

Negative Refractive Index in Hydrodynamical Systems

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Abstract

We discuss the presence of exotic electromagnetic phenomena in systems with finite charge density which are described by hydrodynamics. We show that such systems generically have negative refractive index for low frequencies electromagnetic waves, i.e. the energy flux and the phase velocity of the wave propagate in opposite directions. We comment on possible phenomenological applications, focusing on the Quark Gluon Plasma.

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Introduction

Electromagnetic waves in continuous media have been a large field of investigation in the last years. Indeed light waves have an amazing behavior in a new class of artificial materials (metamaterials), created in laboratories about a decade ago [1, 2]. One of the most attractive properties is the negative refractive index: namely the energy flux and the phase velocity of a wave packet propagate in opposite directions for a certain range of frequencies. This behavior led both to promising physical applications and to interesting theoretical developments.

In this paper we will discuss the existence of negative refraction in hydrodynamics. More precisely we show that *negative refraction seems to be a generic phenomenon in homogeneous and isotropic systems that have a finite non zero charge density and that admit a description in term of hydrodynamical equations*, at least for certain range of their parameters. This conclusion is valid both for normal and super conducting fluids, and for relativistic or non relativistic systems. The only necessary ingredients to reach this goal are linear hydrodynamics, classical electrodynamics and linear response theory.

Finally we speculate on the connection of our result with phenomenological and experimental applications. Specifically we concentrate on the possible presence of negative refraction in Quark Gluon Plasma, a strongly coupled plasma at finite charge density that should be created in high energy physics colliding experiments.

Note that in [3] it was already argued that negative refraction is an ubiquitous phenomenon in hydrodynamical charged systems, supporting the claim by the inspection of a strongly coupled theory where the transport coefficients can be explicitly computed through the gauge/gravity correspondence. In [4] the same topic was investigated for an holographic superconductor in the probe limit, in [5] for a charged black hole in four dimensions, and in [6, 7] for systems with D7 flavor branes. In particular this last development seems promising for more realistic application to the Quark Gluon Plasma physics.

The structure of this paper is the following: in Section (1) we explain the physics of negative refraction by reviewing the basic aspects of linear response theory and of electromagnetism in spatial dispersive media. In Section (2) we study the two points correlation function of the transverse current for a system described by hydrodynamical equations. In Section (3) we make our main claim: negative refraction is generic in charged liquids. In Section (4) we look for possible phenomenological examples and in particular we speculate about the QGP.

1 Negative Refraction and Linear Response Theory

The propagation of transverse polarized electromagnetic waves in a medium is usually described in terms of the refractive index n defined as $n^2 = \epsilon\mu$, where ϵ is the electric permittivity, while μ is the magnetic permeability. In real media they are both complex functions of the frequency w . A medium is told to have negative refractive index [8, 9] if for certain range of frequencies the phase velocity of a wave packet propagates in the opposite direction to the energy flux.

The phase velocity of a wave is defined as $\vec{v}_{ph} = 1/\text{Re}(n)\hat{k}$, where \hat{k} is the unit vector in the direction of the wave vector \vec{k} , while the direction of the energy flux, for not too dissipative and dispersive media, is given by the Poynting vector, that for long wavelength is $\vec{S} = \text{Re}(n/\mu)\hat{k}|E_T|^2$ in the transverse channel, where E_T is the component of the electric field transverse to \vec{k} . In this setup negative refraction is equivalent to ask that the phase velocity and the Poynting vector are in opposite direction, namely $\text{Re}(n) < 0$ and $\text{Re}(n/\mu) > 0$. It is possible to show [10] that in passive media the energy flow and the phase velocity are opposite if and only if

$$n_{DL}(w) = |\epsilon(w)|\text{Re}(\mu(w)) + |\mu(w)|\text{Re}(\epsilon(w)) \quad (1)$$

is negative. Hence negative refraction is equivalent to $n_{DL}(w) < 0$ for certain range of frequencies w . Our task is to find a general class of media for which this condition is satisfied. This aim can be reached only after the computation of the response functions ϵ and μ .

These two quantities can be obtained by studying the retarded two point correlation function of the electromagnetic current in the medium. Here we concentrate on the propagation of transverse waves in homogeneous and isotropic media that admit an hydrodynamical description for a certain range of the physical parameters. Hydrodynamical media are able to transmit external perturbation quite easily, implying that their response functions generically depends from the wave vector k and not only from the frequency w . This property is called spatial dispersion.

