Hierarchically Organized Minority Games

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Abstract

In this paper a hierarchical extension of the Minority Game is defined and studied. Numerical simulations show a special type of emergent global behavior between separated parts of the hierarchical structure, connected only through a normalized mean field quantity.

Keywords: minority game, hierarchy

1 INTRODUCTION

In this paper a simple, single-level strategic game, namely the Minority Game (MG), will be extended to form a hierarchical organization. The individual MGs will form a group, and individual players within each MG will make decisions which depend on the average behavior of their own and other MGs. Calculations show that under certain conditions the elements of the individual MGs "know" much more about each other than one would expect.

"...Hierarchical organization is a common way to structure a group of people, where members chiefly communicate with their immediate superior and with their immediate subordinates. Structuring organizations in this way is useful partly because it can reduce the communication overhead." [1].

One of the most interesting questions in the theory of hierarchical systems is how much information a subsystem has about the performance of the whole. In addition to study this problem the top-down influence of the "whole" system to the elements will also be quantitatively studied.

2 HIERARCHICAL EXTENSION of MG

MG is a simple model of inductively rational systems [2]. This game is a system of N interacting agents. Each agent brings a decision $a_i\left(s_i(t), \mu(t)\right)$ between $+1$ and

-1 based on her $s_i(t)$ strategy and the knowledge of the history of the game $\mu(t)$. In all simulations each agent chooses from one of two stategies which are randomly chosen for each agent in the beginning of the game. A strategy is a look-up table which assigns the actual choice to the M component vector $\mu(t)$ of the previous M outcome signs, $sgn(A(t))$. $A(t)$ is defined as $A(t) = \sum_i a_i\left(s_i(t), \mu(t)\right)$. Each agent chooses the s^{th} strategy from the possible ones with probability

$$Prob\{s_i\left(t\right) = s\} = Z_i e^{\Gamma_i U_{is}(t)} \tag{1}$$

Where $Z_i^{-1} = \sum_{s'} e^{\Gamma_i U_{is'}(t)}$, Γ_i is the inverse temperature, and the performance of the i^{th} agent's s^{th} strategy is evaluated by a cumulative score $U_{is}(t)$. The updating rule for the evaluation:

$$U_{is}\left(t+1\right) = U_{is}\left(t\right) - a_i\left(s_i(t), \mu(t)\right) A(t) \tag{2}$$

The second term represents the gain of the individual agents.

The original model is extended to form a hierarchical structure. This extension is done in the following way. N' original MGs are connected into a new MG, where the original MGs are the agents of this new game. In this hierarchical organization there are N' ($j = 1...N'$) original MGs, in each of which N ($i = 1...N$) (low level) agents play and where their decision is $a_{ij}(s_{ij}(t), \mu_j(t))$. We use μ_j since each local MG's history contains the local and global minority sign hence it is different for every local MG. For simplicity we will denote $a_{ij}\left(s_i(t), \mu_j(t)\right)$ by $a_{ij}\left(t\right)$.

In each low level MG we have an outcome minority sign, $-sgn(A_j)$. From the set of these signs a global sum and a minority sign, is defined as $A = \sum_j sgn(A_j)$ and $-sgn(A)$. Thus the score updating dynamics of the strategies are modified as:

$$U_{ijs}(t+1) = U_{ijs}(t) - a_{ij}(t)\left(A_j(t) + C\sum_k sgn\left(A_k(t)\right)\right), \tag{3}$$

The constant, C, expresses the ratio of the contribution of the local and global environments. The local environment consists of the agents playing in the original MG, while the global one is derived from the connection of the MGs. The game among the MGs, however, differs from the original. The MGs, as individual agents, don't have strategies and memories explicitly. Instead the global connection between the MGs is made only through their low level agents.

When $C \to 0$ we obtain the original definition. It is often assumed in the theory of hierarchical structures that the local interactions should be much stronger than the global ones [3]. However, the violation of this assumption implies non-trivial results.

