

STABLE CATEGORIES OF COHEN-MACAULAY MODULES AND CLUSTER CATEGORIES

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Dedicated to Ragnar-Olaf Buchweitz on the occasion of his sixtieth birthday

ABSTRACT. By Auslander's algebraic McKay correspondence, the stable category of Cohen-Macaulay modules over a simple singularity is equivalent to the 1-cluster category of the path algebra of a Dynkin quiver (i.e. the orbit category of the derived category by the action of the Auslander-Reiten translation). In this paper we give a systematic method to construct a similar type of triangle equivalence between the stable category of Cohen-Macaulay modules over a Gorenstein isolated singularity R and the generalized (higher) cluster category of a finite dimensional algebra Λ . The key role is played by a bimodule Calabi-Yau algebra, which is the higher Auslander algebra of R as well as the higher preprojective algebra of an extension of Λ . As a byproduct, we give a triangle equivalence between the stable category of graded Cohen-Macaulay R -modules and the derived category of Λ . Our main results apply in particular to a class of cyclic quotient singularities and to certain toric affine threefolds associated with dimer models.

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INTRODUCTION

There has recently been a lot of interest centered around **Hom**-finite triangulated Calabi-Yau categories over a field k , especially in dimension two. The work on 2-Calabi-Yau categories was originally motivated by trying to categorify the ingredients in the definition of the cluster algebras introduced by Fomin and Zelevinsky [FZ02]. It started in [BMR+06] through the cluster categories together with a special class of objects called cluster tilting objects, and in [GLS06, BIRS09, GLS07, IO09] through the investigation of preprojective algebras and their higher analogs.

The generalized n -cluster categories associated with finite dimensional algebras of global dimension at most n were introduced in [Ami09, Guo10]. In these categories, special objects called n -cluster tilting play an important role. The cluster categories are a special case of the generalized 2-cluster categories, and the 2-cluster tilting objects are then the cluster tilting objects. The generalized n -cluster categories can be considered to be the canonical ones among n -Calabi-Yau triangulated categories having n -cluster tilting objects.

On the other hand, a well-known example of Calabi-Yau triangulated categories was given in old work by Auslander [Aus78], where the stable category of (maximal) Cohen-Macaulay modules over commutative isolated d -dimensional local Gorenstein singularities are shown to be $(d-1)$ -Calabi-Yau. Recently they are studied from the viewpoint of higher analog of Auslander-Reiten theory, and the existence of $(d-1)$ -cluster tilting objects is shown for quotient singularities in [Iya07a] and for some three dimensional hypersurface singularities in [BIKR08]. They are further investigated in [IY08, KR08, KMV11].

It is of interest to understand the relationship between these two classes of Calabi-Yau triangulated categories, i.e. the stable categories of Cohen-Macaulay modules and the generalized n -cluster categories. A well-known example is given by Kleinian singularities. They are given as hypersurfaces $R = k[x, y, z]/(f)$ as well as invariant subrings $R = S^G$ of G , where $S = k[X, Y]$ is a polynomial algebra over an algebraically closed field k of characteristic zero and G is a finite subgroup of $\mathrm{SL}_2(k)$. The correspondence between f

and G is given as follows.

type	A_n	D_n	E_6	E_7	E_8
f	$x^{n+1} + yz$	$x^{n-1} + xy^2 + z^2$	$x^4 + y^3 + z^2$	$x^3y + y^3 + z^2$	$x^5 + y^3 + z^2$
G	cyclic	binary dihedral	binary tetrahedral	binary octahedral	binary icosahedral

In this case the stable category $\underline{\mathbf{CM}}(R)$ is equivalent to the mesh category $M(\overline{Q})$ of the Auslander-Reiten quiver of $\underline{\mathbf{CM}}(R)$, which is the double \overline{Q} of a Dynkin quiver Q [Rei87, RV89]. On the other hand, $M(\overline{Q})$ is equivalent to the 1-cluster category $\mathcal{C}_1(kQ)$ of Q , i.e. the orbit category $\mathcal{D}^b(kQ)/\tau$ of the derived category $\mathcal{D}^b(kQ)$ by the action of τ . Hence we can deduce an equivalence

$$(0.0.1) \quad \underline{\mathbf{CM}}(R) \simeq \mathcal{C}_1(kQ).$$

One of the aims of this paper is to prove this type of equivalence for a more general class of quotient singularities. Some crucial observations in the above setting are the following:

- [Her78, Aus86] R is representation-finite in the sense that there are only finitely many indecomposable Cohen-Macaulay modules. More precisely $\mathbf{CM}(R) = \text{add } S$ holds.
- [Aus86] The Auslander algebra $\text{End}_R(S)$ is isomorphic to the skew group algebra $S * G$. In particular, the AR quiver of $\mathbf{CM}(\hat{R})$ (where \hat{R} is the completion of the algebra R) is isomorphic to the McKay quiver of G , which is the double of an extended Dynkin quiver \tilde{Q} . (Note that in this case one has an equivalence between the stable categories $\underline{\mathbf{CM}}(R) \simeq \underline{\mathbf{CM}}(\hat{R})$)
- [Rei87, RV89, BSW10] $S * G$ is Morita-equivalent to the preprojective algebra Π of \tilde{Q} . Hence $k\tilde{Q}$ is the degree zero part of a certain grading of Π .

In particular the equivalence (0.0.1) is a direct consequence of the above observations. Also we have the following bridge between R and kQ , where e is the idempotent of $\text{End}_R(S) \simeq S * G$ corresponding to the summand R of S :

$$R \begin{array}{c} \xrightarrow{\text{Auslander algebra}} \\ \xleftarrow{e(-)e} \end{array} S * G \begin{array}{c} \xrightarrow{\text{Morita}} \\ \xleftarrow{\text{preprojective algebra}} \end{array} \Pi \begin{array}{c} \xrightarrow{\text{degree 0 part}} \\ \xleftarrow{\text{preprojective algebra}} \end{array} k\tilde{Q} \xrightarrow{-/(e)} kQ$$

We will deal with the more general class of quotient singularities S^G , where $S = k[x_1, \dots, x_d]$ and G is a finite cyclic subgroup of the special linear subgroup $\mathbf{SL}_d(k)$ with additional conditions, where no $g \neq 1$ has eigenvalue 1. We will construct in Theorem 5.1 a triangle equivalence

$$(0.0.2) \quad \underline{\mathbf{CM}}(S^G) \simeq \mathcal{C}_{d-1}(\underline{A})$$

for the generalized $(d-1)$ -cluster category $\mathcal{C}_{d-1}(\underline{A})$ of some algebra \underline{A} of global dimension at most $d-1$, which we describe. This is shown as a special case of our main Theorem 4.1. There we start from a bimodule d -Calabi-Yau graded algebra B of Gorenstein parameter 1 (e.g. B is the skew group algebra $S * G$ when we deal with quotient singularities with additional conditions). For an idempotent e satisfying certain axioms, we have a similar

picture as above:

$$eBe \begin{array}{c} \xrightarrow{(d-1)\text{-Auslander algebra}} \\ \xleftarrow{e(-)e} \end{array} B \begin{array}{c} \xleftarrow{\text{degree 0 part}} \\ \xrightarrow{d\text{-preprojective algebra}} \end{array} B_0 \xrightarrow{-/\langle e \rangle} B_0/\langle e \rangle$$

Our main result asserts that there exists a triangle equivalence

$$\underline{\mathbf{CM}}(eBe) \simeq \mathcal{C}_{d-1}(B_0/\langle e \rangle).$$

In addition to the quotient singularities already mentioned, this also applies to some examples coming from dimer models.

The main step of the proof consists of constructing a triangle equivalence

$$\underline{\mathbf{CM}}^{\mathbb{Z}}(eBe) \simeq \mathcal{D}^b(B_0/\langle e \rangle)$$

where $\underline{\mathbf{CM}}^{\mathbb{Z}}(eBe)$ is the category of graded Cohen-Macaulay eBe -modules. This intermediate result in the case where $B = S * G$ recovers a result due to Kajiura-Saito-Takahashi [KST07] and Lenzing-de la Peña [LP11] for $d = 2$ and due to Ueda [Ued08] for any d and G cyclic. Moreover the equivalence (0.0.2) was already shown in [KR08] for the case $d = 3$ and $G = \text{diag}(\omega, \omega, \omega)$ where ω is a primitive third root of unity. It would be interesting to generalize our result to non-cyclic quotient singularities. This could then be regarded as an analog of a triangle equivalence $\underline{\mathbf{CM}}^{\mathbb{Z}}(S^G) \simeq \mathcal{D}^b(\Lambda)$ for some finite dimensional algebra Λ given in [IT10].

Results of a similar flavor have been shown in previous papers. In [Ami09, ART11, AIRT12], it was shown that the 2-Calabi-Yau categories \mathcal{C}_w associated with elements w in Coxeter groups in [BIRS09] are triangle equivalent to generalized 2-cluster categories $\mathcal{C}_2(\underline{A})$ for some algebras \underline{A} of global dimension at most two. In [IO09], it was shown that the stable categories of modules over d -preprojective algebras of $(d-1)$ -representation-finite algebras are triangle equivalent to generalized d -cluster categories of stable $(d-1)$ -Auslander algebras. We were able to use some of the ideas in these papers for $d \geq 2$.

We refer to [TV10] for similar independent results based on the language of quivers with potential. We thank Michel Van den Bergh for informing us about his work with Thanhoffer de Völcséy.

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In section 1 we give some background material on n -cluster tilting subcategories in n -Calabi-Yau categories and on generalized n -cluster categories. Let B be a bimodule d -Calabi-Yau algebra (see Definition 2.1) with an idempotent e , and let $C = eBe$. In section 2, under certain conditions on B and e , we show that C is an Iwanaga-Gorenstein algebra (see Definition 1.1), and that Be is a $(d-1)$ -cluster tilting object in the category $\mathbf{CM}(C)$ of Cohen-Macaulay C -modules. In section 3, which is independent of section 2, we assume that $B = \bigoplus_{\ell \geq 0} B_\ell$ is graded, and give sufficient conditions for B to be the d -preprojective algebra of $A = B_0$. In particular A is a $(d-1)$ -representation-infinite algebra in the sense of [HIO12] and a quasi extremely-Fano algebra in the sense of [MM10]. In section 4, we use results from sections 2 and 3 to prove our main result, which gives

sufficient conditions for the stable category $\underline{\mathbf{CM}}(C)$ to be triangle equivalent to a generalized $(d - 1)$ -cluster category. The application to C being an invariant ring is given in section 5. In Section 6 we apply our main result to Jacobian algebras constructed from dimer models on the torus.

Notation. Let k be a field. We denote by $D = \text{Hom}_k(-, k)$ the k -dual. All modules are right modules.

For a k -algebra A , we denote by $\text{Mod } A$ the category of A -modules, by $\text{mod } A$ the category of finitely generated A -modules and by $\text{fd } A$ the category of finite dimensional A -modules. We let $\otimes := \otimes_k$ and $A^e := A^{\text{op}} \otimes A$. For a \mathbb{Z} -graded k -algebra B , we denote by $\text{Gr } B$ the category of all \mathbb{Z} -graded B -modules, by $\text{gr } B$ the category of finitely generated \mathbb{Z} -graded B -modules and by $\text{grproj } B$ the category of finitely generated \mathbb{Z} -graded projective B -modules. We often regard B^e in a natural way as a \mathbb{Z} -graded algebra, and consider the category $\text{Gr } B^e$ of \mathbb{Z} -graded B^e -modules.

For an abelian category \mathcal{A} , we denote by $\mathcal{C}(\mathcal{A})$ the category of chain complexes, by $\mathcal{K}(\mathcal{A})$ the homotopy category and by $\mathcal{D}(\mathcal{A})$ the derived category. We denote by $\mathcal{C}^b(\mathcal{A})$ the category of bounded chain complexes, by $\mathcal{K}^b(\mathcal{A})$ the bounded homotopy category and by $\mathcal{D}^b(\mathcal{A})$ the bounded derived category.

For a k -algebra A , we let $\mathcal{D}(A) := \mathcal{D}(\text{Mod } A)$. We denote by $\text{per } A$ the thick subcategory of $\mathcal{D}(A)$ generated by A . We denote by $\mathcal{D}^{\text{fd}}(A)$ the full subcategory of $\mathcal{D}(A)$ consisting of objects X satisfying $\dim_k(H^*(X)) < \infty$. For a noetherian k -algebra A , we denote by $\mathcal{D}^b(A)$ the full subcategory of $\mathcal{D}(A)$ consisting of objects X satisfying $H^*(X) \in \text{mod } A$.

We denote by gf the composition of morphisms (or arrows) $f : X \rightarrow Y$ and $g : Y \rightarrow Z$.

1. BACKGROUND MATERIAL

In this section we give some background material on cluster tilting subcategories and on generalized cluster categories.

1.1. Cohen-Macaulay modules over Iwanaga-Gorenstein algebras. The following class of noetherian algebras was given by Iwanaga [Iwa79].

Definition 1.1. A noetherian algebra C is called *Iwanaga-Gorenstein* if $\text{inj.dim}_C C < \infty$ and $\text{inj.dim}_{C^{\text{op}}} C < \infty$.

For example, commutative local Gorenstein algebras and finite dimensional selfinjective algebras are clearly Iwanaga-Gorenstein. Iwanaga-Gorenstein algebras have a distinguished class of modules defined as follows.

Definition 1.2. Let C be an Iwanaga-Gorenstein algebra. The category $\mathbf{CM}(C)$ of (*maximal*) *Cohen-Macaulay C -modules* is defined by

$$\mathbf{CM}(C) := \{X \in \text{mod } C \mid \text{Ext}_C^i(X, C) = 0 \text{ for any } i > 0\}.$$

The *stable category* $\underline{\mathbf{CM}}(C)$ has the same objects as $\mathbf{CM}(C)$, and the morphisms spaces are given by

$$\text{Hom}_{\underline{\mathbf{CM}}(C)}(X, Y) := \text{Hom}_C(X, Y) / [C](X, Y)$$

where $[C](X, Y)$ consists of morphisms factoring through the smallest full subcategory $\text{add } C$ of $\text{mod } C$ stable under direct summands and containing C .

If C is a local commutative Gorenstein algebra, then $\mathbf{CM}(C)$ is exactly the category of maximal Cohen-Macaulay C -modules. If C is a finite dimensional selfinjective algebra, then $\mathbf{CM}(C)$ is just $\text{mod } C$.

Let us give basic properties of the category $\mathbf{CM}(C)$.

Proposition 1.3. *Let C be an Iwanaga-Gorenstein algebra.*

- (a) $\mathbf{CM}(C)$ is a Frobenius category and $\underline{\mathbf{CM}}(C)$ is a triangulated category [Hap88, Thm 2.6].
- (b) We have dualities $\mathbf{CM}(C) \begin{array}{c} \xrightarrow{\text{Hom}_C(-,C)} \\ \xleftarrow{\text{Hom}_{C^{\text{op}}}(-,C)} \end{array} \mathbf{CM}(C^{\text{op}})$ which are mutually quasi-inverse and preserve the extension groups.
- (c) We have a triangle equivalence $\underline{\mathbf{CM}}(C) \simeq \mathcal{D}^b(C)/\text{per } C$. [Buc87, Thm 4.4.1],[KV87, Ric89]

When an Iwanaga-Gorenstein algebra C is a \mathbb{Z} -graded algebra, the category $\mathbf{CM}^{\mathbb{Z}}(C)$ of graded Cohen-Macaulay C -modules is defined by

$$\mathbf{CM}^{\mathbb{Z}}(C) := \{X \in \text{gr } C \mid \text{Ext}_C^i(X, C) = 0 \text{ for any } i > 0\}.$$

Then the stable category $\underline{\mathbf{CM}}^{\mathbb{Z}}(C)$ is defined similarly as above.

We have the following parallel results.

Proposition 1.4. *Let C be a \mathbb{Z} -graded Iwanaga-Gorenstein algebra.*

- (a) $\mathbf{CM}^{\mathbb{Z}}(C)$ is a Frobenius category and $\underline{\mathbf{CM}}^{\mathbb{Z}}(C)$ is a triangulated category.
- (b) We have dualities $\mathbf{CM}^{\mathbb{Z}}(C) \begin{array}{c} \xrightarrow{\text{Hom}_C(-,C)} \\ \xleftarrow{\text{Hom}_{C^{\text{op}}}(-,C)} \end{array} \mathbf{CM}^{\mathbb{Z}}(C^{\text{op}})$ which are mutually quasi-inverse and preserve the extension groups.
- (c) We have a triangle equivalence $\underline{\mathbf{CM}}^{\mathbb{Z}}(C) \simeq \mathcal{D}^b(\text{gr } C)/\text{gr per } C$.

1.2. d -Calabi-Yau categories and d -cluster tilting objects.

