POSITIVITY AND PERIODICITY OF Q-SYSTEMS IN THE WZW FUSION RING

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Abstract. We study properties of solutions of Q-systems in the WZW fusion ring obtained by the Kirillov-Reshetikhin modules. We make a conjecture about their positivity and periodicity and give a proof of it in some cases. We also construct a positive solution of the level k restricted Q-system of classical types in the WZW fusion ring. As an application, we prove some conjectures on the level k restricted Q-systems including the Kuniba-Nakanishi-Suzuki (KNS) conjecture.

1. introduction

Many parts of modern mathematics are influenced by mathematical physics. Conformal field theory provided a huge area of applications for the representation theory of the affine Kac-Moody algebras. The study of integrable systems and solvable lattice models gave rise to the development of the theory of quantum groups.

As shown in [Fin96], it is now a well-known fact that there is a close relationship between the representation theory of the affine Kac-Moody algebra $\hat{\mathfrak{g}}$ at the level k and and the representation theory of the quantum group $U_q(\mathfrak{g})$ for the root of unity $q = \exp(\frac{\pi\sqrt{-1}}{t(k+h^\vee)})$ where t is the ratio of a long and a short root of $\mathfrak{g}$ squared. This relationship has been formalized as modular tensor categories ([BK01]). It is interesting to see the appearance of the number $k + h^\vee$ in both subjects where k and $h^\vee$ denote the level and the dual Coxeter number, respectively.

The (level k restricted) Y-systems and T-systems also originated from mathematical physics closely related to conformal field theory and integrable systems ([Zam91, KNS94]). They have provided an active area of mathematical research, especially in the theory of cluster algebras. As a kind of sequences defined by certain recurrence relations, they both possess various remarkable properties like positivity and periodicity ([Kel10, IIK+10]). The number $k + h^\vee$ again reveals its presence as a period of these objects.

There exists another closely related object called the (level k restricted) Q-systems originated from the representation theory of the Yangian([KR90]). As close relatives of Y-systems and T-systems, it is desirable to have similar notions of positivity and periodicity in their study. This is the main goal of this paper. In this paper, we will explore how these interesting phenomenon can show up when we bring the Kirillov-Reshetikhin (KR) modules in quantum group theory into the WZW fusion ring. We can see a very concrete and beautiful interaction between

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the representation theory of the affine Kac-Moody algebras and that of quantum groups. The number $k + h^\vee$ again plays an important role here.

Nahm's conjecture[Nah07] attempts to give a partial answer to the question of when a certain form of q-hypergeometric series can be a modular function. In this conjecture, one is faced with a system of algebraic equations associated to the q-hypergeometric series, which is seemingly quite distant from the world of 'modularity'. When this q-hypergeometric series has a particularly nice form based on the Cartan matrix of $\mathfrak{g}$, the algebraic equations can be recast into the level k restricted Q-system associated to $\mathfrak{g}$. Our approach brings the business of solving algebraic equations in Nahm's conjecture into the world of 'modularity' at least in this special case.

It will be topics of further research to interpret our results in terms of the theory of representations of quantum groups at roots of unity in a more straightforward way and to understand how they fit into the theory of cluster algebras. An application to solvable vertex and RSOS models can be found in [KNS11, Section 3.7].

The outline of this paper is as follows. In Section 2, we give a brief review on the notions of the modular S-matrix, the WZW fusion ring, Q-systems and KR modules. In Section 3, we propose the main conjecture about the positivity and periodicity of solutions of the Q-system in the WZW fusion ring obtained by the KR modules. We also prove the main conjecture in some simple cases. In Section 4, we introduce the level k restricted Q-systems and prove some simple properties of their solutions mainly as a preparation for the next section. In Section 5, we present positive solutions of the level k restricted Q-systems of all classical types in the WZW fusion ring. In Section 6, we will give some applications of our results in the previous section. We provide an explanation for some phenomenon observed by Kuniba and Nakanishi in their formulation of dilogarithm identities for conformal field theories and also prove the Kuniba-Nakanishi-Suzuki conjecture on the positivity and the level truncation properties of the quantum dimension solutions of Q-systems for all classical types.

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2. Notations and reviews

In this section, we fix notations related to a simple Lie algebra $\mathfrak{g}$ and recall some results about the modular S-matrix and the WZW fusion ring. For a reference, see [FMS97, Chapter 14 and Chapter 16]. For Q-systems and KR modules in Section 2.4, see [KNS11, Section 13] and the references therein.

2.1. **Notations.** Throughout the paper, we will use the following notations :

- Dynkin diagram X
- $I = \{1, 2, \cdots, r\}$ the index set of X as in Table 1
- $\hat{I} = \{0\} \cup I$ as in Table 1
- $C = (C_{ab})_{a,b \in I}$ the Cartan matrix
- $\mathfrak{g}$ the finite dimensional simple Lie algebra defined by C
- $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$
- $\mathfrak{h}^*$ the dual space of $\mathfrak{h}$
- $\alpha_i, i \in I$ the simple roots
- $\omega_i, i \in I$ the fundamental weights

- $\alpha_i^\vee, i \in I$ the simple coroots
- $\Pi = \{\alpha_i | i \in I\}$ the simple system
- Δ the root system
- Δ_+ the set of positive roots
- $Q = \oplus_{i\in I}\mathbb{Z}\alpha_i$ the root lattice
- $P = \oplus_{i\in I}\mathbb{Z}\omega_i$ the weight lattice
- $P_+ = \{\sum_{i=1}^r \lambda_i\omega_i \in P | \lambda_i \geq 0\}$
- $Q^\vee = \oplus_{i\in I}\mathbb{Z}\alpha_i^\vee$ the coroot lattice
- θ the highest root

$$\theta = \sum_{i=1}^r a_i\alpha_i = \sum_{i=1}^r c_i\alpha_i^\vee$$

- $a_0 = 1$ and $a_i, i \in I$ the marks
- $c_0 = 1$ and $c_i, i \in I$ the comarks
- $(\cdot|\cdot)$ standard symmetric bilinear form on $\mathfrak{h}^*$ satisfying $(\alpha_i^\vee|\alpha_j) = C_{ij}$, $(\omega_i|\alpha_i^\vee) = \delta_{ij}$ and the normalization condition $(\theta|\theta) = 2$
- $h = \sum_{i=0}^r a_i$ the Coxeter number
- $h^\vee = \sum_{i=0}^r c_i$ the dual Coxeter number
- $e^\lambda, \lambda \in P$ the function $e^\lambda : \mathfrak{h}^* \to \mathbb{C}$ defined by

$$\mu \mapsto \exp 2\pi i(\lambda|\mu)$$

- $\chi_\lambda, \lambda \in P^+$ the character of an irreducible representation V of highest weight λ

$$\chi_\lambda = \sum_{\lambda' \in \mathfrak{h}^*} (\dim V_{\lambda'}) e^{\lambda'}$$

  where $V_{\lambda'}$ denotes the weight space corresponding to $\lambda' \in \mathfrak{h}^*$.
- $\hat{P} = \oplus_{i\in\hat{I}}\mathbb{Z}\hat{\omega}_i$ the affine weight lattice
- $\hat{P}^k = \{\sum_{i=0}^r \lambda_i\hat{\omega}_i \in \hat{P} | \sum_{i=0}^r c_i\lambda_i = k\}$
- $\hat{P}^k_+ = \{\sum_{i=0}^r \lambda_i\hat{\omega}_i \in \hat{P}^k | \lambda_i \geq 0\}$
- $\alpha_0 = -\theta$
- $\hat{\alpha}_j = \sum_{i=0}^r (\alpha_j|\alpha_i^\vee)\hat{\omega}_i$, $j \in \hat{I}$
- $s_i, i \in I$ the fundamental reflections on P defined by

$$s_i\lambda = \lambda - (\alpha_i^\vee|\lambda)\alpha_i$$

- $s_i, i \in \hat{I}$ the fundamental reflections on $\hat{P}$ defined linearly by

$$s_i\hat{\omega}_j = \hat{\omega}_j - \delta_{ij}\hat{\alpha}_i$$

  where δ_{ij} denotes the Kronecker delta
- W the finite Weyl group generated by $s_i, i \in I$ (acting on $\mathfrak{h}^*$)
- $\hat{W}$ the affine Weyl group generated by $s_i, i \in \hat{I}$ (acting on $\hat{P}$)
- $\ell(w)$ the length of $w \in W$, or $\hat{W}$
- $\rho = \sum_{i=1}^r \omega_i \in P$ the Weyl vector
- $\hat{\rho} = \sum_{i=0}^r \hat{\omega}_i \in \hat{P}$ the affine Weyl vector
- $w \cdot \hat{\lambda}$ for $w \in \hat{W}$ and $\hat{\lambda} \in \hat{P}$ the shifted affine Weyl group action

$$w \cdot \hat{\lambda} := w(\hat{\lambda} + \hat{\rho}) - \hat{\rho}$$

- $H = \{(a, m) | a \in I, m \in \mathbb{Z}_{\geq 0}\}$
- $H_k = \{(a, m) \in H | a \in I, 0 \leq m \leq t_a k\}$

