

Equilibrium Assignment Model with Uncertainties in Traffic Demands

Aiwu Kuang and Zhongxiang Huang

Department of Traffic and Transportation Engineering, Changsha University of Science and Technology, Changsha 410004, China

Abstract: In this study, we present an equilibrium traffic assignment model considering uncertainties in traffic demands. The link and route travel time distributions are derived based on the assumption that OD traffic demand follows a log-normal distribution. We postulate that travelers can acquire the variability of route travel times from past experiences and factor such variability into their route choice considerations in the form of mean route travel time. Furthermore, all travelers want to minimize their mean route travel times. We formulate the assignment problem as a variational inequality, which can be solved by a route-based heuristic solution algorithm. Some numerical studies on a small test road network are carried out to validate the proposed model and algorithm, at the same time, some reasonable results are obtained.

Keywords: Log-normal distribution, stochastic demand, traffic assignment, user equilibrium

INTRODUCTION

Traffic assignment is an important issue of urban transport planning and has been widely studied by many scholars in the field of transportation, mathematics and physics, such as the user equilibrium principle proposed by Wardrop (1952) and the stochastic user equilibrium principle addressed by Daganzo and Sheffi (1977) are broadly used in transportation planning practices. However, in the past years, the studies on traffic assignment were mainly limited to deterministic road network, which is not sufficient for modeling an actual road network in real world.

In contrast to deterministic network, the essential feature of a stochastic road network is that the link (or route) travel times are stochastic and travelers cannot ensure their actual travel times (Shao *et al.*, 2006a). For transportation network analysis, the variability of travel times could arise from various uncertain factors, which can be divided into two categories, i.e., uncertainties in supplies and uncertainties in demands. Uncertainties due to variations in link capacities are caused from severe natural disasters to daily traffic incidents, such as earthquakes, floods, vehicle stall, road maintenance, traffic accident and so on. On the demand side, uncertainties in traffic demands are usually regarded as the traffic demand fluctuations. In general, the traffic demand between a given OD pair is an aggregate of the trip decisions of individual travelers and will vary between times of day, days of the week and seasons of the year. So far, there has been substantial development in the modeling of traveler's route choice behavior in uncertain road network. Lo *et al.* formulated a travel

time budget-based model to capture the presence of travel time variations due to degradable link capacities (Lo *et al.*, 2006). Shao *et al.* (2006b) established a demand driven travel time reliability-based traffic assignment model, in which the traffic demands were described as normal distribution random variables. More recently, Siu and Lo (2008) developed a methodology to model doubly uncertain transportation network with stochastic link capacity degradation and stochastic demand. These existing studies greatly enriched the theory of uncertain road network equilibrium analysis, at the same time, they also indicated that the conventional traffic assignment model used in deterministic network can no longer applicable in a stochastic environment. In order to obtain more reasonable road network traffic assignment results, in this study, we propose a novel traffic assignment model from the point of view of the day-to-day demand fluctuations. Assume that the OD traffic demand follows a log-normal distribution, the mean and variance of route travel times are determined using the BPR (Bureau of Public Roads) type link impedance function in conjunction with the probability characteristics of log-normal distribution. And then a Variation Inequality (VI) formulation is proposed. A heuristic solution algorithm is developed and some numerical studies are carried out to demonstrate the applications of the proposed model and solution algorithm.

FORMULATION

Route and link flow distributions: Consider a road network represented by a directed graph $G(N, A)$,

where, N is the set of nodes and A is the set of links. Let W be the set of OD pairs and R_w be the set of routes between OD pair w denoted the demand of OD pair w by Q_w , which is treated as a random variable and assumed to follow a log-normal distribution. Let q^w be the mean and ε^w be the variance of OD pair w . Then:

$$Q^w \sim LN(\mu_q^w, \sigma_q^w) \tag{1}$$

where, μ_q^w and σ_q^w are the mean and standard deviation (SD) of $\ln(Q_w)$, which can be computed by:

$$\begin{aligned} \mu_q^w &= \ln(q^w) - \frac{1}{2} \ln \left[1 + \frac{\varepsilon^w}{(q^w)^2} \right] \\ \sigma_q^w &= \left\{ \ln \left[1 + \frac{\varepsilon^w}{(q^w)^2} \right] \right\}^{\frac{1}{2}} \end{aligned} \tag{2}$$