Definition 1.5. A k -linear triangulated category \mathcal{T} is said to be d -Calabi-Yau if it is Hom-finite and there is a functorial isomorphism

$$\text{Hom}_{\mathcal{T}}(X, Y) \simeq D\text{Hom}_{\mathcal{T}}(Y, X[d]) \quad \text{for all } X, Y \in \mathcal{T}.$$

Definition 1.6. [BMR+06],[Iya07a, 2.2],[KR07, 2.1] A d -cluster tilting subcategory \mathcal{V} in a triangulated category \mathcal{T} is a functorially finite subcategory of \mathcal{T} such that

$$\begin{aligned} \mathcal{V} &= \{X \in \mathcal{T}, \text{Hom}_{\mathcal{T}}(X, \mathcal{V}[i]) = 0, \forall 1 \leq i \leq d-1\} \\ &= \{X \in \mathcal{T}, \text{Hom}_{\mathcal{T}}(\mathcal{V}, X[i]) = 0, \forall 1 \leq i \leq d-1\}. \end{aligned}$$

An object $T \in \mathcal{T}$ is called d -cluster tilting if the subcategory $\text{add}(T) \subset \mathcal{T}$ is d -cluster tilting.

Cluster tilting subcategories are interesting because they determine the triangulated category in the following sense:

Proposition 1.7. *Let \mathcal{T} and \mathcal{T}' be triangulated categories and $\mathcal{V} \subset \mathcal{T}$ and $\mathcal{V}' \subset \mathcal{T}'$ be d -cluster tilting subcategories. If $F : \mathcal{T} \longrightarrow \mathcal{T}'$ is a triangle functor such that its restriction $F|_{\mathcal{V}}$ to \mathcal{V} is an equivalence $F|_{\mathcal{V}} : \mathcal{V} \longrightarrow \mathcal{V}'$, then F is an equivalence.*

Proof. The proposition is clear for $d = 1$ since $\mathcal{T} = \mathcal{V}$ and $\mathcal{T}' = \mathcal{V}'$ hold in this case. It is proved in [KR08, Lemma 4.5] for $d \geq 2$. Note that the proof in [KR08] does not use the fact that \mathcal{T} and \mathcal{T}' are d -Calabi-Yau. \square

1.3. Generalized cluster categories. Let $n \geq 1$ be an integer.

Let Λ be a finite dimensional algebra of global dimension at most n . Denote by $\Theta = \Theta_n(\Lambda)$ a projective resolution of

$$\mathbf{RHom}_{\Lambda}(D\Lambda, \Lambda)[n] \simeq \mathbf{RHom}_{\Lambda^e}(\Lambda, \Lambda^e)[n] \simeq \mathbf{RHom}_{\Lambda^{\text{op}}}(D\Lambda, \Lambda)[n] \quad \text{in } \mathcal{D}(\Lambda^e).$$

Definition 1.8. [Kel11, IO09] We denote by \mathcal{A} the differential graded category (DG category for short) of bounded complexes of finitely generated projective Λ -modules. We define a DG functor by

$$F := - \otimes_{\Lambda} \Theta : \mathcal{A} \rightarrow \mathcal{A}.$$

The DG orbit category \mathcal{A}/F has the same objects as \mathcal{A} , and

$$\text{Hom}_{\mathcal{A}/F}(X, Y) := \text{colim}(\bigoplus_{\ell \geq 0} \text{Hom}_{\mathcal{A}}(F^{\ell}X, Y) \rightarrow \bigoplus_{\ell \geq 0} \text{Hom}_{\mathcal{A}}(F^{\ell}X, FY) \rightarrow \bigoplus_{\ell \geq 0} \text{Hom}_{\mathcal{A}}(F^{\ell}X, F^2Y) \rightarrow \dots).$$

We denote by $\mathcal{D}(\mathcal{A}/F)$ the derived category of \mathcal{A}/F . The *generalized n -cluster category* $\mathcal{C}_n(\Lambda)$ is defined as the smallest thick subcategory of $\mathcal{D}(\mathcal{A}/F)$ containing all representable functors of \mathcal{A}/F .

Let $\mathbb{S} = - \otimes_{\Lambda}^{\mathbf{L}} D\Lambda$ be the Serre functor of the category $\mathcal{D}^b(\Lambda)$, and denote by \mathbb{S}_n the composition $\mathbb{S}_n := \mathbb{S} \circ [-n]$. Then we have an isomorphism $\mathbb{S}_n^{-1} \simeq - \otimes_{\Lambda} \Theta$ of functors on $\mathcal{D}^b(\Lambda)$. From the construction of the generalized cluster category $\mathcal{C}_n(\Lambda)$, we have a triangle functor $\pi_{\Lambda} : \mathcal{D}^b(\Lambda) \rightarrow \mathcal{C}_n(\Lambda)$ which induces a fully faithful functor $\mathcal{D}^b(\Lambda)/\mathbb{S}_n \rightarrow \mathcal{C}_n(\Lambda)$ for the orbit category $\mathcal{D}^b(\Lambda)/\mathbb{S}_n$.

Remark 1.9. • For $n = 2$ and an algebra Λ of global dimension 1, one gets the usual cluster category $\mathcal{D}^b(\Lambda)/\mathbb{S}_2$ constructed in [BMR+06].

- For $n = 2$, and an algebra Λ of global dimension 2, the construction is given in [Ami09] in the case where $\mathcal{C}_2(\Lambda)$ is Hom-finite.
- The generalization of results of [Ami09] from 2 to $n \geq 2$ is described in [Guo10].

The functor $\pi : \mathcal{D}^b(\Lambda) \rightarrow \mathcal{C}_n(\Lambda)$ is also described by a universal property (cf [Kel05, Ami09]). Here is the version we will use in this paper (see appendix [IO09]).

Proposition 1.10. [Kel05, Ami09],[IO09, Thm A.20] *Let Λ be a finite dimensional algebra of global dimension at most n . Let C be an Iwanaga-Gorenstein algebra and T be in $\mathcal{D}^b(\Lambda^{\text{op}} \otimes C)$. If there exists a morphism $T \rightarrow \Theta \otimes_{\Lambda} T$ in $\mathcal{D}^b(\Lambda^{\text{op}} \otimes C)$ whose cone is perfect as an object in $\mathcal{D}^b(C)$, then there exists a commutative diagram of triangle functors*

$$\begin{array}{ccc} \mathcal{D}^b(\Lambda) & \xrightarrow{- \otimes_{\Lambda}^{\mathbf{L}} T} & \mathcal{D}^b(C) \\ \downarrow \pi & & \downarrow \text{nat.} \\ \mathcal{C}_n(\Lambda) & \longrightarrow & \underline{\mathbf{CM}}(C). \end{array}$$

Generalized cluster categories also have a nice description using certain DG algebras called *derived preprojective algebras*.

Definition 1.11. [Kel11, IO09] Let Λ be a finite dimensional algebra of global dimension at most n . The *derived $(n + 1)$ -preprojective algebra* of Λ is defined as the tensor DG algebra

$$\mathbf{\Pi}_{n+1}(\Lambda) := \mathbf{T}_\Lambda(\Theta_n(\Lambda)) = \Lambda \oplus \Theta \oplus (\Theta \otimes_\Lambda \Theta) \oplus \dots$$

The $(n + 1)$ -preprojective algebra of Λ is defined as the tensor algebra

$$\Pi_{n+1}(\Lambda) := \mathbf{T}_\Lambda \mathbf{Ext}_\Lambda^n(D\Lambda, \Lambda) \simeq H^0(\mathbf{\Pi}_{n+1}(\Lambda)).$$

The next result is shown in [Ami09, Thm 4.10] for $n = 2$. The generalization to $n \geq 2$ is done in [Guo10].

Theorem 1.12. [Ami09, Guo10],[Iya11, Thm 1.23] *Let Λ be a finite dimensional algebra of global dimension at most n . Then the generalized n -cluster category $\mathcal{C}_n(\Lambda)$ is Hom-finite if and only if the $(n + 1)$ -preprojective algebra $\Pi_{n+1}(\Lambda)$ is finite dimensional. In this case, we have the following properties.*

- (a) *The category $\mathbf{add} \{\mathbb{S}_n^i \Lambda \mid i \in \mathbb{Z}\}$ is an n -cluster tilting subcategory of $\mathcal{D}^b(\Lambda)$.*
- (b) *The category $\mathcal{C}_n(\Lambda)$ is n -Calabi-Yau, and the object $\pi(\Lambda)$ is n -cluster tilting with endomorphism algebra $\Pi_{n+1}(\Lambda)$.*
- (c) *We have a triangle equivalence $\mathcal{C}_n(\Lambda) \simeq \mathbf{per} \Pi_{n+1}(\Lambda) / \mathcal{D}^{\text{fd}}(\Pi_{n+1}(\Lambda))$.*

2. CALABI-YAU ALGEBRAS AS HIGHER AUSLANDER ALGEBRAS

Under certain conditions on a bimodule d -Calabi-Yau algebra B and an idempotent $e \in B$, we show in this section that $C := eBe$ is an Iwanaga-Gorenstein algebra, and that Be is a $(d - 1)$ -cluster tilting object in the category $\mathbf{CM}(C)$ of Cohen-Macaulay C -modules.

Definition 2.1. [Gin06, (3.2.5)] Fix an integer $d \geq 2$. We say that a k -algebra B is *bimodule d -Calabi-Yau* if $B \in \mathbf{per} B^e$ and $\mathbf{RHom}_{B^e}(B, B^e)[d] \simeq B$ in $\mathcal{D}(B^e)$.

Note that if B is bimodule d -Calabi-Yau, then so is B^{op} .

Example 2.1. Let $R = k[x_1, \dots, x_d]$ be a polynomial algebra. If an R -algebra B is a finitely generated free R -module and satisfies $\mathbf{Hom}_R(B, R) \simeq B$ as B^e -modules, then it is bimodule d -Calabi-Yau [Gin06, Thm 7.2.14],[IR08, Thm 3.2].

Let B be a k -algebra, and e an idempotent in B . Assume that B and $e(\neq 1)$ satisfy the following conditions.

- (A1) B is bimodule d -Calabi-Yau.
- (A2) B is noetherian.
- (A3) $\underline{B} := B/\langle e \rangle$ is a finite dimensional k -algebra.

The aim of this section is to prove the following results.

Theorem 2.2. *Let B be a k -algebra, $e \in B$ be an idempotent and $C := eBe$. Under assumptions (A1), (A2) and (A3), we have the following.*

- (a) *C is an Iwanaga-Gorenstein algebra.*

- (b) Be is a Cohen-Macaulay C -module.
- (c) We have natural isomorphisms $\text{End}_C(Be) \simeq B$ and $\text{End}_{C^{\text{op}}}(eB) \simeq B^{\text{op}}$ which induce isomorphisms $\text{End}_{\underline{\text{CM}}(C)}(Be) \simeq \underline{B}$ and $\text{End}_{\underline{\text{CM}}(C^{\text{op}})}(eB) \simeq \underline{B}^{\text{op}}$.
- (d) Be is $(d-1)$ -cluster tilting in $\text{CM}(C)$.

The above statements (c) and (d) show that B is a higher Auslander algebra of C in the sense of [Iya07b, Section 1].

If moreover B is a graded k -algebra, we have the following additional information.

Proposition 2.3. *In addition to assumptions (A1), (A2) and (A3), assume that $B = \bigoplus_{\ell \geq 0} B_\ell$ is a graded k -algebra such that $\dim_k B_\ell$ is finite for all $\ell \in \mathbb{Z}$. Then we have the following.*

- (a) Be is a graded Cohen-Macaulay C -module.
- (b) The isomorphisms in Theorem 2.2 preserve the grading, i.e. they induce isomorphisms

$$\text{Hom}_{\text{Gr } C}(Be, Be(\ell)) \simeq B_\ell, \quad \text{Hom}_{\text{Gr}(C^{\text{op}})}(eB, eB(\ell)) \simeq B_\ell^{\text{op}},$$

$$\text{Hom}_{\underline{\text{CM}}^{\mathbb{Z}}(C)}(Be, Be(\ell)) \simeq \underline{B}_\ell \quad \text{and} \quad \text{Hom}_{\underline{\text{CM}}^{\mathbb{Z}}(C^{\text{op}})}(eB, eB(\ell)) \simeq \underline{B}_\ell^{\text{op}}.$$

- (c) The category $\text{add}\{Be(i) \mid i \in \mathbb{Z}\}$ is a $(d-1)$ -cluster tilting subcategory of $\text{CM}^{\mathbb{Z}}(C)$.

The proof of Theorem 2.2 is given in the next two subsections. Assertions (a), (b) and (c) are proved in subsection 2.1. Subsection 2.2 is devoted to the proof of (d).

2.1. C is Iwanaga-Gorenstein. In the rest of the section we assume that the algebra B satisfies (A1), (A2) and (A3).

The following is a basic property of bimodule d -Calabi-Yau algebras.

Proposition 2.4. *Let B be a bimodule d -Calabi-Yau algebra.*

- (a) [Gin06, Prop 3.2.4][Kel08, Lemma 4.1] *For any $X \in \mathcal{D}(B)$ and $Y \in \mathcal{D}^{\text{fd}}(B)$, we have a functorial isomorphism*

$$\text{Hom}_{\mathcal{D}(B)}(X, Y) \simeq D\text{Hom}_{\mathcal{D}(B)}(Y, X[d]).$$

In particular, $\mathcal{D}^{\text{fd}}(B)$ is a d -Calabi-Yau triangulated category.

- (b) *We have $\text{gl.dim } B = d$.*

Proof. (b) For any $X, Y \in \mathcal{D}(B)$, it is easy to see that we have

$$\begin{aligned} \mathbf{R}\text{Hom}_B(X, Y) &\simeq \mathbf{R}\text{Hom}_{B^e}(B, \text{Hom}_k(X, Y)) \simeq \text{Hom}_k(X, Y) \otimes_{B^e}^{\mathbf{L}} \mathbf{R}\text{Hom}_{B^e}(B, B^e) \\ &\simeq \text{Hom}_k(X, Y) \otimes_{B^e}^{\mathbf{L}} B[-d]. \end{aligned}$$

In particular, for any $X, Y \in \text{Mod } B$, we have

$$\text{Ext}_B^{d+1}(X, Y) \simeq H^{d+1}(\text{Hom}_k(X, Y) \otimes_{B^e}^{\mathbf{L}} B[-d]) = 0.$$

Hence the global dimension of B is at most d . It is exactly d since $\text{Ext}_B^d(\underline{B}, B) \simeq D\text{Hom}_B(B, \underline{B}) \neq 0$ holds by (A3) and (a). \square

Let us make the following easy observations.

Lemma 2.5. (a) For any $X \in \text{fd } B$, we have $\text{Ext}_B^i(X, B) = 0$ for any $i \neq d$.
 (b) For any $X \in \text{mod } \underline{B}$, we have $\text{Ext}_B^i(X, eB) = 0$ for any $i \in \mathbb{Z}$.

Proof. We only prove (b) since (a) is simpler. Since $\dim_k X < \infty$ by (A3), we have

$$\text{Ext}_B^i(X, eB) \simeq D\text{Ext}_B^{d-i}(eB, X)$$

by Proposition 2.4. If $i \neq d$, then $\text{Ext}_B^{d-i}(eB, X)$ is zero since eB is projective. If $i = d$, then it is zero since $X \in \text{mod } \underline{B}$. \square

Proposition 2.6. We have

$$\text{Ext}_C^i(Be, C) \simeq \begin{cases} 0 & \text{if } i \neq 0 \\ eB & \text{if } i = 0 \end{cases} \quad \text{and} \quad \text{Ext}_C^i(Be, Be) \simeq \begin{cases} 0 & \text{if } 1 \leq i \leq d-2 \\ B & \text{if } i = 0. \end{cases}$$

Proof. We consider the triangle

$$Be \overset{\mathbf{L}}{\otimes}_C eB \xrightarrow{f} B \longrightarrow X \longrightarrow Be \overset{\mathbf{L}}{\otimes}_C eB[1] \quad \text{in } \mathcal{D}(B^e),$$

where f is the composition $Be \overset{\mathbf{L}}{\otimes}_C eB \rightarrow Be \otimes_C eB \xrightarrow{\text{mult.}} B$ of natural maps. Applying $-\overset{\mathbf{L}}{\otimes}_B Be$, we have an isomorphism $f \overset{\mathbf{L}}{\otimes}_B Be$. Thus $X \overset{\mathbf{L}}{\otimes}_B Be = 0$ holds. This means that $H^i(X)e = 0$ and hence $H^i(X) \in \text{mod } \underline{B}$ for any $i \in \mathbb{Z}$.

By Lemma 2.5(b), we have $\mathbf{RHom}_B(X, eB) = 0$. Applying $\mathbf{RHom}_B(-, eB)$ to the above triangle, we get

$$\begin{aligned} eB &= \mathbf{RHom}_B(B, eB) \simeq \mathbf{RHom}_B(Be \overset{\mathbf{L}}{\otimes}_C eB, eB) \\ &\simeq \mathbf{RHom}_C(Be, \mathbf{RHom}_B(eB, eB)) \simeq \mathbf{RHom}_C(Be, C) \quad \text{in } \mathcal{D}(C^{\text{op}} \otimes B). \end{aligned}$$

Thus the first assertion follows.

