Class expansion of some symmetric functions in Jucys-Murphy elements

Michel Lassalle
Centre National de la Recherche Scientifique
Institut Gaspard-Monge, Université de Marne-la-Vallée
77454 Marne-la-Vallée Cedex, France
lassalle@univ-mlv.fr
http://igm.univ-mlv.fr/~lassalle

Abstract

We present a method to compute the class expansion of a symmetric function specialized at the Jucys-Murphy elements of the symmetric group. We apply this method to one-row Hall-Littlewood symmetric functions, which interpolate between power sums and complete symmetric functions.

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1 Introduction

Let S_n be the group of permutations of n letters, $\mathbb{C}[S_n]$ its group algebra and $\mathbb{Z}_n$ the center of $\mathbb{C}[S_n]$. Given a partition μ with weight n, denote by C_μ the conjugacy class of permutations having cycle-type μ, viewed as the formal sum of its elements. Since Farahat and Higman [5] it is known that these classes form a basis of $\mathbb{Z}_n$.

For $i = 1, \ldots, n$ the Jucys-Murphy elements J_i are defined by $J_i = \sum_{j<i}(ji)$, where (ji) is a transposition. These commutative elements were introduced independently in [8] and [25, 26]. They do not belong to $\mathbb{Z}_n$. However Jucys and Murphy proved, by different means, that $\mathbb{Z}_n$ is the abstract symmetric algebra $\mathbb{S}$ specialized at the J_i's, i.e. that we have $\mathbb{S}[J_1, \ldots, J_n] = \mathbb{Z}_n$.

Given a symmetric function f, it is therefore a natural problem to study the class expansion

$$f(J_1, \ldots, J_n) = \sum_{|\mu|=n} a_\mu(n)\, C_\mu,$$

i.e. the development of its specialization $f(J_1, \ldots, J_n)$ in terms of the basis C_μ of $\mathbb{Z}_n$. The purpose of this paper is to present a general method to compute such an expansion.

This problem had been solved before by Jucys [8] for $f = e_k$, the elementary symmetric function, and by Lascoux and Thibon [13] for $f = p_k$, the power-sum symmetric function.

Our method provides a new proof for these classical results, but also allows to handle many new cases.

In this paper we consider $f = h_k$, the complete symmetric function and $f = P_k(z)$ the one-row Hall-Littlewood symmetric function, which interpolates between h_k and p_k. But, among others, our method also works with the product $f = h_k e_l$ or with $f = s_{(k,1^l)}$, the Schur function associated with hooks.

It should be emphasized that our method is not algebraic, but analytic. We do not work in the symmetric algebra $\mathbb{C}[S_n]$, but rather in the shifted symmetric algebra [11, 29]. Actually given a partition λ and χ^λ the character of the corresponding irreducible representation, by a celebrated result of Jucys and Murphy we have

$$f(J_1, \ldots, J_n)\, \chi^\lambda = f(A_\lambda)\, \chi^\lambda,$$

with A_λ the alphabet of "contents" of λ.

This fundamental property allows us to translate the class expansion of the central element $f(J_1, \ldots, J_n)$ in terms of the content evaluation $f(A_\lambda)$. Namely we may write *equivalently*

$$f(J_1, \ldots, J_n) = \sum_{|\mu|=n} a_\mu(n)\, C_\mu \quad \text{or} \quad f(A_\lambda) = \sum_{|\mu|=n} a_\mu(n)\, \theta_\mu^\lambda,$$

with θ_μ^λ the central character of λ taken at μ. In this paper we consider the second equality, which connects two shifted symmetric functions and can be studied by analytic means.

This method presents two advantages. Firstly its proofs keep totally elementary. Secondly it has a very natural extension in the framework of Jack polynomials. In that context the symmetric algebra and the Jucys-Murphy elements have not yet been generalized, but the algebra of α-shifted symmetric functions is very well known.

The paper is organized as follows. Section 2 is devoted to general facts about the symmetric group. Section 3 recalls results about the transition measure. Section 4 presents our tools and a summary of our method. The latter is used in Sections 5 and 6 to recover the classical results of Jucys [8] and Lascoux-Thibon [13]. Sections 7 and 8 are respectively devoted to the new cases of complete symmetric functions and Hall-Littlewood symmetric functions. The generating functions associated with these class expansions are considered in Section 9. An extension of our method in the framework of Jack polynomials is briefly sketched at the end.

2 Generalities and notations

We recall some notions about the symmetric group and its representations, referring the reader to [4] for an introduction. In this paper n is a fixed positive integer.

2.1 Permutations and partitions

A partition $\lambda = (\lambda_1, ..., \lambda_r)$ is a finite weakly decreasing sequence of nonnegative integers, called parts. The number $l(\lambda)$ of positive parts is called the length of λ, and $|\lambda| = \sum_{i=1}^r \lambda_i$ the weight of λ. For any integer $i \geq 1$, $m_i(\lambda) = \mathrm{card}\{j : \lambda_j = i\}$ is the multiplicity of the part i in λ. We write $\lambda = (1^{m_1(\lambda)}, 2^{m_2(\lambda)}, 3^{m_3(\lambda)}, \ldots)$, and $\lambda \vdash n$ for $|\lambda| = n$.

We denote by λ' the partition conjugate to λ, with parts given by $\lambda'_i = \sum_{j \geq i} m_j(\lambda)$. We identify λ with its Ferrers diagram $\{(i, j) : 1 \leq i \leq l(\lambda), 1 \leq j \leq \lambda_i\}$. We set

$$z_\lambda = \prod_{i \geq 1} i^{m_i(\lambda)} m_i(\lambda)!, \qquad H_\lambda = \prod_{(i,j) \in \lambda} (\lambda_i + \lambda'_j - i - j + 1).$$

For any partition λ and any integer $1 \leq i \leq l(\lambda) + 1$, we denote by $\lambda^{(i)}$ the partition μ (if it exists) such that $\mu_j = \lambda_j$ for $j \neq i$ and $\mu_i = \lambda_i + 1$. Similarly for any integer $1 \leq i \leq l(\lambda)$, we denote by $\lambda_{(i)}$ the partition ν (if it exists) such that $\nu_j = \lambda_j$ for $j \neq i$ and $\nu_i = \lambda_i - 1$.

Given some positive integers p, q and a partition λ, we denote by $\lambda \setminus (p) \cup (q)$ the partition (if it exists) obtained by removing a part p and adding a part q to λ.

Given a partition ρ, we denote by $\overline{\rho}$ the partition obtained by erasing all parts 1 of ρ. Thus $m_1(\overline{\rho}) = 0$ and $\rho = \overline{\rho} \cup 1^{|\rho| - |\overline{\rho}|}$. Conversely we denote by $\tilde{\rho}$ the partition obtained by adding parts 1 to ρ up to the weight n. Thus $m_1(\tilde{\rho}) = m_1(\rho) + n - |\rho|$ and $\tilde{\rho} = \rho \cup 1^{n-|\rho|}$.

Let S_n be the group of permutations of n letters, $\mathbb{C}[S_n]$ its group algebra and $\mathbb{Z}_n$ the center of $\mathbb{C}[S_n]$. Each permutation $\sigma \in S_n$ factorizes uniquely as a product of disjoint cycles, whose respective lengths are ordered such as to form a partition $\mu = (\mu_1, \ldots, \mu_r)$ with weight n. This partition, called the cycle-type of σ, determines each permutation up to conjugacy in S_n. Given a partition $\mu \vdash n$, we denote by C_μ the conjugacy class of permutations having cycle-type μ.

We view any central function χ on S_n as the formal sum $\sum_\sigma \chi(\sigma)\sigma \in \mathbb{Z}_n$. In particular we identify each C_μ with the formal sum of its elements. The set $\{C_\mu, \mu \vdash n\}$ forms a basis of $\mathbb{Z}_n$.

2.2 Symmetric functions

Let $A = \{a_1, a_2, a_3, \ldots\}$ a (possibly infinite) set of independent indeterminates, called an alphabet. The generating functions

$$E_z(A) = \prod_{a \in A} (1 + za) = \sum_{k \geq 0} z^k e_k(A), \qquad H_z(A) = \prod_{a \in A} (1 - za)^{-1} = \sum_{k \geq 0} z^k h_k(A)$$

define symmetric functions known as respectively elementary and complete. The power sum symmetric functions are defined by $p_k(A) = \sum_{i \geq 1} a_i^k$. For any partition μ, we define functions e_μ, h_μ or p_μ by

$$f_\mu = \prod_{i=1}^{l(\mu)} f_{\mu_i} = \prod_{k \geq 1} f_k^{m_k(\mu)},$$

where f_i stands for e_i, h_i or p_i.

When A is infinite, each of the three sets of functions e_i, h_i or p_i forms an algebraic basis of $\mathbb{S}$, the symmetric algebra with coefficients in $\mathbf{R}$. Each of the sets of functions e_μ, h_μ, p_μ is a linear basis of this algebra. Two other linear bases are formed by the Schur functions s_λ and by the monomial symmetric functions m_λ, defined as the sum of all distinct monomials whose exponent is a permutation of λ.

2.3 Shifted symmetric functions

Although the theory of symmetric functions goes back to the early 19th century, shifted symmetric functions are quite recent. They were introduced and studied in [11, 29].

Being given a finite alphabet $A = \{a_1, a_2, \ldots, a_r\}$, a polynomial in A is "shifted symmetric" if it is symmetric in the shifted variables $a_i - i$. When $A = \{a_1, a_2, a_3, \ldots\}$ is infinite, in analogy with symmetric functions, a "shifted symmetric function" f is a family $\{f_i, i \geq 1\}$ such that f_i is shifted symmetric in $(a_1, a_2, \ldots, a_i)$, together with the stability property $f_{i+1}(a_1, a_2, \ldots, a_i, 0) = f_i(a_1, a_2, \ldots, a_i)$.

This defines $\mathbb{S}^*$, the shifted symmetric algebra with coefficients in $\mathbf{R}$, which is algebraically generated by the "shifted power sums"

$$p_k^*(A) = \sum_{i \geq 1} \Big((a_i - i + 1)_k - (-i + 1)_k \Big).$$

Here for an indeterminate z and any positive integer p, the *lowering* factorial

$$(z)_p = z(z-1)\ldots(z-p+1) = \sum_{i=1}^{p} s(p,i)\, z^i,$$

is the generating function of the Stirling numbers of the first kind $s(p,i)$. Conversely

$$z^p = \sum_{i=1}^{p} S(p,i)(z)_i$$

is the generating function of the Stirling numbers of the second kind $S(p,i)$.

An element $f \in \mathbb{S}^*$ may be evaluated at any sequence $(a_1, a_2, \ldots)$ with finitely many non zero terms, hence at any partition λ. Moreover by analyticity, f is entirely determined by its restriction $f(\lambda)$ to partitions. This identification is usually performed and $\mathbb{S}^*$ is considered as a function algebra on the set of partitions.

2.4 Contents

Given a partition λ, the content of any node $(i,j) \in \lambda$ is defined as $j - i$. Denote by $A_\lambda = \{j - i,\ (i,j) \in \lambda\}$ the finite alphabet of the contents of λ. The symmetric algebra $\mathbb{S}[A_\lambda]$ is generated by the power sums

$$p_k(A_\lambda) = \sum_{(i,j) \in \lambda} (j-i)^k = \sum_{i=1}^{l(\lambda)} \sum_{j=1}^{\lambda_i} (j-i)^k.$$

It is well known [11, 29] that the quantities $p_k(A_\lambda)$ are shifted symmetric polynomials of λ. Indeed for any integer $k \geq 1$, applying the identity $r(z)_{r-1} = (z+1)_r - (z)_r$, we have

$$\begin{aligned} p_k(A_\lambda) &= \sum_{r=1}^{k} \sum_{(i,j)\in\lambda} S(k,r)\,(j-i)_r \\ &= \sum_{r=1}^{k} \frac{S(k,r)}{r+1} \sum_{i=1}^{l(\lambda)} \Big((\lambda_i - i + 1)_{r+1} - (-i+1)_{r+1}\Big) \\ &= \sum_{r=1}^{k} \frac{S(k,r)}{r+1}\, p^*_{r+1}(\lambda). \end{aligned}$$

As a straightforward consequence, the shifted symmetric algebra $\mathbb{S}^*$ is algebraically generated by the functions $p_k(A_\lambda), k \geq 1$ together with $p_1^*(\lambda) = |\lambda|$. The latter corresponds to the cardinal of the alphabet A_λ.

In other words, any shifted symmetric function may be written $f(A_\lambda)$, with $f \in \mathbf{R}[\text{card}, p_1, p_2, p_3, \ldots]$. Moreover *this expression is unique*. As mentioned in [32, Proposition 2.4], this fact was already known to Kerov.

2.5 Representations

The irreducible representations of S_n and their characters are labelled by partitions $\lambda \vdash n$. Given such a partition, we denote by χ^λ the corresponding irreducible character. We write χ^λ_μ for its value $\chi^\lambda(\sigma)$ at any permutation σ of cycle-type μ, and $\dim \lambda = \chi^\lambda_{1^n}$ for the dimension of the representation λ. Then

$$\hat{\chi}^\lambda_\mu = \frac{\chi^\lambda_\mu}{\dim \lambda}, \qquad \theta^\lambda_\mu = \frac{n!}{z_\mu}\, \hat{\chi}^\lambda_\mu,$$

are the normalized character and central character of the representation λ, respectively. The dimension of the representation λ is given by

$$\dim \lambda = \frac{n!}{H_\lambda} = \frac{n!}{\prod_{i=1}^{l(\lambda)} (\lambda_i + l(\lambda) - i)!} \prod_{1 \leq i < j \leq l(\lambda)} (\lambda_i - \lambda_j + j - i).$$

2.6 Jucys-Murphy elements

For $1 \leq i \leq n$ the Jucys-Murphy elements J_i are defined by

$$J_i = \sum_{1 \leq j < i} (ji),$$

where (ji) denotes a transposition. These elements generate a maximal commutative subalgebra of $\mathbb{C}[S_n]$. Jucys [8] and Murphy [26] proved the following fundamental property.

Theorem. *The center* $\mathbb{Z}_n$ *is spanned by the elements* $f(J_1,\ldots,J_n)$*, with* f *a symmetric function. These elements act on irreducible characters by*

$$f(J_1,\ldots,J_n)\,\chi^\lambda = f(A_\lambda)\,\chi^\lambda.$$

This result has the following important consequence.

Corollary. *For any symmetric functions* f, g *(possibly depending on* n*) the following statements are equivalent:*

$$\begin{array}{rl} \text{(i)} & f = g, \\ \text{(ii)} & f(J_1,\ldots,J_n) = g(J_1,\ldots,J_n) \quad \textit{for any } n \geq 1, \\ \text{(iii)} & f(A_\lambda) = g(A_\lambda) \quad \textit{for any partition } \lambda. \end{array}$$

Proof. In view of the theorem the implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are obvious. The implication $(iii) \Rightarrow (i)$ is a consequence of Section 2.4 and the unicity result stated at the end. □

Now given some symmetric function $f \in \mathbb{S}$, consider

$$f(J_1,\ldots,J_n) = \sum_{|\mu|=n} a_\mu(n)\, C_\mu, \tag{2.1}$$

the class expansion of its Jucys-Murphy specialization $f(J_1,\ldots,J_n) \in \mathbb{Z}_n$. By definition we have $C_\mu\,\chi^\lambda = \theta_\mu^\lambda\,\chi^\lambda$. By taking eigenvalues we get

$$f(A_\lambda) = \sum_{|\mu|=n} a_\mu(n)\,\theta_\mu^\lambda. \tag{2.2}$$

for any partition $\lambda \vdash n$. It is well known [11, 29, 3, 18] that the central character θ_μ^λ is a shifted symmetric function of λ. Thus the previous equality holds in $\mathbb{S}^*$. We may *equivalently* study this decomposition in $\mathbb{S}^*$, rather than the original one in $\mathbb{Z}_n$.

2.7 Inverse problem

We may also consider the inverse problem, and look for an expression of each class C_μ as the Jucys-Murphy specialization

$$C_\mu = f_\mu(J_1,\ldots,J_n)$$

of some symmetric function f_μ (depending on n). This amounts to write the corresponding central character as the content evaluation of f_μ, namely

$$\theta_\mu^\lambda = f_\mu(A_\lambda).$$

Moreover it is equivalent to write (2.1), (2.2) or

$$f = \sum_{|\mu|=n} a_\mu(n)\, f_\mu. \tag{2.3}$$

In other words, the symmetric functions f_μ form a basis of $\mathbb{S}$ (depending on n) [11, 3].

The functions f_μ have been made explicit in [18], up to a constant factor. Actually the central character θ^λ_μ may be written as

$$\theta^\lambda_\mu = n!\, z_\mu^{-1} \hat{\chi}^\lambda_\mu = n(n-1)\cdots(n-|\overline{\mu}|+1) z_{\overline{\mu}}^{-1} \hat{\chi}^\lambda_\mu = z_{\overline{\mu}}^{-1} g_\mu(A_\lambda),$$

with g_μ some symmetric function (depending on n), explicitly given in [18] in terms of auxiliary symmetric functions.

Thus we have $f_\mu = z_{\overline{\mu}}^{-1} g_\mu$. Tables giving g_μ for $|\mu| - l(\mu) \leq 14$ are available on a web page [21].

2.8 Dependence on n

The notion of partial permutation of $\{1, \ldots, n\}$ has been introduced in [7]. It leads to define an abstract algebra $\mathbb{B}$, a basis of which is formed by elements B_ρ indexed by all partitions.

There is an isomorphism ι between this algebra and the shifted symmetric algebra $\mathbb{S}^*$, which may be described as follows [7, Theorem 9.1]. For any partition ρ we have

$$\begin{aligned}
\iota(B_\rho)(\lambda) &= \frac{(n)_{|\rho|}}{z_\rho}\, \hat{\chi}^\lambda_{\tilde{\rho}} \\
&= \binom{n - |\rho| + m_1(\rho)}{m_1(\rho)} \frac{n!}{z_{\tilde{\rho}}}\, \hat{\chi}^\lambda_{\tilde{\rho}} \\
&= \binom{n - |\overline{\rho}|}{m_1(\rho)}\, \theta^\lambda_{\tilde{\rho}},
\end{aligned}$$

with $\lambda \vdash n$. This isomorphism implies that the decomposition (2.2) of the shifted symmetric function $f(A_\lambda)$ takes the form

$$f(A_\lambda) = \sum_{|\rho|\leq n} c_\rho \binom{n - |\overline{\rho}|}{m_1(\rho)}\, \theta^\lambda_{\tilde{\rho}},$$

the coefficients c_ρ being independent of n. Equivalently (2.1) writes as

$$f(J_1, \ldots, J_n) = \sum_{|\rho|\leq n} c_\rho \binom{n - |\overline{\rho}|}{m_1(\rho)}\, C_{\tilde{\rho}}.$$

In other words $a_\mu(n)$ may be written as

$$a_\mu(n) = \sum_\rho c_\rho \binom{n - |\overline{\rho}|}{m_1(\rho)},$$

summed over partitions ρ satisfying $\overline{\rho} = \overline{\mu}$, or equivalently $\tilde{\rho} = \mu$.