- $\mathring{H}_k = \{(a,m) \in H | a \in I, 1 \le m \le t_a k - 1\}$
- $\lfloor x \rfloor$ the greatest integer not exceeding x
- $t_a = \frac{2}{(\alpha_a|\alpha_a)}$

We give a remark on the bilinear form $(\cdot|\cdot)$. Whenever we have an affine weight $\hat{\lambda} \in \hat{P}$, $(\hat{\lambda}|\cdot)$ is defined to be $(\lambda|\cdot)$ where $\lambda \in P$ is the image of the projection defined by

$$\sum_{i=0}^{r} \lambda_i \hat{\omega}_i \mapsto \sum_{i=1}^{r} \lambda_i \omega_i.$$

2.2. **Modular S-matrix.** Let X be a Dynkin diagram and $k \ge 2$ be fixed throughout the paper. For a pair of weights $\lambda, \mu \in P$, we consider the following quantity

$$S_{\hat{\lambda},\hat{\mu}} = \frac{i^{|\Delta_+|}}{\sqrt{|P/Q^\vee|(k+h^\vee)^r}} \sum_{w \in W} (-1)^{\ell(w)} \exp\left(-\frac{2\pi i (w(\lambda+\rho)|\mu+\rho)}{k+h^\vee}\right) \tag{2.1}$$

where $\hat{\lambda}$ and $\hat{\mu}$ are the level k affinizations of λ and μ, respectively. Note that $S_{\hat{\lambda},\hat{\mu}} = S_{\hat{\mu},\hat{\lambda}}$.

We call the matrix $S = (S_{\hat{\lambda},\hat{\mu}})_{\hat{\lambda},\hat{\mu} \in \hat{P}_+^k}$ the modular S-matrix. It appears when one tries to describe the modular behavior of the characters of the representations of the affine Kac-Moody algebras. Let us review some of their properties.

For the modular S-matrix, we have the following orthogonality relation

$$SS^\dagger = I_n \tag{2.2}$$

where $S^\dagger$ denotes the transpose of the complex conjugate of S and I_n is the identity matrix of size $n = |\hat{P}_+^k|$. In other words, S is a unitary matrix.

The modular S-matrix satisfies various symmetries which will be heavily used in the following sections. Let $w \in \hat{W}$. Then the shifted action of the affine Weyl group gives

$$S_{w\cdot\hat{\lambda},\hat{\mu}} = (-1)^{\ell(w)} S_{\hat{\lambda},\hat{\mu}}. \tag{2.3}$$

Let $O(\hat{\mathfrak{g}})$ be the outer automorphism group of $\hat{\mathfrak{g}}$ consisting of some diagram automorphisms of the extended Dynkin diagram. See Table 2 for a description. Note that $\tau \in O(\hat{\mathfrak{g}})$ can be uniquely determined by $\tau\hat{\omega}_0$. We use the standard notations for permutations and cycles. For example, for the permutation

$$\tau = \begin{pmatrix} 0 & 1 & \cdots & r-1 & r \\ \tau(0) & \tau(1) & \cdots & \tau(r-1) & \tau(r) \end{pmatrix},$$

we define $\tau\hat{\omega}_a := \hat{\omega}_{\tau(a)}$ for each $a \in \hat{I}$. The cycle $(i \quad j \quad \cdots \quad k)$ sends $\hat{\omega}_i$ to $\hat{\omega}_j$ and $\hat{\omega}_k$ to $\hat{\omega}_i$.

For $\tau \in O(\hat{\mathfrak{g}})$, we have

$$\tau S_{\hat{\lambda},\hat{\mu}} := S_{\tau\hat{\lambda},\hat{\mu}} = S_{\hat{\lambda},\hat{\mu}} e^{-2\pi i (\tau\hat{\omega}_0|\mu)}. \tag{2.4}$$

Thus the action of the outer automorphism group results in the multiplication of a root of unity on the entries of S-matrix.

For a finite weight $\lambda \in P$, let us consider the conjugate weight $-w_0\lambda \in P$ where w_0 is the longest element of the finite Weyl group W. The conjugate weight $\hat{\lambda}^*$ of the affine weight $\hat{\lambda}$ is defined to be the level k affinization of $\lambda^* \in P$. This conjugation can be regarded as a diagram automorphism of the extended Dynkin

TABLE 1. The Dynkin diagrams and the extended Dynkin diagrams (reproduced from [HKO[+]99])

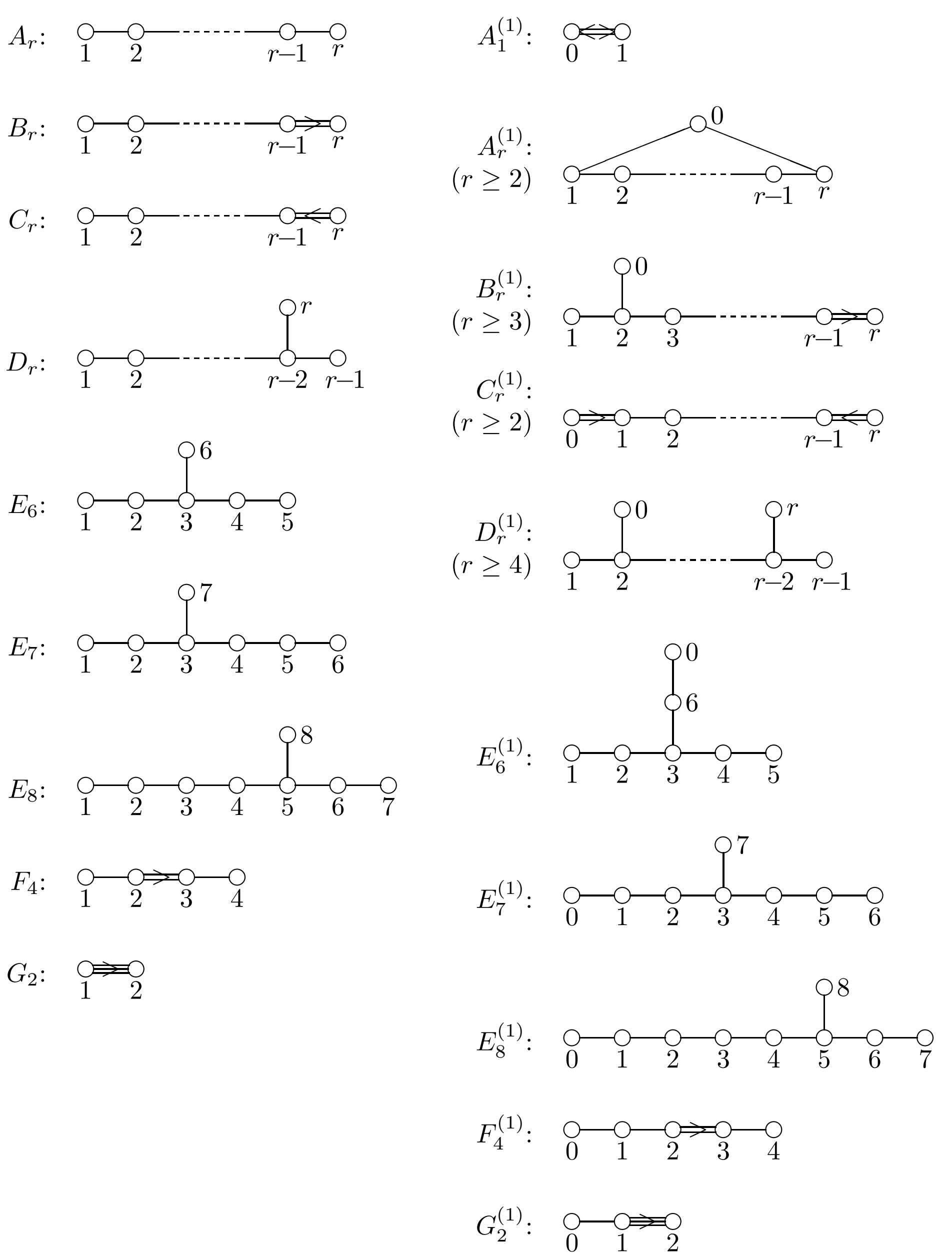

diagram preserving the vertex 0. For concreteness, we give their action in Table 3 using the permutation notation. For $\hat{\lambda}, \hat{\mu} \in \hat{P}_+^k$, we have