Similarly we have

$$\mathbf{RHom}_B(Be \overset{\mathbf{L}}{\otimes}_C eB, B) \simeq \mathbf{RHom}_C(Be, \mathbf{RHom}_B(eB, B)) \simeq \mathbf{RHom}_C(Be, Be) \quad \text{in } \mathcal{D}(C^{\text{op}} \otimes B).$$

Since Be and eB are concentrated in degree 0, $H^i(Be \overset{\mathbf{L}}{\otimes}_C eB)$ vanishes for $i > 0$, and then $H^i(X) = 0$ for any $i > 0$. Hence we have $H^i(\mathbf{RHom}_B(X, B)) = 0$ for any $i < d$ again by Lemma 2.5(a). Applying $\mathbf{RHom}_B(-, B)$ to the above triangle, we have an exact sequence

$$\text{Hom}_{\mathcal{D}(B)}(X, B[i]) \longrightarrow \text{Hom}_{\mathcal{D}(B)}(B, B[i]) \longrightarrow \text{Hom}_{\mathcal{D}(B)}(Be \overset{\mathbf{L}}{\otimes}_C eB, B[i]) \longrightarrow \text{Hom}_{\mathcal{D}(B)}(X, B[i+1]).$$

In particular, for any i with $0 \leq i \leq d-2$, we have isomorphisms

$$\text{Ext}_C^i(Be, Be) \simeq \text{Hom}_{\mathcal{D}(B)}(Be \overset{\mathbf{L}}{\otimes}_C eB, B[i]) \simeq \text{Hom}_{\mathcal{D}(B)}(B, B[i])$$

which show the second assertion. \square

Now we are ready to prove Theorem 2.2(a), (b) and (c).

(i) First we show that C is noetherian.

This follows from (A2) by the following easy argument: Any right ideal I of C gives a right ideal $\tilde{I} := IB$ of B satisfying $\tilde{I}e = I$. Thus any strictly ascending chain of right ideals of C gives a strictly ascending chain of right ideals of B . Thus C is right noetherian. Similarly C is left noetherian.

(ii) Next we show that C is an Iwanaga-Gorenstein algebra.

For any $X \in \text{Mod } C$, we shall show $\text{Ext}_C^{d+1}(X, C) = 0$. Let $Y := X \otimes_C eB$ and P_\bullet be a projective resolution of the B -module Y . Then $P_\bullet e$ is a bounded complex in $\text{add}_C(Be)$ which is quasi-isomorphic to $Ye \simeq X$. Since by Proposition 2.6 $\text{Ext}_C^i(Be, C)$ vanishes for any $i > 0$, we have

$$\text{Ext}_C^{d+1}(X, C) \simeq H^{d+1}(\text{Hom}_C(P_\bullet e, C)).$$

Since we have isomorphisms

$$\text{Hom}_C(P_\bullet e, C) \simeq \text{Hom}_C(P_\bullet \otimes_B Be, C) \simeq \text{Hom}_B(P_\bullet, \text{Hom}_C(Be, C)) \simeq \text{Hom}_B(P_\bullet, eB),$$

we get

$$\text{Ext}_C^{d+1}(X, C) \simeq H^{d+1}(\text{Hom}_B(P_\bullet, eB)) \simeq \text{Ext}_B^{d+1}(Y, eB) = 0$$

by Proposition 2.4.

(iii) We show that Be is a Cohen-Macaulay C -module.

By Proposition 2.6, we only have to show that Be is a finitely generated C -module. By (A2), the right ideal $\langle e \rangle = BeB$ of B is finitely generated. There exists a finite generating set of the B -module BeB which is contained in Be . Clearly it gives a finite generating set of the C -module Be .

(iv) We show Theorem 2.2(c).

We have $\text{End}_C(Be) \simeq B$ by Proposition 2.6. Hence we have an equivalence

$$\text{Hom}_C(Be, -) : \text{add}_C(Be) \rightarrow \text{proj } B$$

which sends C to eB . Thus we have

$$\text{End}_{\underline{\text{CM}}(C)}(Be) = \text{End}_C(Be)/[C] \simeq \text{End}_B(B)/[eB] \simeq B/BeB = \underline{B}.$$

Here we denote by $[C]$ (respectively, $[eB]$) the ideal of $\text{End}_C(Be)$ (respectively, $\text{End}_B(B)$) consisting of morphisms factoring through $\text{add } C$ (respectively, $\text{add } eB$).

Similarly we have $B^{\text{op}} \simeq \text{End}_{C^{\text{op}}}(eB)$ and $\underline{B}^{\text{op}} \simeq \text{End}_{\underline{\text{CM}}(C^{\text{op}})}(eB)$. \square

We end this subsection with the following observation (which will not be used in this paper) asserting that C enjoys the bimodule d -Calabi-Yau property except that C may not be perfect as a bimodule over itself.

Remark 2.7. We have $\mathbf{R}\text{Hom}_{C^e}(C, C^e)[d] \simeq C$ in $\mathcal{D}(C^e)$.

Proof. Let P_\bullet be a projective resolution of the B^e -module B . Applying $eB \otimes_B - \otimes_B Be$, we get an isomorphism $eP_\bullet e \simeq C$ in $\mathcal{D}(C^e)$. By Proposition 2.6, we have

$$\begin{aligned} \mathbf{R}\text{Hom}_{C^e}(eB \otimes_B Be, C^e) &= \mathbf{R}\text{Hom}_{C^{\text{op}}}(eB, C) \otimes \mathbf{R}\text{Hom}_C(Be, C) \\ &= \text{Hom}_{C^{\text{op}}}(eB, C) \otimes \text{Hom}_C(Be, C) = \text{Hom}_{C^e}(eB \otimes_B Be, C^e). \end{aligned}$$

Thus each term $eP_i e$ in $eP_\bullet e$ satisfies $\text{Ext}_{C^e}^i(eP_i e, C^e) = 0$ for any $i > 0$, and we have

$$\mathbf{R}\text{Hom}_{C^e}(C, C^e) \simeq \text{Hom}_{C^e}(eP_\bullet e, C^e).$$

Since the functor

$$eB \otimes_B - \otimes_B Be : \text{proj } B^e \rightarrow \text{mod } C^e$$

is fully faithful by Theorem 2.2(c), we have

$$\text{Hom}_{C^e}(eP_i e, C^e) \simeq \text{Hom}_{B^e}(P_i, Be \otimes_B eB) = e\text{Hom}_{B^e}(P_i, B^e)e$$

Consequently we have

$$\begin{aligned} \mathbf{R}\mathrm{Hom}_{C^e}(C, C^e) &\simeq \mathrm{Hom}_{C^e}(eP_\bullet e, C^e) \\ &\simeq e\mathrm{Hom}_{B^e}(P_\bullet, B^e)e \\ &\simeq e\mathbf{R}\mathrm{Hom}_{B^e}(B, B^e)e \\ &\simeq e(B[-d])e = C[-d]. \end{aligned}$$

Thus the assertion follows. \square

2.2. Be is $(d-1)$ -cluster tilting. In this subsection we prove Theorem 2.2(d).

By Proposition 2.6, we have $\mathrm{Ext}_C^i(Be, Be) = 0$ for any i with $1 \leq i \leq d-2$. The assertion follows from the following lemmas.

Lemma 2.8. *For any $X \in \mathrm{mod} C$, we have $\mathrm{proj.dim}_{B^{\mathrm{op}}}\mathrm{Hom}_C(X, Be) \leq d-2$.*

Proof. Let $P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$ be a projective presentation of X in $\mathrm{mod} C$. Applying $\mathrm{Hom}_C(-, Be)$, we have an exact sequence

$$0 \longrightarrow \mathrm{Hom}_C(X, Be) \longrightarrow \mathrm{Hom}_C(P_0, Be) \longrightarrow \mathrm{Hom}_C(P_1, Be)$$

of B^{op} -modules. Then $\mathrm{Hom}_C(P_i, Be)$ is a projective B^{op} -module for $i = 0, 1$. Since $\mathrm{gl.dim} B^{\mathrm{op}} = d$ by Proposition 2.4, we have $\mathrm{proj.dim}_{B^{\mathrm{op}}}\mathrm{Hom}_C(X, Be) \leq d-2$ \square

Lemma 2.9. *If $X \in \mathrm{CM}(C)$ satisfies $\mathrm{Ext}_C^i(X, Be) = 0$ for any i with $1 \leq i \leq d-2$, then we have $X \in \mathrm{add}_C(Be)$.*

Proof. Let

$$0 \longrightarrow \Omega^{d-2}X \longrightarrow P_{d-3} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

be a projective resolution of the C -module X . Applying $\mathrm{Hom}_C(-, Be)$, we get an exact sequence

$$0 \longrightarrow \mathrm{Hom}_C(X, Be) \longrightarrow \mathrm{Hom}_C(P_0, Be) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_C(P_{d-3}, Be) \longrightarrow \mathrm{Hom}_C(\Omega^{d-2}X, Be) \longrightarrow 0$$

of B^{op} -modules, where we used that $\mathrm{Ext}_C^i(X, Be) = 0$ for any i with $1 \leq i \leq d-2$. By Lemma 2.8, we have $\mathrm{proj.dim}_{B^{\mathrm{op}}}\mathrm{Hom}_C(\Omega^{d-2}X, Be) \leq d-2$. Since each $\mathrm{Hom}_C(P_i, Be)$ is a projective B^{op} -module, it follows that $\mathrm{Hom}_C(X, Be)$ is a projective B^{op} -module. Thus we have $\mathrm{Hom}_C(X, C) = e\mathrm{Hom}_C(X, Be) \in \mathrm{add}_{C^{\mathrm{op}}}(eB)$ and

$$X \simeq \mathrm{Hom}_{C^{\mathrm{op}}}(\mathrm{Hom}_C(X, C), C) \in \mathrm{add}_C \mathrm{Hom}_{C^{\mathrm{op}}}(eB, C) = \mathrm{add}_C(Be)$$

by Propositions 1.3 and 2.6. \square

Lemma 2.10. *If $X \in \mathrm{CM}(C)$ satisfies $\mathrm{Ext}_C^i(Be, X) = 0$ for any $1 \leq i \leq d-2$, then we have $X \in \mathrm{add}_C(Be)$.*

Proof. Let $(-)^* := \mathrm{Hom}_C(-, C) : \mathrm{CM}(C) \rightarrow \mathrm{CM}(C^{\mathrm{op}})$ be the duality in Proposition 1.3. Then we have $(Be)^* = eB$ by Proposition 2.6. Since the duality $(-)^*$ preserves the extension groups, we have $\mathrm{Ext}_{C^{\mathrm{op}}}^i(X^*, eB) = 0$ for any i with $1 \leq i \leq d-2$. Applying Lemma 2.9 to $(B, C, Be, X) := (B^{\mathrm{op}}, C^{\mathrm{op}}, eB, X^*)$, we have $X^* \in \mathrm{add}_{C^{\mathrm{op}}}(eB)$. Applying $(-)^*$ again, we have $X \in \mathrm{add}_C(Be)$. \square

Now Theorem 2.2(d) is a direct consequence of Lemmas 2.9 and 2.10.

3. GRADED CALABI-YAU ALGEBRAS AS HIGHER PREPROJECTIVE ALGEBRAS

In this section, which is independent of Section 2, we work with a graded algebra $B = \bigoplus_{\ell \geq 0} B_\ell$ such that $\dim_k B_0$ is finite. We show under assumptions of d -Calabi-Yau type on \overline{B} , that B is isomorphic to the d -preprojective algebra of $A := B_0$.

3.1. Basic setup and main result.

Definition 3.1. Let $d \geq 2$. Assume that $B = \bigoplus_{\ell \geq 0} B_\ell$ is a positively \mathbb{Z} -graded k -algebra. We say that B is *bimodule d -Calabi-Yau of Gorenstein parameter 1* if $B \in \mathbf{per} B^e$ and there exists a graded projective resolution P_\bullet of B as a bimodule and an isomorphism

$$(3.1.1) \quad P_\bullet \simeq P_\bullet^\vee[d](-1) \quad \text{in } \mathcal{C}^b(\mathbf{grproj} B^e),$$

where we denote by $(-)^\vee = \mathbf{Hom}_{B^e}(-, B^e) : \mathcal{C}^b(\mathbf{grproj} B^e) \rightarrow \mathcal{C}^b(\mathbf{grproj} (B^e)^{\text{op}}) \simeq \mathcal{C}^b(\mathbf{grproj} B^e)$ the natural duality induced by a canonical isomorphism $(B^e)^{\text{op}} \simeq B^e$.

Remark 3.2. If for any $\ell \in \mathbb{N}$ the homogenous part B_ℓ is finite dimensional, then the category $\mathbf{gr} B$ is \mathbf{Hom} -finite and Krull-Schmidt. Hence the graded algebra B is bimodule d -Calabi-Yau of Gorenstein parameter 1 if and only if there exists an isomorphism

$$\mathbf{RHom}_{B^e}(B, B^e)[d](-1) \simeq B \quad \text{in } \mathcal{D}(\mathbf{Gr} B^e).$$

In this case, the minimal projective resolution P_\bullet of B as a B -bimodule satisfies (3.1.1)

Throughout this section we assume

$$(A1^*) \quad B \text{ is bimodule } d\text{-Calabi-Yau of Gorenstein parameter 1.}$$

The aim of this section is to prove the following.

Theorem 3.3. *Let B be as above with $A := B_0$ finite dimensional. Then we have the following.*

- (a) *A is a finite dimensional k -algebra with $\mathbf{gl.dim} A \leq d - 1$.*
- (b) *The derived d -preprojective algebra $\mathbf{\Pi}_d(A)$ is concentrated in degree zero.*
- (c) *There exists an isomorphism $\mathbf{\Pi}_d(A) \simeq B$ of \mathbb{Z} -graded algebras, where $\mathbf{\Pi}_d(A)$ is the d -preprojective algebra of A .*

Note that as a consequence of this Theorem, we obtain that $\dim_k B_\ell$ is finite for all $\ell \geq 0$ since $B_\ell \simeq \underbrace{\mathbf{Ext}_A^{d-1}(DA, A) \otimes_A \cdots \otimes_A \mathbf{Ext}_A^{d-1}(DA, A)}_{\ell \text{ times}}$.

The main step of the proof consists of the following intermediate result.

Proposition 3.4. *Let B be as above, $A := B_0$ and $\Theta = \Theta_{d-1}(A)$ be a projective resolution of $\mathbf{RHom}_{A^e}(A, A^e)[d - 1]$ in $\mathcal{D}(A^e)$. Then there exists a triangle*

$$\Theta \otimes_A B(-1) \xrightarrow{\alpha} B \xrightarrow{a} A \longrightarrow \Theta \otimes_A B(-1)[1] \quad \text{in } \mathcal{D}(\mathbf{Gr}(A^{\text{op}} \otimes B))$$

where $a : B \rightarrow A$ is the natural surjection.

Before proving Proposition 3.4 and Theorem 3.3, let us give an application.

Definition 3.5. [HIO12] Let n be a positive integer. A finite dimensional algebra A is called n -representation infinite if $\text{gl.dim } A \leq n$ and $\mathbb{S}_n^{-i}A$ belongs to $\text{mod } A$ for any $i \geq 0$.

Clearly an algebra A with $\text{gl.dim } A \leq n$ is n -representation infinite if and only if $\mathbf{II}_{n+1}(A)$ is concentrated in degree zero. Thus we have the following immediate consequence.

Corollary 3.6. *Let B be a graded algebra which is bimodule d -Calabi-Yau of Gorenstein parameter 1, with $\dim_k B_0 < \infty$. Then B_0 is $(d-1)$ -representation infinite.*

The n -representation infinite algebras are also called *extremely quasi n -Fano* and studied from the viewpoint of non-commutative algebraic geometry in [MM10]. In particular, Corollary 3.6 was proved in [MM10, Thm 4.12] using quite different methods. We note that combining with Keller's result [Kel11, Thm 4.8], we have a bijection between bimodule d -Calabi-Yau algebras of Gorenstein parameter 1 and $(d-1)$ -representation infinite algebras (see [HIO12, Thm 4.35]).

3.2. Splitting the graded projective resolution. Let us start with the following observation.

Lemma 3.7. *Let Q_\bullet be a complex in $\mathcal{C}^b(\text{grproj } B^e)$ such that each term is generated in degree zero.*

- (a) *The degree zero part $(Q_\bullet)_0$ is isomorphic to $A \otimes_B Q_\bullet \otimes_B A$ in $\mathcal{C}^b(\text{proj } A^e)$.*
- (b) *We have isomorphisms $B \otimes_A A \otimes_B Q_\bullet \simeq Q_\bullet \simeq Q_\bullet \otimes_B A \otimes_A B$ in $\mathcal{C}^b(\text{grproj } B^e)$.*

Let B , P_\bullet , and $A = B_0$ be as in subsection 3.1. The following observation is crucial.

Lemma 3.8. *In the setup above, the following assertions hold.*

- (a) *There exist complexes*

$$\begin{aligned} Q_\bullet &= (Q_{d-1} \longrightarrow \cdots \longrightarrow Q_1 \longrightarrow Q_0) \quad \text{and} \\ R_\bullet &= (R_{d-1} \longrightarrow \cdots \longrightarrow R_1 \longrightarrow R_0) \quad \text{in } \mathcal{C}^b(\text{grproj } B^e) \end{aligned}$$

and a morphism $f : R_\bullet(-1) \longrightarrow Q_\bullet$ in $\mathcal{C}^b(\text{grproj } B^e)$ such that P_\bullet is the mapping cone of f and each Q_i and R_i are generated in degree zero.