3 The transition measure

Given a partition $\lambda \vdash n$, the transition measure ω_λ is a probability measure on the real line, studied by Kerov [9, 10] and others.

For any $i = 1, \dots, l(\lambda)+1$ we define the transition probabilities

$$c_i(\lambda) = \frac{H_\lambda}{H_{\lambda^{(i)}}} = \frac{1}{n+1}\,\frac{\dim \lambda^{(i)}}{\dim \lambda},$$

if the partition $\lambda^{(i)}$ exists, and 0 otherwise. We have easily

$$c_i(\lambda) = \frac{1}{\lambda_i + l(\lambda) - i + 2} \prod_{\substack{j=1\\ j\neq i}}^{l(\lambda)+1} \frac{\lambda_i - \lambda_j + j - i + 1}{\lambda_i - \lambda_j + j - i}.$$

We consider the discrete measure

$$\omega_\lambda = \sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, \delta_{\lambda_i - i + 1},$$

where δ_u is the Dirac measure at u. It is a probability measure, supported by the points $\lambda_i - i + 1$ such that $\lambda^{(i)}$ exists.

We denote the moments of ω_λ by

$$\sigma_k(\lambda) = \sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, (\lambda_i - i + 1)^k. \tag{3.1}$$

Thus the moment generating series of ω_λ is given by

$$\mathcal{M}_\lambda(z) = \sum_{k\geq 0} \sigma_k(\lambda) z^{-k-1} = \sum_{i=1}^{l(\lambda)+1} \frac{c_i(\lambda)}{z - \lambda_i + i - 1}. \tag{3.2}$$

In the more general context of Jack polynomials, we have shown that $\mathcal{M}_\lambda$ may be alternatively written

$$\mathcal{M}_\lambda(z) = z^{-1} \frac{C_\lambda(-z)}{C_\lambda(-z-1)}\, \frac{C_\lambda(-z)}{C_\lambda(-z+1)}, \tag{3.3}$$

where $C_\lambda(z)$ denotes the content polynomial

$$C_\lambda(z) = \prod_{(i,j)\in\lambda} (z + j - i).$$

The proof is given in [17, Theorem 8.1, p. 3470] (written for $\alpha = 1$) by using Lagrange interpolation.

We may identify the developments of (3.2) and (3.3) in descending powers of z. By [17, Corollary 5.2, p. 3464] (written for $y = -1$) we obtain

$$\sigma_k(\lambda) = f_k(A_\lambda),$$

where

$$f_k = \sum_{\substack{q,r\geq 0\\ q+2r\leq k}} \sum_{s=0}^{\min(r,k-2r)} \binom{n+r-1}{r-s} \sum_{|\mu|=k-2r} \left\langle {\mu \atop q} \right\rangle_s z_\mu^{-1} p_\mu \tag{3.4}$$

is a symmetric function depending on n. Here $\left\langle {\mu \atop q} \right\rangle_s$ is some positive integer explicitly known (see [16], [17, p. 3459] or [18, p. 392]), in particular by a generating function.

In view of Section 2.4, the moments $\sigma_k(\lambda)$ are shifted symmetric functions. In this paper we shall mainly need the following elementary values

$$\sigma_0(\lambda) = 1, \qquad \sigma_1(\lambda) = 0, \qquad \sigma_2(\lambda) = n, \qquad \sigma_3(\lambda) = 2p_1(A_\lambda). \tag{3.5}$$

But we also mention

$$\begin{aligned}
\sigma_4(\lambda) &= 3p_2(A_\lambda) + \binom{n+1}{2}, \\
\sigma_5(\lambda) &= 4p_3(A_\lambda) + 2(n+1)p_1(A_\lambda), \\
\sigma_6(\lambda) &= 5p_4(A_\lambda) + 3(n+1)p_2(A_\lambda) + 2p_2(A_\lambda) + 2p_1^2(A_\lambda) + \binom{n+2}{3}.
\end{aligned} \tag{3.6}$$

Since the moments may be written $\sigma_k(\lambda) = f_k(A_\lambda)$ with f_k a symmetric function depending on n, we may also consider the central element

$$M_n^{(k)} = f_k(J_1, \dots, J_n).$$

Biane [1, 2] has shown that $M_n^{(k)} = \pi(J_{n+1}^k)$, with π the orthogonal projection of $\mathbb{C}[S_{n+1}]$ onto $\mathbb{C}[S_n]$. By Jucys' result, for any $\lambda \vdash n$ we have

$$M_n^{(k)}\, \chi^\lambda = \sigma_k(\lambda)\, \chi^\lambda.$$

It is natural to study the equivalent expansions

$$\begin{aligned}
M_n^{(k)} &= \sum_{|\mu|=n} s_\mu^{(k)}(n)\, C_\mu = \sum_\rho \mathbf{s}_\rho^{(k)} \binom{n-|\overline{\rho}|}{m_1(\rho)} C_{\tilde{\rho}}, \\
\sigma_k(\lambda) &= \sum_{|\mu|=n} s_\mu^{(k)}(n)\, \theta_\mu^\lambda = \sum_\rho \mathbf{s}_\rho^{(k)} \binom{n-|\overline{\rho}|}{m_1(\rho)} \theta_{\tilde{\rho}}^\lambda.
\end{aligned}$$

We shall give them explicitly at the end of Section 6.

4 Tools and method

In this paper we make a crucial use of some linear relations between central characters. The following auxiliary material is needed.

4.1 Differential operators

In the space $\mathbf{R}^N$ of N variables $(x_1, \ldots, x_N)$, for any integer $k \geq 0$ we introduce the differential operators

$$
\begin{aligned}
E_k &= \sum_{i=1}^{N} x_i^k \, \frac{\partial}{\partial x_i}, \\
D_k &= \frac{1}{2} \sum_{i=1}^{N} x_i^k \, \frac{\partial^2}{\partial x_i^2} + \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \frac{x_i^k}{x_i - x_j} \frac{\partial}{\partial x_i}.
\end{aligned}
$$

The following result is proved by an easy induction on N.

Lemma. *For any integer $r \geq 2$, we have*

$$
2 \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \frac{x_i^r}{x_i - x_j} = \sum_{i=1}^{r-2} p_i \, p_{r-i-1} + (2N - r) p_{r-1}.
$$

After some easy computation, for any integer $k \geq 2$ and any partition μ, this property yields

$$
\begin{aligned}
2D_k \, p_\mu = & \sum_{r,s \geq 1} rs m_r(\mu)(m_s(\mu) - \delta_{rs}) \, p_{\mu \setminus (r,s) \cup (r+s+k-2)} \\
& + \sum_{r \geq 1} r m_r(\mu) \sum_{i=1}^{r+k-3} p_{\mu \setminus (r) \cup (i, r-i+k-2)} + (2N - k) \sum_{r \geq 1} r m_r(\mu) \, p_{\mu \setminus (r) \cup (r+k-2)}.
\end{aligned} \tag{4.1}
$$

We have also

$$
\begin{aligned}
E_0 \, p_\mu &= \sum_{r \geq 2} r m_r(\mu) \, p_{\mu \setminus (r) \cup (r-1)} + N m_1(\mu) p_{\mu \setminus (1)}, \\
E_2 \, p_\mu &= \sum_{r \geq 1} r m_r(\mu) \, p_{\mu \setminus (r) \cup (r+1)}.
\end{aligned} \tag{4.2}
$$

Let $\Delta_0 = p_1$ considered as a multiplication operator on the symmetric algebra $\mathbb{S}$. For any $k \geq 1$ define the k-th nested commutator

$$
\Delta_k = [D_2, [D_2, \cdots, [D_2, p_1] \cdots]].
$$

After some computation we have easily

$$\begin{aligned} \Delta_1 &= E_2 + (N-1)p_1, \\ \Delta_2 &= 2D_3 + E_2 + (N-1)^2 p_1. \end{aligned} \tag{4.3}$$

It is known that the Schur functions $s_\lambda(x_1, \ldots, x_N)$ are eigenfunctions of D_2, namely

$$D_2 s_\lambda = \big(p_1(A_\lambda) + |\lambda|(N-1)\big) s_\lambda. \tag{4.4}$$

This result and the Pieri formula

$$p_1 s_\lambda = \sum_{i=1}^{l(\lambda)+1} s_{\lambda^{(i)}},$$

imply inductively

$$\Delta_k s_\lambda = \sum_{i=1}^{l(\lambda)+1} (\lambda_i + N - i)^k \, s_{\lambda^{(i)}}.$$

4.2 Linear relations

We are now in a position to obtain some linear relations between central characters. Our purpose is to evaluate

$$\sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, (\lambda_i - i + 1)^k \, \theta_\mu^{\lambda^{(i)}},$$

at least for the first values of k.

Theorem 4.1. *For any partitions $\lambda \vdash n$ and $\mu \vdash n+1$, we have*

$$\sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, \theta_\mu^{\lambda^{(i)}} = \theta^\lambda_{\mu \setminus (1)}, \tag{4.5}$$

$$\sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, (\lambda_i - i + 1)\, \theta_\mu^{\lambda^{(i)}} = \sum_{r \geq 1} r(m_r(\mu)+1)\, \theta^\lambda_{\mu \setminus (r+1) \cup (r)}, \tag{4.6}$$

$$\begin{aligned} \sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, (\lambda_i - i + 1)^2\, \theta_\mu^{\lambda^{(i)}} &= (2n - m_1(\mu) + 1)\, \theta^\lambda_{\mu \setminus 1} \\ &+ \sum_{r,s \geq 1} rs(m_r(\mu)+1)(m_s(\mu)+\delta_{rs}+1)\, \theta^\lambda_{\mu \setminus (r+s+1) \cup (r,s)} \\ &+ \sum_{r,s \geq 2} (r+s-1)(m_{r+s-1}(\mu)+1)\, \theta^\lambda_{\mu \setminus (r,s) \cup (r+s-1)}. \end{aligned} \tag{4.7}$$

Proof. Since $\theta^\lambda_\mu = H_\lambda z_\mu^{-1}\chi^\lambda_\mu$, we may write the classical Frobenius formula

$$s_\lambda = \sum_\mu z_\mu^{-1}\chi^\lambda_\mu\, p_\mu$$

under the equivalent form

$$H_\lambda s_\lambda = \sum_\mu \theta^\lambda_\mu\, p_\mu.$$

We apply the differential operator Δ_k on both sides. Since $H_\lambda = c_i(\lambda) H_{\lambda^{(i)}}$ we obtain

$$\begin{aligned} H_\lambda\, \Delta_k\, s_\lambda &= \sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\,(\lambda_i+N-i)^k\, H_{\lambda^{(i)}}\, s_{\lambda^{(i)}} \\ &= \sum_\nu \Big(\sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\,(\lambda_i+N-i)^k\,\theta_\nu^{\lambda^{(i)}}\Big)\, p_\nu \\ &= \sum_\mu \theta^\lambda_\mu\,\Delta_k\, p_\mu. \end{aligned}$$

Now we write this identity for $k=2$. By (4.1)–(4.3) we get

$$\begin{aligned} &\sum_\nu \Big(\sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\,(\lambda_i+N-i)^2\,\theta_\nu^{\lambda^{(i)}}\Big)\, p_\nu \\ &\qquad = \sum_\mu \theta^\lambda_\mu \Big(\sum_{r,s\geq 1} rsm_r(\mu)(m_s(\mu)-\delta_{rs})\, p_{\mu\setminus(r,s)\cup(r+s+1)} \\ &\qquad + \sum_{r\geq 1} rm_r(\mu) \sum_{i=1}^{r} p_{\mu\setminus(r)\cup(i,r-i+1)} + (2N-3)\sum_{r\geq 1} rm_r(\mu)\, p_{\mu\setminus(r)\cup(r+1)} \\ &\qquad + \sum_{r\geq 1} rm_r(\mu)\, p_{\mu\setminus(r)\cup(r+1)} + (N-1)^2\, p_{\mu\cup(1)}\Big). \end{aligned}$$

If we denote by $L^{(k)}_\mu$, $(k=0,1,2)$, the respective left-hand sides of (4.5)–(4.7), we have

$$\sum_\nu \Big(\sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\,(\lambda_i+N-i)^2\,\theta_\nu^{\lambda^{(i)}}\Big)\, p_\nu = \sum_\nu \Big(L^{(2)}_\nu + 2(N-1)L^{(1)}_\nu + (N-1)^2 L^{(0)}_\nu \Big)\, p_\nu.$$

Since these quantities are independent of N, we obtain $\sum_\nu L^{(k)}_\nu p_\nu$ by identification of the coefficients of $N-1$.

Finally we identify the coefficients of power sums on both sides. Relations (4.5) and (4.6) are straightforward. For (4.7) some attention is needed with the term

$$\sum_{r\geq 1} rm_r(\mu) \sum_{i=1}^{r} p_{\mu\setminus(r)\cup(i,r-i+1)}, \tag{4.8}$$

which should be written as

$$\sum_{r,s\geq 2}(r+s-1)m_{r+s-1}(\mu)\,p_{\mu\setminus(r+s-1)\cup(r,s)}+(2|\mu|-m_1(\mu))\,p_{\mu\cup(1)}.$$

The last term $(2|\mu|-m_1(\mu))p_{\mu\cup(1)}$ is justified as follows. In (4.8) for $r\neq 1$, each case $i=1$ and $i=r$ contributes by $p_{\mu\cup(1)}$. For $r=1$ there is only one such contribution, obtained for $i=r=1$. Hence a total of $m_1(\mu)+2\sum_{r\geq 2}rm_r(\mu)=2|\mu|-m_1(\mu)$. □

Remark: Féray (private communication) has observed that

$$\sum_{i=1}^{l(\lambda)+1}c_i(\lambda)\,(\lambda_i-i+1)^k\,\theta_\mu^{\lambda^{(i)}}=\hat{\chi}^\lambda(\pi(J_{n+1}^kC_\mu)),$$

with π the orthogonal projection of $\mathbb{C}[S_{n+1}]$ onto $\mathbb{C}[S_n]$ and $C_\mu\in\mathbb{C}[S_{n+1}]$ the formal sum of permutations with cycle-type μ. This formula provides an interesting connection between the analytic and combinatorial points of view. The proof is an extension of the one given by Biane [1, Proposition 3.3] for $\mu=1^{n+1}$.

4.3 Our method

Consider a symmetric function f and its central character expansion

$$f(A_\lambda)=\sum_{|\mu|=n}a_\mu(n)\,\theta_\mu^\lambda.$$

We sketch the main steps of our method to compute $a_\mu(n)$.
First step: By definition we have $A_{\lambda^{(i)}}=A_\lambda\cup\{\lambda_i-i+1\}$. Therefore we may write

$$f(A_{\lambda^{(i)}})=f(A_\lambda)+\sum_{k\geq 1}g_k(A_\lambda)(\lambda_i-i+1)^k,\tag{4.9}$$

for a finite family of symmetric functions g_k. This development may be found explicitly when f is specified. But this is a general fact [22, Example 1.5.3 (b), p. 75]: actually $g_k=h_k^\perp f$.
Second step: Using (3.1) the previous expansion implies for $r=0,1,2$,

$$\sum_{i=1}^{l(\lambda)+1}c_i(\lambda)\,(\lambda_i-i+1)^r\,f(A_{\lambda^{(i)}})=\sigma_r(\lambda)f(A_\lambda)+\sum_{k\geq 1}\sigma_{k+r}(\lambda)g_k(A_\lambda).$$

Then it may be possible (but not always) to eliminate the quantities $\sigma_i(\lambda)$. This can be done by performing some linear combinations and using the explicit expressions (3.5)–(3.6). This elimination depends strongly on the specific form of f.

In the most elementary situation (always encountered in this paper), this elimination transforms the previous relation into

$$\sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\,(\lambda_i - i + 1)^r\, f(A_{\lambda^{(i)}}) = F(A_\lambda) + \sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\,(\lambda_i - i + 1)^s\, G(A_{\lambda^{(i)}}), \qquad (4.10)$$

for some $s = 0, 1, 2$ and some symmetric functions F, G (all three depending on r).
Third step: Writing the content evaluation of these functions as

$$F(A_\lambda) = \sum_{|\mu|=n} F_\mu(n)\,\theta_\mu^\lambda, \quad G(A_\lambda) = \sum_{|\mu|=n} G_\mu(n)\,\theta_\mu^\lambda,$$

the previous relation becomes

$$\begin{aligned} &\sum_{|\mu|=n+1} a_\mu(n+1)\Big(\sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\,(\lambda_i - i + 1)^r\,\theta_\mu^{\lambda^{(i)}}\Big) \\ &= \sum_{|\nu|=n} F_\nu(n)\,\theta_\nu^\lambda + \sum_{|\mu|=n+1} G_\mu(n+1)\Big(\sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\,(\lambda_i - i + 1)^s\,\theta_\mu^{\lambda^{(i)}}\Big). \end{aligned}$$

Applying (4.5)–(4.7), the quantities between brackets can be evaluated in terms of the characters θ_ν^λ.
Fourth step: By identification of the coefficients of the θ_ν^λ's, we obtain some linear relations between $a_\mu(n+1)$ and the $F_\mu(n)$, $G_\mu(n+1)$'s.
Final step: These relations may be used to define $a_\mu(n)$ inductively.

This method will appear much clearer below, when applied to $f = e_k, p_k, h_k$ and the Hall-Littlewood function $f = P_k(z)$.

5 Elementary functions

As an easy example, let us first apply our method to recover the classical result of Jucys [8].

Theorem 5.1. *For any positive integer k we have*

$$e_k(J_1, \ldots, J_n) = \sum_{\substack{|\mu|=n \\ l(\mu)=n-k}} C_\mu.$$

Proof. Writing

$$e_k(A_\lambda) = \sum_{|\mu|=n} a_\mu^{(k)}(n)\theta_\mu^\lambda,$$

we must equivalently prove that $a_\mu^{(k)}(n) = \delta_{l(\mu),n-k}$.

First step: Denoting $u_i = \lambda_i - i + 1$, we have

$$E_z(A_{\lambda^{(i)}}) = E_z(A_\lambda)(1 + zu_i).$$

Hence the expansion (4.9) takes the very simple form

$$e_k(A_{\lambda^{(i)}}) = e_k(A_\lambda) + e_{k-1}(A_\lambda)u_i.$$

Second step: Using (3.1) and (3.5) this yields

$$\begin{aligned}
\sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, e_k(A_{\lambda^{(i)}}) &= \sum_{i=1}^{l(\lambda)+1} c_i(\lambda) \big(e_k(A_\lambda) + e_{k-1}(A_\lambda)u_i\big) = e_k(A_\lambda),\\
\sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, u_i\, e_k(A_{\lambda^{(i)}}) &= \sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, u_i\big(e_k(A_\lambda) + e_{k-1}(A_\lambda)u_i\big) = ne_{k-1}(A_\lambda),\\
\sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, u_i^2\, e_k(A_{\lambda^{(i)}}) &= \sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, u_i^2\big(e_k(A_\lambda) + e_{k-1}(A_\lambda)u_i\big) = (ne_k + 2e_1e_{k-1})(A_\lambda).
\end{aligned} \tag{5.1}$$

With the notations of Section 4.3, we have $G = 0$ and $F = e_k,\ ne_{k-1},\ ne_k + 2e_1e_{k-1}$, respectively.