The macroscopic electromagnetism in media with spatial dispersion is defined in term of the three fields D , E , B , and the relation $D_i = \epsilon_{ij}(w, k)E_j$. These fields satisfy the macroscopic Maxwell equations and the tensor function $\epsilon_{ij}(w, k)$ describes the linear response of the medium [11]. For an isotropic medium: $\epsilon_{ij}(w, k) = \epsilon_T(w, k)P_{ij}^T + \epsilon_L(w, k)P_{ij}^L$

where P_{ij}^T and P_{ij}^L are the transverse and longitudinal projectors⁴ with respect to the wave vector k , while ϵ_T and ϵ_L are two scalar functions of k and w .

The propagation of the transverse part of the electromagnetic field requires that the dispersion relation $\epsilon_T(w, k) = k^2/w^2$ is satisfied. If we probe the medium with an electromagnetic field with large enough wave-length we can expand ϵ_T at the second order in k : $\epsilon_T(w, k) = \epsilon(w) + \frac{k^2}{w^2} \left(1 - \frac{1}{\mu(w)}\right)$, where $\epsilon(w)$ the electric permittivity, and $\mu(w)$ is the effective magnetic permeability⁵.

The electric permittivity and the magnetic permeability can be obtained from linear response theory [12]. In the linear response theory the electromagnetic current J_i is proportional to the vector potential A_j , by $J_i = G_{ij}A_j$, where G_{ij} is the retarded correlator of the currents in the medium.

We can decompose G_{ij} in its transverse and longitudinal part, using the projectors P_{ij}^T and P_{ij}^L , and obtain the relation⁶: $\epsilon_T(w, k) = 1 + \frac{4\pi}{w^2} G_T(w, k)$. Expanding G_T to second order in k : $G_T(w, k) = G_T^{(0)}(w) + k^2 G_T^{(2)}(w)$, we find the electric permittivity and effective magnetic permeability

$$\begin{aligned}\epsilon(w) &= 1 + \frac{4\pi}{w^2} G_T^{(0)}(w) \\ \mu(w) &= \frac{1}{1 - 4\pi G_T^{(2)}(w)}\end{aligned}\tag{2}$$

In summary, for long enough wavelength, the propagation of the transverse part of the electric field in the medium can be described in terms of the first orders expansion in k of the retarded correlator of transverse currents.

2 Considerations from Hydrodynamics

In this section we want to discuss some generic properties of the propagation of electromagnetic waves in a charged hydrodynamical system. By hydrodynamical system we mean every system that, for a certain range of its parameters, admits a description in term of hydrodynamical equations for large enough distance and long enough time. Hydrodynamics is indeed the effective theory describing the real-time dynamics of a system at scales larger than the mean free path and time scales larger than the mean free time of the microscopic elements composing the system. At these scales only a finite number of degrees of freedom survives: densities of conserved charges and phases of order parameters.

⁴ $P_{ij}^T = \delta_{ij} - \frac{k_i k_j}{|k|^2}$ and $P_{ij}^L = \frac{k_i k_j}{|k|^2}$.

⁵It is important to observe that the effective magnetic permeability we use here is different from the usual magnetic permeability [11], but it is the right quantity to describe the propagation of transverse waves in spatially dispersive media [8].

⁶Here we follow the opposite convention of [3] for the sign of the Green function to be consistent with the conventions common in the literature (es. [13]). This changes the sign of the q^2 contribution in ϵ_T .

Our main interest is to determine the generic form of the retarded two points correlation function of the transverse part of the electromagnetic current. This result will be obtained by combining hydrodynamical equations and linear dynamics response theory.

For simplicity we focus on the case of a non-relativistic fluids in the normal phase, but, as we will comment at the end of the section, the results can be generalized to other interesting cases. A generic non-relativistic fluid can be described by the Navier-Stokes equations. We just consider the linearized version of the Navier-Stokes equations because we are interested in linear response. They essentially come from the conservation of the particle density, energy density and the momentum density in addition to some constitutive relations and thermodynamic relations. The conservation for the density n takes the form: $\partial_t n + \nabla_i p_i/m = 0$, where p_i is the momentum density and m the characteristic mass of the particles. If the system has another conserved charge associated to some U(1) symmetry, that we will reinterpret later as the electromagnetic charge, and if every particle in the system has charge e under this symmetry, than the conservation equation for the charge density of this symmetry $\rho = en$ is just proportional to the particle density conservation: $\partial_t \rho + \nabla_i j_i = 0$, with j_i the current associated to the conserved charge density. This observation suggests that the properties of the transverse part of the current density j_i are related to the dynamics of the transverse part of the momentum density p_i .