- (b) *We have $R_\bullet \simeq Q_\bullet^\vee[d-1]$ and $Q_\bullet \simeq R_\bullet^\vee[d-1]$ in $\mathcal{C}^b(\text{grproj } B^e)$.*

Proof. (a) Since the resolution P_\bullet of B is minimal, and since $B_i = 0$ for any $i < 0$, each P_i is generated in non-negative degrees. If P_i has a generator in degree $a \geq 0$, then by the isomorphism (3.1.1) P_{d-i} has a generator in degree $1-a$, which implies $1-a \geq 0$. Therefore a has to be 0 or 1, and each P_i is generated in degree 0 or 1.

For each $i = 0, \dots, d$ we write $P_i := P_i^0 \oplus P_i^1(-1)$, where all the indecomposable summands of P_i^0 and P_i^1 are generated in degree zero. By the isomorphism (3.1.1), we have $P_i^1 \simeq (P_{d-i}^0)^\vee$ for any $i \in \mathbb{Z}$. Since the B^e -module B is generated in degree zero, we have $P_0^1 = 0$ and so $P_d^0 = 0$. Then the map $d_i : P_i \rightarrow P_{i-1}$ can be written

$$d_i : P_i^0 \oplus P_i^1(-1) \xrightarrow{\begin{bmatrix} a_i & b_i \\ 0 & -c_i \end{bmatrix}} P_{i-1}^0 \oplus P_{i-1}^1(-1)$$

Therefore we have

$$\begin{array}{ccccccccccc}
 P_{\bullet} = (P_d & \longrightarrow & P_{d-1} & \longrightarrow & \dots & \longrightarrow & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0) \\
 & & \uparrow & & & & \uparrow & & \uparrow & & \uparrow \\
 Q_{\bullet} := (0 & \longrightarrow & P_{d-1}^0 & \longrightarrow & \dots & \longrightarrow & P_2^0 & \xrightarrow{a_2} & P_1^0 & \xrightarrow{a_1} & P_0^0) \\
 & & \uparrow & & & & \uparrow & & \uparrow & & \uparrow \\
 R_{\bullet}(-1) := (0 & \longrightarrow & P_d^1(-1) & \longrightarrow & \dots & \longrightarrow & P_3^1(-1) & \xrightarrow{c_3} & P_2^1(-1) & \xrightarrow{c_2} & P_1^1(-1))
 \end{array}$$

Hence P_{\bullet} is the mapping cone of the morphism $f : R_{\bullet}(-1) \rightarrow Q_{\bullet}$.

(b) We have an exact sequence

$$0 \longrightarrow Q_{\bullet} \longrightarrow P_{\bullet} \longrightarrow R_{\bullet}(-1)[1] \longrightarrow 0 \quad \text{in } \mathcal{C}^b(\text{grproj } B^e).$$

Applying $(-)^{\vee}(-1)[d]$ and using the isomorphism (3.1.1), we have an exact sequence

$$0 \longrightarrow R_{\bullet}^{\vee}[d-1] \longrightarrow P_{\bullet} \longrightarrow Q_{\bullet}^{\vee}(-1)[d] \longrightarrow 0 \quad \text{in } \mathcal{C}^b(\text{grproj } B^e).$$

Since Q_{\bullet} is generated in degree zero and the degree zero part of $Q_{\bullet}^{\vee}(-1)[d]$ is zero, we have $\text{Hom}_{\mathcal{C}^b(\text{grproj } B^e)}(Q_{\bullet}, Q_{\bullet}^{\vee}(-1)[d]) = 0$. Similarly $\text{Hom}_{\mathcal{C}^b(\text{grproj } B^e)}(R_{\bullet}^{\vee}[d-1], R_{\bullet}(-1)[1]) = 0$ holds. Thus we have a commutative diagram

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & Q_{\bullet} & \longrightarrow & P_{\bullet} & \longrightarrow & R_{\bullet}(-1)[1] & \longrightarrow & 0 \\
 & & \updownarrow & & \parallel & & \updownarrow & & \\
 0 & \longrightarrow & R_{\bullet}^{\vee}[d-1] & \longrightarrow & P_{\bullet} & \longrightarrow & Q_{\bullet}^{\vee}(-1)[d] & \longrightarrow & 0
 \end{array}$$

which implies $Q_{\bullet} \simeq R_{\bullet}^{\vee}[d-1]$ and $R_{\bullet} \simeq Q_{\bullet}^{\vee}[d-1]$. \square

Lemma 3.9. *Let Q_{\bullet} be as defined in Lemma 3.8. We have the following isomorphisms.*

- (a) $A \otimes_B Q_{\bullet} \otimes_B A \simeq A$ in $\mathcal{D}(A^e)$.
- (b) $A \otimes_B Q_{\bullet} \simeq B$ in $\mathcal{D}(\text{Gr } A^{\text{op}} \otimes B)$.

Proof. (a) Since P_{\bullet} is isomorphic to the mapping cone of $f : R_{\bullet}(-1) \rightarrow Q_{\bullet}$, we have an isomorphism

$$(P_{\bullet})_0 \simeq \text{Cone}((R_{\bullet})_{-1} \rightarrow (Q_{\bullet})_0) \quad \text{in } \mathcal{C}^b(\text{proj } A^e)$$

where $(X)_{\ell}$ is the degree ℓ part of the complex $X \in \mathcal{C}^b(\text{grproj } B^e)$. Since B is only in non-negative degrees, then so is R_{\bullet} . Hence we have

$$(P_{\bullet})_0 \simeq (Q_{\bullet})_0 \quad \text{in } \mathcal{C}^b(\text{proj } A^e).$$

Since $P_{\bullet} \simeq B$ in $\mathcal{D}(\text{Gr } B^e)$, we have $(P_{\bullet})_0 \simeq B_0 = A$ in $\mathcal{D}(A^e)$. Therefore we get $A \otimes_B Q_{\bullet} \otimes_B A \simeq (Q_{\bullet})_0 \simeq A$ in $\mathcal{D}(A^e)$ by Lemma 3.7.

(b) We have the following isomorphisms in $\mathcal{D}(\text{Gr}(A^{\text{op}} \otimes B))$:

$$\begin{aligned}
 A \otimes_B Q_{\bullet} &\simeq (A \otimes_B Q_{\bullet} \otimes_B A) \otimes_A B && \text{by Lemma 3.7} \\
 &\simeq A \overset{\mathbf{L}}{\otimes}_A B \simeq B && \text{by (a).}
 \end{aligned}$$

\square

Proposition 3.10. *We have $\text{gl.dim } A \leq d - 1$.*

Proof. By Lemma 3.9, $A \otimes_B Q_\bullet \otimes_B A$ is a projective resolution of the A^e -module A . Thus we have $\text{gl.dim } A \leq \text{proj.dim}_{A^e} A \leq d - 1$. \square

Lemma 3.11. *Let R_\bullet be as defined in Lemma 3.8. Then we have the following isomorphisms.*

- (a) $A \otimes_B R_\bullet \otimes_B A \simeq \Theta$ in $\mathcal{D}(A^e)$.
- (b) $A \otimes_B R_\bullet \simeq \Theta \otimes_A B$ in $\mathcal{D}(\text{Gr } A^{\text{op}} \otimes B)$.

Proof. (a) We have the following isomorphisms in $\mathcal{D}(A^e)$:

$$\begin{aligned}
 A \otimes_B R_\bullet \otimes_B A[1-d] &\simeq A \otimes_B Q_\bullet^\vee \otimes_B A && \text{by Lemma 3.8} \\
 &\simeq A \otimes_B \text{Hom}_{B^e}(Q_\bullet, B^e) \otimes_B A \\
 &\simeq \text{Hom}_{B^e}(Q_\bullet, A^e) \\
 &\simeq \text{Hom}_{B^e}(B \otimes_A A \otimes_B Q_\bullet \otimes_B A \otimes_A B, A^e) && \text{by Lemma 3.7} \\
 &\simeq \text{Hom}_{A^e}(A \otimes_B Q_\bullet \otimes_B A, A^e) \\
 &\simeq \mathbf{R}\text{Hom}_{A^e}(A, A^e) && \text{by Lemma 3.9.}
 \end{aligned}$$

(b) We get the following isomorphisms in $\mathcal{D}(\text{Gr}(A^{\text{op}} \otimes B))$:

$$\begin{aligned}
 A \otimes_B R_\bullet &\simeq (A \otimes_B R_\bullet \otimes_B A) \otimes_A B && \text{by Lemma 3.7} \\
 &\simeq \Theta \otimes_A B && \text{by (a).}
 \end{aligned}$$

\square

Now we are ready to prove Proposition 3.4.

By Lemma 3.8 there exists a triangle $R_\bullet(-1) \longrightarrow Q_\bullet \longrightarrow P_\bullet \longrightarrow R_\bullet(-1)[1]$ in $\mathcal{D}(\text{Gr } B^e)$.

Applying the functor $A \overset{\mathbf{L}}{\otimes}_B -$ to this triangle we get the triangle

$$A \otimes_B R_\bullet(-1) \longrightarrow A \otimes_B Q_\bullet \longrightarrow A \otimes_B P_\bullet \longrightarrow A \otimes_B R_\bullet(-1)[1] \quad \text{in } \mathcal{D}(\text{Gr}(A^{\text{op}} \otimes B)).$$

By Lemmas 3.9 and 3.11, we get a commutative diagram

$$\begin{array}{ccccccc}
 A \otimes_B R_\bullet(-1) & \longrightarrow & A \otimes_B Q_\bullet & \longrightarrow & A \otimes_B P_\bullet & \longrightarrow & A \otimes_B R_\bullet[1](-1) \\
 \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\
 \Theta \otimes_A B(-1) & \longrightarrow & B & \xrightarrow{a} & A & \longrightarrow & \Theta \otimes_A B(-1)[1]
 \end{array}$$

in $\mathcal{D}(\text{Gr}(A^{\text{op}} \otimes B))$ with the natural surjection a . \square

We end this subsection with recording the following observation, which is not used in this paper and follows easily from Lemmas 3.9 and 3.11.

Remark 3.12. We have isomorphisms $Q_\bullet \simeq B \overset{\mathbf{L}}{\otimes}_A B$ and $R_\bullet \simeq B \otimes_A \Theta \otimes_A B$ in $\mathcal{D}(A^e)$.

3.3. Proof of Theorem 3.3. From Proposition 3.4, we have a triangle

$$\Theta \otimes_A B(-1) \xrightarrow{\alpha} B \xrightarrow{a} A \longrightarrow \Theta \otimes_A B(-1)[1] \quad \text{in } \mathcal{D}(\text{Gr}(A^{\text{op}} \otimes B)).$$

Since a is the natural surjection, α is an isomorphism except for the degree zero part.

For any $\ell \geq 1$ we use the following notation:

$$\Theta^\ell := \underbrace{\Theta \otimes_A \Theta \otimes_A \cdots \otimes_A \Theta}_{\ell \text{ times}} \in \mathcal{D}(A^e).$$

Definition 3.13. Let $\alpha_\ell : \Theta^\ell \otimes_A B \rightarrow B(\ell)$ be a morphism in $\mathcal{D}(\text{Gr}(A^{\text{op}} \otimes B))$ defined as the composition

$$\alpha_\ell : \Theta^\ell \otimes_A B \xrightarrow{1_{\Theta^{\ell-1}} \otimes_A \alpha(1)} \Theta^{\ell-1} \otimes_A B(1) \xrightarrow{1_{\Theta^{\ell-2}} \otimes_A \alpha(2)} \cdots \longrightarrow \Theta \otimes_A B(\ell-1) \xrightarrow{\alpha(\ell)} B(\ell).$$

Let $\beta_\ell : H^0(\Theta^\ell) \rightarrow B_\ell$ be a morphism in $\text{Mod}(A^e)$ defined as

$$\beta_\ell := H^0(\alpha_\ell)_0 : H^0(\Theta^\ell) \xrightarrow{\sim} B_\ell.$$

Now we are ready to prove Theorem 3.3.

(a) This is already shown in Proposition 3.10.

(b) Looking at the degree zero part of α_ℓ , we have an isomorphism

$$\Theta^\ell = (\Theta^\ell \otimes_A B)_0 \xrightarrow{\sim} B_\ell \quad \text{in } \mathcal{D}(A^e)$$

for any $\ell \geq 0$. Thus $\Pi_d(A) = \Gamma_A \Theta$ is concentrated in degree zero.

(c) Consider the following diagram for any $\ell, m \in \mathbb{Z}$:

$$\begin{array}{ccccc} H^0(\Theta^\ell) \otimes_A H^0(\Theta^m) & \xrightarrow[\sim]{1_{H^0(\Theta^\ell)} \otimes_A \beta_m} & H^0(\Theta^\ell) \otimes_A B_m & \xrightarrow[\sim]{\beta_\ell \otimes_A 1_{B_m}} & B_\ell \otimes_A B_m \\ \downarrow \wr & & \searrow & & \downarrow \text{mult.} \\ H^0(\Theta^{\ell+m}) & \xrightarrow[\sim]{\beta_{\ell+m}} & & \xrightarrow{H^0(\alpha_\ell)_m} & B_{\ell+m} \end{array}$$

The left square commutes since $\alpha_{\ell+m} = \alpha_\ell(m) \circ (1_{\Theta^\ell} \otimes_A \alpha_m)$ holds, and the right triangle commutes since $H^0(\alpha_\ell) : H^0(\Theta^\ell) \otimes_A B \rightarrow B(\ell)$ is a morphism of right B -modules. In particular, the k -linear isomorphism

$$\bigoplus_{\ell \geq 0} \beta_\ell : \Pi_d(A) = \bigoplus_{\ell \geq 0} H^0(\Theta^\ell) \xrightarrow{\sim} B = \bigoplus_{\ell \geq 0} B_\ell$$

is compatible with the multiplication. □

The next lemma, which we will use later, follows immediately from the definitions of α_ℓ and β_ℓ .

Lemma 3.14. (a) *The following diagram is commutative:*

$$\begin{array}{ccc}
 H^0(\Theta^\ell) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{D}(A)}(A, \Theta^\ell) \\
 \downarrow \beta_\ell & & \downarrow -\mathbf{L}_{\otimes_A} B \\
 & & \mathrm{Hom}_{\mathcal{D}(\mathrm{Gr} B)}(B, \Theta^\ell \otimes_A B) \\
 & & \downarrow \alpha_\ell \\
 B_\ell & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{D}(\mathrm{Gr} B)}(B, B(\ell))
 \end{array}$$

(b) β_ℓ is equal to the composition

$$\beta_\ell : H^0(\Theta^\ell) \xrightarrow{\sim} H^0(\Theta) \otimes_A \cdots \otimes_A H^0(\Theta) \xrightarrow[\sim]{\beta_1 \otimes_A \cdots \otimes_A \beta_1} B_1 \otimes_A \cdots \otimes_A B_1 \xrightarrow{\mathrm{mult.}} B_\ell.$$

4. MAIN RESULTS

Let $B = \bigoplus_{\ell \geq 0} B_\ell$ be a positively \mathbb{Z} -graded algebra such that $\dim_k B_0 < \infty$. Let $A := B_0$ and let $e \in A$ be an idempotent. Assume that the conditions (A1*), (A2) and (A3) are satisfied, and in addition

$$(A4) \quad eA(1 - e) = 0.$$

That is, we have an isomorphism of algebras $A \simeq \begin{bmatrix} eAe & 0 \\ (1 - e)Ae & \underline{A} \end{bmatrix}$. Combining Proposition 3.10 and (A4) we immediately get that $\mathrm{gl.dim} \underline{A} \leq d - 1$. Moreover recall from Section 2 that $C := eBe$ is also noetherian and that we have $Be \in \underline{\mathrm{CM}}(C)$ and $eB \in \underline{\mathrm{CM}}(C^{\mathrm{op}})$.

The aim of this section is to prove the following result.

Theorem 4.1. *Under assumptions (A1*), (A2), (A3) and (A4), we have the following.*

- (a) *The functor $F : \mathcal{D}^b(\underline{A}) \xrightarrow{\mathrm{Res.}} \mathcal{D}^b(A) \xrightarrow[-\mathbf{L}_{\otimes_A} Be]{\mathrm{Res.}} \mathcal{D}^b(\mathrm{gr} C) \longrightarrow \underline{\mathrm{CM}}^{\mathbb{Z}}(C)$ is a triangle equivalence. Moreover Be is a tilting object in $\underline{\mathrm{CM}}^{\mathbb{Z}}(C)$.*
- (b) *There exists a triangle equivalence $G : \mathcal{C}_{d-1}(\underline{A}) \rightarrow \underline{\mathrm{CM}}(C)$ making the diagram*

$$\begin{array}{ccc}
 \mathcal{D}^b(\underline{A}) & \xrightarrow[\sim]{F} & \underline{\mathrm{CM}}^{\mathbb{Z}}(C) \\
 \downarrow \pi & & \downarrow \mathrm{nat.} \\
 \mathcal{C}_{d-1}(\underline{A}) & \xrightarrow[\sim]{G} & \underline{\mathrm{CM}}(C)
 \end{array}$$

commutative, where $\mathcal{C}_{d-1}(\underline{A})$ is the generalized $(d - 1)$ -cluster category of \underline{A} .