$$S_{\hat{\lambda}^*, \hat{\mu}} = S^*_{\hat{\lambda}, \hat{\mu}}. \tag{2.5}$$

TABLE 2. Generators of the outer automorphism group $O(\hat{\mathfrak{g}})$

$\mathfrak{g}$	$O(\hat{\mathfrak{g}})$	generators
A_r	$\mathbb{Z}_{r+1}$	$(0 \quad r \quad r-1 \quad \cdots \quad 1)$
B_r	$\mathbb{Z}_2$	$(0 \quad 1)$
C_r	$\mathbb{Z}_2$	$\begin{pmatrix} 0 & 1 & \cdots & r-1 & r \\ r & r-1 & \cdots & 1 & 0 \end{pmatrix}$
$D_{r=2l}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(0 \quad 1)(r-1 \quad r)$, $\begin{pmatrix} 0 & 1 & 2 & \cdots & r-1 & r \\ r & r-1 & r-2 & \cdots & 1 & 0 \end{pmatrix}$
$D_{r=2l+1}$	$\mathbb{Z}_4$	$\begin{pmatrix} 0 & 1 & 2 & \cdots & r-1 & r \\ r-1 & r & r-2 & \cdots & 1 & 0 \end{pmatrix}$
E_6	$\mathbb{Z}_3$	$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 5 & 4 & 3 & 6 & 0 & 2 \end{pmatrix}$
E_7	$\mathbb{Z}_2$	$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 5 & 4 & 3 & 2 & 1 & 0 & 7 \end{pmatrix}$

TABLE 3. The diagram automorphism corresponding to $*$

$\mathfrak{g}$	$*$
A_r	$\begin{pmatrix} 0 & 1 & \cdots & r \\ 0 & r & \cdots & 1 \end{pmatrix}$
$D_{r=2l+1}$	$(r-1 \quad r)$
E_6	$(1 \quad 5)(2 \quad 4)$
otherwise	trivial

Let us also recall some properties of the affine Weyl group. For $\lambda \in P$ and its image $\lambda' \in P$ under the reflection through the affine hyperplane $H_{\alpha,n} = \{x \in \mathfrak{h}^*_{\mathbb{R}} | (\alpha|x) = n\}$ where n is an integer, we can find an odd element $w \in \hat{W}$ such that

$$\hat{\lambda}' = w\hat{\lambda} \in \hat{P}^n. \tag{2.6}$$

For $\hat{\lambda} \in \hat{P}^k$, we define the quantum dimension $\mathcal{D}_{\hat{\lambda}}$ by

$$\mathcal{D}_{\hat{\lambda}} := \frac{S_{\hat{\lambda},\hat{0}}}{S_{\hat{0},\hat{0}}} = \frac{\prod_{\alpha\in\Delta_+} \sin\frac{\pi(\lambda+\rho|\alpha)}{k+h^\vee}}{\prod_{\alpha\in\Delta_+} \sin\frac{\pi(\rho|\alpha)}{k+h^\vee}} \tag{2.7}$$

where the equaltiy can be justified by the Weyl character formula.

Proposition 2.1. *Let $\hat{\lambda} \in \hat{P}^k$. The following conditions are equivalent :*

(i) $\mathcal{D}_{\hat{\lambda}} = 0$
(ii) there exists $w \in \hat{W}$ of odd signature such that $w \cdot \hat{\lambda} = \hat{\lambda}$
(iii) $S_{\hat{\lambda},\hat{\mu}} = 0$ for all $\hat{\mu} \in \hat{P}^k_+$

Proof. Let us prove (i)⇒(ii). Assume that $\mathcal{D}_{\hat{\lambda}} = 0$. From (2.7), we can find $\alpha \in \Delta$ such that $(\lambda+\rho|\alpha)$ is $n(k+h^\vee)$ for some integer n. This implies that $\lambda+\rho$ is fixed under the reflection through the affine hyperplane $H_{\alpha,n(k+h^\vee)} = \{x \in \mathfrak{h}^*_{\mathbb{R}} | (\alpha|x) = n(k+h^\vee)\}$. By (2.6), there exists $w \in \hat{W}$ of odd signature such that $w \cdot \hat{\lambda} = \hat{\lambda}$. (ii)⇒(iii) follows from (2.3). (iii)⇒(i) is obvious from (2.7). □

We also note that if $\mathcal{D}_{\hat\lambda} \neq 0$ for $\hat\lambda \in \hat P^k$, then we can find a unique element $\hat\lambda' \in \hat P^k_+$ such that

$$\hat\lambda' = w \cdot \hat\lambda \tag{2.8}$$

for some $w \in \hat W$.

If $\hat\lambda \in \hat P^k_+$, then $\mathcal{D}_{\hat\lambda} > 0$ from (2.7) and thus

$$S_{\hat\lambda,\hat 0} = \mathcal{D}_{\hat\lambda} S_{\hat 0,\hat 0} \neq 0. \tag{2.9}$$

2.3. **WZW fusion ring.** The WZW fusion ring R is a free $\mathbb{Z}$-module equipped with the basis $\{V_{\hat\omega} | \hat\omega \in \hat P^k_+\}$ and the fusion product is defined by

$$V_{\hat\lambda} \cdot V_{\hat\mu} = \sum_{\hat\nu \in \hat P^k_+} N^{\hat\nu}_{\hat\lambda\hat\mu} V_{\hat\nu}$$

where the fusion coefficient $N^{\hat\nu}_{\hat\lambda\hat\mu}$ can be computed by the Verlinde formula

$$N^{\hat\nu}_{\hat\lambda\hat\mu} = \sum_{\hat\omega \in \hat P^k_+} \frac{S_{\hat\lambda,\hat\omega} S_{\hat\mu,\hat\omega} S^*_{\hat\nu,\hat\omega}}{S_{\hat 0,\hat\omega}}. \tag{2.10}$$

As a ring, it is commutative and associative with unity $V_{k\hat\omega_0}$. There exists an involution $* : R \to R$ given by

$$V^*_{\hat\omega} := V_{\hat\omega^*}.$$

For general $\hat\lambda \in \hat P^k$, which are not necessarily elements of $\hat P^k_+$, we will use the following identification in R :

$$V_{\hat\lambda} := \begin{cases} 0 & \text{if } \mathcal{D}_{\hat\lambda} = 0 \\ (-1)^{\ell(w)} V_{\hat\lambda'} & \text{if } \mathcal{D}_{\hat\lambda} \neq 0 \end{cases} \tag{2.11}$$

where $\hat\lambda' \in \hat P^k_+$ is as in (2.8).

For classical weights $\lambda, \mu \in P_+$, if we have an identity

$$\chi_\lambda \chi_\mu = \sum_{\nu \in P_+} L^\nu_{\lambda\mu} \chi_\nu$$

among charaters, then we obtain the fusion ring identity

$$\begin{aligned} V_{\hat\lambda} \cdot V_{\hat\mu} &= \sum_{\nu \in P_+} L^\nu_{\lambda\mu} V_{\hat\nu} \\ &= \sum_{\hat\nu \in \hat P^k_+} N^{\hat\nu}_{\hat\lambda\hat\mu} V_{\hat\nu} \end{aligned} \tag{2.12}$$

under the identification (2.11). (2.12) is called the Kac-Walton formula.

Let $V = \sum_{\hat\lambda \in \hat P^k_+} Z_{\hat\lambda} V_{\hat\lambda} \in R$. If $Z_{\hat\lambda} \geq 0$ for all $\hat\lambda \in \hat P^k_+$, then we call V non-negative. If V is non-negative with at least one $Z_{\hat\lambda}$ nonzero, then we call V positive. We define non-positive and negative elements in a similar way.

For $\hat\mu \in \hat P^k_+$ and $V = \sum_{\hat\lambda \in \hat P^k_+} Z_{\hat\lambda} V_{\hat\lambda}$, we define the generalized quantum dimension by

$$\operatorname{qdim}_{\hat\mu} V := \sum_{\hat\lambda \in \hat P^k_+} Z_{\hat\lambda} \frac{S_{\hat\lambda,\hat\mu}}{S_{\hat 0,\hat\mu}}.$$

For $\hat{\mu} = \hat{0}$, we will use the notation

$$\mathrm{qdim}_{\hat{0}} V := \mathrm{qdim}\, V.$$

Let us summarize various properties of the generalized quantum dimensions, many of which follow from properties of the modular S-matrix.

Proposition 2.2. *Let $\hat{\mu} \in \hat{P}^k_+$ and $V \in R$. The following properties hold :*

(i) $\mathrm{qdim}_{\hat{\mu}} : R \to \mathbb{C}$ *is a homomorphism,*
(ii) If V is positive, then $\mathrm{qdim}\, V > 0$,
(iii) $\mathrm{qdim}_{\hat{\mu}} V^* = (\mathrm{qdim}_{\hat{\mu}} V)^*$,
(iv) $\mathrm{qdim}_{\hat{\mu}} V_{k(\tau\hat{\omega}_0)} = e^{-2\pi i(\tau\hat{\omega}_0|\mu)}$,
(v) $\mathrm{qdim}_{\hat{\mu}} \tau V = (\mathrm{qdim}_{\hat{\mu}} V_{k(\tau\hat{\omega}_0)})(\mathrm{qdim}_{\hat{\mu}} V)$,
(vi) $V = 0$ if and only if $\mathrm{qdim}_{\hat{\mu}} V = 0$ *for all* $\hat{\mu} \in \hat{P}^k_+$.