Let σ^w be the SD of the traffic demand between OD pair w define the COV of demand as the ratio of the SD σ^w to the mean q^w as follows:

$$COV^w = \sigma^w / q^w \tag{3}$$

According to the studies in Shao, we also assume that:

- The route flow follows the same type of probability distribution as OD demand.
- The route flow COV is equal to that of OD demand.
- The route flows are mutually independent.

Then, the flow conservation equations can be expressed as:

$$Q^w = \sum_k F_k^w, F_k^w \geq 0, X_a = \sum_w \sum_k F_k^w \delta_{a,k}^w \tag{4}$$

where,

- F_k^w : The traffic flow on route k between OD pair w
- X_a : The flow on link a
- $\delta_{a,k}^w$: The link-route incidence parameter whose value is one if link a is on route k and zero otherwise

It is easy to see that F_k^w and X_a are random variables. Let f_k^w and x_a be the mean route flow and link flow, respectively. It follows from Eq. (4) that:

$$q^w = \sum_k f_k^w, f_k^w \geq 0, x_a = \sum_w \sum_k f_k^w \delta_{a,k}^w \tag{5}$$

According to the assumption (a) and (b), the route flows also follow log-normal distributions:

$$F_k^w \sim LN(\mu_{k,f}^w, \sigma_{k,f}^w) \tag{6}$$

$$\begin{aligned} \mu_{k,f}^w &= \ln(f_k^w) - \frac{1}{2} \ln \left[1 + \frac{\varepsilon_k^w}{(f_k^w)^2} \right] \\ \sigma_{k,f}^w &= \left\{ \ln \left[1 + \frac{\varepsilon_k^w}{(f_k^w)^2} \right] \right\}^{\frac{1}{2}} \end{aligned} \tag{7}$$

$$\varepsilon_k^w = (\sigma^w / q^w)^2 (f_k^w)^2 \tag{8}$$

where, $\mu_{k,f}^w$ and $\sigma_{k,f}^w$ are the the mean and S.D. of $\ln(f_k^w)$, ε_k^w is the variance of the route flow.

According to the route-link relationship in Eq. (4), the distribution parameters μ_a^x and σ_a^x of the random link flows can be approximated as follows:

$$X_a \sim LN(\mu_a^x, \sigma_a^x) \tag{9}$$

$$\begin{aligned} \mu_a^x &= \ln(x_a) - \frac{1}{2} \ln \left[1 + \frac{\varepsilon_a^x}{(x_a)^2} \right] \\ \sigma_a^x &= \left\{ \ln \left[1 + \frac{\varepsilon_a^x}{(x_a)^2} \right] \right\}^{\frac{1}{2}} \end{aligned} \tag{10}$$

where, ε_a^x is the variance of link flow X_a . Based on the assumption (c), ε_a^x can be calculated by:

$$\varepsilon_a^x = \sum_w \sum_k (f_k^w)^2 (\sigma^w / q^w)^2 \delta_{a,k}^w \tag{11}$$

Link and route travel time distributions: In this study, we also adopt the BPR link performance function, which defines the travel time, T_a , under flow X_a using two deterministic parameters α and β as:

$$T_a(X_a) = t_a^0 (1 + \alpha (X_a / c_a)^\beta) \tag{12}$$

where, t_a^0 , c_a are the deterministic free-flow travel time and capacity on link a , respectively.