As a consequence we obtain that $\underline{\mathrm{CM}}(C)$ is $(d - 1)$ -Calabi-Yau.

4.1. Notations and plan of the proof. Let us start with some notations which we use in the proof.

We denote as before by $\Theta = \Theta_{d-1}(A)$ a projective resolution of $\mathbf{RHom}_{A^e}(A, A^e)[d-1]$ in $\mathcal{D}(A^e)$, and by $\underline{\Theta} = \Theta_{d-1}(\underline{A})$ a projective resolution of $\mathbf{RHom}_{\underline{A}^e}(\underline{A}, \underline{A}^e)[d-1]$ in $\mathcal{D}(\underline{A}^e)$. For $\ell \geq 1$ we put

$$\Theta^\ell := \underbrace{\Theta \otimes_A \Theta \otimes_A \cdots \otimes_A \Theta}_{\ell \text{ times}} \in \mathcal{D}(A^e) \quad \text{and} \quad \underline{\Theta}^\ell := \underbrace{\underline{\Theta} \otimes_{\underline{A}} \underline{\Theta} \otimes_{\underline{A}} \cdots \otimes_{\underline{A}} \underline{\Theta}}_{\ell \text{ times}} \in \mathcal{D}(\underline{A}^e).$$

We denote by Θ^{-1} a projective resolution of $DA[1-d]$ in $\mathcal{D}(A^e)$, and by $\underline{\Theta}^{-1}$ a projective resolution of $D\underline{A}[1-d]$ in $\mathcal{D}(\underline{A}^e)$. For $\ell \geq 1$ we put

$$\Theta^{-\ell} = \underbrace{\Theta^{-1} \otimes_A \cdots \otimes_A \Theta^{-1}}_{\ell \text{ times}} \in \mathcal{D}(A^e) \quad \text{and} \quad \underline{\Theta}^{-\ell} = \underbrace{\underline{\Theta}^{-1} \otimes_{\underline{A}} \cdots \otimes_{\underline{A}} \underline{\Theta}^{-1}}_{\ell \text{ times}} \in \mathcal{D}(\underline{A}^e).$$

Then for any $\ell, m \in \mathbb{Z}$ we have isomorphisms $\Theta^\ell \otimes_A \Theta^m \simeq \Theta^{\ell+m}$ in $\mathcal{D}(A^e)$ and $\underline{\Theta}^\ell \otimes_{\underline{A}} \underline{\Theta}^m \simeq \underline{\Theta}^{\ell+m}$ in $\mathcal{D}(\underline{A}^e)$.

The proof of Theorem 4.1 is given in the next subsections. It consists of several steps which we outline here for the convenience of the reader.

In subsection 4.2, we construct for all $\ell \geq 0$ an isomorphism

$$(4.1.1) \quad \mathrm{Hom}_{\mathcal{D}(\underline{A})}(\underline{A}, \underline{\Theta}^\ell) \simeq \underline{B}_\ell \quad (\text{Lemma 4.3})$$

compatible with composition in $\mathcal{D}(\underline{A})$ and product in \underline{B} .

In subsection 4.3 we construct a map $\underline{A} \overset{\mathbf{L}}{\otimes}_A Be(1) \rightarrow \underline{\Theta} \overset{\mathbf{L}}{\otimes}_A Be$ in $\mathcal{D}(\mathrm{Gr}(\underline{A}^{\mathrm{op}} \otimes C))$ whose cone is perfect as an object in $\mathcal{D}(\mathrm{Gr}C)$ (Proposition 4.8). With F as in Theorem 4.1(a), it gives us a commutative square for any $\ell \in \mathbb{Z}$

$$(4.1.2) \quad \begin{array}{ccc} \mathcal{D}^b(\underline{A}) & \xrightarrow{F} & \underline{\mathbf{CM}}^{\mathbb{Z}}(C) \\ \downarrow -\otimes_{\underline{A}} \underline{\Theta}^\ell & & \downarrow (\ell) \\ \mathcal{D}^b(\underline{A}) & \xrightarrow{F} & \underline{\mathbf{CM}}^{\mathbb{Z}}(C) \end{array} \quad (\text{Proposition 4.9})$$

and an isomorphism

$$(4.1.3) \quad F(\underline{\Theta}^\ell) \simeq Be(\ell) \quad (\text{Proposition 4.12}).$$

Moreover we can use this to show that F induces a triangle functor $G : \mathcal{C}_{d-1}(\underline{A}) \rightarrow \underline{\mathbf{CM}}(C)$ (Proposition 4.10).

In subsection 4.4 we show that the isomorphisms (4.1.1) and (4.1.3) are compatible with the map $F_{\underline{A}, \underline{\Theta}^\ell}$ for any $\ell \geq 0$, that is, there is a commutative diagram

$$(4.1.4) \quad \begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}^b(\underline{A})}(\underline{A}, \underline{\Theta}^\ell) & \xrightarrow{F_{\underline{A}, \underline{\Theta}^\ell}} & \mathrm{Hom}_{\underline{\mathbf{CM}}^{\mathbb{Z}}(C)}(F(\underline{A}), F(\underline{\Theta}^\ell)) \\ (4.1.1) \downarrow \wr & & (4.1.3) \downarrow \wr \\ \underline{B}_\ell & \xrightarrow[\mathrm{Prop.2.3(b)}]{\sim} & \mathrm{Hom}_{\underline{\mathbf{CM}}^{\mathbb{Z}}(C)}(Be, Be(\ell)) \end{array}$$

It implies that the map $F_{\underline{A}, \underline{\Theta}^\ell}$ is an isomorphism (Proposition 4.13).

The last step of the proof consists of using $(d-1)$ -cluster tilting subcategories in the categories $\mathcal{D}^b(\underline{A})$ and $\underline{\mathbf{CM}}^{\mathbb{Z}}(C)$, (resp. $\mathcal{C}_{d-1}(\underline{A})$ and $\underline{\mathbf{CM}}(C)$) and Proposition 1.7 to show that $F : \mathcal{D}(\underline{A}) \rightarrow \underline{\mathbf{CM}}^{\mathbb{Z}}(C)$ (resp. $G : \mathcal{C}_{d-1}(\underline{A}) \rightarrow \underline{\mathbf{CM}}(C)$) is an equivalence.

4.2. Preprojective algebras. Using the following observation, we identify $\underline{A} \otimes_A \Theta \otimes_A \underline{A}$ and $\underline{\Theta}$ in the rest of this section.

Lemma 4.2. *We have an isomorphism $\underline{A} \otimes_A \Theta \otimes_A \underline{A} \rightarrow \underline{\Theta}$ in $\mathcal{D}(\underline{A}^e)$.*

Proof. We have the following isomorphism

$$\underline{A} \otimes_A \Theta \simeq \mathbf{RHom}_A(DA, \underline{A}) \quad \text{in } \mathcal{D}(\underline{A}^{\text{op}} \otimes A).$$

Let I_{\bullet} be an injective resolution of \underline{A} as an \underline{A}^e -module. It follows from (A4) that I_{\bullet} is also an injective resolution of \underline{A} as an A -module. Hence we have the following isomorphisms in $\mathcal{D}(\underline{A}^e)$:

$$\begin{aligned} \underline{A} \otimes_A \Theta \otimes_A \underline{A} &\simeq \mathbf{RHom}_A(DA, \underline{A}) \otimes_A^{\mathbf{L}} \underline{A} \\ &\simeq \mathbf{Hom}_A(DA, I_{\bullet}) \otimes_A \underline{A} \\ &\simeq \mathbf{Hom}_{A^{\text{op}}}(DI_{\bullet}, A) \otimes_A \underline{A} \\ &\simeq \mathbf{Hom}_{A^{\text{op}}}(DI_{\bullet}, \underline{A}) \\ &\simeq \mathbf{Hom}_{\underline{A}^{\text{op}}}(DI_{\bullet}, \underline{A}) \\ &\simeq \mathbf{Hom}_{\underline{A}}(D\underline{A}, I_{\bullet}) \simeq \underline{\Theta}. \end{aligned}$$

□

Denote by $p_0 : A \rightarrow \underline{A}$ the natural projection in $\mathbf{Mod}(A^e)$. For $\ell \geq 1$ we define the map $p_{\ell} : \Theta^{\ell} \rightarrow \underline{\Theta}^{\ell}$ in $\mathcal{D}(A^e)$ as the following composition:

$$\begin{array}{c} \Theta^{\ell} \simeq A \otimes_A \Theta \otimes_A A \otimes_A \Theta \otimes_A \cdots \otimes_A \Theta \otimes_A A \\ \downarrow p_0 \otimes_A 1_{\Theta} \otimes_A p_0 \otimes_A \cdots \otimes_A p_0 \\ \underline{A} \otimes_A \Theta \otimes_A \underline{A} \otimes_A \Theta \otimes_A \cdots \otimes_A \Theta \otimes_A \underline{A} \\ \downarrow \wr \\ (\underline{A} \otimes_A \Theta \otimes_A \underline{A}) \otimes_{\underline{A}} (\underline{A} \otimes_A \Theta \otimes_A \cdots \otimes_{\underline{A}} (\underline{A} \otimes_A \Theta \otimes_A \underline{A})) \simeq \underline{\Theta}^{\ell}. \end{array}$$

Lemma 4.3. *Let $\beta_{\ell} : H^0(\Theta^{\ell}) \xrightarrow{\sim} B_{\ell}$ be as in Definition 3.13. Then there exists an isomorphism $H^0(\underline{\Theta}^{\ell}) \xrightarrow{\sim} \underline{B}_{\ell}$ making the following diagram commutative.*

$$\begin{array}{ccc} H^0(\Theta^{\ell}) & \xrightarrow[\sim]{\beta_{\ell}} & B_{\ell} \\ \downarrow H^0(p_{\ell}) & & \downarrow \text{nat.} \\ H^0(\underline{\Theta}^{\ell}) & \xrightarrow[\sim]{} & \underline{B}_{\ell}. \end{array}$$

Proof. Let $E := H^0(\Theta)$, $\underline{E} := H^0(\underline{\Theta})$ and for $\ell \geq 1$

$$E^{\ell} := \underbrace{E \otimes_A E \otimes_A \cdots \otimes_A E}_{\ell \text{ times}} \quad \text{and} \quad \underline{E}^{\ell} := \underbrace{\underline{E} \otimes_{\underline{A}} \underline{E} \otimes_{\underline{A}} \cdots \otimes_{\underline{A}} \underline{E}}_{\ell \text{ times}}.$$

(i) We show that $\beta_1 : E \xrightarrow{\sim} B_1$ induces an isomorphism $\underline{E} \xrightarrow{\sim} \underline{B}_1$.

Taking H^0 of the isomorphism $\underline{\Theta} \simeq \underline{A} \otimes_A \Theta \otimes_A \underline{A}$ constructed in Lemma 4.2, we obtain isomorphisms

$$\underline{E} \simeq \underline{A} \otimes_A E \otimes_A \underline{A} \simeq \frac{E}{AeE + EeA} \simeq \frac{B_1}{AeB_1 + B_1eA} \simeq \underline{B}_1 \quad \text{in } \text{Mod}(\underline{A}^e).$$

(ii) We show that $\underline{E} \xrightarrow{\sim} \underline{B}_1$ in (i) induces an isomorphism $\underline{E}^\ell \xrightarrow{\sim} \underline{B}_\ell$ for any $\ell \geq 1$.

Note that for M and N in $\text{Mod}(\underline{A}^e)$ we have a canonical isomorphism $M \otimes_A N \simeq M \otimes_{\underline{A}} N$. Thus we have the following isomorphisms

$$\underline{E}^\ell \simeq \frac{E}{AeE + EeA} \otimes_A \cdots \otimes_A \frac{E}{AeE + EeA} \simeq \frac{E^\ell}{\sum_{i=0}^{\ell-1} E^i e E^{\ell-i}} \simeq \left(\frac{T_A E}{(e)} \right)_\ell.$$

Using the isomorphism of \mathbb{Z} -graded algebras $T_A E \simeq B$ in Theorem 3.3, we obtain

$$\underline{E}^\ell \simeq \left(\frac{T_A E}{(e)} \right)_\ell \simeq \left(\frac{B}{(e)} \right)_\ell \simeq \underline{B}_\ell.$$

(iii) We show that the natural map

$$\text{nat.} : E^\ell \xrightarrow{H^0(p_1) \otimes_A \cdots \otimes_A H^0(p_1)} \underline{E} \otimes_A \cdots \otimes_A \underline{E} \simeq \underline{E} \otimes_{\underline{A}} \cdots \otimes_{\underline{A}} \underline{E} = \underline{E}^\ell$$

makes the following diagram commutative:

$$\begin{array}{ccccc} H^0(\Theta^\ell) & \xrightarrow{\sim} & E^\ell & \xrightarrow{\beta_1 \otimes_A \cdots \otimes_A \beta_1} & B_1 \otimes_A \cdots \otimes_A B_1 & \xrightarrow{\text{mult.}} & B_\ell \\ \downarrow H^0(p_\ell) & & \downarrow \text{nat.} & & & & \downarrow \text{nat.} \\ H^0(\underline{\Theta}^\ell) & \xrightarrow{\sim} & \underline{E}^\ell & \xrightarrow{\sim} & \underline{B}_\ell & & \end{array} \quad \text{(ii)}$$

The right pentagon is clearly commutative since both horizontal maps are induced by the isomorphism of \mathbb{Z} -graded algebras $T_A E \simeq B$.

We then show that the left square is commutative. Since the square

$$\begin{array}{ccc} A \otimes_A A & \xrightarrow{p_0 \otimes_A p_0} & \underline{A} \otimes_A \underline{A} \xrightarrow{\sim} \underline{A} \otimes_{\underline{A}} \underline{A} \\ \downarrow \wr & & \downarrow \wr \\ A & \xrightarrow{p_0} & \underline{A} \end{array}$$

is clearly commutative, we have the assertion from the following isomorphisms:

$$\begin{aligned} (H^0(p_1))^{\otimes_{A^e} \ell} &\simeq (H^0(p_0 \otimes_A 1_\Theta \otimes_A p_0))^{\otimes_{A^e} \ell} \\ &\simeq H^0(p_0) \otimes_A (1_{H^0(\Theta)} \otimes_A H^0(p_0 \otimes_A p_0))^{\otimes_{A^e} \ell-1} \otimes_A 1_{H^0(\Theta)} \otimes_A H^0(p_0) \\ &\simeq H^0(p_0) \otimes_A (1_{H^0(\Theta)} \otimes_A H^0(p_0))^{\otimes_{A^e} \ell-1} \otimes_A 1_{H^0(\Theta)} \otimes_A H^0(p_0) \\ &\simeq H^0(p_\ell). \end{aligned}$$

(iv) Now the assertion follows from the commutative diagram in (iii) since the upper horizontal map is β_ℓ by Proposition 3.14. \square

From Lemma 4.3, we immediately get the following consequence.

Corollary 4.4. *We have an isomorphism $\Pi_d(\underline{A}) \simeq \underline{B}$ of \mathbb{Z} -graded algebras.*

By hypothesis (A3), the algebra \underline{B} is finite dimensional. Therefore we get the following consequence of Theorem 1.12.

Corollary 4.5. *Let $\mathcal{C}_{d-1}(\underline{A})$ be the generalized $(d-1)$ -cluster category associated to \underline{A} . Then the following hold.*

- (a) $\mathcal{C}_{d-1}(\underline{A})$ is a $(d-1)$ -Calabi-Yau triangulated category.
- (b) The object $\pi(\underline{A})$ is a $(d-1)$ -cluster tilting object in $\mathcal{C}_{d-1}(\underline{A})$.
- (c) The category $\text{add}\{\underline{\Theta}^\ell \mid \ell \in \mathbb{Z}\} \subset \mathcal{D}^b(\underline{A})$ is a $(d-1)$ -cluster tilting subcategory of $\mathcal{D}^b(\underline{A})$.

4.3. Compatibility of gradings. For any $\ell \geq 0$, we consider the map

$$q_\ell := p_\ell \otimes_A 1_{Be} : \Theta^\ell \otimes_A Be \rightarrow \underline{\Theta}^\ell \otimes_A Be \quad \text{in } \underline{\mathbf{CM}}^{\mathbb{Z}}(C).$$

Lemma 4.6. *The cone of the map $q_\ell : \Theta^\ell \otimes_A Be \rightarrow \underline{\Theta}^\ell \otimes_A Be$ in $\mathcal{D}(\text{Gr}(\underline{A}^{\text{op}} \otimes C))$ is perfect as an object in $\mathcal{D}(\text{Gr}C)$.*

Proof. The cone of $p_0 : A \rightarrow \underline{A}$ is in $\text{thick}(AeA) \subset \mathcal{D}(A)$. Moreover, since A is of finite global dimension and is finite dimensional, Θ is in $\text{per}A^e$. Then the cone of the map $p_1 = p_0 \otimes_A 1_\Theta \otimes p_0$ is in $\text{thick}(AeA) \subset \mathcal{D}(A)$. By an easy induction, we get that the cone of p_ℓ is in $\text{thick}(AeA) \subset \mathcal{D}(A)$. Since AeA is in $\text{thick}(eA) \subset \mathcal{D}(A)$ and $eA \otimes_A Be = C$, then the cone of $q_\ell = p_\ell \otimes_A 1_{Be}$ is in $\text{thick}(C) \subset \mathcal{D}(\text{Gr}C)$. \square

For $\ell \geq 1$ we consider the map

$$\gamma_\ell := \alpha_\ell \otimes_A 1_{Be} : \Theta^\ell \otimes_A Be \rightarrow Be(\ell) \quad \text{in } \mathcal{D}(\text{Gr}(A^{\text{op}} \otimes C)).$$

Lemma 4.7. *The morphism $1_{\underline{A}} \otimes_A \gamma_1 : \underline{A} \otimes_A \Theta \otimes_A Be \rightarrow \underline{A} \otimes_A Be(1)$ is an isomorphism in $\mathcal{D}(\text{Gr}(\underline{A}^{\text{op}} \otimes C))$.*

Proof. The cone of this morphism is $\underline{A} \otimes_A A(1) \otimes_B Be = \underline{A} \otimes_B Be(1) = \underline{A}e(1) = 0$, so we have the assertion. \square

From Lemmas 4.6 and 4.7 we get the following fundamental consequences.