Proof. (i) is a consequence of the Verlinde formula (2.10). (ii) follows from (2.7). (iii) is a consequence of (2.5). For (iv) and (v), we can use (2.4). If $V = \sum_{\hat{\lambda} \in \hat{P}^k_+} Z_{\hat{\lambda}} V_{\hat{\lambda}}$, then

$$S_{\hat{0},\hat{\mu}} \cdot \mathrm{qdim}_{\hat{\mu}} V = \sum_{\hat{\lambda} \in \hat{P}^k_+} Z_{\hat{\lambda}} S_{\hat{\lambda},\hat{\mu}}.$$

Then (vi) follows from (2.2). □

An interesting consequence of the above proposition is that for $\tau \in O(\hat{\mathfrak{g}})$,

$$\tau V = V_{k(\tau\hat{\omega}_0)} \cdot V, \tag{2.13}$$

which allows us to interpret the action of $O(\hat{\mathfrak{g}})$ in R as multiplications by certain elements of R.

Some results on non-negative elements in R can be translated into results on certain non-negative integral matrices as follows :

Proposition 2.3. *Let $V \in R$ be non-negative and $M = (M_{\hat{\lambda}\hat{\mu}})_{\hat{\lambda},\hat{\mu} \in \hat{P}^k_+}$ be the matrix defined by*

$$V \cdot V_{\hat{\lambda}} = \sum_{\hat{\mu} \in \hat{P}^k_+} M_{\hat{\lambda}\hat{\mu}} V_{\hat{\mu}}.$$

It has the following properties :

(i) M is a non-negative integral matrix,
(ii) $\mathrm{qdim}_{\hat{\mu}} V$ *is an eigenvalue of M with an eigenvector* $(\mathrm{qdim}_{\hat{\mu}} V_{\hat{\lambda}})_{\hat{\lambda} \in \hat{P}^k_+}$,
(iii) $\mathrm{qdim}\, V$ *is the Perron-Frobenius eigenvalue of M.*

Proof. It follows from the fact that the fusion coefficient $N^{\hat{\nu}}_{\hat{\lambda}\hat{\mu}}$ is a non-negative integer and Proposition 2.2. □

2.4. Q-systems and KR modules.

Definition 2.4. Let us assume that $Q^{(a)}_{-1} = 0$ for all $a \in I$. For a family of commuting variables $\{Q^{(a)}_m | a \in I, m \in \mathbb{Z}_{\geq 0}\}$ in a ring, consider recurrences given by

$$\left(Q^{(a)}_m\right)^2 = Q^{(a)}_{m-1} Q^{(a)}_{m+1} + \prod_{b \sim a} \prod_{j=0}^{-C_{ab}-1} Q^{(b)}_{\lfloor \frac{C_{ba} m - j}{C_{ab}} \rfloor}, \quad m \geq 0. \tag{2.14}$$

where $b \sim a$ means $C_{ab} < 0$. We call this system of recurrence relations the unrestricted Q-system of type X.

When X is of simply-laced type, it is of the form

$$\left(Q_m^{(a)}\right)^2 = Q_{m-1}^{(a)} Q_{m+1}^{(a)} + \prod_{b\sim a} Q_m^{(b)}.$$

Let q be a non-zero complex number which is not a root of unity. There exists a special class of finite dimensional modules of the quantum affine algebra $U_q(\hat{\mathfrak{g}})$ called the Kirillov-Reshetikhin (KR) modules. For a given KR module $W_m^{(a)}(u)$ parametrized by $(a,m) \in H$ and $u \in \mathbb{C}$, we can obtain a finite dimensional $U_q(\mathfrak{g})$-module $\operatorname{res} W_m^{(a)}(u)$ by restriction which allows us to drop the dependence on the spectral parameter u.

As proved in [Nak03, Her06], the classical characters $Q_m^{(a)}$ of $\operatorname{res} W_m^{(a)}(u)$ gives a solution of the unrestricted Q-system. In general, the $U_q(\mathfrak{g})$-module $\operatorname{res} W_m^{(a)}(u)$ is not irreducible and the character $Q_m^{(a)}$ can be written as a sum of characters of irreducible modules of $\mathfrak{g}$ as

$$Q_m^{(a)} = \sum_{\omega\in P_+} Z(a,m,\omega)\chi_\omega \tag{2.15}$$

for some non-negative integers $Z(a,m,\omega) \in \mathbb{Z}$. See [HKO$^+$99, Appendix] for a thorough treatment of this topic. We give an explicit description of (2.15) for all classical types in Appendix B. Now we can define the main objects in our study.

Definition 2.5. For each $(a,m) \in H$, we define an element $W_m^{(a)}$ of R by

$$W_m^{(a)} := \sum_{\lambda\in P_+} Z(a,m,\lambda)V_{\hat{\lambda}}.$$

where $Z(a,m,\lambda) \in \mathbb{Z}$ is as in (2.15). By (2.11), we can write $W_m^{(a)}$ as

$$W_m^{(a)} = \sum_{\hat{\lambda}\in \hat{P}_+^k} \hat{Z}(a,m,\hat{\lambda})W_{\hat{\lambda}}$$

for some integers $\hat{Z}(a,m,\hat{\lambda}) \in \mathbb{Z}$. For $\hat{\mu} \in \hat{P}_+^k$, we will denote the generalized quantum dimension of $W_m^{(a)}$ by

$$\mathcal{D}_{m,\hat{\mu}}^{(a)} := \operatorname{qdim}_{\hat{\mu}} W_m^{(a)}.$$

Let us take a look at an example.

Example 2.6. Let $X = D_5$ and $k = 4$. From (B.8), we can find

$$W_4^{(2)} = V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} + V_{2\hat{\omega}_2} + V_{-2\hat{\omega}_0+3\hat{\omega}_2} + V_{-4\hat{\omega}_0+4\hat{\omega}_2}.$$

The shifted action of the affine Weyl group gives us

$$s_0\cdot(-2\hat{\omega}_0 + 3\hat{\omega}_2) = 2\hat{\omega}_2,$$
$$s_0\cdot(-4\hat{\omega}_0 + 4\hat{\omega}_2) = 2\hat{\omega}_0 + \hat{\omega}_2.$$

Thus $W_4^{(2)} = V_{4\hat{\omega}_0}$ from (2.11).

The fact that (2.15) solves the unrestricted Q-system implies

$$\left(W_m^{(a)}\right)^2 = W_{m-1}^{(a)} W_{m+1}^{(a)} + \prod_{b\sim a} \prod_{j=0}^{-C_{ab}-1} W^{(b)}_{\lfloor \frac{C_{ba}m-j}{C_{ab}} \rfloor}, \quad m \geq 0. \tag{2.16}$$

In other words, $\left(W_m^{(a)}\right)_{(a,m)\in H}$ is a solution of the unrestricted Q-system in the WZW fusion ring R.

3. Positivity and periodicity conjectures in Q-systems

3.1. **Main conjecture on $W_m^{(a)}$.** Having defined $W_m^{(a)}$, we start with an easy fact about them.

Proposition 3.1. *If X is one of types A, B, C and D, then $W_m^{(a)}$ is positive for each $a \in I$ and $0 \leq m \leq \lfloor \frac{t_a k}{2} \rfloor$.*

Proof. We give the explicit form of $W_m^{(a)}$ for $0 \leq m \leq \lfloor \frac{t_a k}{2} \rfloor$ in Appendix B when X is a Dynkin diagram of classical types. □

The main problem for us is to understand $W_m^{(a)}$ or $\hat{Z}(a, m, \hat{\lambda})$ for all $(a, m) \in H$ and $\hat{\lambda} \in \hat{P}_+^k$. Now we state the main conjectures on $W_m^{(a)}$, which gives a quite remarkable answer to our problem.

Conjecture 3.2. *For $a \in I$, let $\tau_a \in O(\hat{\mathfrak{g}})$ and $\sigma_a = e^{-2\pi i n(\tau_a \hat{\omega}_0|\rho)}$ be as in Table 4 and 5, respectively. The following properties hold :*

(i) $W_m^{(a)}$ *is positive for* $0 \leq m \leq t_a k$,
(ii) $W_{t_a k - m}^{(a)} = \tau_a(W_m^{(a)*})$ *for* $0 \leq m \leq t_a k$,
(iii) $W_{t_a k}^{(a)} = V_{k(\tau_a \hat{\omega}_0)}$,
(iv) $W_{t_a k+1}^{(a)} = W_{t_a k+2}^{(a)} = \cdots = W_{t_a(k+h^\vee)-1}^{(a)} = 0$,
(v) $W_{m+nt_a(k+h^\vee)}^{(a)} = \sigma_a^n \tau_a^n W_m^{(a)}$ *for* $0 \leq m \leq t_a(k+h^\vee)-1$ *and* $n \in \mathbb{Z}_{\geq 0}$.