In the Navier-Stokes system also the momentum density is conserved: $\partial_t p_i + \nabla_j \tau_{ij} = 0$, where τ_{ij} is the stress tensor. The momentum density can be divided into its longitudinal $\nabla \cdot p^L = 0$ and transverse part $\nabla \cdot p^T = 0$. We are interested to the transverse part p^T only. The equation describing the dynamics of the transverse momentum decouples from all the other equations [13, 14]:

$$(\partial_t - \frac{\eta}{nm} \nabla^2) p^T = 0 \quad (3)$$

where η is the shear viscosity of the system. The decoupling of the transverse momentum largely simplifies our analysis and it is valid also for relativistic, non-relativistic, superconducting or normal fluids. Indeed temperature and chemical potential inhomogeneities, and the derivative of the phase of the order parameter for the superfluid phase, only couple to the longitudinal part of the momentum density⁷. Equation (3) is solved by Fourier transforming in the space variable x and by Laplace transforming in the time variable t . After we project on the real frequencies we have

$$p^T(w, k) = \frac{p^T(t = 0, k)}{-iw + \frac{\eta}{nm} k^2} \quad (4)$$

where $p^T(t = 0, k)$ is the Fourier transform of the momentum density at time $t = 0$. This solution describes the evolution of the transverse part of the momentum density:

⁷For a charged fluid the electromagnetic field couples to the particle system and, at the linear level, for the case we are interested in, one has to solve the coupled equations: $(\partial_t - \frac{\eta}{nm} \nabla^2) p^T = en \partial_t A^T$, $(\partial_t^2 - \nabla^2) A^T = J^T$, with the constraint $J^T = (ep^T - e^2 n A^T)/m$. However, in the limit of small coupling e , it is possible to show that this effect gives a subleading contribution to the retarded correlator of the transverse currents. It is important to observe that the order of limits matters; in all the paper we consider small e before then small k and then for last small w .

the system is displaced from its equilibrium configuration by an external force and then it is left to evolve freely in time. The same information is contained in the long wave, low frequencies behavior of the two point retarded correlation function of the transverse momentum $G_{p^T}(w, k)$. The correlation functions can be typically computed using the linear response theory. Linear hydrodynamics and linear response theory at long time and large scale are two different descriptions for the same physics, and they must coincide [13, 14].

Indeed the evolution of a degree of freedom $\mathcal{O}(t, x)$ of the system due to the perturbation of the equilibrium state with an external field $\phi(t, x)$, can be described solving the initial value problem associated to the hydrodynamical equation of the system: $\mathcal{O}_H(w, k)$, or computing the two points retarded correlation function $G_{\mathcal{O}}(w, k)$ of the observable \mathcal{O} according to: $\mathcal{O}_{LR}(w, k) = G_{\mathcal{O}}(w, k)\phi(w, k)$. A very general result from linear response theory says that for adiabatically applied perturbation we have:

$$\mathcal{O}_{LR}(w, k) = \frac{G_{\mathcal{O}}(w, k) - G_{\mathcal{O}}(k)}{iw G_{\mathcal{O}}(k)} \mathcal{O}(t=0, k) \quad (5)$$

where $G_{\mathcal{O}}(k)$ is the $w \rightarrow 0$ limit of $G_{\mathcal{O}}(w, k)$. This formula eliminates the explicit dependence on the external field and it describes the relaxation process as an initial value problem. For small enough frequencies and wave vectors: $\mathcal{O}_{LR}(w, k) = \mathcal{O}_H(w, k)$ or in other words we can use hydrodynamics to infer the long time and large distance behavior of the retarded correlator, in the regime in which it is usually difficult to compute it directly, due to the collective interactions of the system.

From this fundamental result we can now study the case of the transverse momentum. Equating the expression (5) for $\mathcal{O} = p^T$ with the hydrodynamical result (4), we obtain:

$$G_{p^T}(w, k) = \frac{iwmn}{-iw + \frac{n}{mn}k^2} + mn \quad (6)$$

where we took the $k \rightarrow 0$ limit and we used equilibrium statistical mechanics to show that $G_{p^T}(0, 0) = mn$ [13].

Starting from (6), we compute the retarded correlator of the transverse current: $G_{J^T}(w, k)$. The induced transverse current J^T is the response of the system to an external electromagnetic field A_{ext}^μ . The standard interacting hamiltonian density is: $\delta\mathcal{H} = -J_\mu A_{ext}^\mu$.