Proposition 4.8. *The cone of the composition map*

$$\underline{A} \otimes_A Be(1) \xrightarrow{(1_{\underline{A}} \otimes_A \gamma_1)^{-1}} \underline{A} \otimes_A \Theta \otimes_A Be \xrightarrow{(1_{\underline{A}} \otimes_A \Theta) \otimes_{Ap_0} \otimes_A 1_{Be}} \underline{A} \otimes_A \Theta \otimes_A \underline{A} \otimes_A Be \simeq \underline{\Theta} \otimes_A Be$$

in $\mathcal{D}(\text{Gr}(\underline{A}^{\text{op}} \otimes C))$ is perfect as an object in $\mathcal{D}(\text{Gr}C)$.

Proposition 4.9. *The following diagrams of triangle functors are commutative up to isomorphism.*

$$\begin{array}{ccc} \mathcal{D}^b(\underline{A}) & \xrightarrow{F} & \underline{\mathbf{CM}}^{\mathbb{Z}}(C) \\ -\otimes_{\underline{A}} \underline{\Theta} \downarrow & & \downarrow (1) \\ \mathcal{D}^b(\underline{A}) & \xrightarrow{F} & \underline{\mathbf{CM}}^{\mathbb{Z}}(C) \end{array} \quad \begin{array}{ccc} \mathcal{D}^b(\underline{A}) & \xrightarrow{F} & \underline{\mathbf{CM}}^{\mathbb{Z}}(C) \\ -\otimes_{\underline{A}} \underline{\Theta}^{-1} \downarrow & & \downarrow (-1) \\ \mathcal{D}^b(\underline{A}) & \xrightarrow{F} & \underline{\mathbf{CM}}^{\mathbb{Z}}(C). \end{array}$$

Combining Proposition 4.8 with the universal property of the generalized cluster category (Proposition 1.10), we get the following consequence.

Proposition 4.10. *There exists a triangle functor $G : \mathcal{C}_{d-1}(\underline{A}) \rightarrow \underline{\mathbf{CM}}(C)$ such that we have a commutative diagram*

$$\begin{array}{ccc} \mathcal{D}^b(\underline{A}) & \xrightarrow{F} & \underline{\mathbf{CM}}^{\mathbb{Z}}(C) \\ \pi \downarrow & & \downarrow \text{nat.} \\ \mathcal{C}_{d-1}(\underline{A}) & \xrightarrow{G} & \underline{\mathbf{CM}}(C). \end{array}$$

The following isomorphism in $\underline{\mathbf{CM}}^{\mathbb{Z}}(C)$ plays an important role.

Proposition 4.11. *The morphism in Proposition 4.8 gives an isomorphism*

$$\delta : F(\underline{\Theta}) = \underline{\Theta} \otimes_A^{\mathbf{L}} Be \xrightarrow{\sim} \underline{A} \otimes_A^{\mathbf{L}} Be(1) = F(\underline{A}) \quad \text{in } \underline{\mathbf{CM}}^{\mathbb{Z}}(C)$$

such that the following diagram commutes:

$$\begin{array}{ccc} \underline{\Theta} \otimes_A Be & \xrightarrow{q_1} & \underline{\Theta} \otimes_A^{\mathbf{L}} Be \\ \downarrow \gamma_1 & & \downarrow \delta \\ A \otimes_A Be(1) & \xrightarrow{q_0(1)} & \underline{A} \otimes_A^{\mathbf{L}} Be(1) \end{array}$$

Proof. The assertion follows from the following commutative diagram:

$$\begin{array}{ccccc} \underline{\Theta} \otimes_A Be & \xrightarrow{q_1} & & & \underline{\Theta} \otimes_A^{\mathbf{L}} Be \\ \downarrow \gamma_1 & \searrow^{p_0 \otimes_A (1_{\underline{\Theta} \otimes_A Be})} & & & \downarrow \delta \\ & \underline{A} \otimes_A \underline{\Theta} \otimes_A Be & \xrightarrow{(1_{\underline{A} \otimes_A \underline{\Theta}}) \otimes_{A p_0} \otimes_A^{\mathbf{L}} 1_{Be}} & \underline{A} \otimes_A \underline{\Theta} \otimes_A \underline{A} \otimes_A^{\mathbf{L}} Be & \\ & & \searrow^{1_{\underline{A} \otimes_A \gamma_1}} & & \\ A \otimes_A Be(1) & \xrightarrow{q_0(1)} & & & \underline{A} \otimes_A^{\mathbf{L}} Be(1) \end{array}$$

□

For any $\ell \geq 1$, let $\delta_\ell : \underline{\Theta}^\ell \otimes_A^{\mathbf{L}} Be \rightarrow \underline{A} \otimes_A^{\mathbf{L}} Be(\ell)$ be an isomorphism in $\underline{\mathbf{CM}}^{\mathbb{Z}}(C)$ defined as the composition

$$\delta_\ell : \underline{\Theta}^\ell \otimes_A^{\mathbf{L}} Be \xrightarrow{1_{\underline{\Theta}^{\ell-1}} \otimes_A^{\mathbf{L}} \delta} \underline{\Theta}^{\ell-1} \otimes_A^{\mathbf{L}} Be(1) \xrightarrow{1_{\underline{\Theta}^{\ell-2}} \otimes_A^{\mathbf{L}} \delta(1)} \dots \longrightarrow \underline{\Theta} \otimes_A^{\mathbf{L}} Be(\ell-1) \xrightarrow{\delta(\ell-1)} Be(\ell).$$

The isomorphism δ_ℓ and Lemma 4.6 for $\ell = 0$ give us in particular the following consequence.

Proposition 4.12. *For any $\ell \in \mathbb{Z}$ we have $F(\underline{\Theta}^\ell) \simeq Be(\ell)$ in $\underline{\mathbf{CM}}^{\mathbb{Z}}(C)$.*

4.4. **F and G are equivalences.** The following result is the key step for proving that the triangle functors F and G are equivalences.

Proposition 4.13. *The map $F_{\underline{A}, \underline{\Theta}^\ell} : \mathrm{Hom}_{\mathcal{D}(\underline{A})}(\underline{A}, \underline{\Theta}^\ell) \rightarrow \mathrm{Hom}_{\underline{\mathrm{CM}}^{\mathbb{Z}}(C)}(\underline{A} \otimes_A^{\mathbf{L}} Be, \underline{\Theta}^\ell \otimes_A^{\mathbf{L}} Be)$ is an isomorphism for any $\ell \in \mathbb{Z}$.*

In order to prove this we need the following intermediate lemmas.

Lemma 4.14. *The isomorphism $\underline{B}_\ell \simeq \mathrm{Hom}_{\underline{\mathrm{CM}}^{\mathbb{Z}}(C)}(Be, Be(\ell))$ of Proposition 2.3(b) makes the following diagram commutative:*

$$\begin{array}{ccc}
 H^0(\Theta^\ell) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{D}(A)}(A, \Theta^\ell) \\
 \downarrow \beta_\ell & & \downarrow -\otimes_A^{\mathbf{L}} Be \\
 B_\ell & & \mathrm{Hom}_{\underline{\mathrm{CM}}^{\mathbb{Z}}(C)}(Be, \Theta^\ell \otimes_A Be) \\
 \downarrow \text{nat.} & & \downarrow \gamma_\ell \\
 \underline{B}_\ell & \xrightarrow{\sim} & \mathrm{Hom}_{\underline{\mathrm{CM}}^{\mathbb{Z}}(C)}(Be, Be(\ell))
 \end{array}$$

Proof. The above diagram is a part of the following:

$$\begin{array}{ccccc}
 H^0(\Theta^\ell) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{D}(A)}(A, \Theta^\ell) & & \\
 \downarrow \beta_\ell & & \downarrow -\otimes_A^{\mathbf{L}} B & & \\
 B_\ell & & \mathrm{Hom}_{\mathcal{D}(\mathrm{Gr} B)}(B, \Theta^\ell \otimes_A B) & \xrightarrow{-\otimes_B Be} & \mathrm{Hom}_{\mathcal{D}(\mathrm{Gr} C)}(Be, \Theta^\ell \otimes_A Be) \\
 \downarrow \text{nat.} & & \downarrow \alpha_\ell & & \downarrow \gamma_\ell \\
 \underline{B}_\ell & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{D}(\mathrm{Gr} B)}(B, B(\ell)) & \xrightarrow{-\otimes_B Be} & \mathrm{Hom}_{\mathcal{D}(\mathrm{Gr} C)}(Be, Be(\ell)) \\
 & & & & \downarrow \\
 \underline{B}_\ell & \xrightarrow{\sim} & & & \mathrm{Hom}_{\underline{\mathrm{CM}}^{\mathbb{Z}}(C)}(Be, Be(\ell))
 \end{array}$$

The upper left pentagon is commutative by Lemma 3.14. The upper right square is commutative since by definition $\gamma_\ell = \alpha_\ell \otimes_A^{\mathbf{L}} 1_{Be}$. The lower pentagon is commutative since the isomorphism of \mathbb{Z} -graded algebras $\underline{B} \simeq \bigoplus_{\ell \in \mathbb{Z}} \mathrm{Hom}_{\underline{\mathrm{CM}}^{\mathbb{Z}}(C)}(Be, Be(\ell))$ is induced by the isomorphism of \mathbb{Z} -graded algebras $B \simeq \bigoplus_{\ell \in \mathbb{Z}} \mathrm{Hom}_{\mathrm{Gr} B}(B, B(\ell))$ (Proposition 2.3(b)). Hence the original diagram is commutative. \square

Lemma 4.15. *For any $\ell \geq 0$ the following diagram commutes.*

$$\begin{array}{ccc}
 H^0(\Theta^\ell) & \xrightarrow{H^0(p_\ell)} & H^0(\underline{\Theta}^\ell) \\
 \downarrow \wr & & \downarrow \wr \\
 \mathrm{Hom}_{\mathcal{D}(A)}(A, \Theta^\ell) & \xrightarrow{p_\ell} & \mathrm{Hom}_{\mathcal{D}(A)}(A, \underline{\Theta}^\ell) \\
 \downarrow -\mathbb{L}_{\otimes_A Be} & & \uparrow \cdot p_0 \\
 & & \mathrm{Hom}_{\mathcal{D}(\underline{A})}(\underline{A}, \underline{\Theta}^\ell) \\
 & & \downarrow -\mathbb{L}_{\otimes_A Be} \\
 \mathrm{Hom}_{\underline{\mathrm{CM}}^{\mathbb{Z}}(C)}(Be, \Theta^\ell \otimes_A Be) & \xrightarrow{q_\ell \cdot q_0^{-1}} & \mathrm{Hom}_{\underline{\mathrm{CM}}^{\mathbb{Z}}(C)}(\underline{A} \otimes_A Be, \underline{\Theta}^\ell \otimes_A Be)
 \end{array}$$

Proof. This is clear since by definition $q_\ell = p_\ell \otimes_A 1_{Be}$. \square

Lemma 4.16. *We have the following commutative diagram in $\underline{\mathrm{CM}}^{\mathbb{Z}}(C)$:*

$$\begin{array}{ccc}
 \Theta^\ell \otimes_A Be & \xrightarrow{q_\ell} & \underline{\Theta}^\ell \otimes_A Be \\
 \downarrow \gamma_\ell & & \downarrow \delta_\ell \\
 Be(\ell) & \xrightarrow{q_0(\ell)} & \underline{A} \otimes_A Be(\ell)
 \end{array}$$

Proof. For the case $\ell = 1$, the assertion is shown in Proposition 4.11. Assume that the assertion is true for $\ell - 1$. Consider the following commutative diagram:

$$\begin{array}{ccccc}
 \Theta \otimes_A \Theta^{\ell-1} \otimes_A Be & \xrightarrow{1_{\Theta \otimes_A} q_{\ell-1}} & \Theta \otimes_A \underline{\Theta}^{\ell-1} \otimes_A Be & \xrightarrow{p_0 \otimes_A (1_{\Theta \otimes_A} 1_{\underline{\Theta}^{\ell-1} \otimes_A Be})} & \underline{\Theta} \otimes_A \underline{\Theta}^{\ell-1} \otimes_A Be \\
 \downarrow 1_{\Theta \otimes_A} \gamma_{\ell-1} & & \downarrow 1_{\Theta \otimes_A} \delta_{\ell-1} & & \downarrow 1_{\underline{\Theta} \otimes_A} \delta_{\ell-1} \\
 \Theta \otimes_A Be(\ell-1) & \xrightarrow{1_{\Theta \otimes_A} q_0(\ell-1)} & \Theta \otimes_A \underline{A} \otimes_A Be(\ell-1) & \xrightarrow{p_0 \otimes_A (1_{\Theta \otimes_A} 1_{\underline{A} \otimes_A Be(\ell-1)})} & \underline{\Theta} \otimes_A \underline{A} \otimes_A Be(\ell-1) \\
 \downarrow \gamma_1(\ell-1) & & & & \downarrow \delta_1(\ell-1) \\
 Be(\ell) & \xrightarrow{q_0(\ell)} & & & \underline{A} \otimes_A Be(\ell)
 \end{array}$$

Clearly the upper right square is commutative. The upper left square is commutative by our induction assumption, and the lower pentagon is commutative for the case $\ell = 1$. Thus the commutativity for the case ℓ follows from the biggest square. \square

Proof of Proposition 4.13. For $\ell < 0$, the spaces $\mathrm{Hom}_{\mathcal{D}(\underline{A})}(\underline{A}, \underline{\Theta}^\ell)$ and $\mathrm{Hom}_{\underline{\mathrm{CM}}^{\mathbb{Z}}(C)}(F(\underline{A}), F(\underline{\Theta}^\ell)) \simeq \underline{B}_\ell$ vanish, hence $F_{\underline{A}, \underline{\Theta}^\ell}$ is an isomorphism in this case.

For $\ell \geq 0$ consider the following diagram:

$$\begin{array}{ccccc}
 B_\ell & \xleftarrow{\beta_\ell} & H^0(\Theta^\ell) & \xrightarrow{H^0(p_\ell)} & H^0(\underline{\Theta}^\ell) \\
 \downarrow \text{nat.} & & \downarrow \wr & & \downarrow \wr \\
 & & \text{Hom}_{\mathcal{D}(\underline{A})}(A, \Theta^\ell) & & \text{Hom}_{\mathcal{D}(\underline{A})}(\underline{A}, \underline{\Theta}^\ell) \\
 & & \downarrow -\mathbb{L}_{\otimes_A} Be & & \downarrow -\mathbb{L}_{\otimes_A} Be = F_{\underline{A}, \underline{\Theta}^\ell} \\
 & & \text{Hom}_{\underline{\mathbf{CM}}^{\mathbb{Z}}(C)}(Be, \Theta^\ell \otimes_A Be) & \xrightarrow{q_\ell \cdot q_0^{-1}} & \text{Hom}_{\underline{\mathbf{CM}}^{\mathbb{Z}}(C)}(\underline{A} \otimes_A Be, \underline{\Theta}^\ell \otimes_A Be) \\
 & & \downarrow \gamma_\ell \cdot & & \downarrow \delta_\ell \cdot \\
 \underline{B}_\ell & \xrightarrow{\sim} & \text{Hom}_{\underline{\mathbf{CM}}^{\mathbb{Z}}(C)}(Be, Be(\ell)) & \xrightarrow{q_0(\ell) \cdot q_0^{-1}} & \text{Hom}_{\underline{\mathbf{CM}}^{\mathbb{Z}}(C)}(\underline{A} \otimes_A Be, \underline{A} \otimes_A Be(\ell))
 \end{array}$$

By Lemma 4.14 the left hexagon is commutative, by Lemma 4.15 the upper right hexagon is commutative, and by Lemma 4.16 the lower square is commutative. Hence the whole diagram commutes.