Third step: Applying (4.5) and (4.6) the two first relations write as

$$\sum_{|\mu|=n+1} a_\mu^{(k)}(n+1)\theta^\lambda_{\mu\setminus(1)} = \sum_{|\nu|=n} a_\nu^{(k)}(n)\theta^\lambda_\nu,$$

$$\sum_{|\mu|=n+1} a_\mu^{(k)}(n+1) \sum_{r\geq 1} r\big(m_r(\mu)+1\big)\theta^\lambda_{\mu\setminus(r+1)\cup(r)} = n\sum_{|\nu|=n} a_\nu^{(k-1)}(n)\theta^\lambda_\nu.$$

Fourth step: By identification of coefficients on both sides, for any $\mu \vdash n$ we obtain

$$\begin{aligned}
a^{(k)}_{\mu\cup(1)}(n+1) &= a^{(k)}_\mu(n),\\
\sum_{r\geq 1} rm_r(\mu)\, a^{(k)}_{\mu\setminus(r)\cup(r+1)}(n+1) &= na^{(k-1)}_\mu(n).
\end{aligned} \tag{5.2}$$

Final step: The previous recurrence relations allow us to conclude by a triple induction: firstly on k, then on n and finally on the lowest part of $\mu \vdash n$.

We warn the reader that we shall use such a multiple induction several times in this paper. Details are given here, but will not be repeated below.

(i) Assume $a^{(i)}_\mu(n) = \delta_{l(\mu),n-i}$ for any n and μ, and $i \leq k-1$.

(ii) Assume $a^{(k)}_\mu(m-1) = \delta_{l(\mu),m-k-1}$ for any $\mu \vdash m-1$. By the first relation (5.2) we have obviously $a^{(k)}_\mu(m) = \delta_{l(\mu),m-k}$ for those $\mu \vdash m$ whose lowest part is 1.

(iii) Now assume this property to be true for those $\mu \vdash m$ whose lowest part is $p-1$. Let $\nu \vdash m$ having lowest part p. Then $\mu = \nu \setminus (p) \cup (p-1) \vdash m-1$ has lowest part $p-1$ with multiplicity 1.

Writing the second relation (5.2) for μ determines $a_\nu^{(k)}(m)$, since all partitions on the left-hand side have lowest part $p-1$, but $\nu = \mu \setminus (p-1) \cup (p)$. Namely

$$\begin{aligned}(p-1)\, a_\nu^{(k)}(m) &= (m-1) a_\mu^{(k-1)}(m-1) - \sum_{r>p-1} r m_r(\mu)\, a^{(k)}_{\mu\setminus(r)\cup(r+1)}(m) \\ &= \left(m-1-\sum_{r>p-1} r m_r(\mu)\right) \delta_{l(\mu),m-k} \\ &= (p-1)\delta_{l(\nu),m-k}.\end{aligned}$$

□

Up to now we have only used the two first equations (5.1). But the third one has also an interesting consequence.

Proposition 5.2. *For any positive integer k we have*

$$(e_1e_k)(J_1,\ldots,J_n) = \sum_{\substack{|\mu|=n \\ l(\mu)=n-k-1}} a_\mu C_\mu + \sum_{\substack{|\mu|=n \\ l(\mu)=n-k+1}} \left(\binom{n}{2} - a_\mu\right) C_\mu,$$

with $a_\mu = \sum_{r\geq 2} m_r(\mu)\binom{r}{2}$.

Proof. The last relation (5.1) writes as

$$2(e_ke_1)(A_\lambda) := \sum_{|\mu|=n} 2a_\mu^{(k,1)}(n)\theta_\mu^\lambda = -ne_{k+1}(A_\lambda) + \sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\,(\lambda_i - i + 1)^2\, e_{k+1}(A_{\lambda^{(i)}}).$$

Using (4.7), the expansion of the right-hand side is

$$\begin{aligned}-n \sum_{\substack{|\mu|=n \\ l(\mu)=n-k-1}} \theta_\mu^\lambda + \sum_{\substack{|\mu|=n+1 \\ l(\mu)=n-k}} \Big(&(2n - m_1(\mu)+1)\, \theta^\lambda_{\mu\setminus 1} \\ &+ \sum_{r,s\geq 1} rs(m_r(\mu)+1)(m_s(\mu)+\delta_{rs}+1)\, \theta^\lambda_{\mu\setminus(r+s+1)\cup(r,s)} \\ &+ \sum_{r,s\geq 2} (r+s-1)(m_{r+s-1}(\mu)+1)\, \theta^\lambda_{\mu\setminus(r,s)\cup(r+s-1)}\Big).\end{aligned}$$

By identification of coefficients on both sides, we obtain

$$\begin{aligned}2a_\mu^{(k,1)}(n) = -n\delta_{l(\mu),n-k-1} &+ \sum_{r,s\geq 1} rsm_r(\mu)(m_s(\mu)-\delta_{rs})\, \delta_{l(\mu)-1,n-k} \\ &+ \sum_{r,s\geq 1} (r+s-1) m_{r+s-1}(\mu)\, \delta_{l(\mu)+1,n-k}.\end{aligned}$$

The coefficient of $\delta_{l(\mu),n-k-1}$ is

$$-n+\sum_{r,s\geq 1}(r+s-1)m_{r+s-1}(\mu)=-n+\sum_{t\geq 1}t^2 m_t(\mu)=2a_\mu.$$

The coefficient of $\delta_{l(\mu),n-k+1}$ is

$$\sum_{r,s\geq 1} rs m_r(\mu)(m_s(\mu)-\delta_{rs})=n^2-\sum_{t\geq 1}t^2 m_t(\mu)=n^2-n-2a_\mu.$$

□

6 Power sums

We are now in a position to give an alternative proof of the classical result of Lascoux-Thibon [13]. We study the expansion

$$p_k(A_\lambda)=\sum_{|\mu|=n} a_\mu^{(k)}(n)\theta_\mu^\lambda$$

in two steps: firstly a recurrence between coefficients, secondly a generating function.

6.1 Recurrence

Denoting $u_i=\lambda_i-i+1$, the expansion (4.9) takes the very simple form

$$p_k(A_{\lambda^{(i)}})=p_k(A_\lambda)+u_i^k.$$

Therefore (3.1) and (3.5) yield

$$\begin{aligned}
\sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, p_k(A_{\lambda^{(i)}}) &= p_k(A_\lambda)+\sigma_k(\lambda),\\
\sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, u_i\, p_k(A_{\lambda^{(i)}}) &= \sigma_{k+1}(\lambda),\\
\sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, u_i^2\, p_k(A_{\lambda^{(i)}}) &= np_k(A_\lambda)+\sigma_{k+2}(\lambda).
\end{aligned}$$

By elimination we get immediately

$$\begin{aligned}
\sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, p_k(A_{\lambda^{(i)}}) &= p_k(A_\lambda)+\sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, u_i\, p_{k-1}(A_{\lambda^{(i)}}),\\
\sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, u_i\, p_k(A_{\lambda^{(i)}}) &= -np_{k-1}(A_\lambda)+\sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, u_i^2\, p_{k-1}(A_{\lambda^{(i)}}).
\end{aligned}$$

Applying (4.5)–(4.7) these relations write respectively as

$$\sum_{|\mu|=n+1} a_\mu^{(k)}(n+1)\theta^\lambda_{\mu\backslash(1)} = \sum_{|\nu|=n} a_\nu^{(k)}(n)\,\theta^\lambda_\nu + \sum_{|\mu|=n+1} a_\mu^{(k-1)}(n+1) \sum_{r\geq 1} r\big(m_r(\mu)+1\big)\theta^\lambda_{\mu\backslash(r+1)\cup(r)},$$

$$\begin{aligned}\sum_{|\mu|=n+1} a_\mu^{(k)}(n+1) \sum_{r\geq 1} r\big(m_r(\mu)+1\big)\theta^\lambda_{\mu\backslash(r+1)\cup(r)} &= -n\sum_{|\nu|=n} a_\nu^{(k-1)}(n)\,\theta^\lambda_\nu \\ &+ \sum_{|\mu|=n+1} a_\mu^{(k-1)}(n+1)\,\Big((2n-m_1(\mu)+1)\,\theta^\lambda_{\mu\backslash 1} \\ &+ \sum_{r,s\geq 1} rs(m_r(\mu)+1)(m_s(\mu)+\delta_{rs}+1)\,\theta^\lambda_{\mu\backslash(r+s+1)\cup(r,s)} \\ &+ \sum_{r,s\geq 2}(r+s-1)(m_{r+s-1}(\mu)+1)\,\theta^\lambda_{\mu\backslash(r,s)\cup(r+s-1)}\Big).\end{aligned}$$

By identification of coefficients on both sides, for any $\mu\vdash n$ we get

$$a^{(k)}_{\mu\cup(1)}(n+1) = a^{(k)}_\mu(n) + \sum_{r\geq 1} rm_r(\mu)\,a^{(k-1)}_{\mu\backslash(r)\cup(r+1)}(n+1), \tag{6.1}$$

$$\begin{aligned}\sum_{r\geq 1} rm_r(\mu)\,a^{(k)}_{\mu\backslash(r)\cup(r+1)}(n+1) &= -na^{(k-1)}_\mu(n) \\ &+ \sum_{r,s\geq 1} rsm_r(\mu)(m_s(\mu)-\delta_{rs})\,a^{(k-1)}_{\mu\backslash(r,s)\cup(r+s+1)}(n+1) \\ &+ \sum_{r,s\geq 1}(r+s-1)m_{r+s-1}(\mu)\,a^{(k-1)}_{\mu\backslash(r+s-1)\cup(r,s)}(n+1).\end{aligned} \tag{6.2}$$

In the second sum on the right-hand side, the cases $r=1$ or $s=1$ give the total contribution $(2n-m_1(\mu))\,a^{(k-1)}_{\mu\cup(1)}(n+1)$.

These two recurrence relations determine the coefficients $a^{(k)}_\mu(n)$ by a triple induction: on k, on n and on the lowest part of μ. Firstly (6.1) gives $a^{(k)}_\mu(n+1)$ for those μ whose lowest part is 1. Then (6.2) determines $a^{(k)}_\mu(n+1)$ by induction on the lowest part of μ. The proof is strictly parallel to the final step of Theorem 5.1 and is left to the reader.

These recurrence relations also imply that $a^{(k)}_\mu(n)$ writes as

$$a^{(k)}_\mu(n) = \sum_{\overline{\rho}=\overline{\mu}} c^{(k)}_\rho \binom{n-|\overline{\rho}|}{m_1(\rho)},$$

with $c^{(k)}_\rho$ independent of n (a property already known in view of Section 2.8). By substitution into (6.1)–(6.2), we obtain the corresponding recurrence relations for $c^{(k)}_\rho$.

This computation will be used several times in this paper. Since it is rather technical, we postpone it to an Appendix. Using the Lemma given there with $z=1$, we obtain the following result.

Theorem 6.1. *We have the class expansion*

$$p_k(J_1,\dots,J_n)=\sum_\rho c_\rho^{(k)}\binom{n-|\overline{\rho}|}{m_1(\rho)}C_{\tilde\rho},$$

where the coefficients $c_\rho^{(k)}$ *are determined by the two recurrence relations*

$$c^{(k)}_{\rho\cup(1)}=\sum_{r\geq 1} rm_r(\rho)\,c^{(k-1)}_{\rho\backslash(r)\cup(r+1)}, \tag{6.3}$$

$$\begin{aligned}\sum_{r\geq 1} rm_r(\rho)\,c^{(k)}_{\rho\backslash(r)\cup(r+1)}=|\rho|\,c^{(k-1)}_\rho&+\sum_{r,s\geq 1} rsm_r(\rho)(m_s(\rho)-\delta_{rs})\,c^{(k-1)}_{\rho\backslash(r,s)\cup(r+s+1)}\\&+\sum_{r,s\geq 1}(r+s-1)m_{r+s-1}(\rho)\,c^{(k-1)}_{\rho\backslash(r+s-1)\cup(r,s)}.\end{aligned} \tag{6.4}$$

Remarks: (i) In the second sum on the right-hand side of (6.4), the cases $r=1$ or $s=1$ give the total contribution $(2|\rho|-m_1(\rho))\,c^{(k-1)}_{\rho\cup(1)}$.
(ii) As above, $c_\rho^{(k)}$ is inductively defined by (6.3) for ρ having lowest part 1, and by (6.4) for ρ having lowest part >1.
(iii) By induction on k and the lowest part of ρ, the coefficients $c_\rho^{(k)}$ are non zero for $|\rho|+l(\rho)=k+2-2i$ for some $i\geq 0$.

6.2 Generating functions

We now look for a generating function of the coefficients $c_\rho^{(k)}$, written under the form

$$\phi_\rho(t)=\sum_{k\geq 0} c_\rho^{(k)}\,\frac{t^k}{k!}.$$

It will be useful to collect the ϕ_ρ's for partitions ρ having the same weight. For that purpose, we introduce a set of N auxiliary variables $X=(x_1,\dots,x_N)$, and we define

$$\Phi_m(t;X)=\sum_{|\rho|=m}\phi_\rho(t)\,z_\rho^{-1}p_\rho(X).$$

We shall translate the recurrence relations (6.3)–(6.4) in terms of Φ_m. This will allow us to evaluate Φ_m, hence ϕ_ρ.

In $\mathbf{R}^N$ we introduce the differential operators

$$\mathbf{E}=E_0-N\frac{\partial}{\partial p_1},\qquad \mathbf{D}=2D_1-(2N-1)\mathbf{E}-N(N-1)\frac{\partial}{\partial p_1}.$$

Their actions on symmetric functions are independent of N. Actually denoting $\widehat{p}_\mu = z_\mu^{-1} p_\mu$ they act on the power sums by

$$\begin{aligned}
\mathbf{E}\, \widehat{p}_\mu &= \sum_{r\geq 1} r(m_r(\mu)+1)\, \widehat{p}_{\mu\setminus(r+1)\cup(r)},\\
\mathbf{D}\, \widehat{p}_\mu &= \sum_{r,s\geq 1} (r+s-1)(m_{r+s-1}(\mu)+1)\, \widehat{p}_{\mu\setminus(r,s)\cup(r+s-1)}\\
&\quad + \sum_{r,s\geq 1} rs(m_r(\mu)+1)(m_s(\mu)+1+\delta_{rs})\, \widehat{p}_{\mu\setminus(r+s+1)\cup(r,s)}.
\end{aligned}$$

This is a direct consequence of (4.1)–(4.2) and

$$\begin{aligned}
z_\mu^{-1} z_{\mu\setminus(r+1)\cup(r)} &= \frac{r(m_r(\mu)+1)}{(r+1)m_{r+1}(\mu)},\\
z_\mu^{-1} z_{\mu\setminus(r,s)\cup(r+s-1)} &= \frac{(r+s-1)(m_{r+s-1}(\mu)+1)}{rsm_r(\mu)(m_s(\mu)-\delta_{rs})},\\
z_\mu^{-1} z_{\mu\setminus(r+s+1)\cup(r,s)} &= \frac{rs(m_r(\mu)+1)(m_s(\mu)+1+\delta_{rs})}{(r+s+1)m_{r+s+1}(\mu)}.
\end{aligned}$$

Proposition 6.2. *The recurrence relations (6.3)–(6.4) are equivalent with*

$$\frac{d}{dt}\frac{\partial}{\partial p_1}\Phi_m = \mathbf{E}\,\Phi_m, \qquad \frac{d}{dt}\mathbf{E}\,\Phi_{m+1} = \mathbf{D}\,\Phi_{m+1} + m\Phi_m.$$

Proof. We have

$$\begin{aligned}
\frac{d}{dt}\frac{\partial}{\partial p_1}\Phi_m &= \sum_{|\rho|=m}\sum_{k\geq 0} c_\rho^{(k)} \frac{t^{k-1}}{(k-1)!}\, \widehat{p}_{\rho\setminus(1)},\\
\mathbf{E}\,\Phi_m &= \sum_{|\rho|=m}\sum_{k\geq 0} c_\rho^{(k)} \frac{t^k}{k!} \sum_{r\geq 1} r(m_r(\rho)+1)\, \widehat{p}_{\rho\setminus(r+1)\cup(r)}\\
\mathbf{D}\,\Phi_{m+1} &= \sum_{|\rho|=m+1}\sum_{k\geq 0} c_\rho^{(k)} \frac{t^k}{k!}\Big(\sum_{r,s\geq 1} (r+s-1)(m_{r+s-1}(\rho)+1)\, \widehat{p}_{\rho\setminus(r,s)\cup(r+s-1)}\\
&\quad + \sum_{r,s\geq 1} rs(m_r(\rho)+1)(m_s(\rho)+1+\delta_{rs})\, \widehat{p}_{\rho\setminus(r+s+1)\cup(r,s)}\Big).
\end{aligned}$$

We conclude by identification of coefficients of $t^{k-1}\widehat{p}_\rho$. ∎

Instead of power sums, we may alternatively decompose $\Phi_m(t;X)$ in terms of Schur functions, and write

$$\Phi_m(t;X) = \sum_{|\rho|=m} \psi_\rho(t)\, s_\rho(X).$$

As an easy consequence of $2D_1 = [E_0, D_2]$ together with (4.3) and

$$E_0 s_\lambda = \sum_{i=1}^{l(\lambda)} (N + \lambda_i - i) s_{\lambda_{(i)}}, \qquad \frac{\partial}{\partial p_1} s_\lambda = \sum_{i=1}^{l(\lambda)} s_{\lambda_{(i)}},$$

we have

$$\mathbf{E} s_\lambda = \sum_{i=1}^{l(\lambda)} (\lambda_i - i) s_{\lambda_{(i)}}, \qquad \mathbf{D} s_\lambda = \sum_{i=1}^{l(\lambda)} (\lambda_i - i)^2 s_{\lambda_{(i)}}. \tag{6.5}$$

Therefore, by identification of coefficients of Schur functions on both sides, Proposition 6.2 writes equivalently as

$$\begin{aligned} \sum_{i=1}^{l(\rho)+1} \frac{d}{dt}\, \psi_{\rho^{(i)}}(t) &= \sum_{i=1}^{l(\rho)+1} (\rho_i - i + 1)\, \psi_{\rho^{(i)}}(t), \\ \sum_{i=1}^{l(\rho)+1} (\rho_i - i + 1) \frac{d}{dt}\, \psi_{\rho^{(i)}}(t) &= \sum_{i=1}^{l(\rho)+1} (\rho_i - i + 1)^2\, \psi_{\rho^{(i)}}(t) + |\rho| \psi_\rho(t). \end{aligned} \tag{6.6}$$

Because $p_0(J_1, \ldots, J_n) = n$, this first order (overdeterminate) differential system must be solved with the initial conditions $\psi_\rho(0) = \delta_{\rho,(1)}$, equivalently $c_\rho^{(0)} = \delta_{\rho,(1)}$.