It is important to stress that in presence of an electromagnetic field the transverse velocity density field of the system is no more p^T/m but $(p^T - enA^T)/m$, due to the minimal coupling of the system to the vector potential. The transverse current is proportional to the transverse field velocity v^T of the system [13–16] and in particular:

$$J^T = env^T = \frac{e}{m}p^T - \frac{e^2n}{m}A^T \quad (7)$$

The Green function $G_{J^T}(w, k)$ is obtained by assuming that all the electromagnetic fields in the system are external; explicitly we put $A = A_{ext}$ everywhere [13]. This hypothesis

may appear too strong and it amounts to treat even the internally generated electromagnetic field as an external perturbation; in this case the equation (3) is exact. We believe that this assumption can be justified in various systems and regimes. We take it as a working hypothesis, and we will justify it in the following, when we will discuss possible phenomenological applications. With this approximation the problem simplifies and the interacting hamiltonian density becomes $\delta\mathcal{H} = -j^T A_{ext}^T$, where $j^T = p^T e/m$ is the translational current density, and:

$$j^T(w, k) = G_{j^T}(w, k)A_{ext}^T = \frac{e^2}{m^2}G_{p^T}(w, k)A_{ext}^T \quad (8)$$

Using equations (7), (6) and the fact that $J^T = G_{J^T}A_{ext}^T$ we obtain:

$$G_{J^T}(w, k) = \frac{iw\frac{e^2n}{m}}{-iw + \frac{\eta}{nm}k^2} \quad (9)$$

This result is valid for a normal, non-relativistic fluid. Analogously a similar result also for relativistic and/or superfluid systems can be deduced. Indeed from linearized hydrodynamics and linear response theory the generic form of the retarded correlator for the transverse part of the electromagnetic current is:

$$G_{J^T}(w, k) = \frac{iw\mathcal{B}}{-iw + \mathcal{D}k^2} - \mathcal{C} \quad (10)$$

with \mathcal{B} , \mathcal{C} and \mathcal{D} three real positive constants depending on the particular system. The \mathcal{C} constant is a contribution coming from the superfluid phase and it is zero in the normal phase. An important example that we analyze in the following section is a relativistic charged fluid for which $\mathcal{B} = \rho^2/(\epsilon + P)$ and $\mathcal{D} = \eta/(\epsilon + P)$, where ρ is the charge density, ϵ the energy density and P the pressure. For completeness we give also the explicit expressions for the superconducting phase: in the non-relativistic case $\mathcal{B} = e^2n_n/m$, $\mathcal{D} = \eta/mn_n$ and $\mathcal{C} = e^2n_s/m$, while in the relativistic case $\mathcal{B} = \rho_n^2/(\epsilon + P - \rho_s\mu)$, $\mathcal{D} = \eta/(\epsilon + P - \rho_s\mu)$ and $\mathcal{C} = \rho_s\rho_n/(\epsilon + P - \rho_s\mu)$, where n_n is the particle density of the normal component of the fluid, while n_s is the particle density of the superfluid component, ρ_n , ρ_s is the normal and superconducting charge density and μ is the electromagnetic chemical potential.

3 Hydrodynamics and Negative Refraction

From the generic form of (10) we can obtain $G^{(0)}(w)$ and $G^{(2)}(w)$ and hence the generic form of $\epsilon(w)$ and $\mu(w)$:

$$\epsilon(w) = 1 - \frac{4\pi}{w^2}(\mathcal{B} + \mathcal{C}), \quad \mu(w) = \frac{1}{1 - 4\pi\frac{i\mathcal{B}\mathcal{D}}{w}} \quad (11)$$

From the equations (11) we observe that for small enough frequencies $n_{DL}(w)$ is negative. In particular $n_{DL}(w) < 0$ for $w < 2\sqrt{\pi(\mathcal{B} + \mathcal{C})}$. Hence we can conclude that *an isotropic*

and homogeneous system, in local thermal and mechanical equilibrium, invariant under relativistic or non-relativistic space time transformations, at finite charge density and in the normal or superconducting phase, described for long enough time and large enough distance by hydrodynamical equations, has negative refractive index for small enough frequencies.