Moreover by Lemma 4.3 the map $\beta_\ell : H^0(\Theta^\ell) \simeq B_\ell$ induces an isomorphism $H^0(\underline{\Theta}^\ell) \simeq \underline{B}_\ell$. Therefore the following diagram is commutative

$$\begin{array}{ccc}
 \text{Hom}_{\mathcal{D}(\underline{A})}(\underline{A}, \underline{\Theta}^\ell) & \xrightarrow{F_{\underline{A}, \underline{\Theta}^\ell}} & \text{Hom}_{\underline{\mathbf{CM}}^{\mathbb{Z}}(C)}(F(\underline{A}), F(\underline{\Theta}^\ell)) \\
 \wr \downarrow & & \wr \downarrow q_0(\ell)^{-1} \delta_\ell \cdot q_0 \\
 \underline{B}_\ell & \xrightarrow{\sim} & \text{Hom}_{\underline{\mathbf{CM}}^{\mathbb{Z}}(C)}(Be, Be(\ell))
 \end{array}$$

and $F_{\underline{A}, \underline{\Theta}^\ell}$ is an isomorphism. □

Proof of Theorem 4.1. By Proposition 4.12, the functor F restricted to the subcategory $\text{add}\{\underline{\Theta}^\ell \mid \ell \in \mathbb{Z}\} \subset \mathcal{D}^b(\underline{A})$ induces a functor:

$$\text{add}\{\underline{\Theta}^\ell \mid \ell \in \mathbb{Z}\} \rightarrow \text{add}\{Be(\ell) \mid \ell \in \mathbb{Z}\} \subset \underline{\mathbf{CM}}^{\mathbb{Z}}(C).$$

This is an equivalence by Propositions 4.12 and 4.13. These subcategories are $(d - 1)$ -cluster tilting subcategories by Corollary 4.5 and Proposition 2.3(c). Thus F is an equivalence by Proposition 1.7.

Similarly, the map $G_{\pi_A, \pi_{\underline{A}}}$ is an isomorphism since $F_{\underline{A}, \underline{\Theta}^\ell}$ is an isomorphism for any $\ell \in \mathbb{Z}$. Therefore we deduce that G is an equivalence using the same criterion. □

5. APPLICATION TO QUOTIENT SINGULARITIES

In this section we apply the main theorem in the previous section to invariant rings.

5.1. Setup and main result. Let S be the polynomial ring $k[x_1, \dots, x_d]$ over an algebraically closed field k of characteristic zero, and G be a finite subgroup of $\mathbf{SL}_d(k)$ acting freely on $k^d \setminus \{0\}$. The group G acts on S in a natural way. We denote by $R := S^G$ the invariant ring and by $S * G$ the skew group algebra. Then R is a Gorenstein isolated singularity of Krull dimension d . We assume that G is a cyclic group generated by $g = \text{diag}(\zeta^{a_1}, \dots, \zeta^{a_d})$ with a primitive n -th root ζ of unity and integers a_j satisfying

(B1) $0 < a_j < n$ and $(n, a_j) = 1$ for any j with $1 \leq j \leq d$.

(B2) $a_1 + \cdots + a_d = n$.

We regard $S = k[x_1, \dots, x_d]$ as a $\frac{\mathbb{Z}}{n}$ -graded ring $\bigoplus_{\ell \in \mathbb{Z}} S_\ell$ by putting $\deg x_j = \frac{a_j}{n}$. Since G acts on S by $g \cdot x_i = \zeta^{a_i} x_i$, the invariant subring is given by

$$S^G = \bigoplus_{\ell \in \mathbb{Z}} S_\ell.$$

Now we define graded S^G -modules for each i with $0 \leq i < n$ by

$$T^i := \bigoplus_{\ell \in \mathbb{Z}} S_{\ell + \frac{i}{n}},$$

where the degree ℓ part of T^i is $S_{\ell + \frac{i}{n}}$. Then we have $T^0 = S^G$. Let

$$T := \bigoplus_{i=0}^{n-1} T^i \quad \text{and} \quad T' := \bigoplus_{i=1}^{n-1} T^i.$$

Note that we have $T \simeq S$ as (ungraded) S^G -modules. Define k -algebras by

$$\begin{aligned} A &:= \mathbf{End}_{\mathbf{Gr}(S^G)}(T), & \underline{A} &:= \mathbf{End}_{\underline{\mathbf{CM}}^{\mathbb{Z}}(S^G)}(T) \\ B &:= \mathbf{End}_{S^G}(T), & \underline{B} &:= \mathbf{End}_{\underline{\mathbf{CM}}(S^G)}(T). \end{aligned}$$

Then B and \underline{B} are graded algebras such that $A = B_0$ and $\underline{A} = \underline{B}_0$. We will give explicit presentations of B , A and \underline{A} in terms of quivers with relations in Proposition 5.5.

Let e be the idempotent of $B = \mathbf{End}_{S^G}(T)$ associated with the direct summand T^0 of T . Then we have $eBe \simeq S^G$, $\underline{A} \simeq A/\langle e \rangle$ and $\underline{B} \simeq B/\langle e \rangle$.

Our main result in this section is the following.

Theorem 5.1. *Under the assumptions and notations above, we have the following.*

- (a) The functor $F : \mathcal{D}^b(\underline{A}) \xrightarrow{\text{Res.}} \mathcal{D}^b(A) \xrightarrow{-\overset{\mathbf{L}}{\otimes}_A Be} \mathcal{D}^b(\mathbf{gr} S^G) \longrightarrow \underline{\mathbf{CM}}^{\mathbb{Z}}(S^G)$ is a triangle equivalence. Moreover $T \simeq Be$ is a tilting object in $\underline{\mathbf{CM}}^{\mathbb{Z}}(S^G)$.
- (b) There exists a triangle equivalence $G : \mathcal{C}_{d-1}(\underline{A}) \rightarrow \underline{\mathbf{CM}}(S^G)$ making the diagram

$$\begin{array}{ccc} \mathcal{D}^b(\underline{A}) & \xrightarrow[\sim]{F} & \underline{\mathbf{CM}}^{\mathbb{Z}}(S^G) \\ \downarrow \pi & & \downarrow \text{nat.} \\ \mathcal{C}_{d-1}(\underline{A}) & \xrightarrow[\sim]{G} & \underline{\mathbf{CM}}(S^G) \end{array}$$

commutative, where $\mathcal{C}_{d-1}(\underline{A})$ is the generalized $(d-1)$ -cluster category of \underline{A} .

As a consequence, we recover the following results.

Corollary 5.2. *In the setup above, the following assertions hold.*

- (a) [Aus78, III.1] The stable category $\underline{\mathbf{CM}}(S^G)$ of maximal Cohen-Macaulay R -modules is a $(d-1)$ -Calabi-Yau triangulated category.
- (b) [Iya07a, Thm 2.5] The S^G -module S is a $(d-1)$ -cluster tilting object in $\underline{\mathbf{CM}}(S^G)$.

As a special case of Theorem 5.1 we have the following.

Corollary 5.3. *Let $G \subset \mathrm{SL}_3(k)$ be a finite cyclic subgroup satisfying (B1). Then the stable category $\underline{\mathbf{CM}}(S^G)$ of maximal Cohen-Macaulay modules is triangle equivalent to the generalized 2-cluster category $\mathcal{C}_2(\underline{A})$ for a finite dimensional algebra \underline{A} of global dimension at most 2.*

Proof. We only have to check the condition (B2). Let $g = \mathrm{diag}(\zeta^{a_1}, \zeta^{a_2}, \zeta^{a_3})$ be a generator of G . Since $0 < a_i < n$ and $g \in \mathrm{SL}_3(k)$, we have $a_1 + a_2 + a_3 = n$ or $2n$. If this is n , then (B2) is satisfied. If this is $2n$, then $g^{-1} = \mathrm{diag}(\zeta^{n-a_1}, \zeta^{n-a_2}, \zeta^{n-a_3})$ satisfies (B2) since $(n - a_1) + (n - a_2) + (n - a_3) = n$. \square

Remark 5.4. (a) The equivalence $F : \mathcal{D}^b(\underline{A}) \rightarrow \underline{\mathbf{CM}}^{\mathbb{Z}}(S^G)$ is obtained by Ueda [Ued08]. Our proof is very different since he uses a strong theorem due to Orlov [Orl05].

(b) The equivalence $G : \mathcal{C}_{d-1}(\underline{A}) \rightarrow \underline{\mathbf{CM}}(S^G)$ is an analog of an independent result proved by Thanhoffer de Völcsey and Van den Bergh [TV10, Proposition 1.3]. They use generalized cluster categories associated with quivers with potential instead of those associated with algebras of finite global dimension.

5.2. Proof of Theorem 5.1. Let G be a finite cyclic subgroup of $\mathrm{SL}_d(k)$ generated by $g = \mathrm{diag}(\zeta^{a_1}, \dots, \zeta^{a_d})$ as above, and let S^G , B , \underline{B} , A and \underline{A} be as defined in the previous subsection. Then $B = \mathrm{End}_{S^G}(S)$ is isomorphic to the skew group algebra $S * G$ by [Aus86, Yos90], which is known to have global dimension d . We want to show that conditions (A1*) to (A4) in the previous section are satisfied in this case. We start with condition (A1*), and here we need some notation.

First we give a concrete description of the McKay quiver Q of the cyclic group G [McK80]. The set Q_0 of vertices is $\mathbb{Z}/n\mathbb{Z}$. The arrows are

$$x_j^i = x_j : i \rightarrow i + a_j \quad (i \in \mathbb{Z}/n\mathbb{Z}, 1 \leq j \leq d).$$

Proposition 5.5. (a) *A presentation of B is given by the McKay quiver with commutative relations*

$$x_{j'}^{i+a_j} x_j^i = x_j^{i+a_{j'}} x_{j'}^i \quad (i \in \mathbb{Z}/n\mathbb{Z}, 1 \leq j, j' \leq d).$$

(b) *A presentation of A is obtained from that of B by removing all arrows $x_j^i : i \rightarrow i'$ with $i > i'$.*

(c) *A presentation of \underline{A} is obtained from that of A by removing the vertex 0.*

Proof. (a) This is known (e.g. [CMT07, Prop. 2.8(3)], [BSW10, Cor. 4.2]).

(b) By our grading on T , the degree of the morphism $x_j^i : T^i \rightarrow T^{i'}$ is 0 if $i < i'$, and 1 otherwise. Thus we have the assertion.

(c) This is clear. \square

We denote by Q_ℓ the set of paths of length ℓ , and by kQ_ℓ the k -vector space with basis Q_ℓ . Then kQ_0 is a k -algebra which we denote by $L := kQ_0$. Clearly we have

$$kQ_\ell = \underbrace{kQ_1 \otimes_L \cdots \otimes_L kQ_1}_{\ell \text{ times}}.$$

Define a vector space U_ℓ as the factor space of kQ_ℓ modulo the subspace generated by

$$v \otimes x_i \otimes x_j \otimes v' + v \otimes x_j \otimes x_i \otimes v'.$$

We denote by $v_1 \wedge v_2 \wedge \cdots \wedge v_\ell$ the image of $v_1 \otimes v_2 \otimes \cdots \otimes v_\ell$ in U_ℓ . Then U_ℓ has a basis consisting of

$$x_{j_\ell} \wedge x_{j_{\ell-1}} \wedge \cdots \wedge x_{j_1}$$

where

$$i \xrightarrow{x_{j_1}} i + a_{j_1} \xrightarrow{x_{j_2}} \cdots \xrightarrow{x_{j_\ell}} i + a_{j_1} + \cdots + a_{j_\ell}$$

is a path of length ℓ satisfying $j_1 < j_2 < \cdots < j_\ell$. Now let

$$P_\bullet := (B \otimes_L U_d \otimes_L B \xrightarrow{\delta_d} B \otimes_L U_{d-1} \otimes_L B \xrightarrow{\delta_{d-1}} \cdots \xrightarrow{\delta_1} B \otimes_L U_0 \otimes_L B),$$

where δ_ℓ is defined by

$$\begin{aligned} & \delta_\ell(b \otimes (x_{j_1} \wedge x_{j_2} \wedge \cdots \wedge x_{j_{\ell-1}} \wedge x_{j_\ell}) \otimes b') \\ & := \sum_{i=1}^{\ell} (-1)^{i-1} (bx_{j_i} \otimes (x_{j_1} \wedge \cdots \overset{\vee}{x_{j_i}} \cdots \wedge x_{j_\ell}) \otimes b' + b \otimes (x_{j_1} \wedge \cdots \overset{\vee}{x_{j_i}} \cdots \wedge x_{j_\ell}) \otimes x_{j_i} b'). \end{aligned}$$

Then we have the following result which implies the condition (A1*).

Theorem 5.6. *The complex P_\bullet is a projective resolution of the graded B^e -module B satisfying $P_\bullet \simeq P_\bullet^\vee[d](-1)$ in $\mathcal{C}^b(\mathbf{gproj} B^e)$. In particular B is a bimodule d -Calabi-Yau algebra of Gorenstein parameter 1.*

Proof. The assertion not involving the grading is known and easy to check (e.g. [BSW10, Thm 6.2]). We will show that each δ_ℓ is homogeneous of degree 0 by introducing a certain grading on P_\bullet . Define the degree map $g : Q_1 \rightarrow \mathbb{Z}$ by

$$g(i \xrightarrow{x_j} i') := \begin{cases} 0 & 0 \leq i < i' < n, \\ 1 & 0 \leq i' < i < n. \end{cases}$$

Then we have a well-defined degree map

$$g(x_{j_1} \wedge \cdots \wedge x_{j_\ell}) := g(x_{j_1}) + \cdots + g(x_{j_\ell})$$

on basis vectors of U_ℓ . Since the value is always 0 or 1 by the condition (B2) $a_1 + \cdots + a_d = n$, we have a decomposition

$$U_\ell = U_\ell^0 \oplus U_\ell^1$$

where U_ℓ^0 (respectively, U_ℓ^1) is the subspace spanned by the elements of degree 0 (respectively, 1). We regard U_ℓ^0 as having degree 0 and U_ℓ^1 as having degree 1. Then each map δ_ℓ is homogeneous of degree 0. \square

We proceed to show the other conditions.

Lemma 5.7. *The graded algebra $S * G$ satisfies the conditions (A1*), (A2), (A3) and (A4) in Theorem 4.1.*

Proof. (A1*) This was shown in the previous theorem.

(A2) The ring $B = S * G$ is clearly noetherian.

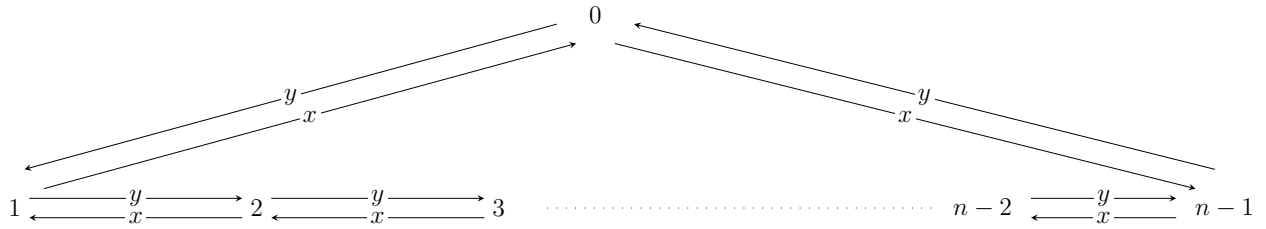
(A3) S^G is an isolated singularity by (B1). Then the stable category $\underline{\mathbf{CM}}(S^G)$ has finite dimensional homomorphism spaces [Aus78, Yos90]. Hence $\dim_k \underline{B}$ is finite.

(A4) It is a direct consequence of the definition of A that the vertex 0 in the McKay quiver is a source. We use the idempotent e corresponding to this vertex. \square

Now Theorem 5.1 is an immediate consequence of Theorem 4.1 and Lemma 5.7. \square

In the subsections 5.3, 5.4 and 5.5, which are devoted to examples, we use the notation $\frac{1}{n}(a_1, \dots, a_d)$ for the element $\text{diag}(\zeta^{a_1}, \dots, \zeta^{a_d}) \in \text{SL}_d(k)$, where $a_1 + \dots + a_d = n$ and ζ is a primitive n -root of unity.

5.3. Example: Case $d = 2$. Let $G \subset \text{SL}_2(k)$ be a finite cyclic subgroup. Then there exists a generator of the form $\frac{1}{n}(1, n - 1)$. The algebra $S * G$ is presented by the McKay quiver



with the commutativity relation $xy = yx$. The grading induced by the generator $\frac{1}{n}(1, n - 1)$ makes the arrows x of degree 0 and the arrows y of degree 1. The idempotent corresponding to the direct summand T_0 of T corresponds to the vertex 0 of the McKay quiver. Hence, the algebra $\underline{A} = \text{End}_{\underline{\text{CM}}^{\mathbb{Z}}(S^G)}(T)$ is isomorphic to kQ where Q is A_{n-1} with the linear orientation. Hence by Theorem 5.1, we obtain a triangle equivalence $\underline{\text{CM}}(S^G) \simeq \mathcal{C}_1(A_{n-1})$.