All these properties but the last one are a kind of reformulations and generalizations of several conjectures suggested in [KN92] , [Kun93], and [KNS11]. See Remark 3.5 also for more details. The periodicity (v) has been pointed out in [Lee12b] in a less precise form than given here. It allows us to describe $W_m^{(a)}$ for all $m \geq 0$ completely from $W_m^{(a)}$ for $0 \leq m \leq \lfloor \frac{t_a k}{2} \rfloor$.

The conjecture is supported by many symbolic calculations involving the affine Weyl group. In Section 3.2, we will give a proof of Conjecture 3.2 for some special cases. See Theorem 3.10 for a precise statement.

Let us collect a few simple consequences of Conjecture 3.2.

Theorem 3.3. *Assume that Conjecture 3.2 is true. For $(a, m) \in H$,*

(i) $W_m^{(a)}$ *is always either positive, negative or zero.*
(ii) $W_{m+Mt_a(k+h^\vee)}^{(a)} = W_m^{(a)}$ *where M is given by the table*

	A_r	B_r	C_r	D_{2r}	D_{2r+1}	E_6	E_7	E_8	F_4	G_2
M	$r+1$	2	2	2	4	3	2	1	1	1

Proof. (i) is a consequence of Conjecture 3.2 (i), (iv) and (v). To prove (ii), we can use Conjecture 3.2 (v), together with the fact $\sigma_a^M = \tau_a^M = 1$. □

TABLE 4. The diagram automorphism $\tau_a \in O(\hat{\mathfrak{g}})$ in Conjecture 3.2

	$\tau_a\hat{\omega}_0$
A_r	$\hat{\omega}_a$
B_r	$\begin{cases} \hat{\omega}_1 & \text{if } a \equiv 1 \pmod 2 \\ \hat{\omega}_0 & \text{if } a \equiv 0 \pmod 2 \end{cases}$
C_r	$\begin{cases} \hat{\omega}_0 & \text{if } 1 \le a \le r-1 \\ \hat{\omega}_r & \text{if } a = r \end{cases}$
D_r	$\begin{cases} \hat{\omega}_1 & \text{if } 1 \le a \le r-2 \text{ and } a \equiv 1 \pmod 2 \\ \hat{\omega}_0 & \text{if } 1 \le a \le r-2 \text{ and } a \equiv 0 \pmod 2 \\ \hat{\omega}_a & \text{if } a = r-1 \text{ or } a = r \end{cases}$
E_6	$\begin{array}{c\|c\|c\|c\|c\|c\|c} a & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline \tau_a\hat{\omega}_0 & \hat{\omega}_1 & \hat{\omega}_5 & \hat{\omega}_0 & \hat{\omega}_1 & \hat{\omega}_5 & \hat{\omega}_0 \end{array}$
E_7	$\begin{array}{c\|c\|c\|c\|c\|c\|c\|c} a & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline \tau_a\hat{\omega}_0 & \hat{\omega}_0 & \hat{\omega}_0 & \hat{\omega}_0 & \hat{\omega}_6 & \hat{\omega}_0 & \hat{\omega}_6 & \hat{\omega}_6 \end{array}$
E_8, F_4, G_2	$\hat{\omega}_0$

Theorem 3.4. *Assume that Conjecture 3.2 is true. Then the following properties hold for each $a \in I$ and $\hat{\mu} \in \hat{P}^k_+$:*

(i) $\mathcal{D}^{(a)}_m > 0$ *for* $0 \le m \le t_a k$,

(ii) $D^{(a)}_{t_a k - m,\hat{\mu}} = D^{(a)}_{t_a k,\hat{\mu}} D^{(a)*}_{m,\hat{\mu}}$ *for* $0 \le m \le t_a k$,

(iii) $\mathcal{D}^{(a)}_{t_a k,\hat{\mu}} = e^{-2\pi i(\tau_a\hat{\omega}_0|\mu)}$,

(iv) $\mathcal{D}^{(a)}_{t_a k+1,\hat{\mu}} = \mathcal{D}^{(a)}_{t_a k+2,\hat{\mu}} = \cdots = \mathcal{D}^{(a)}_{t_a(k+h^\vee)-1,\hat{\mu}} = 0$,

(v) $\mathcal{D}^{(a)}_{m+nt_a(k+h^\vee),\hat{\mu}} = e^{-2\pi i n(\tau_a\hat{\omega}_0|\mu+\rho)}\mathcal{D}^{(a)}_{m,\hat{\mu}}$ *for* $0 \le m \le t_a(k+h^\vee)-1$ *and* $n \in \mathbb{Z}_{\ge 0}$.

Proof. Each statement follows directly from Conjecture 3.2 and Proposition 2.2. □

Remark 3.5. In Theorem 6.2, we will prove some statements in Theorem 3.4 for all classical types without assuming the validity of Conjecture 3.2 under some mild assumptions. The relation

$$D^{(a)}_{t_a k-m,\hat{\mu}} = D^{(a)}_{t_a k} D^{(a)*}_{m,\hat{\mu}}$$

in Theorem 3.4 (ii) has been observed in [KN92] from numerical tests but has not been proved. Our approach using the modular S-matrix and the WZW fusion ring now gives a way to explain this phenomenon. The KNS conjecture [KNS11, Conjecture 14.2] follows from Theorem 3.4. See Section 6 for its precise statement where we give a proof of it for all classical types without using Conjecture 3.2.

3.2. **Proof of positivity and periodicity in some cases.** The following property of the modular S-matrix has been noticed in [Spi90]. We make the statement in a precise form.

Proposition 3.6. *Let λ, μ and σ be elements of P. If $\lambda' = \lambda + (k+h^\vee)\sigma$, then*

$$S_{\hat{\lambda}',\hat{\mu}} = e^{-2\pi i(\sigma|\mu+\rho)} S_{\hat{\lambda},\hat{\mu}}.$$

TABLE 5. The sign factor $\sigma_a = \pm 1$ in Conjecture 3.2

$\mathfrak{g}$	$\sigma_a = e^{-2\pi i(\tau_a \hat{\omega}_0 \mid \rho)}$
A_r	-1 if a and r are both odd
B_r	-1 if a is odd
C_r	-1 if $r \equiv 1, 2 \pmod 4$ and $a = r$
D_r	-1 if $r \equiv 2, 3 \pmod 4$ and $a = r, r-1$
E_7	-1 if $a = 4, 6, 7$
otherwise	1

TABLE 6. Composition of τ_a and $*$

X	$\tau_a \circ *$
A_r	$\begin{pmatrix} 0 & 1 & \cdots & a-1 & a & a+1 & \cdots & r \\ a & a-1 & \cdots & 1 & 0 & r & \cdots & a+1 \end{pmatrix}$
B_r	$(0 \quad 1)$ if $a \equiv 1 \pmod 2$
C_r	$\begin{pmatrix} 0 & 1 & 2 & \cdots & r-2 & r-1 & r \\ r & r-1 & r-2 & \cdots & 2 & 1 & 0 \end{pmatrix}$ if $a = r$
$D_{r=2l}$	$\begin{cases} (0 \quad 1)(r-1 \quad r) & \text{if } 1 \le a \le r-2 \text{ and } a \equiv 1 \pmod 2 \\ \begin{pmatrix} 0 & 1 & 2 & \cdots & r-2 & r-1 & r \\ r-1 & r & r-2 & \cdots & 2 & 0 & 1 \end{pmatrix} & \text{if } a = r-1 \\ \begin{pmatrix} 0 & 1 & 2 & \cdots & r-2 & r-1 & r \\ r & r-1 & r-2 & \cdots & 2 & 1 & 0 \end{pmatrix} & \text{if } a = r \end{cases}$
$D_{r=2l+1}$	$\begin{cases} (0 \quad 1) & \text{if } 1 \le a \le r-2 \text{ and } a \equiv 1 \pmod 2 \\ \begin{pmatrix} 0 & 1 & 2 & \cdots & r-2 & r-1 & r \\ r-1 & r & r-2 & \cdots & 2 & 0 & 1 \end{pmatrix} & \text{if } a = r-1 \\ \begin{pmatrix} 0 & 1 & 2 & \cdots & r-2 & r-1 & r \\ r & r-1 & r-2 & \cdots & 2 & 1 & 0 \end{pmatrix} & \text{if } a = r \end{cases}$
E_6	$\begin{cases} (0 \quad 1)(2 \quad 6) & \text{if } a = 1, 4 \\ (0 \quad 5)(4 \quad 6) & \text{if } a = 2, 5 \\ (1 \quad 5)(2 \quad 4) & \text{if } a = 3, 6 \end{cases}$
E_7	$(0 \quad 6)(1 \quad 5)(2 \quad 4)$ if $a = 4, 6, 7$
otherwise	trivial