As we explained we computed the response functions after expanding $G_T(w, k)$ for small k . This expansion is valid if $|k^2| \ll w/D$, otherwise the effects of spatial dispersion are very strong and the whole k dependence has to be taken into account. This gives origin to multiple solutions for the dispersion relation $n^2(w) = \epsilon(w)\mu(w)$, the so-called additional light waves (ALWs) [17, 18]. In the rest of this note we work in the regime $|k^2| \ll w/D$ neglecting the existence of ALWs. In this regime the large wavelength constraint becomes $|n^2(w)| \ll \frac{1}{wD}$, and it imposes a constraint on the frequency range that we can explore. Moreover, because we are always working in the hydrodynamical limit w and k must be small compared to the temperature and the chemical potential of the system. This requirement introduce an additional constraint on the allowed frequency range.

Another interesting quantity to compute is the ratio $\left| \frac{\text{Re}[n(w)]}{\text{Im}[n(w)]} \right|$. If it is larger than one means that the propagation of the electromagnetic waves dominates over the dissipation. In our example unfortunately this ratio is smaller than one in the frequency range in which we have negative refraction, meaning that the dissipation is an important phenomenon. However this seems to be the normal behavior for isotropic metamaterials that experience negative refraction. In the actual experiments and simulations this problem is typically solved by introducing anisotropies or by considering superconducting materials. We discuss some of these developments in [19], while here we concentrate on the easiest example of a non superconducting and isotropic hydrodynamical system. It is important to observe that, even if the final result shows relevant dissipation effects, the procedure that we have just described is reliable in the regime of small spatial dispersion.

To conclude it is important to observe that in the hydrodynamical derivation we assumed that the external field A_{ext} is the dominating one. This analysis ignored the contribution of the induced internal field. By taking into account its contribution some constraints are imposed on the validity of the q^2 expansion. This requirement enforces another lower bound on the frequency. We checked that there are regimes of parameters in which this bound doesn't affects the whole region of negative refraction. Anyway we leave deeper analysis for further studies.

4 Phenomenology

The arguments we gave in the previous sections bring us to the conclusion that negative refraction is a quite common phenomenon in charged hydrodynamical systems. It becomes important to understand the range of applicability of our conclusion to physically observable systems.

The main properties of the system we are describing is its finite charged density and

the fact that it is in local thermo-mechanical equilibrium. These two facts together lead to important constraints. Indeed the presence of non zero charge density generates forces⁸ that will eventually repel its constituents, showing that the system is actually out of equilibrium. This conclusion is false for infinite or three dimensional periodic systems, that are actually what our formalism is describing. On the contrary side, infinite or three dimensional periodic systems do not exist. For this reason we need to admit that our model describe only the bulk properties of a finite sample, of which we disregard the boundary effects. However the existence of finite systems at finite charge density and in local equilibrium is not obvious.

A possible way out is to consider the systems we are describing as part of a globally neutral system. An intriguing example is to consider the electrons of a metal as a charged fluid. In the approximation in which the lattice ions in the metal provide a static homogeneous charge distribution, the entire system is globally neutral and in mechanical equilibrium, and the system should have negative refraction for certain range of parameters⁹.

In this paper we will not pursue this intriguing line of research related to solid state physics, but we will instead focus on another interesting example, related to high energy physics. Indeed another way out is to think that we are describing the bulk dynamics of the whole finite system, in which there exist other forces that compensate for the electromagnetic repulsion. A natural candidate for this line of thinking is the QGP (Quark Gluon Plasma): a strongly coupled plasma that is believed to form in the high energy collisions of ions inside the particle accelerators. This plasma is globally charged due to charge conservation. Moreover it is believed to be in local thermo-mechanical equilibrium, at least for a small fraction of time, and to be well approximated by hydrodynamics. Finally the QGP dynamics is dominated by the strong interaction among its constituents, so it seems justified to neglect the internal electromagnetic field.

These two examples are clearly in contrast with two common lores: there are no natural materials with negative refraction, the QGP is insensitive to the electromagnetic interaction; however the simplicity and generality of the effects we described in the previous sections suggest that it is maybe worthwhile to investigate more along this direction.

4.1 A calculable example

In this section we perform a more quantitative analysis of negative refraction in hydrodynamical systems and we speculate about its application to the QGP. In order to make a concrete prediction, we need to specify the hydrodynamical coefficients entering the green function and then the electromagnetic parameters (2).

Recent advances in high energy physics offer explicit models that belong to the universality class we have just discussed. In these physical systems, exact values of the thermodynamical parameters and transport coefficients are already at our disposal. These

⁸It is however important to observe that only the longitudinal part of the electric field E^L is influenced by the presence of a finite charge density ρ .