In the context of MG for values of C when the global interactions are stronger than the local ones, the agents don't choose a strategy to maximize their own winning chance only, but also to help their whole group win against other groups.

To be able to analyze our extended multilevel model, we will decompose it into three subsystems. Each subsystem contains binary interlayer interactions only, as shown in Fig.(1). The characteristic quantities, namely the global efficiency and the degree of symmetry, will be properly redefined.

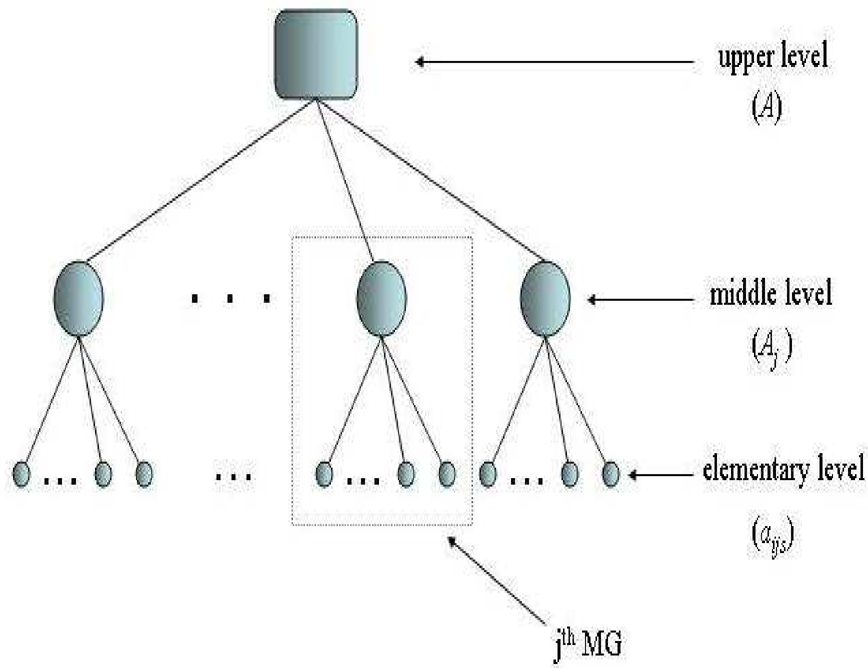

Figure 1: Hierarchical organization of the minority games

3 RESULTS

3.1 Interactions between the elementary and intermediate levels

The new additive part of Eq.(3) is the same for every elementary participant, therefore we expect that the normalized global "waste" σ_j^2/N averaged over the N' MG's (denoted by $< \sigma_j^2/N >_{N'}$) will show the same characteristic pattern as in the original game. A small value of the waste implis a high value of efficiency.

By this extension, however, the total payoff of the j^{th} MG $u_j = -A_j^2 - A_j A$ might be greater than zero, and thus it is no longer a "negative sum game".

Fig.(2) shows the results of the comparative studies between the original and the special features of the extended game by calculating σ_j^2 using the equation

$$\sigma_j^2 \equiv \overline{\langle A_j^2 \rangle} = \sum_{\mu_j=1}^{P_{em}} \rho^{\mu_j} \left\langle \left(\sum_{i=1}^{N} -a_{ij}(t) A_j^{\mu_j}(t) \right)^2 \right\rangle \tag{4}$$

where $P_{em} = 2^M$ is the number of possible memory states, ρ^{μ_j} is the probability of occurrence of the μth state in the jth MG. The $\langle ... \rangle$ refers to the averaging over the possible strategies and the overline stands for the averages over the memory states.