In particular, when $\sigma \in Q^\vee \subseteq P$ is a coroot,

$$S_{\hat{\lambda}', \hat{\mu}} = S_{\hat{\lambda}, \hat{\mu}}.$$

Proof. For $w \in W$, we have

$$(w\sigma|\mu + \rho) = (\sigma|\mu + \rho) \pmod{\mathbb{Z}}.$$

Then from (2.1), we can pull out the phase $e^{-2\pi i(\sigma|\mu+\rho)}$ so that

$$S_{\hat{\lambda}', \hat{\mu}} = e^{-2\pi i(\sigma|\mu+\rho)} S_{\hat{\lambda}, \hat{\mu}}.$$

If $\sigma \in Q^\vee$, then the fact that $(\sigma|\mu + \rho) \in \mathbb{Z}$ implies $e^{-2\pi i(\sigma|\mu+\rho)} = 1$. $\square$

TABLE 7. Vertices corresponding to minuscule fundamental weights

	a
A_r	$1 \le a \le r$
B_r	$a = 1$
C_r	$a = r$
D_r	$a = 1, r, r-1$
E_6	$a = 1, 5$
E_7	$a = 6$

Lemma 3.7. *Let $\tau\hat{\omega}_0 = \hat{\omega}_a$ for some $\tau \in O(\hat{\mathfrak{g}})$ and $a \in I$. Let $\sigma = e^{-2\pi i(\omega_a|\rho)}$. If $\lambda = m\omega_a$ and $\lambda' = \lambda + n(k + h^\vee)\omega_a$, then*

$$S_{\hat{\lambda}',\hat{\mu}} = \sigma^n \tau^n S_{\hat{\lambda},\hat{\mu}}$$

for any $\hat{\mu} \in \hat{P}_+^k$.

Proof. If we apply Proposition 3.6 for $\sigma = n\omega_a$, then we get

$$\begin{aligned} S_{\hat{\lambda}',\hat{\mu}} &= e^{-2\pi i(n\omega_a|\mu+\rho)} S_{\hat{\lambda},\hat{\mu}} \\ &= e^{-2\pi i n(\omega_a|\rho)} e^{-2\pi i n(\omega_a|\mu)} S_{\hat{\lambda},\hat{\mu}} \\ &= \sigma^n \tau^n S_{\hat{\lambda},\hat{\mu}}. \end{aligned}$$

We have used (2.4) to get the equality in the last line. This proves our lemma. □

This result can be applied for any $a \in I$ listed in Table 7.

Proposition 3.8. *Let $a \in I$ be as in Table 7. For each integer l such that $1 \le l \le h^\vee - 1$, there exists a positive root α such that $(\omega_a|\alpha) = 1$ and $(\rho|\alpha) = l$.*

Our proof is based on a case-by-case check for each root system. Since it is straightforward and lengthy, we give it in Appendix A.

Theorem 3.9. *Let $a \in I$ be as in Table 7. Then $V_{(k-m)\hat{\omega}_0 + m\hat{\omega}_a} = 0$ for $k + 1 \le m \le k + h^\vee - 1$.*

Proof. By Proposition 3.8 and the product formula (2.7) for the quantum dimension, we get $\mathcal{D}_{(k-m)\hat{\omega}_0+m\hat{\omega}_a} = 0$. Then by Proposition 2.1,

$$\operatorname{qdim}_{\hat{\mu}} V = \frac{S_{(k-m)\hat{\omega}_0+m\hat{\omega}_a,\hat{\mu}}}{S_{\hat{0},\hat{\mu}}} = 0$$

for all $\hat{\mu} \in \hat{P}_+^k$. Then our statement follows from Proposition 2.2 (vi). □

Note that the vertices in Table 7 are the ones for which the decomposition (2.15) is given by $Q_m^{(a)} = m\omega_a$ (conjecturally for types E_6 and E_7).

Theorem 3.10. *Let $a \in I$ as in Table 7. Under the further assumption that $W_m^{(a)} = V_{(k-m)\hat{\omega}_0+m\hat{\omega}_a}$ for all $m \in \mathbb{Z}_{\ge 0}$, Conjecture 3.2 is true. In particular, it holds for all $a \in I$ in the case of type A_r.*

Proof. First note that for $a \in I$, $t_a = 1$. The positivity (i) and the unit boundary condition (iii) are trivial by our assumptions. Using Table 6, it is straightforward to check

$$\begin{aligned}\tau_a(W_m^{(a)*}) &= \tau_a\left(V^*_{(k-m)\hat{\omega}_0+m\hat{\omega}_a}\right)\\ &= V_{m\hat{\omega}_0+(k-m)\hat{\omega}_a}\\ &= W_{k-m}^{(a)}\end{aligned}$$

for all $0 \le m \le k$. This proves the symmetry condition (ii). (iv) follows from Theorem 3.9.

By Lemma 3.7, for $0 \le m \le k+h^\vee-1$ and $n \in \mathbb{Z}_{\ge 0}$, we have

$$\operatorname{qdim}_{\hat{\mu}} W^{(a)}_{m+n(k+h^\vee)} = \operatorname{qdim}_{\hat{\mu}} \sigma_a^n \tau_a^n W_m^{(a)}$$

for all $\hat{\mu} \in \hat{P}_+^k$. Then by Proposition 2.2 (vi), $W^{(a)}_{m+n(k+h^\vee)} = \sigma_a^n\tau_a^n W_m^{(a)}$. This proves the periodicity condition (v). □

Example 3.11. Let $X = A_3$ and $k = 3$. We illustrate Theorem 3.10 as follows :

$$\begin{bmatrix} W_0^{(1)} & W_0^{(2)} & W_0^{(3)} \\ W_1^{(1)} & W_1^{(2)} & W_1^{(3)} \\ W_2^{(1)} & W_2^{(2)} & W_2^{(3)} \\ W_3^{(1)} & W_3^{(2)} & W_3^{(3)} \\ W_4^{(1)} & W_4^{(2)} & W_4^{(3)} \\ \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} V_{3\hat{\omega}_0} & V_{3\hat{\omega}_0} & V_{3\hat{\omega}_0} \\ V_{2\hat{\omega}_0+\hat{\omega}_1} & V_{2\hat{\omega}_0+\hat{\omega}_2} & V_{2\hat{\omega}_0+\hat{\omega}_3} \\ V_{\hat{\omega}_0+2\hat{\omega}_1} & V_{\hat{\omega}_0+2\hat{\omega}_2} & V_{\hat{\omega}_0+2\hat{\omega}_3} \\ V_{3\hat{\omega}_1} & V_{3\hat{\omega}_2} & V_{3\hat{\omega}_3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -V_{3\hat{\omega}_1} & V_{3\hat{\omega}_2} & -V_{3\hat{\omega}_3} \\ -V_{2\hat{\omega}_1+\hat{\omega}_2} & V_{\hat{\omega}_0+2\hat{\omega}_2} & -V_{\hat{\omega}_2+2\hat{\omega}_3} \\ -V_{\hat{\omega}_1+2\hat{\omega}_2} & V_{2\hat{\omega}_0+\hat{\omega}_2} & -V_{2\hat{\omega}_2+\hat{\omega}_3} \\ -V_{3\hat{\omega}_2} & V_{3\hat{\omega}_0} & -V_{3\hat{\omega}_2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ V_{3\hat{\omega}_2} & V_{3\hat{\omega}_0} & V_{3\hat{\omega}_2} \\ V_{2\hat{\omega}_2+\hat{\omega}_3} & V_{2\hat{\omega}_0+\hat{\omega}_2} & V_{\hat{\omega}_1+2\hat{\omega}_2} \\ V_{\hat{\omega}_2+2\hat{\omega}_3} & V_{\hat{\omega}_0+2\hat{\omega}_2} & V_{2\hat{\omega}_1+\hat{\omega}_2} \\ V_{3\hat{\omega}_3} & V_{3\hat{\omega}_2} & V_{3\hat{\omega}_1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -V_{3\hat{\omega}_3} & V_{3\hat{\omega}_2} & -V_{3\hat{\omega}_1} \\ -V_{\hat{\omega}_0+2\hat{\omega}_3} & V_{\hat{\omega}_0+2\hat{\omega}_2} & -V_{\hat{\omega}_0+2\hat{\omega}_1} \\ -V_{2\hat{\omega}_0+\hat{\omega}_3} & V_{2\hat{\omega}_0+\hat{\omega}_2} & -V_{2\hat{\omega}_0+\hat{\omega}_1} \\ -V_{3\hat{\omega}_0} & V_{3\hat{\omega}_0} & -V_{3\hat{\omega}_0} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ V_{3\hat{\omega}_0} & V_{3\hat{\omega}_0} & V_{3\hat{\omega}_0} \\ V_{2\hat{\omega}_0+\hat{\omega}_1} & V_{2\hat{\omega}_0+\hat{\omega}_2} & V_{2\hat{\omega}_0+\hat{\omega}_3} \\ V_{\hat{\omega}_0+2\hat{\omega}_1} & V_{\hat{\omega}_0+2\hat{\omega}_2} & V_{\hat{\omega}_0+2\hat{\omega}_3} \\ V_{3\hat{\omega}_1} & V_{3\hat{\omega}_2} & V_{3\hat{\omega}_3} \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \end{bmatrix}$$

One can clearly see the appearance of the number $k + h^\vee = 7$ and the level truncation properties described in Conjecture 3.2. A complete period is $(r+1)(k+h^\vee) = 28$. After each 7 terms, there is an action of diagram automorphisms of order 4 on each column.