⁹We thanks Jan Zaanen for discussions related to this point.

models are believed to describe some aspects of the strongly coupled plasma (quark gluon plasma) that is supposed to form in particle colliders and is outside the regime of perturbative QCD¹⁰. They are based on the gauge-gravity correspondence (AdS/CFT), i.e. a relation between a strongly coupled d dimensional field theory and a gravity theory in $d + 1$ dimensions. Here we skip the details of the correspondence, referring the interested reader to the huge literature [20], and we straightforwardly move to the results¹¹.

In [3], using the results of [21], we computed the electromagnetic response function of a four dimensional system dual to an AdS_5 charged black hole, and we found negative refraction at low frequencies. The green function for that setup is

$$G_{JT}(w, k) = \frac{iw\mathcal{B}}{-iw + \mathcal{D}k^2} + iw\mathcal{R} + \mathcal{N}(w^2 - k^2) \quad (12)$$

where the first term is the leading one we derived with pure hydrodynamics (10), whereas the extra two terms are specific of the AdS/CFT computation. They are subleading with respect to the hydrodynamical ones, but they give $O(1/w)$ and constant contribution to the electromagnetic response function ϵ_T , changing in this way the quantitative results. The permittivity and the permeability then read

$$\epsilon(w) = 1 + \frac{4\pi}{w^2} (-\mathcal{B} + iw\mathcal{R} + w^2\mathcal{N}) \quad \mu(w) = \left(1 - 4\pi\left(\frac{i\mathcal{B}\mathcal{D}}{w} - \mathcal{N}\right)\right)^{-1} \quad (13)$$

The various coefficients are

$$\mathcal{B} = \frac{\rho^2}{\epsilon + P} \quad \mathcal{D} = \frac{\eta}{\epsilon + P} \quad \mathcal{R} = 8\pi^2 T^2 \frac{\eta}{(\epsilon + P)^2} \left(\frac{\rho}{\mu}\right)^2 \quad \mathcal{N} = \text{constant} \quad (14)$$

The charge density, energy density and pressure, and shear viscosity are determined to be

$$\begin{aligned} \rho = e &= \frac{\sqrt{3}N_c^2 \left(\pi T + \frac{\sqrt{3\pi^2 T^2 + \mu^2}}{\sqrt{3}}\right)^3 \sqrt{\mu^2 + 2\pi T \left(3\pi T - \sqrt{3}\sqrt{3\pi^2 T^2 + \mu^2}\right)}}{32\pi^2 \mu} \\ \epsilon + P &= \frac{N_c^2 \left(\pi T + \frac{\sqrt{3\pi^2 T^2 + \mu^2}}{\sqrt{3}}\right)^4 \left(3\mu^2 + 4\pi T \left(3\pi T - \sqrt{3}\sqrt{3\pi^2 T^2 + \mu^2}\right)\right)}{32\pi^2 \mu^2} \\ \eta &= \frac{N_c^2 \left(\pi T + \frac{\sqrt{3\pi^2 T^2 + \mu^2}}{\sqrt{3}}\right)^3}{64\pi^2} \end{aligned} \quad (15)$$

Note that the only free parameters in the permittivity and in the permeability are the temperature T , the electromagnetic chemical potential μ and the dimensionless parameters N_c . The electric charge e is fixed to its classical value in natural units $e = 0.3028$.

¹⁰ Despite some of their successes it is important to stress that these models describe a different theory than QCD. For this reason we can just use them to provide an approximate estimate of physical quantities.

¹¹Note that in the context of the AdS/CFT correspondence, the electromagnetic field is by construction an external perturbation.