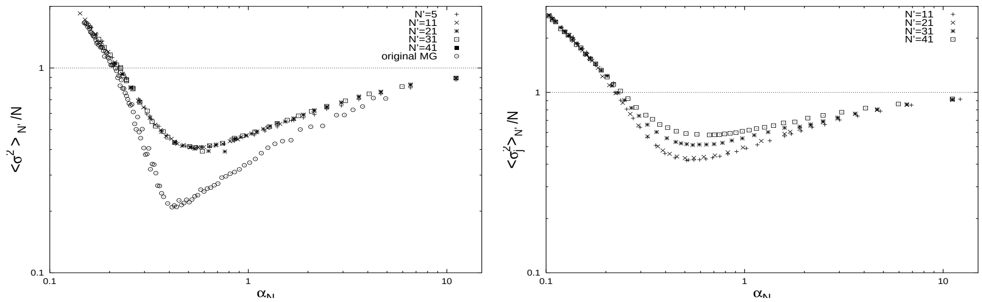

Figure 2: $< \sigma_j^2/N >_{N'}$ as a function of $\alpha_N = P_{em}/N$ for various N's, with $C = 1/N'$ (left) and $C = 1$ (right).$\Gamma_{ij} \sim \infty$, $U_{ijs}(0) = 0$, for $\forall\, i, j, s$.

The left part of Fig. 2 refers to the case of $C = 1/N'$ for various values of N'. Simulations show that the average of σ_j^2/N deviates from that obtained in the original game, but the deviation does not depend on the value of N'. In the right part of Fig.(2) ($C = 1$), the average of σ_j^2/N differs not only from the original characteristics, but is different for each value of N'. It is easy to understand this dependence on N', because the additive part in Eq.(3) does not implement any normalization. (The order of magnitude of the new additive factor is $o(NN')$, while the original part of Eq.(3) is just $o(N)$. For larger values of our new additive part (which is proportional to N') and by the definition of MG we obtain larger values of σ_j^2/N than one would see in the original game.

A more striking result was obtained for the behavior of the symmetry between the two minority signs. Fig.(3) plots the degree of asymmetry (θ_j^2) as a function of α_N (left) and C (right), respectively. The asymmetry is defined by

$$\theta_j = \sqrt{\frac{1}{P_{em}} \sum_{\mu_j=1}^{P_{em}} \langle -sgn(A_j(t))|\mu_j \rangle^2} \tag{5}$$

where $\langle -sgn(A_j(t))|\mu_j \rangle$ is the conditional probability of $-sgn(A_j(t))$ conditioned on μ_j.

Our numerical results show that the system nowhere became symmetrical for the two signs. Indeed in Fig.(3) we can see that for various Cs and N's we obtain the same characteristics reflecting that there is no phase transition in the system with the parameters we investigated. The tendency of the change occurs more slowly than in the original game. The difference appears because our subsystems are not closed because of the interaction between the MGs. Though there exists an ‘optimal strategy’ where the symmetrical phase should appear, it’s efficiency is very low.

3.2 Interactions between the middle and upper levels

Our naive expectation is to find ’random players,’ i.e. players who select their strategies randomly, because the choices of the middle level players are just the results of minority games of the lower level. These ’choices’ cannot be interpreted as the results

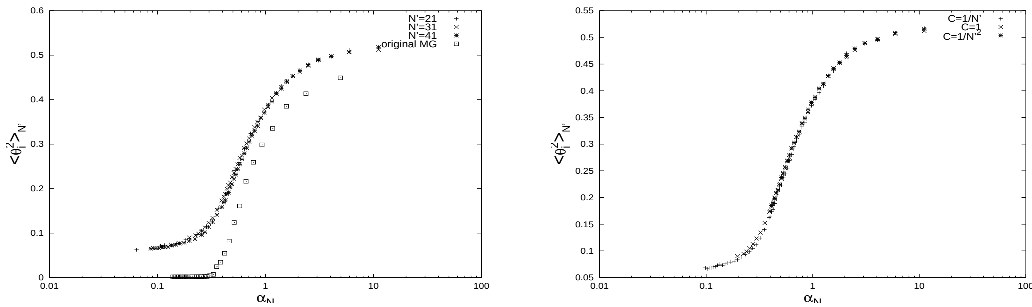

Figure 3: $<\theta_j^2>_{N'}$ as a function of α_N for various N's (left) and Cs (right). $\Gamma_{ij} \sim \infty$, $U_{ijs}(0) = 0$, for $\forall\, i, j, s$.

of 'decisions' because the middle level does not have any direct memories or strategies. They are only mean field averages over the individual MGs. Although the lack of memory implies the break down of the definition of the original scaled parameter, α_N, we still found this quantity to be useful.