4. The level k restricted Q-system

From what we have seen in the previous section, it seems natural to separate and focus only on certain parts of a solution of Q-systems when it satisfies the boundary conditions $Q^{(a)}_{t_a k+1} = 0$ for all $a \in I$. To deal with this situation, let us give the following definition.

Definition 4.1. For commuting variables $\left(Q^{(a)}_m\right)_{(a,m)\in H_k}$ in a ring, we call the system of equations

$$(4.1)\qquad \begin{cases} Q^{(a)}_{-1} = 0 \\ \left(Q^{(a)}_m\right)^2 = Q^{(a)}_{m-1}Q^{(a)}_{m+1} + \prod_{b\sim a}\prod_{j=0}^{-C_{ab}-1} Q^{(b)}_{\lfloor \frac{C_{ba}m-j}{C_{ab}}\rfloor}, \quad 0 \le m \le t_a k \\ Q^{(a)}_{t_a k+1} = 0 \end{cases}$$

the level k restricted Q-system of type X.

Conjecture 3.2 claims that $\left(W^{(a)}_m\right)_{(a,m)\in H_k}$ is a positive solution of the level k restricted Q-system with a certain central symmetry in the fusion ring R.

4.1. The boundary Q-systems and the unit boundary condition.

Definition 4.2. For commuting variables $\left(Q^{(a)}\right)_{a\in I}$ in a ring, we define the boundary Q-system as the system of equations

$$(4.2)\qquad \left(Q^{(a)}\right)^2 = \prod_{b\sim a}\left(Q^{(b)}\right)^{-C_{ab}}, \quad a \in I.$$

Assuming that $Q^{(a)} \neq 0$ for each $a \in I$, it can be rewritten as

$$\prod_{b=1}^{r}\left(Q^{(b)}\right)^{C_{ab}} = 1.$$

We now prove an easy but important property of a solution of the boundary Q-system.

Proposition 4.3. *Let $\left(c^{(a)}\right)_{a\in I}$ be a solution of the boundary Q-system. If $\left(Q^{(a)}_m\right)_{(a,m)\in H}$ is a solution of the level k restricted or unrestricted Q-system, then so is $\left(c^{(a)}Q^{(a)}_m\right)_{(a,m)\in H}$.*

Proof. We are assuming that (4.2)

$$\left(c^{(a)}\right)^2 = \prod_{b\sim a}\left(Q^{(b)}\right)^{-C_{ab}} = \prod_{b\sim a}\prod_{j=0}^{-C_{ab}-1} c^{(b)}$$

and (2.14)

$$\left(Q^{(a)}_m\right)^2 = Q^{(a)}_{m+1}Q^{(a)}_{m-1} + \prod_{b\sim a}\prod_{j=0}^{-C_{ab}-1} Q^{(b)}_{\lfloor \frac{C_{ba}m-j}{C_{ab}}\rfloor}$$

are satisfied for each $a \in I$. Then it easily follows that

$$\left(c^{(a)} Q_m^{(a)}\right)^2 = \left(c^{(a)}\right)^2 Q_{m+1}^{(a)} Q_{m-1}^{(a)} + \prod_{b \sim a} \prod_{j=0}^{-C_{ab}-1} c^{(b)} Q^{(b)}_{\lfloor \frac{C_{ba}m-j}{C_{ab}} \rfloor}.$$

□

4.2. **String of zeros.** In this section, we investigate some conditions under which we have a string of zeros in a solution of Q-systems. For the rest of this section, we assume that $\left(Q_m^{(a)}\right)_{(a,m)\in H}$ is a complex solution of the Q-system of type X.

Lemma 4.4. *Let $m \in \mathbb{Z}_{\geq 0}$. Suppose that $Q^{(a)}_{t_a m+1} = 0$ for all $a \in I$. Then $Q^{(a)}_{t_a m+1} = \cdots = Q^{(a)}_{t_a(m+1)} = 0$ for all $a \in I$.*

Proof. This is a consequence of (2.14). □

Lemma 4.5. *Let $m \in \mathbb{Z}_{\geq 0}$. Suppose $Q^{(a)}_{t_a m} = 0$ for all $a \in I$. Then $Q^{(a)}_{t_a m+1} = \cdots = Q^{(a)}_{t_a(m+1)-1} = 0$ for all $a \in I$.*

Proof. It also follows from (2.14) easily. □

Lemma 4.6. *Let $m \geq 1$. Assume that $Q^{(a)}_{t_a m-1} \neq 0$ for all $a \in I$ and $\left(Q^{(a)}_{t_a m}\right)_{a\in I}$ is a solution of the boundary Q-system. Then $Q^{(a)}_{t_a m+1} = \cdots = Q^{(a)}_{t_a(m+2)-1} = 0$ for all $a \in I$.*

Proof. The equation (2.14)

$$\left(Q^{(a)}_{t_a m}\right)^2 = Q^{(a)}_{t_a m-1} Q^{(a)}_{t_a m+1} + \prod_{b\sim a} \left(Q^{(b)}_{t_b m}\right)^{-C_{ab}}$$

implies $Q^{(a)}_{t_a m+1} = 0$. Then our statement follows from Lemma 4.4 and 4.5. □

Lemma 4.7. *Let $m \in \mathbb{Z}_{\geq 0}$. Suppose $Q^{(a)}_{t_a m} = \cdots = Q^{(a)}_{t_a(m+1)-1} = 0$ for all $a \in I$ and $Q^{(b)}_{t_b(m+1)} = 0$ for at least one vertex $b \in I$. Then $Q^{(a)}_{t_a(m+1)} = \cdots = Q^{(a)}_{t_a(m+2)-1} = 0$ for all $a \in I$.*

Proof. We can show that $Q^{(a)}_{t_a(m+1)} = 0$ for all $a \in I$ using (2.14). Then the lemma follows from Lemma 4.5. □

5. The level k restricted Q-system in the WZW fusion ring

The main result of this section is Theorem 5.5 where we construct a positive solution of the level k restricted Q-system in the WZW fusion ring R.

5.1. **Boundary Q-systems in the WZW fusion ring.**

Theorem 5.1. *Let $(\tau_a)_{a\in I}$ be as in Table 4. Then $\left(V_{k(\tau_a \hat{\omega}_0)}\right)_{a\in I}$ is a solution of the boundary Q-system in R.*

Proof. Note that first that for type A_r, Theorem 3.10 already proves our theorem. For types E_8, G_2 and F_4, it is trivial since τ_a is the identity for any $a \in I$.

By Proposition 2.2 (vi), we can show that $\left(V_{k(\tau_a \hat{\omega}_0)}\right)_{a\in I}$ satisfies the boundary Q-system by proving

$$\prod_{b\in I} \left(\operatorname{qdim}_{\hat{\mu}} V_{k(\tau_b\hat{\omega}_0)}\right)^{C_{ab}} = 1, \quad a \in I$$

for any $\hat{\mu} \in \hat{P}^k_+$. Since

$$\operatorname{qdim}_{\hat{\mu}} V_{k(\tau_a\hat{\omega}_0)} = e^{-2\pi i(\tau_a\hat{\omega}_0|\mu)},$$

one way to verify this is to show

$$\sum_{b\in I} C_{ab}(\tau_a\hat{\omega}_0) \in Q^\vee, \quad a \in I. \tag{5.1}$$

Now we list the conditions (5.1) to be checked for each type explicitly.

For type B_r, we only have to check that $2\omega_1 \in Q^\vee$.

For type C_r, the only condition we need to check is $2\omega_r \in Q^\vee$.