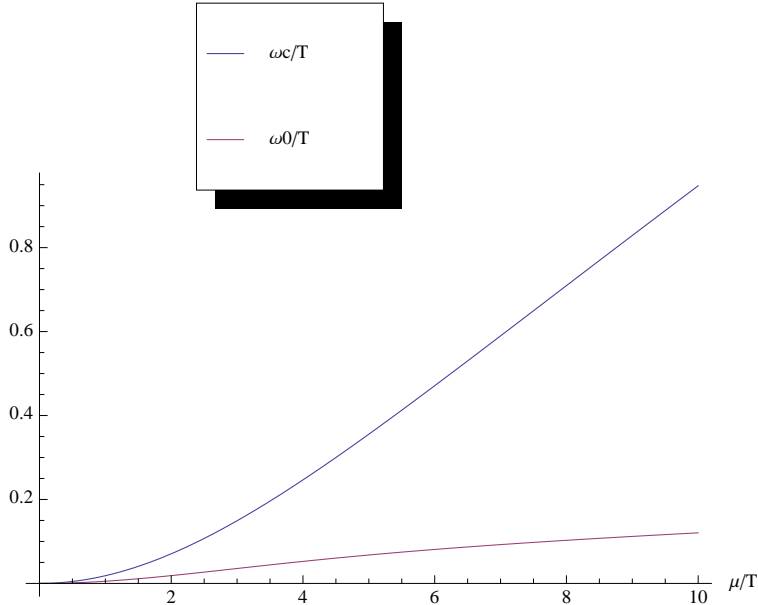


Figure 1: Critical frequency w_c/T and the minimal allowed frequency w_0/T as a function of the chemical potential μ/T .

The parameter \mathcal{N} comes from the UV frequencies through the renormalization procedure, and it is fixed by requiring that the permittivity approaches unity¹² for $w \rightarrow \infty$. The parameter N_c is the rank of the $SU(N_c)$ gauge group of the underlying strongly coupled four dimensional field theory. The expressions (13) are valid in the hydrodynamical regime, where both the frequency and the wave vector are small compared to the other scales of the theory, namely the temperature and/or the chemical potential.

We can proceed by plotting some numerical result for the critical frequency below which we predict negative refraction. Since we are interested in quark gluon plasma physics, we can conventionally set $N_c = 3$ as the number of colors of QCD, from now on. In Figure 1 (blue line) we plot the ratio of the critical frequency w_c over the temperature as a function of the ratio between the chemical potential and the temperature. We see that the region of negative refraction grows with μ/T . However our analysis does not cover the entire region of frequencies below w_c due to the constraints explained in the previous section. The allowed range is restricted to $w_0/T < w/T < w_c/T$. In the Figure 1 we plot also the minimal allowed frequency (red line) w_0/T . The region where our approximations are valid and which exhibits negative refraction is the one in between the two lines.

Our results are supposed to describe some of the electromagnetic bulk properties of the system. However, for this to be reliable, the characteristic wavelength of negative refraction must be smaller than the typical length scale of the experiment.

¹²Namely for very high frequencies the electromagnetic waves is supposed to probe the vacuum.

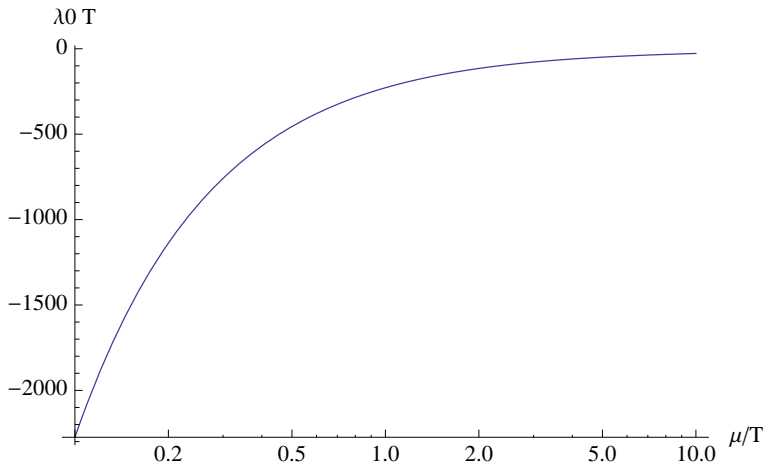


Figure 2: Minimal wavelength λ_0/T with negative refraction as a function of the chemical potential over the temperature.

For this reason we have to analyze the wave length $\lambda(w) = \frac{2\pi}{\text{Re}[n(w)]w}$. Note that the real value of the refractive index $\text{Re}[n(w)] = \text{Re}[\sqrt{\epsilon\mu}]$ vanishes at w_c and increases for smaller frequencies [3]. Thus the wavelength $\lambda(w)$ diverges at w_c and monotonically decreases for smaller frequencies. The minimal length that can be probed by the negative refraction is then set by $\lambda(w_0) \equiv \lambda_0$. In Figure 2 we plot $\lambda_0 T$ as a function of μ/T . It is negative to stress that there is negative refraction.

For a generic system, given the temperature and the electrical chemical potential, one can extract from the Figures 1 and 2 the frequency interval and the smaller wavelength where we expect negative refraction.

Observe that even if the analysis has been made in the hydrodynamical approximation, the actual physical values of the frequency can be large, if the temperature and/or the chemical potential are large. In the following, for instance, we find $w_c \sim 10^{21}$ Hertz for the QGP.

