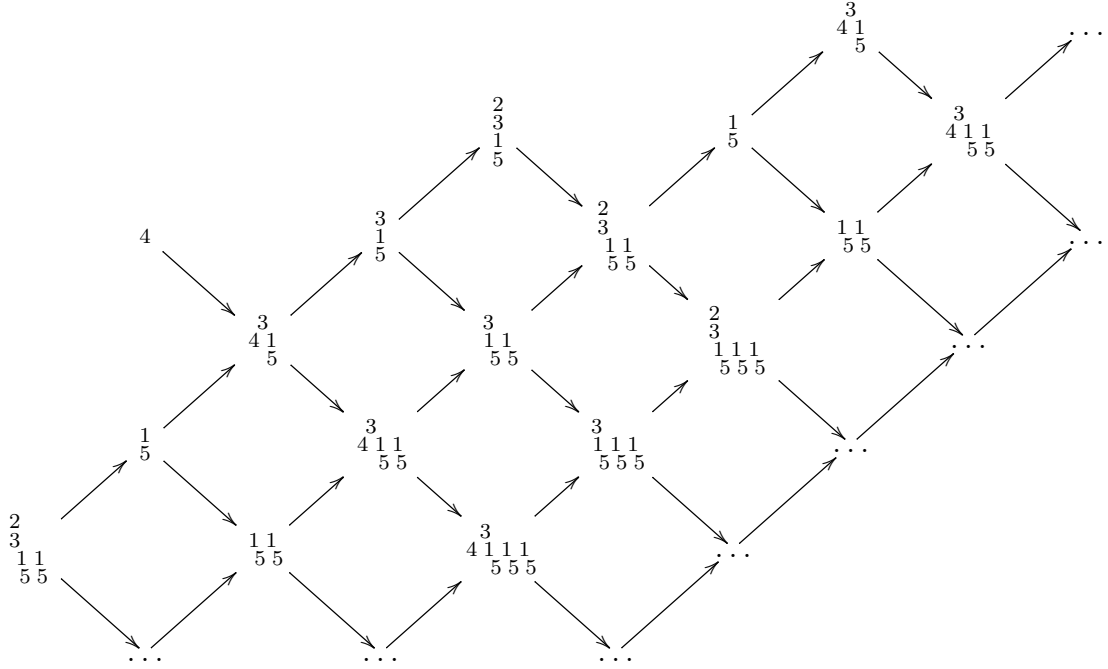
Let $\mathcal{O}_C$ denote the coray in $\Gamma(\operatorname{mod} C)$ ending in the projective at vertex 2. Observe that if we induce the modules in $\mathcal{O}_C$, that is apply $D\operatorname{Hom}_C(-, DB)$, then we obtain a coray in the tube of $\Gamma(\operatorname{mod} B)$ that ends in the projective B-module at vertex 2. Recall that there is an isomorphism of C-modules $D\operatorname{Hom}_C(M, DB) \cong M \oplus D\operatorname{Hom}_C(M, DE)$ for every $M \in \operatorname{mod} C$. On the other hand coinduction $\operatorname{Hom}_C(B, -)$ of modules in $\mathcal{O}_C$ will act as the identity map, because there is a C-module isomorphism $\operatorname{Hom}_C(B, M) \cong M \oplus \operatorname{Hom}_C(E, M)$ and the last summand is zero for all $M \in \mathcal{O}_C$. Therefore, we see the coray $\mathcal{O}_C$ appearing in $\Gamma(\operatorname{mod} B)$. Finally, the only module in the tube above that does not belong to $\mathcal{O}_C$ is the simple at 4. Observe, that this module is also a projective C-module, so inducing it we obtain the corresponding projective B-module $\begin{smallmatrix}4\\2\end{smallmatrix}$.

Similarly, in $\Gamma(\operatorname{mod} C')$ there is a skew tube consisting of four corays and one ray

starting in $\begin{smallmatrix} 1 \\ 5 \\ 3 \\ 4 \end{smallmatrix}$, the injective C'-module at vertex 4. Denote this ray by $\mathcal{R}_{C'}$. Analogous calculations yield that the ray in $\Gamma(\operatorname{mod} B)$ starting in $\begin{smallmatrix} 2 \\ 3 \\ 1 \\ 5 \\ 3 \\ 4 \end{smallmatrix}$, the injective B-module, is coinduced from $\mathcal{R}_{C'}$. But the induction of $\mathcal{R}_{C'}$ acts as the identity map, so we see this exact same ray appearing in $\Gamma(\operatorname{mod} B)$. Observe that every module on this ray is indeed an indecomposable C'-module. Finally, the simple injective C'-module at vertex 2, belongs to this skew tube, and its induction is again the simple module supported at vertex 2.

In particular we observe that the B-modules which make up the mesh emphasized in bold are not induced and not coinduced from any tilted algebra. Moreover, as one goes down the tube in $\Gamma(\operatorname{mod} B)$ there will appear infinitely many such meshes, where no module is induced or coinduced.

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