More generally, if G is a finite subgroup (not necessarily cyclic) of $\text{SL}_2(k)$, the algebra $B = S * G$ is Morita equivalent to the preprojective algebra $\Pi_2(\tilde{Q})$ of an extended Dynkin quiver \tilde{Q} . There exists a \mathbb{Z} -grading on B such that $A := B_0$ is Morita equivalent to the path algebra $k\tilde{Q}$ and B is bimodule 2-Calabi-Yau of Gorenstein parameter 1. Moreover B has an idempotent e such that $eBe = S^G$ and e is the exceptional vertex of \tilde{Q} . Thus by Theorem 4.1 we have a triangle equivalence $\mathcal{C}_1(kQ) \simeq \underline{\text{CM}}(S^G)$ for $Q := \tilde{Q} \setminus \{e\}$.

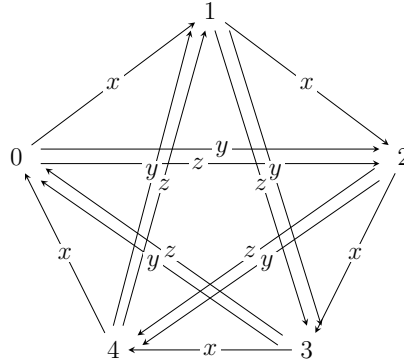
Moreover, the triangulated category $\mathcal{C}_1(kQ)$ is equivalent to the category $\text{proj} \Pi_2(kQ)$, where $\Pi_2(kQ)$ is the preprojective algebra associated to the Dynkin quiver Q . Hence we recover the well-known proposition below.

Proposition 5.8. *Let $G \subset \text{SL}_2(k)$ be a finite subgroup and Q be the corresponding Dynkin quiver.*

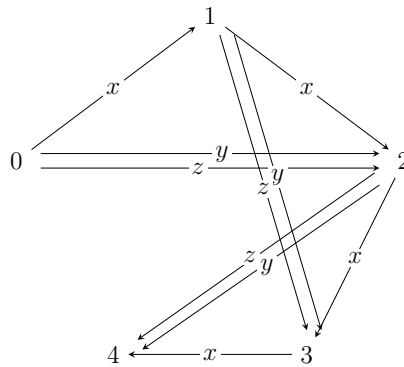
- (a) [Rei87, RV89, BSW10] *We have equivalences $\underline{\text{CM}}(S^G) \simeq \mathcal{C}_1(kQ) \simeq \text{proj} \Pi_2(kQ)$.*
- (b) [KST07, LP11] *We have equivalences $\underline{\text{CM}}^{\mathbb{Z}}(S^G) \simeq \mathcal{D}^b(kQ) \simeq \text{gr} \text{proj} \Pi_2(kQ)$.*

Remark 5.9. Let \underline{A} be a finite-dimensional algebra of global dimension at most 1. Then, if k is algebraically closed, \underline{A} is Morita equivalent to the path algebra kQ of an acyclic quiver Q . The 1-cluster category $\mathcal{C}_1(kQ)$ is **Hom**-finite if and only if Q is of Dynkin type. Thus we obtain a kind of converse of Theorem 4.1 for $d = 2$: every 1-cluster category can be realized as the stable category of Cohen-Macaulay modules over an isolated singularity.

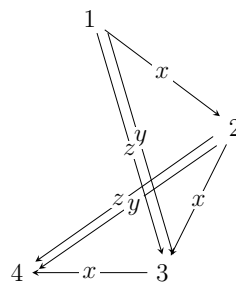
5.4. **Example: Case $d = 3$.** Let $G \subset \mathrm{SL}_3(k)$ be the subgroup generated by $\frac{1}{5}(1, 2, 2)$. Then $B = S * G$ is presented by the McKay quiver



with the commutativity relations $xy = yx$, $yz = zy$, $zx = xz$. By the choice of the grading, the algebra A , which is the degree 0 part of B , is presented by the quiver



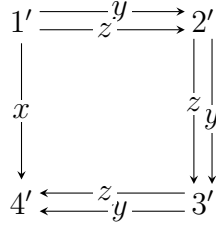
with the commutativity relations. The idempotent e of the algebra B corresponds to the summand S^G which corresponds to the vertex 0. Therefore \underline{A} is presented by the quiver



with the commutativity relations. By Theorem 5.1 the category $\underline{\mathrm{CM}}(S^G)$ is triangle equivalent to the generalized cluster category $\mathcal{C}_2(\underline{A})$.

Now take another generator of the group G given by $\frac{1}{5}(3, 1, 1)$. Then the algebra B is same as the above, but has a different grading. We denote by A' its degree zero subalgebra.

One then easily checks that the algebra \underline{A}' is given in this case by the quiver

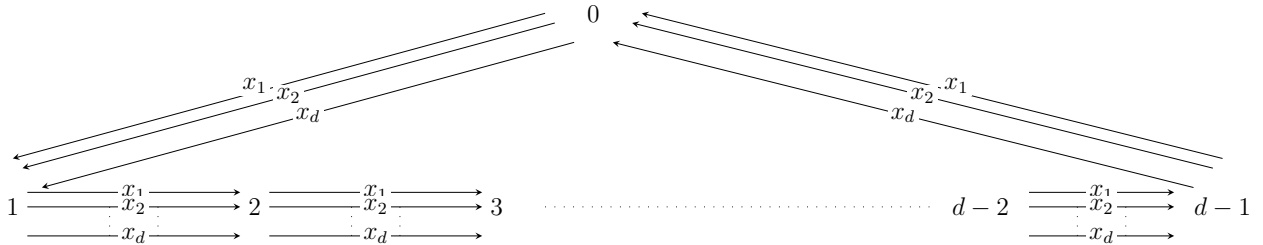


with commutativity relations.

By Theorem 5.1 the category $\underline{\mathbf{CM}}(S^G)$ is triangle equivalent to the generalized cluster category $\mathcal{C}_2(\underline{A}')$. Hence we get a triangle equivalence between the generalized cluster categories $\mathcal{C}_2(\underline{A}) \simeq \mathcal{C}_2(\underline{A}')$ (the algebras \underline{A} and \underline{A}' are cluster equivalent in the sense of [AO10]). However, one can show that the algebras \underline{A} and \underline{A}' are not derived equivalent since they have different Coxeter polynomials. (One can also see this using results of [AO10].) Now we have two different gradings on S^G , which we denote by \mathbb{Z} and \mathbb{Z}' . Then we have

$$\underline{\mathbf{CM}}^{\mathbb{Z}}(S^G) \simeq \mathcal{D}^b(\underline{A}) \not\simeq \mathcal{D}^b(\underline{A}') \simeq \underline{\mathbf{CM}}^{\mathbb{Z}'}(S^G).$$

5.5. Example: General d . Now let $d = n$ and G be generated by $\frac{1}{d}(1, \dots, 1)$. Then, proceeding as before, it is not hard to see that $B = S * G$ is presented by the McKay quiver



with the commutative relations $x_j x_i = x_i x_j$. Then, with the grading corresponding to the generator $\frac{1}{d}(1, \dots, 1)$, one can check that the algebra A is the d -Beilinson algebra and the algebra \underline{A} is given by the quiver



with the commutativity relations.

For the case $d = 3$ the equivalence $\mathcal{C}_2(\underline{A}) \simeq \underline{\mathbf{CM}}(S^G)$ was already proved in [KR08] using a recognition theorem for the acyclic 2-cluster category.

6. EXAMPLES COMING FROM DIMER MODELS

In this section we show that our main theorem applies to examples coming from dimer models which do not come from quotient singularities. This builds upon results from [Bro12, IU09, Dav11, Boc11] which we recall.

6.1. 3-Calabi-Yau algebras from dimer models. Let Γ be a bipartite graph on a torus. We denote by Γ_0 (resp. Γ_1 , and Γ_2) the set of vertices (resp. edges and faces) of the graph. To such a graph we associate a quiver with a potential (Q, W) in the sense of [DWZ08]. The quiver viewed as an oriented graph on the torus is the dual of the graph Γ . Faces of Q dual to white vertices are oriented clockwise and faces of Q dual to black vertices are oriented anti-clockwise. Hence any vertex $v \in \Gamma_0$ corresponds canonically to a cycle c_v of Q . The potential W is defined as

$$W = \sum_{v \text{ white}} c_v - \sum_{v \text{ black}} c_v.$$

Assume that there exists a consistent charge on this graph, that is, a map $R : Q_1 \rightarrow \mathbb{R}_{>0}$ such that

- $\forall v \in \Gamma_0 \quad \sum_{a \in Q_1, a \in c_v} R(a) = 2.$
- $\forall i \in Q_0 \quad \sum_{a \in Q_1, s(a)=i} (1 - R(a)) + \sum_{a \in Q_1, t(a)=i} (1 - R(a)) = 2$

For such a consistent dimer model, there always exists a perfect matching, that is a subset D of Γ_1 such that any vertex of Γ_0 belongs to exactly one edge in D . Since Q is the dual of Γ we regard D as a subset of Q_1 . We define a grading d_D on kQ as follows:

$$d_D(a) = \begin{cases} 1 & \text{if } a \in D \\ 0 & \text{else.} \end{cases}$$

Since D is a perfect matching, for any vertex $v \in \Gamma_0$ the cycle c_v contains exactly one arrow of degree 1, and then the potential W is homogeneous of degree 1. Hence D induces a grading d_D on the Jacobian algebra B . In other words D is a cut of (Q, W) in the sense of [HI11].

Proposition 6.1. *Let B be a Jacobian algebra coming from a consistent dimer model. Any perfect matching induces a grading on B making it bimodule 3-Calabi-Yau of Gorenstein parameter 1.*

Proof. We define the following complex P_\bullet of graded projective B -bimodules:

$$\cdots \longrightarrow 0 \longrightarrow P_3 \xrightarrow{\partial_2} P_2 \xrightarrow{\partial_1} P_1 \xrightarrow{\partial_0} P_0 \longrightarrow 0 \longrightarrow \cdots,$$

where

$$\begin{aligned} P_0 &= \bigoplus_{i \in Q_0} B e_i \otimes e_i B \\ P_1 &= \bigoplus_{a \in Q_1} (B e_{t(a)} \otimes e_{s(a)} B)(-d(a)) \\ P_2 &= \bigoplus_{b \in Q_1} (B e_{s(b)} \otimes e_{t(b)} B)(1 - d(b)) \\ P_3 &= \bigoplus_{i \in Q_0} B e_i \otimes e_i B(-1) \end{aligned}$$

and where the maps ∂_0 , ∂_1 and ∂_2 are defined as follows.

$$\begin{aligned} \partial_2(e_i \otimes e_i) &= \sum_{a, t(a)=i} a \otimes e_i - \sum_{b, s(b)=i} e_i \otimes b; \\ \partial_1(e_{s(b)} \otimes e_{t(b)}) &= \sum_{a \in Q_1} \partial_{b,a} W \text{ where } \partial_{b,a}(bpaq) = p \otimes q \in B e_{t(a)} \otimes e_{s(a)} B; \\ \partial_0(e_{t(a)} \otimes e_{s(a)}) &= a \otimes e_{s(a)} - e_{t(a)} \otimes a. \end{aligned}$$

By [Bro12, Thm 7.7] this complex is a projective resolution of B as a bimodule and satisfies $P_\bullet^\vee \simeq P[3]$ in $\mathcal{C}^b(\mathbf{proj} B^e)$. It is then easy to check that, as a graded complex, it satisfies $P_\bullet^\vee \simeq P[3](-1)$ in $\mathcal{C}^b(\mathbf{gr} \mathbf{proj} B^e)$. Hence the graded algebra B is bimodule 3-Calabi-Yau of Gorenstein parameter 1. \square

Remark 6.2. It is proved in [Bro12, Dav11, Boc11] that the Jacobian algebra $B = \text{Jac}(Q, W)$ is a non commutative crepant resolution of its center $C = Z(B)$ which is the coordinate ring of a Gorenstein affine toric threefold. Moreover the coordinate ring of any Gorenstein affine toric threefold can be obtained from a consistent dimer model [Gul08, IU09].

The following result gives an interpretation of the stable category of Cohen-Macaulay modules over certain Gorenstein affine toric threefold in terms of cluster categories.

Theorem 6.3. *Let Γ be a consistent dimer model, and denote by $B = \text{Jac}(Q, W)$ the associated Jacobian algebra. Assume there exists a perfect matching D and a vertex i of Q with the following properties:*

- *the degree zero part A of B with respect to d_D is finite dimensional*
- *i is a source of the quiver $Q - D$.*
- *the algebra $B/\langle e_i \rangle$ is finite dimensional.*

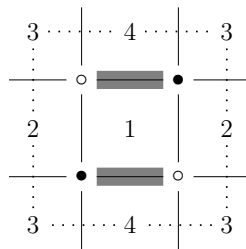
Denote by C the center of the algebra B , and \underline{A} the algebra $A/\langle e_i \rangle$. Then the algebra C is a Gorenstein isolated singularity, and we have the following triangle equivalences

$$\begin{array}{ccc} \mathcal{D}^b(\underline{A}) & \xrightarrow{\sim} & \underline{\text{CM}}^{\mathbb{Z}}(C) \\ \downarrow & & \downarrow \\ \mathcal{C}_2(\underline{A}) & \xrightarrow{\sim} & \underline{\text{CM}}(C) \end{array}$$

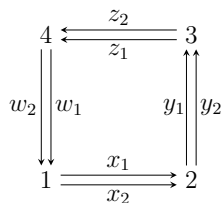
where $\mathcal{C}_2(\underline{A})$ is the generalized 2-cluster category associated to the algebra \underline{A} .

Proof. The algebra B satisfies A1* by Proposition 6.1. The algebra C is a Gorenstein affine toric threefold and B is a finitely generated C -module, hence B is noetherian. The hypothesis on the perfect matching D and the vertex i are clearly equivalent to (A3) and (A4). Moreover by [Bro12, Lemma 5.6], the center of B is isomorphic to eBe for any primitive idempotent e of B . Hence Theorem 6.3 is a consequence of Theorem 4.1. \square

6.2. Examples. Let Γ and D be given by the following picture.



The associated Jacobian algebra B is presented by the quiver



with potential $W = w_1 z_1 y_1 x_1 + w_2 z_2 y_2 x_2 - w_1 z_2 y_1 x_2 - w_2 z_1 y_2 x_1$.

The center C of this algebra is the semigroup algebra $C = \mathbb{C}[\mathbb{Z}^3 \cap \sigma^\vee]$ where σ^\vee is the positive cone

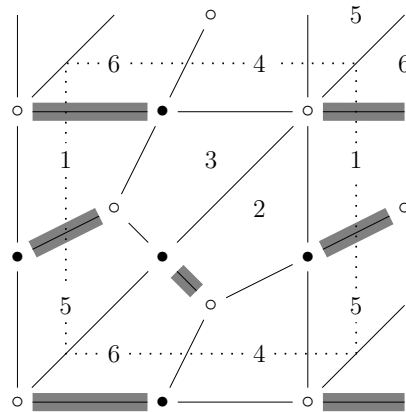
$$\sigma^\vee = \{\lambda_1 n_1 + \lambda_2 n_2 + \lambda_3 n_3 + \lambda_4 n_4, \lambda_i \geq 0\}, \quad n_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, n_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, n_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, n_4 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

The algebra C is a homogenous coordinate algebra of $\mathbb{P}^1 \times \mathbb{P}^1$.

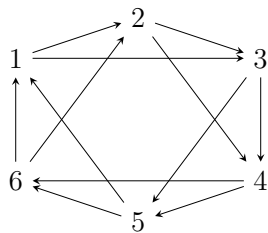
Then the perfect matching D corresponds to $\{x_1, x_2\}$. Thus the quiver of the subalgebra $A = B_0$ is acyclic so A is finite dimensional and the vertex 1 becomes a source in the quiver of A . Moreover, the algebra $\underline{B} = B/\langle e_1 \rangle$ is the path algebra of an acyclic quiver, so it is finite dimensional. Therefore we can apply Theorem 6.3 and we obtain a triangle equivalence $\mathcal{C}_2(\underline{A}) \simeq \underline{\mathbf{CM}}(C)$ where \underline{A} is the path algebra of the quiver $2 \rightrightarrows 3 \rightrightarrows 4$.

We end this paper by giving a non-commutative example. Note that in Theorem 4.1 the algebra C is not necessarily commutative, and the idempotent e is not necessarily primitive.

Let Γ be the following dimer model.



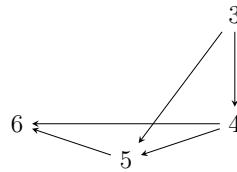
The associated Jacobian algebra B is presented by the quiver



with potential

$$W = a_{65}a_{54}a_{43}a_{32}a_{21}a_{16} + a_{26}a_{64}a_{42} + a_{15}a_{53}a_{31} - a_{16}a_{64}a_{43}a_{31} - a_{65}a_{53}a_{32}a_{26} - a_{21}a_{15}a_{54}a_{42}.$$

In this case it is easy to check that the algebra $B/\langle e \rangle$ is not finite dimensional for any primitive idempotent e , or in other words, the center of B is not an isolated singularity. However, the degree zero subalgebra $A = B_0$ and the algebra $\underline{B} = B/\langle e_1 + e_2 \rangle$ are the path algebras of acyclic quivers, and are therefore finite dimensional. We can apply Theorem 4.1 with the perfect matching D described in the picture above. We obtain a triangle equivalence $\mathcal{C}_2(\underline{A}) \simeq \underline{\mathbf{CM}}(C)$ where \underline{A} is the path algebra of the quiver



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