4.2 QGP and Negative Refraction

Here we give some numerical results about negative refraction in the QGP. In the description of the physics of ion collisions (like nuclei of *Au* or *Pb*), the independent parameters are the temperature and the baryonic chemical potential μ_B . The phase diagram of QCD as a function of these variables is not completely explored. The common lore is that QCD is a plasma for temperature around $T_c \simeq 150 - 170 \text{ MeV}$ and energy density $\epsilon_c \simeq O(\text{GeV}/\text{fm}^3)$. Useful references for the QGP physics and the experiments are [22–24].

We are interested in identifying the typical frequencies and wavelength where we expect negative refraction in the QGP. For this purpose we need the values of T and of the electromagnetic chemical potential μ . We consider the simplifying assumption that the

sample of colliding ions has the same number of protons and neutrons, i.e. we neglect the total isospin charge. In this approximation the electromagnetic chemical potential results 1/2 the baryonic one.

The estimated temperature and baryonic chemical potential vary among the different experiments (e.g. AGS, SPS, RHIC) and depending on the data that are taken into account. For simplicity we can consider three benchmark points for the values of T and μ_B , which correspond approximately to the values at the freeze out¹³ for the three experiments AGS, SPS and RHIC. Modifying the ratio and/or the critical temperature of few percents does not change significantly our conclusions.

Using (15),(14) and (13) we compute the thermodynamical quantities for the three benchmark points, and the frequency range and the minimal wavelength where we expect negative refraction:

	\sim AGS	\sim SPS	\sim RHIC
T	125 MeV	148 MeV	177 MeV
$\mu = \frac{\mu_B}{2}$	270 MeV	200 MeV	14.5 MeV
ρ/e	$0.67/fm^3$	$0.66/fm^3$	$0.066/fm^3$
ϵ	$1.32GeV/fm^3$	$2.29GeV/fm^3$	$4.28GeV/fm^3$
$[w_0, w_c]$	$[4.02, 15.5] \times 10^{21}Hz$	$[2.07, 7.46] \times 10^{21}Hz$	$[9.79, 33.6] \times 10^{18}Hz$
λ_0	168 fm	225 fm	3100 fm

Unfortunately the typical wavelength is larger than the typical dimension of the sample of quark gluon plasma, which is of the order of few fm, so negative refraction can probably not be tested in the actual collider experiments. The probed wavelength becomes smaller with increasing μ/T , but in order to reach the fm, the chemical potential should be very large, $\mu/T \sim 100$, so probably in region beyond the plasma phase of QCD.

In this section we didn't want to give precise numerical results, but just point out that high energy physics can offer examples of finite charged systems where electromagnetic exotic phenomena could appear. It would be appealing to use models which are better approximation of QCD to give more quantitative predictions on the electromagnetic properties of QGP.

5 Conclusions

In this paper we have shown that systems at finite charge density that can be described by hydrodynamics have negative refractive index for low frequencies electromagnetic waves. We supported this claim computing the general properties of the linear response of the system to an external electromagnetic field. We hope that our discussion could stimulate interest on this fascinating topic both at theoretical and phenomenological level. Indeed the negative refraction in the low frequency regime is generic in both relativistic and

¹³The freeze out curve in the T, μ_B plane is determined by the fact that the inelastic collisions cease, and in some cases it corresponds to the critical line of the QCD phase transition between hadrons and the QGP.

non relativistic systems, both in a normal phase or in a superfluid one. We restrict our phenomenological interest to the QGP but in principle the same observation can be valid in many different cases.

There are many directions to pursue the research on this subject.

At theoretical level it would be nice to generalize this effect to more realistic setups, relaxing some of the constraints, to better understand the robustness of our result. Indeed, while the requirement of a finite charge density seems intrinsic in our claim, the other hypothesis of homogeneity and isotropy may, in principle, be relaxed. Moreover, introducing boundaries in our analysis would also be very interesting for concrete applications.

At the level of AdS/CFT it would be very nice to use better approximations of the QGP to see if this electromagnetic effect could have relevant high energy physics phenomenology. Moreover some metamaterial, the so called stereometamaterials [25] seems to be intrinsically strongly coupled and studying the non hydrodynamical modes in AdS/CFT could give new results related to the propagation of electromagnetic waves in strongly coupled plasmas.

Finally, at the phenomenological level, it would be very nice to understand under which approximations some common solid states systems can be described as charged fluid and show in this way negative refraction. Conducting electrons in metal could reveal nice surprises.

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