We define the waste σ_{mu}^2 in terms of the variation in each of the middle level games (mu is the symbolic notation of "between middle and upper"). Explicitly, we have

$$\sigma_{mu}^2 \equiv \overline{\langle A^2 \rangle} = \sum_{j=1}^{N'} \sum_{\mu_j=1}^{P_{em}} \rho^{\mu_j} \left\langle \left(\sum_{j=1}^{N'} -sgn\left(\sum_{i=1}^{N} a_{ij}(t) A_i^{\mu_j}(t) \right) A(t) \right)^2 \right\rangle \qquad (6)$$

Note that the last term of Eq.(6) depends on N'. The lack of normalization will not have any effect, because we are only interested in the sign of the sum of the local agents' performance.

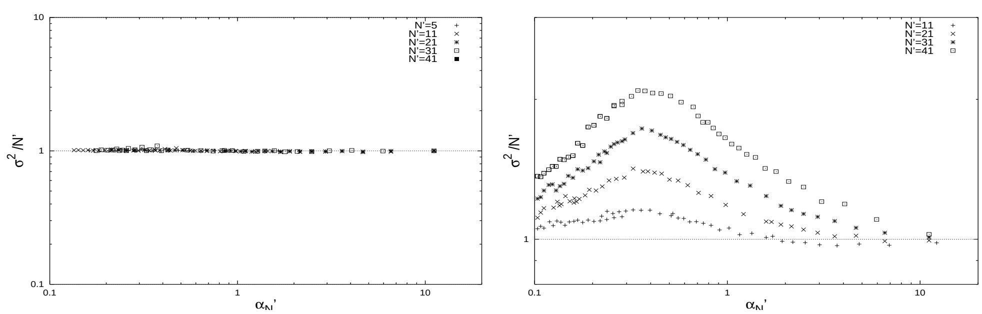

Figure 4: σ_{mu}^2/N' as a function of $\alpha'_N = P/N'$ for various N's, with $C = 1/N'$ (left) and $C = 1$ (right). $\Gamma_{ij} \sim \infty$, $U_{ijs}(0) = 0$, for $\forall\, i, j, s$.

The simulation results reflect the expectations discussed above. For those values of C which implement larger local connections within an MG than to the global one ($C =$

$1/N'$) the system shows random behavior (Fig.4 left). When the global connection gets stronger, i.e. $C = 1$, the system shows an emergent coordination [4], i.e. the system as a whole shows a different than random performance. Fig.(4) (right) shows positive and not negative deviation from the random performance. This phenomenon is the result of the interaction of the two minority games.

3.3 Interactions between the elementary and upper levels

In this case we could ask whether there is any direct connection between the elementary and upper levels. Here we calculated different globally averaged functions σ_{eu} and θ^2_{eu} (where eu the symbolic notation of "between elementary and upper"). We define

$$\sigma^2_{eu} \equiv \overline{\langle A^2 \rangle} = \sum_{\mu=1}^{P_{eu}} \rho^\mu \left\langle \left(\sum_{j=1}^{N'} \sum_{i=1}^{N} a_{ij}(t) A^\mu(t) \right)^2 \right\rangle \tag{7}$$

First, we note that there is now another scaled parameter. For $P_{eu} = (2^{\frac{M}{2}})^{N'+1}$ possible states of the system the new scaled parameter is $\alpha_{N,N'} = \frac{P_{eu}}{NN'}$. And μ refers to the state of the whole system (i.e. $\rho^\mu \in \bigotimes_{j=1}^{N'} \bigoplus_{\mu_j=1}^{P_{em}} \rho^{\mu_j}$).