Proposition 6.3. *The solutions of the system (6.6) are given by*

$$|\rho|!\, \psi_\rho(t) = \begin{cases} (e^t - 1)^{r-1} (e^{-t} - 1)^s & \text{if } \rho \text{ is a hook } (r, 1^s) \text{ with } r \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We begin by the statement for hooks. By linear combination of (6.6) written for $\rho = (r)$ we obtain

$$\begin{aligned} (r+1)\, \psi'_{(r+1)} &= r(r+1)\, \psi_{(r+1)} + r\psi_{(r)}, \\ (r+1)\, \psi'_{(r,1)} &= -(r+1)\, \psi_{(r,1)} - r\psi_{(r)}. \end{aligned}$$

If $\psi_{(r)}$ satisfies the statement, so do $\psi_{(r+1)}$ and $\psi_{(r,1)}$. Similarly by linear combination of (6.6) written for $\rho = (r, 1^s)$ we obtain

$$\begin{aligned} (r+s+1)\, \psi'_{(r+1,1^s)} + (s+1)\, \psi'_{(r,2,1^{s-1})} &= r(r+s+1)\, \psi_{(r+1,1^s)} + (r+s)\psi_{(r,1^s)}, \\ r\, \psi'_{(r,2,1^{s-1})} + (r+s+1)\, \psi'_{(r,1^{s+1})} &= -(s+1)(r+s+1)\, \psi_{(r,1^{s+1})} - (r+s)\psi_{(r,1^s)}. \end{aligned}$$

If $\psi_{(r,1^s)}$ is known for any r and $s \leq s_0$, this gives $\psi_{(r,2,1^{s_0-1})} = 0$ and $\psi_{(r,1^{s_0+1})}$.

In a second step we show that $\psi_\rho = 0$ when ρ is not a hook, i.e. when $\rho = (\rho_1, \ldots, \rho_k)$ with $\rho_2 \geq 2$. This is performed by a double induction on k and on ρ_k. As above by linear combination of (6.6) written for $\sigma = (\rho_1, \ldots, \rho_k - 1)$, we obtain

$$\psi'_\rho = (\rho_k - k)\, \psi_\rho, \qquad \psi'_{(\sigma,1)} = -k\, \psi_{(\sigma,1)},$$

which gives $\psi_\rho = \psi_{(\sigma,1)} = 0$. ☐

We are now in a position to give another proof of the Lascoux-Thibon's result [13]. We shall make a limited, but crucial, use of λ-ring calculus. Here we shall not enter into details, and refer the reader to [12, Chapter 2] or [15, Section 3] for a short survey of this theory.

If f is a symmetric function, we denote by $f[P]$ its λ-ring action on any polynomial P. Let q be some indeterminate, $X = (x_1, \ldots, x_N)$ an alphabet and $X^\dagger = \sum_{i=1}^N x_i$. We have the fundamental Cauchy formulas [12, p.13]

$$\begin{aligned} h_m[(q-1)X^\dagger] &= \sum_{|\rho|=m} p_\rho[q-1]\, z_\rho^{-1} p_\rho(X) \\ &= \sum_{|\rho|=m} s_\rho[q-1]\, s_\rho(X), \end{aligned}$$

with [12, p.11]

$$p_\rho[q-1] = \prod_{i\geq 1} (q^i - 1)^{m_i(\rho)},$$

$$s_\rho[q-1] = \begin{cases} (-1)^s q^{r-1}\,(q-1) & \text{if } \rho \text{ is a hook } (r, 1^s) \text{ with } r \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 6.4. *In the class expansion*

$$p_k(J_1, \ldots, J_n) = \sum_\rho c_\rho^{(k)} \binom{n - |\overline{\rho}|}{m_1(\rho)} C_{\tilde{\rho}},$$

the coefficients $c_\rho^{(k)}$ *have the generating function*

$$\sum_{k\geq 0} c_\rho^{(k)} \frac{t^k}{k!} = \frac{e^{-t}}{|\rho|!} (1-e^{-t})^{|\rho|-2} \prod_{i\geq 1} (e^{it}-1)^{m_i(\rho)}.$$

Proof. The assertion writes as

$$\phi_\rho(t) = \frac{e^{-t}}{|\rho|!} (1-e^{-t})^{|\rho|-2}\, p_\rho[q-1]|_{q=e^t}.$$

In view of the first Cauchy formula this is equivalent with

$$\Phi_m(t; X) = \frac{e^{-t}}{m!} (1-e^{-t})^{m-2}\, h_m[(q-1)X^\dagger]|_{q=e^t}.$$

But Proposition 6.2 asserts that

$$\psi_\rho(t) = \frac{e^{-t}}{|\rho|!} (1-e^{-t})^{|\rho|-2}\, s_\rho[q-1]|_{q=e^t}.$$

The second Cauchy formula allows us to conclude. ∎

6.3 Moments

As a quick by-product of our proof we obtain the explicit form of

$$\sigma_k(\lambda) = \sum_{|\mu|=n} s_\mu^{(k)}(n)\, \theta_\mu^\lambda = \sum_\rho \mathbf{s}_\rho^{(k)} \binom{n-|\overline{\rho}|}{m_1(\rho)} \theta_{\tilde{\rho}}^\lambda,$$

which gives also the class expansion of the central element $M_n^{(k)}$. A very different proof was found independently by Féray [6].

Theorem 6.5. *The class expansions*

$$p_k(J_1,\ldots,J_n) = \sum_\rho c_\rho^{(k)} \binom{n-|\overline{\rho}|}{m_1(\rho)} C_{\tilde{\rho}}, \qquad M_n^{(k)} = \sum_\rho \mathbf{s}_\rho^{(k)} \binom{n-|\overline{\rho}|}{m_1(\rho)} C_{\tilde{\rho}}$$

are connected by

$$\mathbf{s}_\rho^{(k)} = c_{\rho\cup(1)}^{(k)}.$$

Proof. At the beginning of this section, we have seen that

$$\sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, p_k(A_{\lambda^{(i)}}) = p_k(A_\lambda) + \sigma_k(\lambda).$$

By (4.5) it yields

$$\sum_{|\mu|=n+1} a_\mu^{(k)}(n+1)\theta_{\mu\setminus(1)}^\lambda = \sum_{|\nu|=n} (a_\nu^{(k)}(n) + s_\nu^{(k)}(n))\, \theta_\nu^\lambda.$$

Equivalently

$$s_\mu^{(k)}(n) = a_{\mu\cup(1)}^{(k)}(n+1) - a_\mu^{(k)}(n) = \sum_{\overline{\rho}=\overline{\mu}} c_\rho^{(k)} \binom{n-|\overline{\mu}|}{m_1(\rho)-1}.$$

□

As indicated in Section 3, this result gives the class expansion of $f_k(J_1,\ldots,J_n)$ with f_k a symmetric function depending on n, given by (3.4).

7 Complete functions

We now consider the expansion

$$h_k(A_\lambda) = \sum_{|\mu|=n} a_\mu^{(k)}(n)\theta_\mu^\lambda = \sum_\rho c_\rho^{(k)} \binom{n-|\overline{\rho}|}{m_1(\rho)} \theta_{\tilde{\rho}}^\lambda.$$

The proof is very similar, though the situation is much more complicated.

7.1 Recurrence

Denoting $u_i = \lambda_i - i + 1$, we have by definition

$$H_z(A_{\lambda^{(i)}}) = H_z(A_\lambda)(1 - zu_i)^{-1}.$$

Hence the expansion (4.9) writes as

$$h_k(A_{\lambda^{(i)}}) = \sum_{j=0}^{k} h_{k-j}(A_\lambda) u_i^j = h_k(A_\lambda) + \sum_{j=1}^{k} h_{k-j}(A_\lambda) u_i^j.$$

Using (3.1) and (3.5) we obtain

$$\begin{aligned} \sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, h_k(A_{\lambda^{(i)}}) &= \sum_{i=1}^{l(\lambda)+1} c_i(\lambda) \Big(h_k(A_\lambda) + \sum_{j=1}^{k} h_{k-j}(A_\lambda) u_i^j \Big) \\ &= h_k(A_\lambda) + \sum_{j=2}^{k} h_{k-j}(A_\lambda)\, \sigma_j(\lambda). \end{aligned}$$

Similarly we have

$$\begin{aligned} \sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, u_i\, h_k(A_{\lambda^{(i)}}) &= \sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, u_i \Big(h_k(A_\lambda) + \sum_{j=1}^{k} h_{k-j}(A_\lambda) u_i^j \Big) \\ &= \sum_{j=1}^{k} h_{k-j}(A_\lambda)\, \sigma_{j+1}(\lambda) \\ &= \sum_{j=2}^{k+1} h_{k-j+1}(A_\lambda)\, \sigma_j(\lambda). \end{aligned}$$

And also

$$\begin{aligned} \sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, u_i^2\, h_k(A_{\lambda^{(i)}}) &= \sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, u_i^2 \Big(\sum_{j=0}^{k} h_{k-j}(A_\lambda) u_i^j \Big) \\ &= \sum_{j=0}^{k} h_{k-j}(A_\lambda)\, \sigma_{j+2}(\lambda) \\ &= \sum_{j=2}^{k+2} h_{k-j+2}(A_\lambda)\, \sigma_j(\lambda). \end{aligned}$$

By elimination, we get immediately

$$\begin{aligned} \sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, h_k(A_{\lambda^{(i)}}) &= h_k(A_\lambda) + \sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, u_i\, h_{k-1}(A_{\lambda^{(i)}}), \\ \sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, u_i\, h_k(A_{\lambda^{(i)}}) &= \sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, u_i^2\, h_{k-1}(A_{\lambda^{(i)}}). \end{aligned}$$

Applying (4.5)–(4.7) we obtain

$$\sum_{|\mu|=n+1} a_\mu^{(k)}(n+1)\theta^\lambda_{\mu\setminus(1)} = \sum_{|\nu|=n} a_\nu^{(k)}(n)\,\theta^\lambda_\nu + \sum_{|\mu|=n+1} a_\mu^{(k-1)}(n+1) \sum_{r\geq 1} r\big(m_r(\mu)+1\big)\theta^\lambda_{\mu\setminus(r+1)\cup(r)},$$

$$\sum_{|\mu|=n+1} a_\mu^{(k)}(n+1) \sum_{r\geq 1} r\big(m_r(\mu)+1\big)\theta^\lambda_{\mu\setminus(r+1)\cup(r)} = \sum_{|\mu|=n+1} a_\mu^{(k-1)}(n+1)\Big((2n-m_1(\mu)+1)\,\theta^\lambda_{\mu\setminus 1} + \sum_{r,s\geq 1} rs(m_r(\mu)+1)(m_s(\mu)+\delta_{rs}+1)\,\theta^\lambda_{\mu\setminus(r+s+1)\cup(r,s)} + \sum_{r,s\geq 2}(r+s-1)(m_{r+s-1}(\mu)+1)\,\theta^\lambda_{\mu\setminus(r,s)\cup(r+s-1)}\Big).$$

By identification of coefficients on both sides, for any $\mu \vdash n$ we get

$$a^{(k)}_{\mu\cup(1)}(n+1) = a^{(k)}_\mu(n) + \sum_{r\geq 1} r m_r(\mu)\, a^{(k-1)}_{\mu\setminus(r)\cup(r+1)}(n+1), \tag{7.1}$$

$$\sum_{r\geq 1} r m_r(\mu)\, a^{(k)}_{\mu\setminus(r)\cup(r+1)}(n+1) = \sum_{r,s\geq 1} rs m_r(\mu)(m_s(\mu)-\delta_{rs})\, a^{(k-1)}_{\mu\setminus(r,s)\cup(r+s+1)}(n+1) + \sum_{r,s\geq 1}(r+s-1)m_{r+s-1}(\mu)\, a^{(k-1)}_{\mu\setminus(r+s-1)\cup(r,s)}(n+1). \tag{7.2}$$

As for power sums, these two recurrence relations determine the coefficients $a^{(k)}_\mu(n)$ by a triple induction: on k, on n and on the lowest part of μ. Using the Lemma given in the Appendix with $z=0$, we also obtain the following result.

Theorem 7.1. *In the class expansion*

$$h_k(J_1,\ldots,J_n) = \sum_\rho c^{(k)}_\rho \binom{n-|\overline{\rho}|}{m_1(\rho)} C_{\tilde{\rho}},$$

the coefficients $c^{(k)}_\rho$ are determined by the two recurrence relations

$$c^{(k)}_{\rho\cup(1)} = \sum_{r\geq 1} r m_r(\rho)\, c^{(k-1)}_{\rho\setminus(r)\cup(r+1)}, \tag{7.3}$$

$$\sum_{r\geq 1} rm_r(\rho)\, c^{(k)}_{\rho\setminus(r)\cup(r+1)} = 2|\rho|\, c^{(k-1)}_{\rho} + m_1(\rho)\, c^{(k-1)}_{\rho\setminus(1)}$$
$$+ \sum_{r,s\geq 1} rsm_r(\rho)(m_s(\rho)-\delta_{rs})\, c^{(k-1)}_{\rho\setminus(r,s)\cup(r+s+1)} \tag{7.4}$$
$$+ \sum_{r,s\geq 1} (r+s-1)m_{r+s-1}(\rho)\, c^{(k-1)}_{\rho\setminus(r+s-1)\cup(r,s)}.$$

Remarks: (i) Surprisingly relation (7.1) (resp.(7.3)) is identical with (6.1) (resp. (6.3)). *(ii)* In the second sum at the right-hand side of (7.4), the cases $r=1$ or $s=1$ give the total contribution $(2|\rho|-m_1(\rho))\, c^{(k-1)}_{\rho\cup(1)}$.

7.2 Leading terms

The generating function of the coefficients $c^{(k)}_{\rho}$ will be studied at Section 9.7, in a more general context.

Proposition 7.2. *The coefficients $c^{(k)}_{\rho}$ are non zero only if $|\rho|-l(\rho)=k-2i$ for some $i\geq 0$. Moreover if $|\rho|-l(\rho)=k$, they are non zero only if $m_1(\rho)=0$.*

Proof. This is shown by induction on k and the lowest part of ρ. If this lowest part is 1, write (7.3) for $\sigma=\rho\setminus(1)$. The right-hand side is non zero only if $|\sigma|+1-l(\sigma)=k-1-2i$ for some $i\geq 0$.Therefore $c^{(k)}_{\rho}$ is non zero only if $|\rho|-l(\rho)=k-2i$ for some $i\geq 1$.

Similarly if the lowest part of ρ is $p>1$, write (7.4) for $\sigma=\rho\setminus(p)\cup(p-1)$. The two sums at the right-hand side are non zero for respectively $|\sigma|+1-l(\sigma)+1=k-1-2i$ and $|\sigma|+1-l(\sigma)-1=k-1-2i$, with $i\geq 0$. We conclude by induction on p. □

The following result was proved independently by Murray [27] and Novak [28].

Proposition 7.3. *If $|\rho|-l(\rho)=k$ and $m_1(\rho)=0$, we have*

$$c^{(k)}_{\rho} = \prod_{i=1}^{l(\rho)} C(\rho_i-1),$$

with $C(r)$ the Catalan number $(2r)!/(r+1)!r!$.

Proof. As shown above, for $|\sigma|-l(\sigma)=k-1$ we have

$$\sum_{r\geq 1} rm_r(\sigma)\, c^{(k)}_{\sigma\setminus(r)\cup(r+1)} = 2|\sigma|\, c^{(k-1)}_{\sigma} + m_1(\sigma)\, c^{(k-1)}_{\sigma\setminus(1)}$$
$$+ \sum_{r,s\geq 1} (r+s-1)m_{r+s-1}(\sigma)\, c^{(k-1)}_{\sigma\setminus(r+s-1)\cup(r,s)},$$

since the first sum at the right-hand side of (7.4) does not contribute. The proof is done by induction on k and the lowest part p of ρ. If we write the previous equation for $\sigma=\rho\setminus(p)\cup(p-1)$, the left-hand side writes as

$$(p-1)c^{(k)}_{\rho} + \sum_{r\geq p} r(m_r(\rho)-\delta_{rp})\, c^{(k)}_{\rho\setminus(p,r)\cup(p-1,r+1)}.$$

Since any partition appearing on the right-hand side has lowest part $\leq p-1$, we can define $c_\rho^{(k)}$ inductively. To conclude it is therefore sufficient to substitute the statement into (7.4) and to prove

$$\sum_{r\geq 2} rm_r \frac{C(r)}{C(r-1)} = 2\sum_{r\geq 2} rm_r + \sum_{t\geq 2} tm_t \sum_{r+s=t+1} \frac{C(r-1)C(s-1)}{C(t-1)}.$$

But this is an obvious consequence of

$$C(r) = \sum_{i=1}^{r-2} C(i)C(r-i-1) + 2C(r-1),$$

a well known recurrence for Catalan numbers. □

8 Hall-Littlewood functions

Let z be an indeterminate and $P_\lambda(z)$ denote the Hall-Littlewood symmetric functions [22, Chapter 3]. When λ is the row partition (k), it is known that $P_k(z)$ interpolates between the power-sum p_k and the complete function h_k, namely

$$P_k(0) = h_k, \qquad P_k(1) = p_k.$$

In view of our previous results, it is therefore a natural problem to study the development of $P_k(J_1,\ldots,J_n;z)$.

For clarity of display, the parameter z being kept fixed, we shall often omit its dependence, though many quantities introduced below are polynomials in z. We write

$$P_k(A_\lambda) = \sum_{|\mu|=n} a_\mu^{(k)}(n)\theta_\mu^\lambda = \sum_\rho c_\rho^{(k)} \binom{n-|\overline{\rho}|}{m_1(\rho)} \theta_{\tilde{\rho}}^\lambda.$$

8.1 Our equations

Given two alphabets A, B and a partition ρ, we have the fundamental formula

$$P_\rho(A\cup B) = \sum_{\sigma\subset\rho} P_{\rho/\sigma}(B)P_\sigma(A)$$

involving skew Hall-Littlewood functions $P_{\rho/\sigma}$ [22, (5.5'), p. 228]. When B has only one element b, $P_{\rho/\sigma}(B)=0$ unless $\rho\setminus\sigma$ is a horizontal strip. In this case $P_{\rho/\sigma}(B) = \psi_{\rho/\sigma} b^{|\rho|-\sigma|}$, where $\psi_{\rho/\sigma} = \prod_{j\in J}(1-z^{m_j(\sigma)})$ and J is the set of j such that $\rho\setminus\sigma$ has no node in the column j and one node in the column $j+1$ [22, (5.8') and (5.14'), p. 229].

Applying this classical result to the alphabet of contents A_λ, and writing $u_i = \lambda_i - i + 1$, we obtain

$$P_\rho(A_{\lambda^{(i)}}) = \sum_{\sigma\subset\rho} P_\sigma(A_\lambda)\,\psi_{\rho/\sigma}\,u_i^{|\rho|-|\sigma|},$$

summed over partitions σ such that $\rho \setminus \sigma$ is a horizontal strip. This is the form taken by (4.9) for Hall-Littlewood symmetric functions.