Using the probability characteristics of log-normal link flow, the mean and variance of the link travel time based on the BPR function can be derived as follows:

$$t_a = t_a^0 + \frac{\alpha t_a^0}{(c_a)^\beta} e^{\beta \mu_a^x + \frac{1}{2} \beta^2 (\sigma_a^x)^2} \tag{13}$$

$$\varepsilon_a^t = \left(\frac{\alpha t_a^0}{(c_a)^\beta} \right)^2 \times \left[e^{2\beta \mu_a^x + \beta^2 (\sigma_a^x)^2} \right] \times \left[e^{\beta^2 (\sigma_a^x)^2} - 1 \right] \tag{14}$$

where, t_a, ε_a^l are the mean and variance of link travel time T_a , respectively.

To facilitate the presentation of the essential ideas, this paper assumes that the link travel times are independent. Hence, using the route-link incidence relationship, the random travel time T_k^w on route k between OD pair w can be simply expressed as the summation of the independent link travel time variables along route k :

$$T_k^w = \sum_a T_a(X_a)\delta_{a,k}^w \quad (15)$$

According to the Central Limit Theorem, in a network with routes consisting of many links, the route travel time follows a normal distribution regardless of the link travel time distributions. Therefore, the route travel time distribution, the mean route travel time t_k^w and the route travel time standard deviation $\sigma_{k,t}^w$ can be expressed as:

$$T_k^w \sim N(t_k^w, (\sigma_{k,t}^w)^2) \quad (16)$$

$$t_k^w = \sum_a \delta_{a,k}^w \left\{ t_a^0 + \frac{\alpha t_a^0}{(c_a)^\beta} e^{\beta \mu_a^s + \frac{1}{2} \beta^2 (\sigma_a^s)^2} \right\} \quad (17)$$

$$\sigma_{k,t}^w = \sqrt{\sum_a \delta_{a,k}^w \left\{ \frac{(\alpha t_a^0)^2}{(c_a)^{2\beta}} \left[e^{2\beta \mu_a^s + \beta^2 (\sigma_a^s)^2} \right] \times \left[e^{\beta^2 (\sigma_a^s)^2} - 1 \right] \right\}} \quad (18)$$

Route choice behavior and VI formulation: In analytical modeling approaches, three criteria, namely User Equilibrium (UE), System Optimum (SO) and Stochastic User Equilibrium (SUE), are frequently used to model travelers' route choice behaviors. In this paper, it is assumed that the deterministic user equilibrium is reachable under stochastic demands with a log-normal distribution. At the equilibrium point, for each OD pair, no travelers can improve their mean route travel time by unilaterally changing routes. In other words, all used routes between each OD pair have equal mean travel time and no unused route has a lower mean travel time. Let vector t be the mean route travel time $(\dots, t_k^w, \dots)^T$, π^w denotes the minimal mean travel time between OD pair w , f denotes the route flow vector $(\dots, f_k^w, \dots)^T$. Then, the UE conditions can be expressed as:

$$t_k^w - \pi^w \begin{cases} = 0 & f_k^w > 0 \\ \geq 0 & f_k^w = 0 \end{cases} \quad (19)$$

Mathematically, the UE conditions can be formulated as an equivalent VI problem as follows.

Find a mean route flow vector $f^* \in \Omega$, such that:

$$t^T(f^*)(f - f^*) \geq 0, \forall f \in \Omega \quad (20)$$

where, the superscript “*” is used to designate the solution of the variational inequality problem; Ω represents the feasible set for the mean route flow vector defined by Eq. (5).

Solution algorithm: In this section, a solution algorithm is developed. It should be noted that a number of well-known algorithms developed for traditional traffic assignment problems, such as Frank-Wolfe algorithm cannot be directly applied to solve model 20 due to the non-additive property of route travel times. Under this circumstance, a route-based algorithm is available for solving the proposed VI model. It is assumed that during the process of traffic assignment, the assignment route set is fixed all along, which can assure the convergence of the algorithm. The step-by-step procedure of this algorithm is given below:

- Step 0: Initialization:** Define the stopping tolerance, set iteration counter $n = 0$, determine the assignment route set with column generation algorithm, generate an initial route flow pattern $f^{(0)}$ by free flow travel time.
- Step 1: Update:** Update the mean route travel time vector $t^{(n)}$ based on the current route flow $f^{(n)}$.
- Step 2: Direction finding:** Perform the all or nothing assignment in terms of the updated mean route travel time $t^{(n)}$ and get the auxiliary route flow $g^{(n)}$.
- Step 3: Move:** Find a new route flow pattern based on the Method of Successive Averages (MSA), namely, $f^{(n+1)} = f^{(n)} + (g^{(n)} - f^{(n)}) / (n+1)$.
- Step 4: Convergence check:** If a certain equilibrium criterion is reached, then stop and report the solution; otherwise, set $n = n + 1$ and go to Step 1.

NUMERICAL STUDIES

A simple example road network given in Fig. 1 consists of four nodes, five links and one OD pair (from node 1 to node 4). For OD pair Q^{14} , the

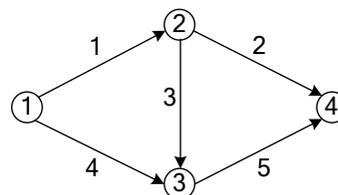


Fig. 1: Example network

Table 1: Link free flow travel time and link capacity

Link no.	Free flow time (min)	Capacity (pcu/h)
1	5	600
2	12	400
3	6	400
4	10	400
5	8	600