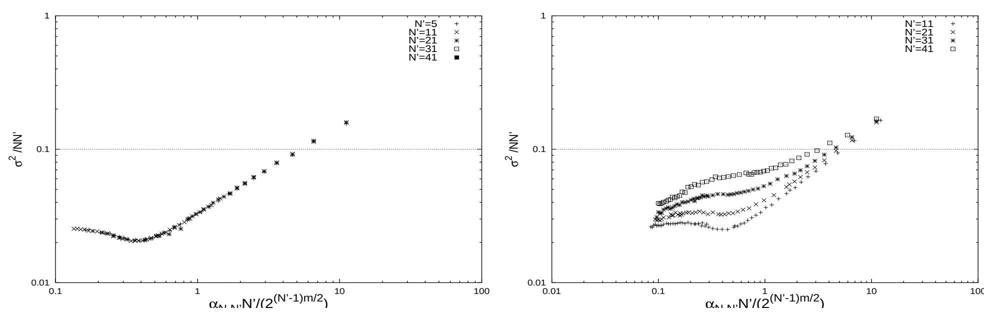

Figure 5: $\sigma^2_{eu}/(NN')$ as a function of $\alpha_{N,N'}$ for various N's, with $C = 1/N$ (left) and $C = 1$ (right). $\Gamma_{ij} \sim \infty$, $U_{ijs}(0) = 0$, for $\forall\, i, j$.

In these numerical simulations (Fig.(5)) the results are qualitatively the same. (Without normalization the value of $\sigma^2_{eu}/(NN')$ increases for greater values of N'.)

Another interesting result of our investigation is the symmetry at this level. One might believe that θ^2_{eu} would not depend on either N' or C, explicitly. However, the simulation result (Fig.6) still shows some dependence on N' and C.

4 Discussion

We studied a hierarchical extension of the MG for three levels and studied the inter-level interactions. First, the local MGs, i.e. the connections between the lower and

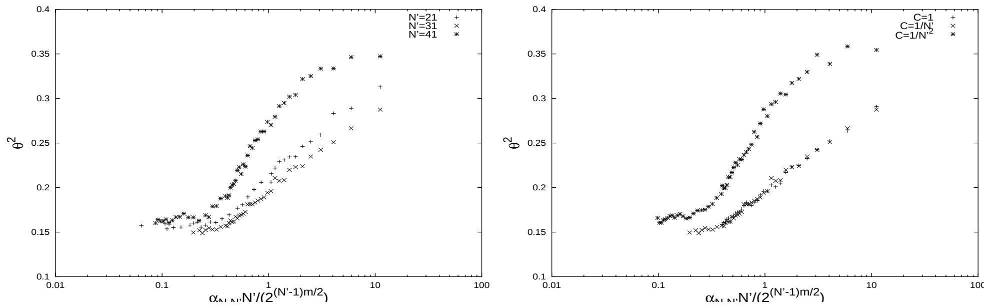

Figure 6: θ^2_{eu} as a function of $\alpha_{N,N'}$ for various values of N' (left) and C (right). $\Gamma_{ij} \sim \infty$, $U_{ijs}(0) = 0$, for $\forall\, i, j, s$.

middle layers were studied. It was interesting to see that for the ratio $C = 1$ the global efficiency depends on the number of individual MGs. By using this formula the individual agents know the number of all MGs. In addition, at least in a broad region of the parameters, the degree of symmetry is never zero. Consequently, in contrast to the original game, there always exists an optimal strategy for winning the game.

Second, the connections between the intermediate and the upper levels were investigated. For the case of relatively strong local connections, the system seems to behave randomly. For strong global coupling the behavior deviates from random behavior, and shows a self-organized global behavior, which can be considered a special type of 'emergent coordination.'

Finally, the results of the study of the connections between the lower and the upper level also exhibits the property seen previously: for the case of $C = 1$ the global efficiency depends on the number of games. In addition, the global efficiency shows interesting dependences both on the values of the ratio number, and on the number of games. In the future we intend to investigate these dependences analytically.

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