When $\rho = (k)$ we obtain

$$P_k(A_{\lambda^{(i)}}) = P_k(A_\lambda) + (1-z)\sum_{m=1}^{k-1} P_m(A_\lambda)\, u_i^{k-m} + u_i^k.$$

Then relations (3.1) and (3.5) yield

$$\begin{aligned}
\sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, P_k(A_{\lambda^{(i)}}) &= P_k(A_\lambda) + (1-z)\sum_{m=1}^{k-2} P_m(A_\lambda)\, \sigma_{k-m}(\lambda) + \sigma_k(\lambda),\\
\sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, u_i\, P_k(A_{\lambda^{(i)}}) &= (1-z)\sum_{m=1}^{k-1} P_m(A_\lambda)\, \sigma_{k-m+1}(\lambda) + \sigma_{k+1}(\lambda),\\
\sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, u_i^2\, P_k(A_{\lambda^{(i)}}) &= P_k(A_\lambda)\sigma_2(\lambda) + (1-z)\sum_{m=1}^{k-1} P_m(A_\lambda)\, \sigma_{k-m+2}(\lambda) + \sigma_{k+2}(\lambda).
\end{aligned}$$

Here we have omitted the details of the computation since it goes exactly as in Section 7.1. By elimination, we get immediately

$$\begin{aligned}
\sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, P_k(A_{\lambda^{(i)}}) &= P_k(A_\lambda) + \sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, u_i\, P_{k-1}(A_{\lambda^{(i)}}),\\
\sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, u_i\, P_k(A_{\lambda^{(i)}}) &= -nzP_{k-1}(A_\lambda) + \sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, u_i^2\, P_{k-1}(A_{\lambda^{(i)}}).
\end{aligned}$$

8.2 Recurrence

Applying (4.5)–(4.7) the previous relations write as

$$\begin{aligned}
\sum_{|\mu|=n+1} a_\mu^{(k)}(n+1)\theta^\lambda_{\mu\setminus(1)} &= \sum_{|\nu|=n} a_\nu^{(k)}(n)\, \theta^\lambda_\nu\\
&\quad + \sum_{|\mu|=n+1} a_\mu^{(k-1)}(n+1) \sum_{r\geq 1} r\big(m_r(\mu)+1\big)\theta^\lambda_{\mu\setminus(r+1)\cup(r)},\\
\sum_{|\mu|=n+1} a_\mu^{(k)}(n+1) \sum_{r\geq 1} r\big(m_r(\mu)+1\big)\theta^\lambda_{\mu\setminus(r+1)\cup(r)} &= -nz \sum_{|\nu|=n} a_\nu^{(k-1)}(n)\, \theta^\lambda_\nu\\
&\quad + \sum_{|\mu|=n+1} a_\mu^{(k-1)}(n+1)\, \Big((2n-m_1(\mu)+1)\, \theta^\lambda_{\mu\setminus 1}\\
&\quad + \sum_{r,s\geq 1} rs(m_r(\mu)+1)(m_s(\mu)+\delta_{rs}+1)\, \theta^\lambda_{\mu\setminus(r+s+1)\cup(r,s)}\\
&\quad + \sum_{r,s\geq 2} (r+s-1)(m_{r+s-1}(\mu)+1)\, \theta^\lambda_{\mu\setminus(r,s)\cup(r+s-1)}\Big).
\end{aligned}$$

By identification of coefficients on both sides, for any $\mu \vdash n$ we get

$$a^{(k)}_{\mu\cup(1)}(n+1) = a^{(k)}_{\mu}(n) + \sum_{r\geq 1} rm_r(\mu)\, a^{(k-1)}_{\mu\setminus(r)\cup(r+1)}(n+1), \tag{8.1}$$

$$\begin{aligned}\sum_{r\geq 1} rm_r(\mu)\, a^{(k)}_{\mu\setminus(r)\cup(r+1)}(n+1) &= -nza^{(k-1)}_{\mu}(n)\\ &+ \sum_{r,s\geq 1} rsm_r(\mu)(m_s(\mu)-\delta_{rs})\, a^{(k-1)}_{\mu\setminus(r,s)\cup(r+s+1)}(n+1)\\ &+ \sum_{r,s\geq 1} (r+s-1)m_{r+s-1}(\mu)\, a^{(k-1)}_{\mu\setminus(r+s-1)\cup(r,s)}(n+1).\end{aligned} \tag{8.2}$$

As previously, these two recurrence relations determine the coefficients $a^{(k)}_{\mu}(n)$ by a triple induction: on k, on n and on the lowest part of μ. Using the Lemma in the Appendix, we also obtain the following result.

Theorem 8.1. *We have the class expansion*

$$P_k(J_1,\ldots,J_n;z) = \sum_{\rho} c^{(k)}_{\rho} \binom{n-|\overline{\rho}|}{m_1(\rho)} C_{\tilde{\rho}},$$

where the coefficients $c^{(k)}_{\rho}$ are determined by the recurrence relations

$$c^{(k)}_{\rho\cup(1)} = \sum_{r\geq 1} rm_r(\rho)\, c^{(k-1)}_{\rho\setminus(r)\cup(r+1)}, \tag{8.3}$$

$$\begin{aligned}\sum_{r\geq 1} rm_r(\rho)\, c^{(k)}_{\rho\setminus(r)\cup(r+1)} &= (2-z)|\rho|\, c^{(k-1)}_{\rho} + (1-z)m_1(\rho)\, c^{(k-1)}_{\rho\setminus(1)}\\ &+ \sum_{r,s\geq 1} rsm_r(\rho)(m_s(\rho)-\delta_{rs})\, c^{(k-1)}_{\rho\setminus(r,s)\cup(r+s+1)}\\ &+ \sum_{r,s\geq 1} (r+s-1)m_{r+s-1}(\rho)\, c^{(k-1)}_{\rho\setminus(r+s-1)\cup(r,s)}.\end{aligned} \tag{8.4}$$

We recover Theorems 6.1 or 7.1 by making $z=1$ or $z=0$.

8.3 Leading terms

The following result is proved exactly as Proposition 7.2.

Proposition 8.2. *The coefficients $c^{(k)}_{\rho}$ are non zero only if $|\rho| - l(\rho) = k - 2i$ for some $i \geq 0$. Moreover if $|\rho| - l(\rho) = k$, they are non zero only if $m_1(\rho) = 0$.*

For any positive integer r we define

$$\mathcal{C}(r) = \sum_{m=0}^{r} z^{r-m}(1-z)^m \binom{r+m}{r-m} \frac{1}{m+1}\binom{2m}{m}. \tag{8.5}$$

In particular $\mathcal{C}(r) = 1$ for $z = 1$ and $\mathcal{C}(r) = C(r)$ for $z = 0$.

The polynomial $\mathcal{C}(r)$ is therefore a generalized Catalan "number", which does not seem to have been introduced before. Its first values are given by

$$\mathcal{C}(0) = \mathcal{C}(1) = 1, \quad \mathcal{C}(2) = 2 - z,$$
$$\mathcal{C}(3) = z^2 - 5z + 5, \quad \mathcal{C}(4) = (2 - z)(z^2 - 7z + 7).$$

Proposition 8.3. *The polynomial $\mathcal{C}(r)$ is given by $\mathcal{C}(1) = 1$ and the recurrence formula*

$$\mathcal{C}(r) = (1 - z)\sum_{i=1}^{r-2} \mathcal{C}(i)\mathcal{C}(r - i - 1) + (2 - z)\mathcal{C}(r - 1).$$

Proof. We consider the generating function $\mathbf{C}(u) = 1 + \sum_{r\geq 1} \mathcal{C}(r)u^r$ with $\mathcal{C}(r)$ defined as in the statement. It satisfies the relation

$$(\mathbf{C}(u) - 1)/u - z\mathbf{C}(u) = (1 - z)\mathbf{C}(u)^2.$$

The only solution regular at $u = 0$ is given by

$$\mathbf{C}(u) = \frac{uz - 1 + \sqrt{z^2u^2 + 2(z - 2)u + 1}}{2(z - 1)u}.$$

By two applications of the binomial formula we expand

$$\sqrt{z^2u^2 + 2(z - 2)u + 1} = (1 - uz)\sqrt{1 + 4\frac{(z - 1)u}{(zu - 1)^2}}$$

as a power series. ☐

As an obvious corollary, $\mathcal{C}(r)$ is divisible by $(2 - z)$ when r is even.

Proposition 8.4. *If $|\rho| - l(\rho) = k$ and $m_1(\rho) = 0$, we have*

$$c_\rho^{(k)} = (1 - z)^{l(\rho)-1} \prod_{i=1}^{l(\rho)} \mathcal{C}(\rho_i - 1).$$

Proof. By Proposition 8.2, for $|\sigma| - l(\sigma) = k - 1$ we have

$$\sum_{r\geq 1} rm_r(\sigma)\, c^{(k)}_{\sigma\setminus(r)\cup(r+1)} = (2 - z)|\sigma|\, c_\sigma^{(k-1)} + (1 - z)m_1(\sigma)\, c^{(k-1)}_{\sigma\setminus(1)}$$
$$+ \sum_{r,s\geq 1} (r + s - 1)m_{r+s-1}(\sigma)\, c^{(k-1)}_{\sigma\setminus(r+s-1)\cup(r,s)},$$

since the first sum at the right-hand side of (8.4) does not contribute. The proof is done by induction on k and the lowest part p of ρ, exactly as in Proposition 7.3. The previous equation written for $\sigma = \rho\setminus(p)\cup(p-1)$ defines $c_\rho^{(k)}$ inductively. To conclude it is therefore sufficient to substitute the statement into (8.4) and to prove

$$\sum_{r\geq 2} rm_r \frac{\mathcal{C}(r)}{\mathcal{C}(r-1)} = (2-z)\sum_{r\geq 2} rm_r + (1-z)\sum_{u\geq 2} um_u \sum_{r+s=u+1} \frac{\mathcal{C}(r-1)\mathcal{C}(s-1)}{\mathcal{C}(u-1)}.$$

But this is an obvious consequence of Proposition 8.3. □

Alternative expressions of $\mathcal{C}(r)$ may be obtained by using a result of Matsumoto and Novak [23], which gives the leading coefficient of $m_\lambda(J_1,\ldots,J_n)$, with m_λ a monomial symmetric function.

Proposition 8.5. *We have*

$$\mathcal{C}(r) = \frac{1}{r+1} P_r(1^{r+1}; z).$$

Proof. Given an alphabet X and using λ-ring notations (see Section 6.2), it is well known ([22, (2.10), p. 209] and [15, p. 240]) that

$$(1-z)P_k(X;z) = h_k[(1-z)X^\dagger] = \sum_{|\lambda|=k} (1-z)^{l(\lambda)} m_\lambda(X).$$

Given a partition ρ with $|\rho| - l(\rho) = k$ and $m_1(\rho) = 0$, let us compare the leading coefficients $c_\rho^{(k)}$ of both sides. For $(1-z)P_k(J_1,\ldots,J_n;z)$, it is given by Proposition 8.4. For the right-hand side, using [23, Theorem 5.3], it is given by

$$\sum_{|\lambda|=k} \sum_{\Lambda_1,\ldots,\Lambda_{l(\rho)}} \prod_{i=1}^{l(\rho)} (1-z)^{l(\Lambda_i)} \frac{m_{\Lambda_i}(1^{\rho_i})}{\rho_i} = \prod_{i=1}^{l(\rho)} \Big(\sum_{|\mu|=\rho_i-1} (1-z)^{l(\mu)} \frac{m_\mu(1^{\rho_i})}{\rho_i} \Big),$$

where $\Lambda_1,\ldots,\Lambda_{l(\rho)}$ is an arrangement of the parts of $\lambda \vdash (|\rho| - l(\rho))$ into $l(\rho)$ partitions $\Lambda_i \vdash \rho_i - 1$. By comparison, we obtain

$$\mathcal{C}(r) = \frac{1}{r+1} \sum_{|\mu|=r} (1-z)^{l(\mu)-1} m_\mu(1^{r+1}) = \frac{1}{r+1} P_r(1^{r+1}; z).$$

□

This result provides many interesting formulas for $\mathcal{C}(r)$, in particular those given in [15, p. 240]. Here we shall only indicate two of them, obtained by two classical ways of computing $\sum_{|\mu|=r, l(\mu)=s} m_\mu$.

By [22, Example 1.2.19, p. 33] together with [22, Example 1.2.1, p. 26] we obtain

$$\mathcal{C}(r) = \frac{1}{r+1} \sum_{m=0}^{r} (-1)^{m-1} \frac{1-z^m}{1-z} \binom{r+1}{m} \binom{2r-m}{r}. \tag{8.6}$$

Using [22, Example 1.4.10, p. 68] together with [22, Example 1.3.4, p. 45] we obtain

$$\mathcal{C}(r) = \frac{1}{r+1} \sum_{m=0}^{r-1} (-z)^m \binom{r-1}{m} \binom{2r-m}{r}. \tag{8.7}$$

We are in lack of a direct proof of the equivalence of (8.5) with (8.6) or (8.7).

9 Generating functions

We now consider the generating functions of the coefficients $c_\rho^{(k)}$ defined by Theorem 8.1. As for $z=1$, we write them under the form

$$\phi_\rho(t) = \sum_{k\geq 0} c_\rho^{(k)} \frac{t^k}{k!}.$$

We present two methods to study these generating functions.

9.1 First method: differential system

This method is strictly parallel to the case of power sums, given in Section 6.2. However the situation is much more complicated, and strong difficulties are encountered, which makes this approach unefficient.

As in Section 6.2 we define

$$\Phi_m(t;X) = \sum_{|\rho|=m} \phi_\rho(t)\, z_\rho^{-1} p_\rho(X) = \sum_{|\rho|=m} \psi_\rho(t)\, s_\rho(X).$$

The following result is proved exactly as Proposition 6.2.

Proposition 9.1. *The recurrence relations (8.3)–(8.4) are equivalent with*

$$\frac{d}{dt}\frac{\partial}{\partial p_1}\Phi_m = \mathbf{E}\,\Phi_m, \qquad \frac{d}{dt}\mathbf{E}\,\Phi_{m+1} = \mathbf{D}\,\Phi_{m+1} + (2-z)m\Phi_m + (1-z)p_1\Phi_{m-1}.$$

Applying (6.5), by identification of coefficients of Schur functions on both sides, Proposition 9.1 writes equivalently as

$$\begin{aligned} \sum_{i=1}^{l(\rho)+1} \frac{d}{dt}\psi_{\rho^{(i)}}(t) &= \sum_{i=1}^{l(\rho)+1} (\rho_i - i + 1)\,\psi_{\rho^{(i)}}(t), \\ \sum_{i=1}^{l(\rho)+1} (\rho_i - i + 1)\frac{d}{dt}\psi_{\rho^{(i)}}(t) &= \sum_{i=1}^{l(\rho)+1} (\rho_i - i + 1)^2\,\psi_{\rho^{(i)}}(t) \\ &\quad + (2-z)|\rho|\psi_\rho(t) + (1-z)\sum_{i=1}^{l(\rho)} \psi_{\rho_{(i)}}(t). \end{aligned} \tag{9.1}$$

This first order (overdeterminate) differential system must be solved with the initial conditions $\psi_\rho(0) = \delta_{\rho,(0)}$, due to $c_\rho^{(0)} = \delta_{\rho,(0)}$.

However, in spite of its very simple structure, a general solution of the differential system (9.1) is as yet unknown. Some qualitative results are easy to obtain. By elementary means we have

$$\begin{aligned} \psi_\rho &= \psi_\rho|_{z=1}, \quad \text{if} \quad |\rho| = 2, \\ \psi_\rho &= (2-z)\,\psi_\rho|_{z=1}, \quad \text{if} \quad |\rho| = 3. \end{aligned}$$

Starting from these cases, three properties may be proved by an easy inductive argument:

(i) $\psi_\rho(t)$ is a polynomial in z with degree $|\rho|-2$. It is divisible by $(2-z)$ when $|\rho|$ is odd.

(ii) Denoting ρ' the partition conjugate to ρ, we have $\psi_{\rho'}(t) = \psi_\rho(-t)$.

(iii) With $a_i^{(\rho)}(t)$ a polynomial in t and in z, $\psi_\rho(t)$ expands as

$$\psi_\rho(t) = \sum_{k=1-l(\rho)}^{\rho_1-1} a_k^{(\rho)}(t)e^{kt}.$$

There is empirical evidence that the degree of $a_k^{(\rho)}$ in t depends on ρ. More precisely, $a_k^{(\rho)}$ has degree d in t if either $\rho_{d+1} \geq d+2$ or $\rho_{d+2} \geq d+1$. For instance

- $a_k^{(\rho)}$ has degree 0 in t for any ρ with weight ≤ 6, but $(3,3)$ or $(2,2,2)$,
- $a_k^{(\rho)}$ has degree ≤ 1 in t for any ρ with weight ≤ 12, but $(4,4,4)$ or $(3,3,3,3)$,
- $a_k^{(\rho)}$ has degree ≤ 2 in t for any ρ with weight ≤ 20, but $(5,5,5,5)$ or $(4,4,4,4,4)$, and so on.

Therefore the structure of $a_k^{(\rho)}$ becomes more and more complicated when $|\rho|$ increases.

A general formula for $\psi_\rho(t)$ is lacking even when ρ is a hook $(r,1^s)$. The case of hooks is only known for $0 \leq s \leq 3$, in which cases the structure of $\psi_{(r,1^s)}(t)$ is already very messy.

We have implemented the differential system (9.1) on computer and obtained $\psi_\rho(t)$ for $|\rho| \leq 14$. In a second step, the generating functions $\phi_\rho(t)$ have been recovered by

$$\phi_\rho(t) = \sum_\sigma \chi_\rho^\sigma \psi_\sigma(t),$$

which is a straightforward consequence of the classical Frobenius formula

$$s_\rho = \sum_\sigma \chi_\sigma^\rho \, z_\sigma^{-1} p_\sigma.$$

Tables giving $\phi_\rho(t)$ for $|\rho| \leq 14$ are available on a web page [20].

9.2 Second method: expansion of ϕ_ρ

The difficulties encountered to solve the differential system (9.1) lead us to a very different approach.

We start from the case $z=1$ where, according to Theorem 6.4, the generating function ϕ_ρ writes as

$$|\rho|!\,\phi_\rho(t) = e^{(1-|\rho|)t}\,(e^t-1)^{|\rho|-2} \prod_{i\geq 1} (e^{it}-1)^{m_i(\rho)}. \tag{9.2}$$

Expanding the right-hand side we obtain

$$|\rho|!\,\phi_\rho(t) = \sum_{k=1}^{|\rho|-1} a_k^{(\rho)} \Big((e^{kt}-1) + (-1)^{|\rho|-l(\rho)}(e^{-kt}-1) \Big),$$

where the coefficients $a_k^{(\rho)}$ are given as follows.

For any integer $r \geq 0$, denote by $\mathbf{I}_r$ the family of nonnegative integers $I = (i_0, i_1, i_2, \ldots)$ such that $i_0 + \sum_{u\geq 1} u i_u = r$. Then for $k = |\rho| - 1 - r$ we have

$$a_k^{(\rho)} = \sum_{I\subset \mathbf{I}_r} (-1)^{|I|} \binom{|\rho|-2}{i_0} \prod_{u\geq 1} \binom{m_u(\rho)}{i_u}.$$

For instance $\mathbf{I}_0 = 0$, $\mathbf{I}_1 = \{(1,0),(0,1)\}$ and $\mathbf{I}_2 = \{(2,0,0),(1,1,0),(0,2,0),(0,0,1)\}$ so that we have

$$\begin{aligned} a_{|\rho|-1}^{(\rho)} &= 1, \qquad a_{|\rho|-2}^{(\rho)} = -(|\rho|-2+m_1(\rho)),\\ a_{|\rho|-3}^{(\rho)} &= \binom{|\rho|-2}{2} + m_1(\rho)(|\rho|-2) + \binom{m_1(\rho)}{2} - m_2(\rho). \end{aligned}$$

For arbitrary z, this development of ϕ_ρ may be generalized as follows.