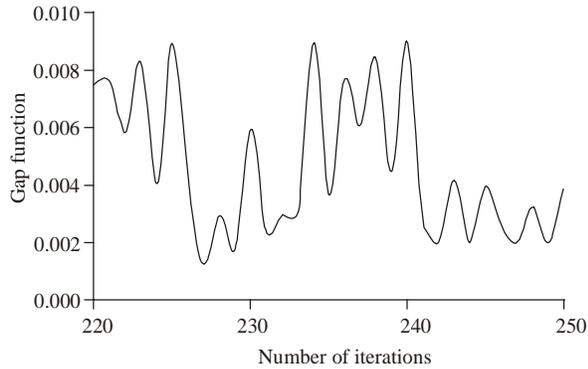


Fig. 2: The convergence of the proposed solution algorithm

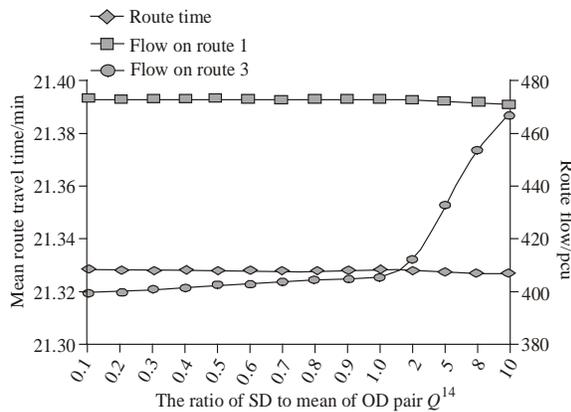


Fig. 3: Equilibrium travel time and flow with different COV

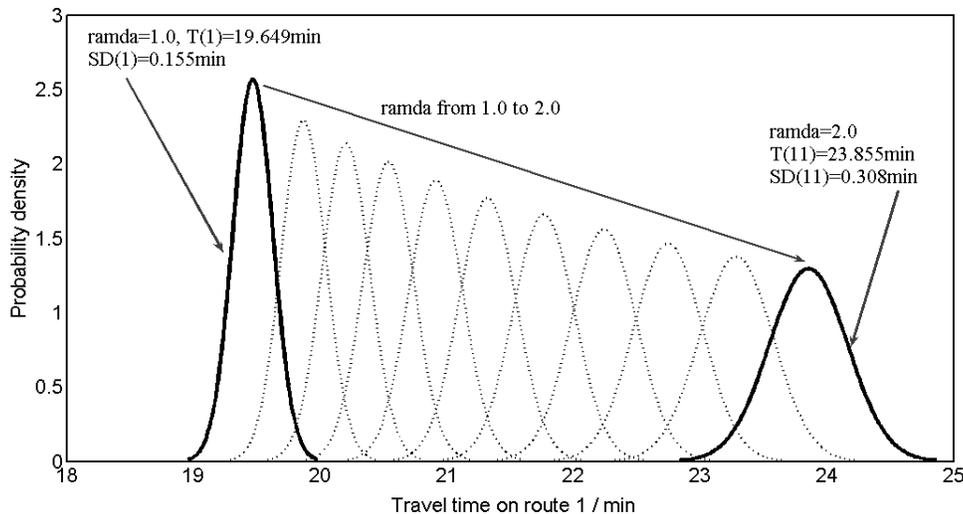


Fig. 4: Travel time probability density of route 1

three associated routes given in link sequence are as follows: route 1 (link 1 → 2), route 2 (link 1 → 3 → 5) and route 3 (link 4 → 5).

It is assumed that the demand of OD pair Q^{14} follows log-normal distribution with mean demand $q^{14} = 1000$ (pcu/h). The link travel time function adopted in this study is the BPR function Eq. (12) with parameters $\alpha = 0.2$ and $\beta = 2$. The network characteristics, including link free-flow travel times and link capacities are given in Table 1.

Firstly, we study the convergence of the proposed solution algorithm. Figure 2 depicts the values of an equilibrium criterion against iteration number with a predetermined $COV^{14} = 0.1$. The adopted equilibrium criterion is defined as a gap function as follows:

$$G = \sum_w \sum_k f_k^{w(n)} (t_k^{w(n)} - t_{min}^{w(n)}) \quad (21)$$

Figure 3 displays the effects of the fluctuation of traffic demand on the mean route travel time and route flow pattern in the network. It can be found that when COV^{14} increases, the equilibrium mean route travel time significantly increases, especially for a relative larger COV. The result is consistent with our anticipation, that is, with the increase of COV, travelers should expend more travel time in order to confront a more uncertain travel environment. In addition, Fig. 3 also shows that travelers will switch from route 1 and route 3 to route 2 with the increasing of COV^{14} . This phenomenon uncertain transportation network.

Finally, we consider the impacts of traffic demand levels on route travel times. First, the base demand of OD pair (1, 4) is set to be $q^{14} = 3000/2$ (pcu/h) and the COV^{14} equals to 0.5. To simplify the presentation, here, we only examine the travel time on route 1 under different demand levels: $q = \lambda \cdot q^{14}$, where, $\lambda = (1.0, 1.1,$

..., 1.9, 2.0) is the demand multiplier that represents the demand level from 1 to 11, respectively. Figure 4 shows the probability density functions of the travel time distribution on route 1 with respect to different demand levels. From the figure, we can conclude that a higher demand will lead to an increase of the mean and variance of the route travel time (and hence the route flow). Therefore, Fig. 4 indeed verifies that the route flow is actually random while taking the stochastic demand into account.

CONCLUSION

We have proposed a modeling approach to study the traffic assignment problem in a transportation system with uncertainties in traffic demands. Due to the day-to-day traffic demand fluctuations, the link and route travel times should be considered as random variables. In this study, we treat the demand of each OD pair as a log-normal distributed random variable, the mean and variance of the link travel time is derived based on the moments of log-normal distribution. In contrast to the conventional user equilibrium, we use the mean route time as a new route choice criterion. The model is formulated as a variational inequality problem and solved by the method of successive averages in conjunction with a column generation procedure. Some numerical analyses of applying the proposed modeling approach and algorithm in a small test network have been carried out to demonstrate its applicability and efficiency. Further studies on this problem include extending the model to account for travelers' perception errors to establish a SUE model and applying the proposed modeling approach into real networks.

ACKNOWLEDGMENT

This research is supported in part by the National Natural Science Foundation of China under grant number 51078044, the Natural Science Foundation of Hunan Province under grant number 10JJ6074 and the Education Department of Hunan Province under research grant 11B004.

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