Theorem 9.2. *For any partition ρ, the generating function $\phi_\rho(t)$ is a polynomial in z with degree $|\rho|-2$, which writes as*

$$|\rho|!\,\phi_\rho(t) = \sum_{k=1}^{|\rho|-1} \Big((a_k^{(\rho)}(t)e^{kt} - a_k^{(\rho)}(0)) + \epsilon_\rho (a_k^{(\rho)}(-t)e^{-kt} - a_k^{(\rho)}(0)) \Big). \tag{9.3}$$

Here $\epsilon_\rho = (-1)^{|\rho|-l(\rho)}$ and for $k = |\rho|-1-r$, the coefficient $a_k^{(\rho)}(t)$ is given by

$$a_k^{(\rho)}(t) = \sum_{I\subset \mathbf{I}_r} (-1)^{|I|} \binom{|\rho|-2}{i_0} \prod_{u\geq 1} \binom{m_u(\rho)}{i_u} F_I(z,t;|\rho|), \tag{9.4}$$

where for any $I \subset \mathbf{I}_r$,

$$F_I(z,t;|\rho|) = \sum_{j=0}^{d} t^j F_I^{(j)}(z;|\rho|)$$

is a polynomial in z and t, depending rationally on $|\rho|$.

Remarks: (i) The degree d depends on $k = |\rho|-1-r$, not on r. We have $d = i$ for $|\rho|/(i+2) - i - 2 < k \leq |\rho|/(i+1) - i - 1$. Therefore we have $d \leq i$ for $(i+1)(i+2) \leq |\rho| < (i+2)(i+3)$. In particular $d = 0$ for $|\rho| \leq 5$, $d \leq 1$ for $6 \leq |\rho| \leq 11$ and $d \leq 2$ for $12 \leq |\rho| \leq 19$. This fact is similar to the one mentioned in Section 9.1 for $\psi_\rho(t)$.
(ii) Proposition 8.2 implies that $\phi_\rho(t)$ is divisible by $t^{|\rho|-l(\rho)}$. This fact was obvious on (9.2), but keeps hidden in the expansion (9.3).

Notation: For a better display, we put $w = |\rho|$ and we write the multiplicities of ρ as a set of indeterminates $\mathbf{m} = (m_1, m_2, \ldots)$ linked by $\sum_{i\geq 1} i m_i = w$. We denote

$$M_I(w, \mathbf{m}) = (-1)^{|I|} \binom{w-2}{i_0} \prod_{u\geq 1} \binom{m_u}{i_u}.$$

Given two indices $u, v \geq 1$, we define $\mathbf{m} \setminus (u) \cup (v) = (m_1, \ldots, m_u - 1, \ldots, m_v + 1, \ldots)$. Of course $\mathbf{m} \setminus (u, u)$ should be understood as $(m_1, \ldots, m_u - 2, \ldots)$, and similarly $\mathbf{m} \cup (u, u) = (m_1, \ldots, m_u + 2, \ldots)$.

Proof. We start from the expansion

$$\phi_\rho(t) = \sum_{k=1-|\rho|}^{|\rho|-1} a_k^{(\rho)}(t) e^{kt}.$$

It is easily written as in (9.3) in view of $\phi_\rho(0) = 0$ and because Proposition 8.2 yields

$$\phi_\rho(-t) = \epsilon_\rho \phi_\rho(t).$$

Let us asssume that (9.4) holds.

By definition, $c_\rho^{(u)}$ is the coefficient of $t^u/u!$ in $\phi_\rho(t)$. But if $f(t) = \sum_{j\geq 0} f_j t^j$ is a polynomial in t, the coefficient of $t^u/u!$ in the t-expansion of $e^{kt} f(t)$ is

$$\sum_{0\leq j\leq u} f_j k^{u-j} (u)_j,$$

with $(u)_j$ the lowering factorial. Therefore we have

$$c_\rho^{(u)} = \frac{1}{w!} \big(1 + (-1)^u \epsilon_\rho\big) \sum_{k=1}^{w-1} \sum_{I \subset \mathbf{I}_{w-k-1}} M_I(w, \mathbf{m}) \sum_{0\leq j\leq u} F_I^{(j)}(z; w) k^{u-j} (u)_j. \tag{9.5}$$

On the other hand, $c_\rho^{(u)}$ is inductively defined by the recurrence relations (8.3) and (8.4). If we substitute (9.5) inside (8.3) and (8.4), we obtain two equations which are valid for *infinitely many values* of u. Therefore it is possible to identify the coefficients of terms $k^{u-j}(u)_j$, for $k = 1, \ldots, w-1$, on both sides.

Using $(u)_j = (u-1)_j + j(u-1)_{j-1}$, we obtain the following equations, which translate (8.3) and (8.4) in terms of the F_I's. With $j = 0, \ldots, d$, the counterpart of (8.3) is

$$\sum_{I \subset \mathbf{I}_r} M_I(w+1, \mathbf{m} \cup (1)) \Big((w-r) F_I^{(j)}(z; w+1) + (j+1) F_I^{(j+1)}(z; w+1) \Big) = \sum_{s\geq 1} s m_s \sum_{I \subset \mathbf{I}_r} M_I(w+1, \mathbf{m} \setminus (s) \cup (s+1))\, F_I^{(j)}(z; w+1). \tag{9.6}$$

And the counterpart of (8.4) is

$$
\begin{aligned}
&\sum_{s\geq 1} sm_s \sum_{I\subset \mathbf{I}_r} M_I(w+1,\mathbf{m}\setminus(s)\cup(s+1)) \\
&\qquad\qquad\qquad \times\Big((w-r)F_I^{(j)}(z;w+1)+(j+1)F_I^{(j+1)}(z;w+1)\Big) = \\
&(2-z)w(w+1)\sum_{I\subset \mathbf{I}_{r-1}} M_I(w,\mathbf{m})\,F_I^{(j)}(z;w) \\
&+(1-z)w(w+1)m_1\sum_{I\subset \mathbf{I}_{r-2}} M_I(w-1,\mathbf{m}\setminus(1))\,F_I^{(j)}(z;w-1) \qquad (9.7)\\
&+\sum_{u,v\geq 1} uvm_u(m_v-\delta_{uv})\sum_{I\subset \mathbf{I}_r} M_I(w+1,\mathbf{m}\setminus(u,v)\cup(u+v+1))\,F_I^{(j)}(z;w+1) \\
&+\sum_{u,v\geq 1}(u+v-1)m_{u+v-1}\sum_{I\subset \mathbf{I}_r} M_I(w+1,\mathbf{m}\setminus(u+v-1)\cup(u,v))\,F_I^{(j)}(z;w+1).
\end{aligned}
$$

We emphasize that these equations only involve w and $(m_1, m_2, \ldots, m_r)$. Actually it is not difficult to check that, due to $\sum_{i\geq 1} im_i = w$ and

$$\sum_{u,v\geq 1} uvm_u(m_v-\delta_{uv}) + (u+v-1)m_{u+v-1} = w^2,$$

the indeterminates $(m_{r+1}, m_{r+2}, \ldots)$ do not appear.

Therefore we may consider (9.6) and (9.7) as identities in the *independent* indeterminates $(m_1, m_2, \ldots, m_r)$. By identification of coefficients on both sides of these identities, we obtain linear relations between the $F_I^{(j)}$'s.

Then we proceed by induction on r and define the family $\{F_I^{(j)}(z;w), I \subset \mathbf{I}_r\}$ as the solutions of this (overdetermined) linear system. These solutions exist because the matrix of this linear system is *independent* of z. Since a solution does exist when $z = 1$, the property keeps true for z arbitrary. □

Remark: In this induction, it is easily seen that the polynomials $\{F_I(z,t;w), I \subset \mathbf{I}_r\}$ are divisible by $2-z$ when r is odd.

9.3 Examples

Let us present some explicit results obtained by this algorithm. For any integer $k \geq 1$, we define $R_1(z) = 1$ and

$$R_k(z) = \frac{1}{(k-1)!}\prod_{j=1}^{k-1}(k-jz).$$

For instance $R_2(z) = 2-z$ and $R_3(z) = (3-z)(3-2z)/2$. Obviously $R_k(1) = 1$.

For clarity of display, till the end of this paper, we use the convention

$$
\begin{aligned}
e^{kt} \pm e^{-kt} &:= (e^{kt}-1) + \epsilon_\rho(e^{-kt}-1),\\
t(e^{kt} \mp e^{-kt}) &:= t(e^{kt} - \epsilon_\rho\, e^{-kt}).
\end{aligned}
$$

For $|\rho| = 2$ we have

$$\phi_\rho(t) = \phi_\rho(t)|_{z=1} = \frac{1}{2}(e^t \pm e^{-t}).$$

For $|\rho| = 3$ we have

$$\phi_\rho(t) = (2-z)\,\phi_\rho(t)|_{z=1} = \frac{1}{6}R_2(z)\big(e^{2t} \pm e^{-2t} - (m_1+1)(e^t \pm e^{-t})\big).$$

For $|\rho| = 4$ we have

$$\begin{aligned} 4!\,\phi_\rho(t) &= R_3(z)(e^{3t} \pm e^{-3t}) - R_2(z)(2-z)(m_1+2)(e^{2t} \pm e^{-2t}) + (e^t \pm e^{-t}) \\ &\quad \times \left((2m_1+1)(z^2 - \frac{5}{2}(z-1)) + \binom{m_1}{2}(z^2 - \frac{11}{2}(z-1)) - m_2(z^2 + \frac{1}{2}(z-1)) \right). \end{aligned}$$

Similarly for $|\rho| = 5$ we have

$$\begin{aligned} 5!\,\phi_\rho(t) &= R_4(z)(e^{4t} \pm e^{-4t}) - R_3(z)(2-z)(m_1+3)(e^{3t} \pm e^{-3t}) + R_2(z)(e^{2t} \pm e^{-2t}) \\ &\quad \times \left((3m_1+3)(z^2 - \frac{28}{9}(z-1)) + \binom{m_1}{2}(z^2 - \frac{16}{3}(z-1)) - m_2(z^2 + \frac{4}{3}(z-1)) \right) \\ &\quad - (2-z)(e^t \pm e^{-t}) \left((3m_1+1)(z^2 - \frac{7}{6}(z-1)) + 3\binom{m_1}{2}(z^2 - \frac{17}{6}(z-1)) \right. \\ &\quad \left. + \binom{m_1}{3}(z^2 - \frac{19}{2}(z-1)) - (m_1m_2 + 3m_2 - m_3)(z^2 + \frac{1}{2}(z-1)) \right). \end{aligned}$$

In the previous examples, all $a_k^{(\rho)}(t)$ have degree 0 in t. For $|\rho| = 6$ a t-component appears for the first time at $k = 1$, $r = 4$. We have

$$\begin{aligned} 6!\,\phi_\rho(t) &= R_5(z)(e^{5t} \pm e^{-5t}) - R_4(z)(2-z)(m_1+4)(e^{4t} \pm e^{-4t}) + R_3(z)(e^{3t} \pm e^{-3t}) \\ &\quad \times \left((4m_1+6)(z^2 - \frac{27}{8}(z-1)) + \binom{m_1}{2}(z^2 - \frac{21}{4}(z-1)) - m_2(z^2 + \frac{9}{4}(z-1)) \right) \\ &\quad - R_2(z)(2-z)(e^{2t} \pm e^{-2t}) \left((6m_1+4)(z^2 - 2z + 2) + 4\binom{m_1}{2}(z^2 - \frac{11}{3}(z-1)) \right. \\ &\quad \left. + \binom{m_1}{3}(z^2 - \frac{26}{3}(z-1)) - (m_1m_2 + 4m_2 - m_3)(z^2 + \frac{4}{3}(z-1)) \right) \\ &\quad + (e^t \pm e^{-t}) \sum_{I \subset \mathbf{I}_4} M_I(6, \mathbf{m})\big(z^4 + a_I z^2(z-1) + b_I(z-1)^2\big) \\ &\quad + t(e^t \mp e^{-t})\,(z^2-1)(2z-1) \sum_{I \subset \mathbf{I}_4} M_I(6, \mathbf{m})c_I, \end{aligned}$$

with a_I, b_I and c_I listed below.

I	a_I	b_I	c_I
(4,0,0,0,0) (3,1,0,0,0)	-7/2	7/4	0
(0,4,0,0,0)	-245/12	2035/24	5/4
(1,3,0,0,0)	-91/16	591/32	-5/8
(2,2,0,0,0)	-47/6	37/12	5/24
(2,0,1,0,0)	0	0	-5/24

I	a_I	b_I	c_I
(0,2,1,0,0) (0,0,0,0,1)	-33/4	-77/8	-5/4
(1,1,1,0,0)	-263/48	-197/96	5/8
(0,0,2,0,0)	115/12	235/24	5/4
(0,1,0,1,0)	-5/12	115/24	5/4
(1,0,0,1,0)	-11/16	-49/32	-5/8

Similarly a t-component appears for $|\rho| = 7$ at $k = 1$, $r = 5$. We have

$$\begin{aligned}
7!\,\phi_\rho(t) = {} & R_6(z)(e^{6t} \pm e^{-6t}) - R_5(z)(2-z)(m_1+5)(e^{5t} \pm e^{-5t}) + R_4(z)(e^{4t} \pm e^{-4t}) \\
& \times \left(5(m_1+2)(z^2 - \frac{88}{25}(z-1)) + \binom{m_1}{2}(z^2 - \frac{26}{5}(z-1)) - m_2(z^2 + \frac{16}{5}(z-1))\right) \\
& - R_3(z)(2-z)(e^{3t} \pm e^{-3t}) \left(10(m_1+1)(z^2 - \frac{99}{40}(z-1)) + 5\binom{m_1}{2}(z^2 - \frac{81}{20}(z-1)) \right. \\
& \left. + \binom{m_1}{3}(z^2 - \frac{33}{4}(z-1)) - (m_1 m_2 + 5m_2 - m_3)(z^2 + \frac{9}{4}(z-1)) \right) \\
& + R_2(z)(e^{2t} \pm e^{-2t}) \sum_{I \subset \mathbf{I}_4} M_I(7, \mathbf{m}) \Big(z^4 + A_I z^2 (z-1) + B_I (z-1)^2\Big) \\
& + (e^t \pm e^{-t})(2-z) \sum_{I \subset \mathbf{I}_5} M_I(7, \mathbf{m}) \Big(z^4 + a_I z^2 (z-1) + b_I (z-1)^2\Big) \\
& + t(e^t \mp e^{-t})\,(2-z)(z^2-1)(2z-1) \sum_{I \subset \mathbf{I}_5} M_I(7, \mathbf{m}) c_I,
\end{aligned}$$

with A_I, B_I, a_I, b_I and c_I listed below.

I	A_I	B_I
(4,0,0,0,0) (3,1,0,0,0)	-47/10	22/5
(0,4,0,0,0)	-47/3	212/3
(1,3,0,0,0)	-172/15	304/15
(2,2,0,0,0)	-88/15	136/15
(2,0,1,0,0)	-71/30	-74/15

I	A_I	B_I
(0,2,1,0,0) (0,0,0,0,1)	-26/3	-40/3
(1,1,1,0,0)	-4/15	-32/15
(0,0,2,0,0)	37/3	44/3
(0,1,0,1,0)	11/6	2/3
(1,0,0,1,0)	-67/15	-116/15

I	a_I	b_I	c_I
(5,0,0,0,0,0) (4,1,0,0,0,0)	-19/10	11/20	0
(0,5,0,0,0,0)	4371/10	6271/20	77/4
(1,4,0,0,0,0)	-4167/50	3573/100	-119/20
(3,2,0,0,0,0)	0	0	7/40
(2,3,0,0,0,0)	0	0	7/8
(3,0,1,0,0,0)	-9/2	3/4	-7/40
(0,3,1,0,0,0)	-1281/20	-3521/40	-49/4
(2,1,1,0,0,0)	0	0	-7/40
(1,2,1,0,0,0)	0	0	63/20

I	a_I	b_I	c_I
(1,0,2,0,0,0)	0	0	-7/20
(0,1,2,0,0,0)	-163/10	717/20	21/4
(0,0,1,1,0,0)	-257/40	-1137/80	-7/4
(2,0,0,1,0,0)	-163/20	17/40	-7/40
(0,2,0,1,0,0)	-3623/40	5417/80	35/4
(1,1,0,1,0,0)	1249/100	-1331/200	-7/4
(1,0,0,0,1,0)	-1363/100	697/200	7/20
(0,1,0,0,1,0)	1849/20	-2091/40	-21/4
(0,0,0,0,0,1)	-593/20	907/40	7/4

9.4 Constant terms

We present the results of the algorithmic method of Section 9.2 for those coefficients $a_k^{(\rho)}(t)$ which do not depend on t. This is the situation encountered most of the time in the examples given above.

Let $k = |\rho| - r - 1$ with $0 \le r \le |\rho| - 2$. We assume that the coefficient $a_k^{(\rho)}(t)$ does not depend on t. This case only occurs for $|\rho|/2 - 2 < k \le |\rho| - 1$, i.e. $0 \le r < |\rho|/2 + 1$.

Then $F_I(z,t;|\rho|)$ writes as

$$F_I(z,t;|\rho|) = R_k(z)\, p_I(z;|\rho|),$$

where $p_I(z;w)$ is a polynomial of degree r in z, depending rationally on w, and divisible by $2-z$ when r is odd.

The families $\{p_I(z;w), I \subset \mathbf{I}_r\}$ are defined inductively by the linear system

$$\begin{aligned}
(w-r)\sum_{I\subset \mathbf{I}_r} & M_I(w+1,\mathbf{m}\cup(1))p_I(z;w+1) = \\
& \sum_{s\geq 1} s m_s \sum_{I\subset \mathbf{I}_r} M_I(w+1,\mathbf{m}\setminus(s)\cup(s+1))\, p_I(z;w+1), \\
(w-r)\sum_{s\geq 1} & s m_s \sum_{I\subset \mathbf{I}_r} M_I(w+1,\mathbf{m}\setminus(s)\cup(s+1))p_I(z;w+1) = \\
& (2-z)w(w+1)\sum_{I\subset \mathbf{I}_{r-1}} M_I(w,\mathbf{m})\, p_I(z;w) \\
& +(1-z)w(w+1)m_1 \sum_{I\subset \mathbf{I}_{r-2}} M_I(w-1,\mathbf{m}\setminus(1))\, p_I(z;w-1) \\
& +\sum_{u,v\geq 1} uv m_u(m_v-\delta_{uv})\sum_{I\subset \mathbf{I}_r} M_I(w+1,\mathbf{m}\setminus(u,v)\cup(u+v+1))\, p_I(z;w+1) \\
& +\sum_{u,v\geq 1}(u+v-1)m_{u+v-1}\sum_{I\subset \mathbf{I}_r} M_I(w+1,\mathbf{m}\setminus(u+v-1)\cup(u,v))\, p_I(z;w+1).
\end{aligned}$$

It is easy to compute $p_I(z;w)$ for the first values of r. For $r\leq 1$ we have

$$p_{(0)}(z;w)=1, \qquad p_{(1,0)}(z;w)=p_{(0,1)}(z;w)=2-z.$$

For $r=2$ we have $p_I(z;w)=z^2+a_I(w)(z-1)$ with $a_I(w)$ given by

$$\begin{aligned}
a_{(2,0,0)}(w) &= a_{(1,1,0)}(w) = -2\frac{(w-3)(2w-3)}{(w-2)^2}, \\
a_{(0,2,0)}(w) &= -\frac{5w-9}{w-2}, \qquad a_{(0,0,1)}(w) = \frac{(w-3)^2}{w-2}.
\end{aligned}$$

For $r=3$ we have $p_I(z;w)=(2-z)\big(z^2+a_I(w)(z-1)\big)$ with $a_I(w)$ given by

$$\begin{aligned}
& a_{(3,0,0,0)}(w) = a_{(2,1,0,0)}(w) = -2\frac{(w-4)^2(2w-3)}{(w-2)(w-3)^2}, \qquad a_{(1,2,0,0)}(w) = -\frac{(w-4)(5w-8)}{(w-2)(w-3)}, \\
& a_{(0,3,0,0)}(w) = -\frac{7w-16}{w-3}, \qquad a_{(0,1,1,0)}(w) = a_{(1,0,1,0)}(w) = a_{(0,0,0,1)}(w) = \frac{(w-4)^2}{w-3}.
\end{aligned}$$

Unfortunately as soon as $r\geq 4$, the expression of $p_I(z;w)$ becomes very messy. For $r=4$ we have

$$p_I(z;w) = z^4 + \frac{a_I(w)}{c_I(w)}z^2(z-1) + \frac{b_I(w)}{c_I(w)}(z-1)^2,$$

where for $w \geq 7$, $a_I(w)$, $b_I(w)$ and $c_I(w)$ are given by the following table.

I	a_I	b_I
(4,0,0,0,0) (3,1,0,0,0)	$-4(w-5)(2w-5)(w^3-10w^2+32w-30)$	$4(w-5)^3(2w-5)(2w-3)$
(0,4,0,0,0)	$-2(7w^3-84w^2+312w-375)$	$43w^3-485w^2+1677w-1875$
(1,3,0,0,0)	$-(w-5)(11w^3-111w^2+330w-300)$	$2(w-5)(14w^3-145w^2+426w-375)$
(2,2,0,0,0)	$-9w^5+171w^4-1249w^3+4361w^2-7214w+4500$	$2(w-5)^2(10w^3-99w^2+274w-225)$
(2,0,1,0,0)	$(w-5)(w^5-23w^4+193w^3-745w^2+1334w-900)$	$-2(w-5)^3(2w^3-19w^2+52w-45)$
(0,2,1,0,0) (0,0,0,0,1)	$(w-5)(w^2-16w+50)$	$-5(w-5)^3$
(1,1,1,0,0)	$w^4-22w^3+167w^2-506w+500$	$-2(w-5)^2(2w^2-17w+25)$
(0,0,2,0,0)	$2(w-5)(w^3-12w^2+51w-75)$	$(w-5)^3(w^2-6w+15)$
(0,1,0,1,0)	$w^4-23w^3+188w^2-636w+750$	$-(w-5)^2(4w^2-39w+75)$
(1,0,0,1,0)	$(w-5)(w^4-20w^3+131w^2-342w+300)$	$-2(w-5)^3(2w^2-12w+15)$

I	c_I	I	c_I
(4,0,0,0,0) (3,1,0,0,0)	$(w-2)(w-3)^2(w-4)^2$	(0,2,1,0,0) (0,0,0,0,1)	$(w-4)(w-6)$
(0,4,0,0,0)	$(w-3)(w-4)(w-6)$	(1,1,1,0,0)	$(w-2)(w-4)(w-6)$
(1,3,0,0,0)	$(w-2)(w-3)(w-4)(w-6)$	(0,0,2,0,0) (0,1,0,1,0)	$(w-3)(w-4)(w-6)$
(2,2,0,0,0) (2,0,1,0,0)	$(w-2)(w-3)^2(w-4)(w-6)$	(1,0,0,1,0)	$(w-2)(w-3)(w-4)(w-6)$

For $r = 5$ we have

$$p_I(z;w) = (2-z)\left(z^4 + \frac{A_I(w)}{C_I(w)}z^2(z-1) + \frac{B_I(w)}{C_I(w)}(z-1)^2\right),$$

where for $w \geq 9$, $A_I(w)$, $B_I(w)$ and $C_I(w)$ are given by the table below.

I	A_I
(5,0,0,0,0,0) (4,1,0,0,0,0)	$-4(w-6)^2(2w-5)(w^3-11w^2+39w-39)$
(0,5,0,0,0,0)	$-18w^4+402w^3-3198w^2+10668w-13104$
(1,4,0,0,0,0)	$-14w^5+358w^4-3486w^3+15968w^2-33784w+26208$
(3,2,0,0,0,0)	$-(w-6)(9w^5-188w^4+1499w^3-5668w^2+10000w-6552)$
(2,3,0,0,0,0)	$-11w^6+328w^5-3921w^4+23888w^3-77488w^2+125328w-78624$
(3,0,1,0,0,0)	$(w-6)^2(w^5-24w^4+211w^3-852w^2+1576w-1092)$
(0,3,1,0,0,0)	$w^5-38w^4+531w^3-3472w^2+10832w-13104$
(2,1,1,0,0,0)	$(w-6)(w^6-34w^5+459w^4-3114w^3+11072w^2-19388w+13104)$
(1,2,1,0,0,0)	$w^6-38w^5+571w^4-4332w^3+17368w^2-34528w+26208$
(1,0,2,0,0,0)	$2(w-6)^2(w^3-15w^2+70w-91)$
(0,1,2,0,0,0)	$2(w-6)(w^4-23w^3+194w^2-739w+1092)$
(0,0,1,1,0,0)	$2(w-6)^2(w^3-16w^2+92w-182)$
(2,0,0,1,0,0)	$(w-6)^2(w^5-28w^4+287w^3-1340w^2+2844w-2184)$
(0,2,0,1,0,0)	$w^5-36w^4+499w^3-3342w^2+10732w-13104$
(1,1,0,1,0,0)	$(w-6)(w^5-31w^4+365w^3-1997w^2+4980w-4368)$
(1,0,0,0,1,0)	$(w-6)^2(w^4-26w^3+219w^2-712w+728)$
(0,1,0,0,1,0)	$(w-6)(w^4-30w^3+323w^2-1428w+2184)$
(0,0,0,0,0,1)	$(w-6)^2(w^2-22w+91)$

I	B_I
(5,0,0,0,0,0) (4,1,0,0,0,0)	$4(w-6)^4(2w-3)(2w-5)$
(0,5,0,0,0,0)	$71w^4-1474w^3+10736w^2-32496w+36288$
(1,4,0,0,0,0)	$(w-6)(43w^4-842w^3+5584w^2-14256w+12096)$
(3,2,0,0,0,0)	$2(w-6)^3(10w^3-107w^2+304w-252)$
(2,3,0,0,0,0)	$2(w-6)^2(14w^4-265w^3+1664w^2-3912w+3024)$
(3,0,1,0,0,0)	$-2(w-6)^4(2w^3-19w^2+50w-42)$
(0,3,1,0,0,0)	$-(w-6)^2(7w^3-116w^2+592w-1008)$
(2,1,1,0,0,0)	$-2(w-6)^3(2w^4-39w^3+252w^2-614w+504)$
(1,2,1,0,0,0)	$-(w-6)^2(5w^4-104w^3+752w^2-2144w+2016)$
(1,0,2,0,0,0)	$(w-6)^4(w^2-8w+14)$
(0,1,2,0,0,0)	$(w-6)^3(w^3-16w^2+78w-168)$
(0,0,1,1,0,0)	$(w-6)^4(w^2-8w+28)$
(2,0,0,1,0,0)	$-2(w-6)^4(2w^3-25w^2+86w-84)$
(0,2,0,1,0,0)	$-(w-6)^2(5w^3-94w^2+572w-1008)$
(1,1,0,1,0,0)	$-(w-6)^3(4w^3-65w^2+292w-336)$
(1,0,0,0,1,0)	$-(w-6)^4(5w^2-40w+56)$
(0,1,0,0,1,0)	$-(w-6)^3(5w^2-68w+168)$
(0,0,0,0,0,1)	$-7(w-6)^4$

I	C_I
(5,0,0,0,0,0) (4,1,0,0,0,0)	$(w-2)(w-3)(w-4)^2(w-5)^2$
(0,5,0,0,0,0)	$(w-4)(w-5)(w-7)(w-8)$
(1,4,0,0,0,0)	$(w-2)(w-4)(w-5)(w-7)(w-8)$
(3,2,0,0,0,0)	$(w-2)(w-3)(w-4)^2(w-5)(w-7)$
(2,3,0,0,0,0)	$(w-2)(w-3)(w-4)(w-5)(w-7)(w-8)$
(3,0,1,0,0,0)	$(w-2)(w-3)(w-4)^2(w-5)(w-7)$
(0,3,1,0,0,0)	$(w-4)(w-5)(w-7)(w-8)$
(2,1,1,0,0,0)	$(w-2)(w-3)(w-4)(w-5)(w-7)(w-8)$
(1,2,1,0,0,0)	$(w-2)(w-4)(w-5)(w-7)(w-8)$
(1,0,2,0,0,0)	$(w-2)(w-5)(w-7)(w-8)$
(0,1,2,0,0,0) (0,0,1,1,0,0)	$(w-4)(w-5)(w-7)(w-8)$
(2,0,0,1,0,0)	$(w-2)(w-3)(w-4)(w-5)(w-7)(w-8)$
(0,2,0,1,0,0)	$(w-4)(w-5)(w-7)(w-8)$
(1,1,0,1,0,0) (1,0,0,0,1,0)	$(w-2)(w-4)(w-5)(w-7)(w-8)$
(0,1,0,0,1,0)	$(w-4)(w-5)(w-7)(w-8)$
(0,0,0,0,0,1)	$(w-5)(w-7)(w-8)$

We have also computed the case $r=6$ where we have

$$p_I(z;w)=z^6+\frac{a_I(w)}{d_I(w)}z^4(z-1)+\frac{b_I(w)}{d_I(w)}z^2(z-1)^2+\frac{c_I(w)}{d_I(w)}(z-1)^3.$$

But since card $\mathbf{I}_6=30$, these tables cannot be given here. They are available upon request.

9.5 Other terms

Let $k = |\rho| - r - 1$ with $0 \le r \le |\rho| - 2$. Section 9.4 was devoted to those coefficients $a_k^{(\rho)}(t)$ which are independent of t. This case occurs only for $|\rho|/2 - 2 < k \le |\rho| - 1$, i.e. $0 \le r < |\rho|/2 + 1$.

For $1 \le k \le |\rho|/2 - 2$, i.e. $|\rho|/2 + 1 \le r \le |\rho| - 2$, the coefficient $a_k^{(\rho)}(t)$ is no longer a constant. In this situation an explicit formula is not yet known.

However there is empirical evidence that $F_I(z, t; |\rho|)$ writes as

$$\begin{aligned} F_I(z,t;|\rho|) = R_k(z)\, G_I^{(0)}(z,|\rho|) &+ t(z-1)(z+k)\big((k+1)z-k\big) R_k(z)^2\, G_I^{(1)}(z;|\rho|) \\ &+ t^2(z-1)^2(2z+k)((k+2)z-k)(z+k)^2((k+1)z-k)^2 R_k(z)^3\, G_I^{(2)}(z;|\rho|) \\ &+ t^3[\ldots]. \end{aligned}$$

Here for $u \ge 0$, $G_I^{(u)}(z;|\rho|)$ is a polynomial in z, depending rationally on $|\rho|$. It has degree $|\rho| - (u+1)k - (u+1)^2$ in z. Thus $G_I^{(u)}(z;|\rho|)$ is nonzero for $1 \le k \le |\rho|/(u+1) - u - 1$. In particular $G_I^{(1)}(z;|\rho|)$ is nonzero for $1 \le k \le |\rho|/2 - 2$ and $G_I^{(2)}(z;|\rho|)$ is nonzero for $1 \le k \le |\rho|/3 - 3$.

Formulas given in Section 9.3 for $|\rho| = 6, k = 1, r = 4$ and $|\rho| = 7, k = 1, r = 5$ are the first cases encountered. For $|\rho| = 8$ there are two possibilities, either $k = 1$, $r = 6$ or $k = 2$, $r = 5$, with $G_I^{(1)}(z;8)$ having respectively degree 2 or 0 in z.

9.6 Residues

It should be emphasized that, when a t-component occurs, i.e. for $|\rho|/2+1 \le r \le |\rho| - 2$, the equality $G_I^{(0)}(z,|\rho|) = p_I(z,|\rho|)$ cannot hold. Actually in this situation, the rational function $p_I(z;w)$ is singular at $w = |\rho|$.

This fact can be checked on the tables of Section 9.4. For $r = 4$ we see that $p_I(z;w)$ has a simple pole at $w = 6$. And for $r = 5$, we see that $p_I(z;w)$ has two simple poles at $w = 7$ and $w = 8$.

In other words, a nonzero t-component $F_I^{(1)}(z;|\rho|)$ occurs exactly at those values of $|\rho|$ which are poles of $p_I(z;w)$. Moreover, denoting $\operatorname{Res}(f,a)$ the residue of a rational function $f(w)$ at $w = a$, it may be proved easily by induction that for $|\rho|/2 + 1 \le r \le |\rho| - 2$, we have

$$F_I^{(1)}(z;|\rho|) = \frac{1}{2} R_k(z)\, \operatorname{Res}(p_I(z;w), |\rho|).$$

For instance we have for $r = 4, 5$ respectively

$$\begin{aligned} \operatorname{Res}(p_I(z;w), 6) &= 2\,(z^2-1)(2z-1)c_I, \\ \operatorname{Res}(p_I(z;w), 7) &= 2\,(z^2-1)(2z-1)(2-z)c_I, \end{aligned}$$

with c_I given at Section 9.3 for $|\rho| = 6, 7$ respectively.

We conjecture that this property is a general fact, keeping valid for any component $F_I^{(u)}(z;|\rho|)$, $u > 1$, which we expect to write as the residue of $p_I(z;w)$ at a multiple pole of order u. If this was true, the structure of $\phi_\rho(t)$ might be totally encoded in the polynomials $p_I(z;w)$.

9.7 Complete functions

An important application is obtained by specializing $z = 0$. Since $P_k(0) = h_k$, this particular case corresponds to complete functions. Then $\phi_\rho(t)$ is the generating function of the coefficients $c_\rho^{(k)}$ of Theorem 7.1.

Denoting $|\rho| = w$, the first terms of the expansion of $\phi_\rho(t)$ are

$$\begin{aligned}
w!\,\phi_\rho(t) &= R_{w-1}(e^{(w-1)t} \pm e^{-(w-1)t}) - 2R_{w-2}(m_1 + w - 2)(e^{(w-2)t} \pm e^{-(w-2)t}) \\
&\quad + \frac{R_{w-3}}{w-2}(e^{(w-3)t} \pm e^{-(w-3)t})\Big((2m_1 + w - 3)(w-3)(2w-3) + (5w-9)\binom{m_1}{2} \\
&\quad + (w-3)^2 m_2\Big) - 2\frac{R_{w-4}}{w-3}(e^{(w-4)t} \pm e^{-(w-4)t})\Big(\frac{1}{3}(3m_1 + w - 4)(w-4)^2(2w-3) \\
&\quad + (w-4)(5w-8)\binom{m_1}{2} + (7w-16)\binom{m_1}{3} + (w-4)^2(m_1 m_2 + (w-2)m_2 - m_3)\Big) \\
&\quad + R_{w-5}(e^{(w-5)t} \pm e^{-(w-5)t}) \sum_{I \subset \mathbf{I}_4} M_I(w, \mathbf{m}) \frac{b_I(w)}{c_I(w)} \\
&\quad + 2R_{w-6}(e^{(w-6)t} \pm e^{-(w-6)t}) \sum_{I \subset \mathbf{I}_5} M_I(w, \mathbf{m}) \frac{B_I(w)}{C_I(w)} + \text{etc} \ldots.
\end{aligned}$$

Here $w \geq 9$ is implicitly assumed. The coefficients $b_I(w)$, $c_I(w)$, $B_I(w)$ and $C_I(w)$ are listed in the tables of Section 9.4 and

$$R_k = k^{k-1}/(k-1)!.$$

For such contributions with $w/2 - 2 < k \leq w - 1$, the coefficient of $e^{kt} \pm e^{-kt}$ does not depend on t. However one must add terms with $1 \leq k \leq w/2 - 2$ whose contributions are not yet explicitly known.

10 Other symmetric functions

Our method may be used in other situations than those presented above. Here are three examples.

Firstly [22, Example 3.2.3, p. 214] the Hall-Littlewood symmetric function writes as

$$P_k(z) = \sum_{a+b=k} (-z)^b s_{(a,1^b)}.$$

In other words, it is the generating function for Schur functions associated with hooks. Therefore the results of Sections 8–9 can be at once translated in terms of hook Schur functions.

Then we can handle the product $h_k e_l$, since the Pieri formula yields

$$h_k e_l = s_{(k,1^l)} + s_{(k+1,1^{l-1})}.$$

And we can also deal with

$$p_{k,l} = \sum_{\substack{|\lambda|=k \\ l(\lambda)=l}} m_\lambda = \sum_{a+b=k} (-1)^{b-l} \binom{b}{l} h_a e_b,$$

the first sum taken on monomial symmetric functions associated with partitions of weight k and length l [22, Example 1.2.19, p. 33]. Observe that $p_{k,1} = p_k$, $p_{k,k} = e_k$ and $\sum_{l=1}^{k} p_{k,l} = h_k$.

Of course our method can be applied directly to $s_{(k,1^l)}$, $h_k e_l$, or $p_{k,l}$ without using the Hall-Littlewood polynomial. Then the recurrence for class expansion coefficients will depend on two parameters k, l. Here we only give our equations (4.10), with $u_i = \lambda_i - i + 1$. For the product $h_k e_l$ they are

$$\sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, (h_k e_l)(A_{\lambda^{(i)}}) = (h_k e_l)(A_\lambda) + \sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, u_i\, (h_{k-1} e_l)(A_{\lambda^{(i)}}),$$

$$\sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, u_i\, (h_k e_l)(A_{\lambda^{(i)}}) = n(h_k e_{l-1})(A_\lambda) + \sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, u_i^2\, (h_{k-1} e_l)(A_{\lambda^{(i)}}).$$

For the Schur functions $s_{(k,1^l)}$ they are

$$\sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, s_{(k,1^l)}(A_{\lambda^{(i)}}) = s_{(k,1^l)}(A_\lambda) + \sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, u_i\, s_{(k-1,1^l)}(A_{\lambda^{(i)}}),$$

$$\sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, u_i\, s_{(k,1^l)}(A_{\lambda^{(i)}}) = n s_{(k,1^{(l-1)})}(A_\lambda) + \sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, u_i^2\, s_{(k-1,1^l)}(A_{\lambda^{(i)}}).$$

And for the partial sums $p_{k,l}$ they are

$$\sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, p_{k,l}(A_{\lambda^{(i)}}) = p_{k,l}(A_\lambda) + \sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, u_i\, p_{k-1,l}(A_{\lambda^{(i)}}),$$

$$\sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, u_i\, p_{k,l}(A_{\lambda^{(i)}}) = n(p_{k-1,l-1} - p_{k-1,l})(A_\lambda) + \sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\, u_i^2\, p_{k-1,l}(A_{\lambda^{(i)}}).$$

The other steps are left to the reader.

Unfortunately our method is not efficient with the one-row Macdonald symmetric function, nor with the products e_μ, p_μ, h_μ. With the latter, two difficulties are quickly encountered. Firstly the computations become very messy. Secondly one needs to extend the results of Theorem 4.1 in order to express

$$\sum_i c_i(\lambda)\, (\lambda_i - i + 1)^k\, \theta_\mu^{\lambda^{(i)}} \quad \text{for } k \geq 3.$$

However for $l \leq 3$ the products $p_k p_l$ and the monomial symmetric functions $m_{(k,l)} = p_k p_l - p_{k+l}$ may be handled without any new ingredient. Actually for $p_k p_l$ we have

$$\sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\,(p_k p_l)(A_{\lambda^{(i)}}) = (p_k p_l + p_k f_l - p_{k-1} f_{l+1})(A_\lambda) + \sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\,u_i\,(p_{k-1} p_l)(A_{\lambda^{(i)}}),$$
$$\sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\,u_i\,(p_k p_l)(A_{\lambda^{(i)}}) = (p_k f_{l+1} - p_{k-1}(f_{l+2} + np_l))(A_\lambda) + \sum_{i=1}^{l(\lambda)+1} c_i(\lambda)\,u_i^2\,(p_{k-1} p_l)(A_{\lambda^{(i)}}).$$

Therefore using (3.6), a recurrence may be defined provided f_l, f_{l+1}, f_{l+2} do not involve products $p_a p_b$, i.e. for $l \leq 3$.

11 Extension to Jack polynomials

Our method has a very natural extension in the framework of Jack polynomials. This generalization will be developed in another paper. Here we quickly present some of its results (omitting the proofs).

Let α be some positive real parameter and $\beta = \alpha - 1$. The family of Jack polynomials $J_\lambda(\alpha)$, indexed by partitions, forms a basis of the algebra of symmetric functions with rational coefficients in α [22, 33]. We consider the transition matrix between this basis and the classical basis of power sums p_μ, i.e. we write

$$J_\lambda(\alpha) = \sum_{|\mu|=|\lambda|} \theta^\lambda_\mu(\alpha)\, p_\mu.$$

As a consequence of the Frobenius formula (see the argument in the introduction of [19]), the quantities $\theta^\lambda_\mu(\alpha)$ generalize the central characters, i.e. we have $\theta^\lambda_\mu(1) = \theta^\lambda_\mu = n!\, z_\mu^{-1} \hat{\chi}^\lambda_\mu$.

Given a partition λ, the α-content of any node $(i,j) \in \lambda$ is defined as $j-1-(i-1)/\alpha$. We denote by $A^{(\alpha)}_\lambda = \{j-1-(i-1)/\alpha,\ (i,j) \in \lambda\}$ the finite alphabet of the α-contents of λ.

Denote by $\mathbf{Q}[\alpha]$ the field of rational functions in α. A polynomial in r indeterminates $\lambda = (\lambda_1, \ldots, \lambda_r)$ with coefficients in $\mathbf{Q}[\alpha]$ is said to be "shifted symmetric" in λ if it is symmetric in the r "shifted variables" $\lambda_i - i/\alpha$. In analogy with symmetric functions, this defines $\mathbf{S}^*(\alpha)$, the algebra of shifted symmetric functions with coefficients in $\mathbf{Q}[\alpha]$. We refer to [29, 30, 31], or to [17, 19] for a short survey.

It is known [19, Proposition 2] that the quantities $\theta^\lambda_\mu(\alpha)$ are shifted symmetric functions of λ, and form a basis of $\mathbf{S}^*(\alpha)$. Moreover [17, Lemma 7.1], given a symmetric function f, its α-content evaluation $f(A^{(\alpha)}_\lambda)$ is also a shifted symmetric function of λ. The argument is similar to the one already given in Section 2.4.

It is therefore a natural problem to consider the expansion

$$f(A^{(\alpha)}_\lambda) = \sum_{|\mu|=n} a_\mu(n)\, \theta^\lambda_\mu(\alpha),$$

with $|\lambda| = n$, and to study the properties of the coefficients $a_\mu(n)$.

The simplest result of this type is the following generalization of Jucys' classical result

$$\alpha^k e_k(A_\lambda^{(\alpha)}) = \sum_{\substack{|\mu|=n \\ l(\mu)=n-k}} \theta_\mu^\lambda(\alpha).$$

This expansion was first obtained in [14, Theorem 5.4], as a consequence of the "Cauchy formula" for Jack polynomials (see also [24, Prop. 8.3]). Another proof may be obtained by generalizing the argument in Section 5, which also yields the expansion

$$\alpha^{k+1}(e_1e_k)(A_\lambda^{(\alpha)}) = \sum_{\substack{|\mu|=n \\ l(\mu)=n-k-1}} a_\mu \theta_\mu^\lambda(\alpha) + \beta \sum_{\substack{|\mu|=n \\ l(\mu)=n-k}} a_\mu \theta_\mu^\lambda(\alpha) + \alpha \sum_{\substack{|\mu|=n \\ l(\mu)=n-k+1}} \left(\binom{n}{2} - a_\mu\right) \theta_\mu^\lambda(\alpha)$$

with a_μ defined in Proposition 5.2.

A second important case is the extension of Lascoux-Thibon's result, i.e. the expansion

$$\alpha^k p_k(A_\lambda^{(\alpha)}) = \sum_{|\mu|=n} a_\mu^{(k)}(n)\, \theta_\mu^\lambda(\alpha).$$

A generalization of the method in Section 6 provides the following result.

Theorem. *In the previous expansion, the coefficients $a_\mu^{(k)}(n)$ are polynomials in n, written as*

$$a_\mu^{(k)}(n) = \sum_{\overline{\rho}=\overline{\mu}} c_\rho^{(k)} \binom{n-|\overline{\rho}|}{m_1(\rho)}.$$

Here the quantities $c_\rho^{(k)}$ are polynomials in (α, β) with nonnegative integer coefficients, determined by the recurrence relations

$$c_{\rho\cup(1)}^{(k)} = \alpha \sum_{r\geq 1} r m_r(\rho)\, c_{\rho\setminus(r)\cup(r+1)}^{(k-1)},$$

$$\begin{aligned}\sum_{r\geq 1} r m_r(\rho)\, c_{\rho\setminus(r)\cup(r+1)}^{(k)} = |\rho|\, c_\rho^{(k-1)} &+ \alpha \sum_{r,s\geq 1} rs m_r(\rho)(m_s(\rho) - \delta_{rs})\, c_{\rho\setminus(r,s)\cup(r+s+1)}^{(k-1)} \\ &+ \sum_{r,s\geq 1} (r+s-1) m_{r+s-1}(\rho)\, c_{\rho\setminus(r+s-1)\cup(r,s)}^{(k-1)} \\ &+ \beta \sum_{r\geq 1} r^2 m_r(\rho) c_{\rho\setminus(r)\cup(r+1)}^{(k-1)}.\end{aligned}$$

By induction on k and the lowest part of ρ, the polynomials $c_\rho^{(k)}$ are non zero for $|\rho| + l(\rho) \leq k+2$. Their generating function

$$\phi_\rho(t) = \sum_{k\geq 0} c_\rho^{(k)} \frac{t^k}{k!}$$

can be determined. However the situation is much more intricate than for $\alpha = 1$. In particular ϕ_ρ cannot be written in factorized form.

The first values are given by

$$\begin{aligned}
\phi_2(t) &= \frac{e^{\alpha t} - e^{-t}}{\alpha+1}, \qquad \phi_{1^2}(t) = \frac{e^{\alpha t} + \alpha e^{-t}}{\alpha+1} - 1,\\
\phi_3(t) &= \frac{e^{2\alpha t} - e^{-t}}{(\alpha+1)(2\alpha+1)} - \frac{e^{\alpha t} - e^{-2t}}{(\alpha+1)(\alpha+2)},\\
\phi_{21}(t) &= \frac{e^{2\alpha t} + 2\alpha e^{-t}}{(\alpha+1)(2\alpha+1)} - \frac{2e^{\alpha t} + \alpha e^{-2t}}{(\alpha+1)(\alpha+2)},\\
\phi_{1^3}(t) &= \frac{e^{2\alpha t} - 4\alpha^2 e^{-t}}{(\alpha+1)(2\alpha+1)} - \frac{4e^{\alpha t} - \alpha^2 e^{-2t}}{(\alpha+1)(\alpha+2)} + 1.
\end{aligned}$$

Generalizing the method of Section 8, we have similar results for the Hall-Littlewood symmetric function $P_k(z)$. Its α-content evaluation writes as

$$\alpha^k P_k(A_\lambda^{(\alpha)}; z) = \sum_\rho c_\rho^{(k)} \binom{n - |\overline{\rho}|}{m_1(\rho)} \theta_{\tilde{\rho}}^\lambda(\alpha),$$

where the coefficients $c_\rho^{(k)}$ are polynomials in (α, β), which are nonzero for $|\rho| - l(\rho) \leq k$. We list them below for $k \leq 4$. The values for h_k and p_k are obtained for $z = 0$ and $z = 1$, in which cases $c_\rho^{(k)}$ has nonnegative integer coefficients in (α, β).

ρ	2
$c_\rho^{(1)}$	1

ρ	3	2^2	2	1^2
$c_\rho^{(2)}$	$2-z$	$1-z$	β	α

ρ	4	32	2^3	2^2	3	21^2	21	2	1^2
$c_\rho^{(3)}$	z^2-5z+5	$(1-z)(2-z)$	$(1-z)^2$	$2\beta(1-z)$	$3\beta(2-z)$	$\alpha(1-z)$	$2\alpha(2-z)$	$\alpha+\beta^2$	$\alpha\beta$

ρ	5	42	3^2	32^2	2^4
$c_\rho^{(4)}$	$(2-z)(z^2-7z+7)$	$(1-z)(z^2-5z+5)$	$(1-z)(2-z)^2$	$(1-z)^2(2-z)$	$(1-z)^3$

4	32	2^3	31^2	31
$\beta(6z^2-29z+29)$	$4\beta(1-z)(2-z)$	$3\beta(1-z)^2$	$\alpha(1-z)(2-z)$	$3\alpha(z^2-5z+5)$

3	2^21^2	2^21	2^2
$(5\alpha+7\beta^2)(2-z)$	$\alpha(1-z)^2$	$4\alpha(1-z)(2-z)$	$4\alpha(z^2-5z+5)+3\beta^2(1-z)$

21^2	21	2	1^4	1^3	1^2
$2\alpha\beta(1-z)$	$6\alpha\beta(2-z)$	$2\alpha\beta+\beta^3$	$3\alpha^2(1-z)$	$4\alpha^2(2-z)$	$\alpha^2+\alpha\beta^2$

However we emphasize that, given a symmetric function f, we are as yet unable to translate its α-content expansion $f(A_\lambda^{(\alpha)})$ in terms of the specialization of f at some generalized Jucys-Murphy elements. Actually, at this moment, we do not know how the symmetric algebra and the Jucys-Murphy elements might be generalized for $\alpha \neq 1$.

The only known exception is for $\alpha = 2$ and $\alpha = 1/2$, where a deep interpretation has been recently found by Matsumoto [24] in terms of odd Jucys-Murphy elements $(J_1, J_3, \ldots, J_{2n-1})$ of S_{2n}.

12 Appendix

Lemma. *Let z be an indeterminate. The quantities*

$$a_\mu^{(k)}(n) = \sum_{\overline{\rho}=\overline{\mu}} c_\rho^{(k)} \binom{n-|\overline{\mu}|}{m_1(\rho)} \tag{12.1}$$

satisfy the recurrence relations

$$a_{\mu\cup(1)}^{(k)}(n+1) = a_\mu^{(k)}(n) + \sum_{r\geq 1} rm_r(\mu)\, a_{\mu\setminus(r)\cup(r+1)}^{(k-1)}(n+1), \tag{12.2}$$

$$\begin{aligned}\sum_{r\geq 1} rm_r(\mu)\, a_{\mu\setminus(r)\cup(r+1)}^{(k)}(n+1) &= -nza_\mu^{(k-1)}(n) \\ &+ \sum_{r,s\geq 1} rsm_r(\mu)(m_s(\mu)-\delta_{rs})\, a_{\mu\setminus(r,s)\cup(r+s+1)}^{(k-1)}(n+1) \\ &+ \sum_{r,s\geq 1} (r+s-1)m_{r+s-1}(\mu)\, a_{\mu\setminus(r+s-1)\cup(r,s)}^{(k-1)}(n+1).\end{aligned} \tag{12.3}$$

if and only if the coefficients $c_\rho^{(k)}$ satisfy the recurrence relations

$$c_{\rho\cup(1)}^{(k)} = \sum_{r\geq 1} rm_r(\rho)\, c_{\rho\setminus(r)\cup(r+1)}^{(k-1)}, \tag{12.4}$$

$$\begin{aligned}\sum_{r\geq 1} rm_r(\rho)\, c_{\rho\setminus(r)\cup(r+1)}^{(k)} &= (2-z)|\rho|\, c_\rho^{(k-1)} + (1-z)m_1(\rho)\, c_{\rho\setminus(1)}^{(k-1)} \\ &+ \sum_{r,s\geq 1} rsm_r(\rho)(m_s(\rho)-\delta_{rs})\, c_{\rho\setminus(r,s)\cup(r+s+1)}^{(k-1)} \\ &+ \sum_{r,s\geq 1} (r+s-1)m_{r+s-1}(\rho)\, c_{\rho\setminus(r+s-1)\cup(r,s)}^{(k-1)}.\end{aligned} \tag{12.5}$$

Proof. When substituting (12.1) into (12.2)–(12.3) we must distinguish the parts 1 of μ since $m_1(\mu) = n - |\overline{\mu}|$ depends on n. Firstly (12.1) yields

$$a_{\mu\cup(1)}^{(k)}(n+1) - a_\mu^{(k)}(n) = \sum_{\overline{\rho}=\overline{\mu}} c_\rho^{(k)} \binom{n-|\overline{\mu}|}{m_1(\rho)-1},$$

so that (12.2) writes as

$$\begin{aligned}\sum_{\overline{\rho}=\overline{\mu}} c_\rho^{(k)} \binom{n-|\overline{\mu}|}{m_1(\rho)-1} &= (n-|\overline{\mu}|) \sum_{\overline{\sigma}=\overline{\mu}\cup(2)} c_\sigma^{(k-1)} \binom{n-|\overline{\mu}|-1}{m_1(\sigma)} \\ &+ \sum_{r\geq 2} rm_r(\mu) \sum_{\overline{\tau}=\overline{\mu}\setminus(r)\cup(r+1)} c_\tau^{(k-1)} \binom{n-|\overline{\mu}|}{m_1(\tau)}.\end{aligned}$$

By identification of the coefficients of $\binom{n-|\overline{\mu}|}{m_1(\rho)}$ on both sides, (12.4) follows.

Secondly we have

$$\sum_{\{r=1\}\cup\{s=1\}} (r+s-1)m_{r+s-1}(\mu)\, a^{(k-1)}_{\mu\setminus(r+s-1)\cup(r,s)}(n+1) = (2n-m_1(\mu))\, a^{(k-1)}_{\mu\cup(1)}(n+1)$$
$$= (n+|\overline{\mu}|)\sum_{\overline{\tau}=\overline{\mu}} c^{(k-1)}_{\tau}\binom{n-|\overline{\mu}|+1}{m_1(\tau)},$$

and (12.3) writes as

$$\begin{aligned}
&(n-|\overline{\mu}|)\sum_{\overline{\sigma}=\overline{\mu}\cup(2)} c^{(k)}_{\sigma}\binom{n-|\overline{\mu}|-1}{m_1(\sigma)} + \sum_{r\geq 2} rm_r(\mu)\sum_{\overline{\tau}=\overline{\mu}\setminus(r)\cup(r+1)} c^{(k)}_{\tau}\binom{n-|\overline{\mu}|}{m_1(\tau)}\\
&= -\, nz\sum_{\overline{\rho}=\overline{\mu}} c^{(k-1)}_{\rho}\binom{n-|\overline{\mu}|}{m_1(\rho)} + (n+|\overline{\mu}|)\sum_{\overline{\rho}=\overline{\mu}} c^{(k-1)}_{\rho}\binom{n-|\overline{\mu}|+1}{m_1(\rho)}\\
&\quad + \sum_{r,s\geq 2}(r+s-1)m_{r+s-1}(\mu)\sum_{\overline{\tau}=\overline{\mu}\setminus(r+s-1)\cup(r,s)} c^{(k-1)}_{\tau}\binom{n-|\overline{\mu}|}{m_1(\tau)}\\
&\quad + \sum_{r,s\geq 2} rsm_r(\mu)(m_s(\mu)-\delta_{rs})\sum_{\overline{\sigma}=\overline{\mu}\setminus(r,s)\cup(r+s+1)} c^{(k-1)}_{\sigma}\binom{n-|\overline{\mu}|}{m_1(\sigma)}\\
&\quad + 2(n-|\overline{\mu}|)\sum_{r\geq 2} rm_r(\mu)\sum_{\overline{\sigma}=\overline{\mu}\setminus(r)\cup(r+2)} c^{(k-1)}_{\sigma}\binom{n-|\overline{\mu}|-1}{m_1(\sigma)}\\
&\quad + (n-|\overline{\mu}|)(n-|\overline{\mu}|-1)\sum_{\overline{\sigma}=\overline{\mu}\cup(3)} c^{(k-1)}_{\sigma}\binom{n-|\overline{\mu}|-2}{m_1(\sigma)}.
\end{aligned}$$

But we have the identity

$$\begin{aligned}
(n+a)\binom{n-a+1}{b} - nz\binom{n-a}{b} &= (1-z)(b+1)\binom{n-a}{b+1} + (2-z)(a+b)\binom{n-a}{b}\\
&\quad + (2a+b-1)\binom{n-a}{b-1},
\end{aligned}$$

which we apply with $a=|\overline{\mu}|=|\overline{\rho}|$ and $b=m_1(\rho)$ so that $a+b=|\rho|$ and $2a+b=2|\rho|-m_1(\rho)$. Now (12.5) follows by identifying the coefficients of $\binom{n-|\overline{\mu}|}{m_1(\rho)}$ on both sides and using

$$\sum_{\{r=1\}\cup\{s=1\}} (r+s-1)m_{r+s-1}(\rho)\, c^{(k-1)}_{\rho\setminus(r+s-1)\cup(r,s)} = (2|\rho|-m_1(\rho))c^{(k-1)}_{\rho\cup(1)}.$$

□

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