For type $D_{r=2l+1}$, we have the following conditions

$$\begin{cases} 2\omega_1 \in Q^\vee \\ \omega_{r-1} + \omega_r - 2\omega_1 \in Q^\vee \\ 2\omega_{r-1} - \omega_1 \in Q^\vee \\ 2\omega_r - \omega_1 \in Q^\vee \end{cases}.$$

For type $D_{r=2l}$, we need to check

$$\begin{cases} 2\omega_1 \in Q^\vee \\ \omega_1 + \omega_{r-1} + \omega_r \in Q^\vee \\ 2\omega_{r-1} \in Q^\vee \\ 2\omega_r \in Q^\vee \end{cases}.$$

For type E_6, the conditions to be checked are given by

$$\begin{cases} 2\omega_1 - \omega_5 \in Q^\vee \\ 2\omega_5 - \omega_1 \in Q^\vee \\ \omega_1 + \omega_5 \in Q^\vee \end{cases}.$$

For type E_7, we only have to check $2\omega_6 \in Q^\vee$.

These can all be verified by a straightforward calculation. □

We now give an analogous result of Proposition 4.3 for a solution in the WZW fusion ring.

Proposition 5.2. *Let $(\tau_a)_{a\in I}$ be as in Table 4. If $\left(W^{(a)}_m\right)_{(a,m)\in H}$ is a solution of the level k restricted or unrestricted Q-system in R, then so is $\left(\tau^{(a)} W^{(a)}_m\right)_{(a,m)\in H}$.*

Proof. In view of Proposition 2.2 (vi), it suffices to show that

$$\left(\operatorname{qdim}_{\hat{\mu}} \tau^{(a)} W^{(a)}_m\right)_{(a,m)\in H}$$

is a solution of the Q-system for any $\hat{\mu} \in \hat{P}^k_+$. This can be easily proven by Theorem 5.1, Proposition 4.3 and Proposition 2.2 (v). □

5.2. **Positive solution of the level k-restricted Q-system in the WZW fusion ring.** Now we prove the most important stepping stone in proving Theorem 5.5.

Lemma 5.3. *Let X be a Dynkin diagram of types A, B, C and D. Let $s = \lfloor \frac{t_a k}{2} \rfloor$. If $t_a k$ is even, then $W_{s+1}^{(a)} = \tau_a(W_{s-1}^{(a)*})$. If $t_a k$ is odd, then $W_{s+1}^{(a)} = \tau_a(W_{s}^{(a)*})$.*

Proof. For type A_r, the statement follows from Theorem 3.10. For type D_r, Theorem 3.10 and the arguments in [Lee12a, Propositions 3.3, 3.4, 3.6, 3.8] can be used to prove the lemma. For the remaining cases, the method of the proof is exactly the same as in the case of type D_r. Since it is mainly a laborious case-by-case check, we will give a proof for types B_r and C_r in Appendix C and D, respectively. □

Now we can construct a positive solution of the level k restricted Q-system in R. The basic idea is to glue two different solutions of the unrestricted Q-system from the opposite directions to form a single solution of the level k restricted Q-system. In order to glue them consistently at the intersection, we need to employ Lemma 5.3. This idea has been used in [Lee12a] to obtain a proof of the KNS conjecture for type D_r.

Definition 5.4. For each $a \in I$, let us define $R_m^{(a)}$ in R by

$$R_m^{(a)} = \begin{cases} W_m^{(a)} & 0 \le m \le \lfloor \frac{t_a k}{2} \rfloor \\ \tau_a(W_{t_a k - m}^{(a)*}) & \lfloor \frac{t_a k+1}{2} \rfloor \le m \le t_a k \end{cases}$$

and $R_{-1}^{(a)} = R_{t_a k+1}^{(a)} = 0$. Recall an elementary identity $\lfloor \frac{t_a k}{2} \rfloor + \lfloor \frac{t_a k+1}{2} \rfloor = t_a k$ to see that the above is well-defined when $t_a k$ is even and thus the equality $\lfloor \frac{t_a k}{2} \rfloor = \lfloor \frac{t_a k+1}{2} \rfloor$ holds.

Theorem 5.5. *Let X be a Dynkin diagram of types A, B, C and D. Then $\left(R_m^{(a)}\right)_{(a,m)\in H_k}$ is a positive solution of the level k restricted Q-system of type X.*

Proof. We have to check that the equality

$$\left(R_m^{(a)}\right)^2 = R_{m-1}^{(a)} R_{m+1}^{(a)} + \prod_{b\sim a} \prod_{j=0}^{-C_{ab}-1} R^{(b)}_{\lfloor \frac{C_{ba} m - j}{C_{ab}} \rfloor} \tag{5.2}$$

holds true for each $0 \le m \le t_a k$.

To prove (5.2) for $0 \le m \le \lfloor \frac{t_a k}{2} \rfloor - 1$, we can use (2.16) directly. For $m = \lfloor \frac{t_a k}{2} \rfloor$, we need to use Lemma 5.3 together with the equation (2.16).

Note that $\left(W_m^{(a)*}\right)_{(a,m)\in H}$ is also a solution of the unrestricted Q-system. By Proposition 4.3, $\left(\tau_a(W_m^{(a)*})\right)_{(a,m)\in H}$ is also a solution of the unrestricted Q-system. In concrete terms, we have

$$\left(\tau_a(W_m^{(a)*})\right)^2 = \left(\tau_a(W_{m-1}^{(a)*})\right)\left(\tau_a(W_{m+1}^{(a)*})\right) + \prod_{b\sim a} \prod_{j=0}^{-C_{ab}-1} \tau_b(W^{(b)*}_{\lfloor \frac{C_{ba} m - j}{C_{ab}} \rfloor}), \quad m \ge 0. \tag{5.3}$$

Since this holds especially for $0 \le m \le \lfloor \frac{t_a k}{2} \rfloor - 1$, (5.2) is true for $\lfloor \frac{t_a k+1}{2} \rfloor + 1 \le m \le t_a k$. The only possible value not verified so far, which happens when $t_a k$ is odd, is $m = \lfloor \frac{t_a k+1}{2} \rfloor$. For $m = \lfloor \frac{t_a k+1}{2} \rfloor$, again we can use Lemma 5.3 together with (5.3) for $\tau_a(W_m^{(a)*})$. □

Remark 5.6. Lemma 5.3 proves that $R_m^{(a)} = W_m^{(a)}$ for $0 \le m \le \lfloor \frac{t_a k}{2} \rfloor + 1$ in all classical types. However, the identity $R_m^{(a)} = W_m^{(a)}$ for $\lfloor \frac{t_a k}{2} \rfloor + 2 \le m \le t_a k$ still remains to be proved in general except the cases proved in Theorem 3.10 and the vertices adjacent to them.

Definition 5.7. Let $(a, m) \in H_k$. For each $R_m^{(a)} \in R$, we define a non-negative integral matrix $\mathcal{A}_m^{(a)}$ as in Proposition 2.3. We call it the admissibility matrix of $R_m^{(a)}$ following [KNS11, Section 3.7].

Corollary 5.8. *The admissibility matrices* $\left(\mathcal{A}_m^{(a)}\right)_{(a,m)\in H_k}$ *is a solution of the level* k *restricted* Q*-system in the ring of square matrices of size* $|\hat{P}_+^k|$ *over* $\mathbb{Z}$.

6. Applications of the main theorem

Let us begin with a simple lemma.

Lemma 6.1. *Let* $\mathbf{w} = \left(w_m^{(a)}\right)_{(a,m)\in H_k}$ *be a complex solution of the level* k *restricted* Q*-system such that* $w_m^{(a)} \neq 0$ *for any* $(a, m) \in H_k$. *If* $\left(Q_m^{(a)}\right)_{(a,m)\in H}$ *is a solution of the unrestricted* Q*-system and* $w_1^{(a)} = Q_1^{(a)}$ *for any* $a \in I$, *then* $w_m^{(a)} = Q_m^{(a)}$ *for* $0 \le m \le t_a k$.

Proof. This is a direct consequence of the recursion (2.14)

$$\left(Q_m^{(a)}\right)^2 = Q_{m-1}^{(a)} Q_{m+1}^{(a)} + \prod_{b\sim a} \prod_{j=0}^{-C_{ab}-1} Q^{(b)}_{\lfloor \frac{C_{ba} m - j}{C_{ab}} \rfloor}, \quad m \ge 1.$$

□

Now we prove a weaker version of Theorem 3.4 without assuming Conjecture 3.2.

Theorem 6.2. *Let* X *be a Dynkin diagram of types* A, B, C *and* D *and let* $\hat{\mu} \in \hat{P}_+^k$. *Assume that* $\mathcal{D}_{m,\hat{\mu}}^{(a)} \neq 0$ *for all* $a \in I$ *and* $0 \le m \le \lfloor \frac{t_a k}{2} \rfloor$. *Then the following properties hold for each* $a \in I$:

- *(i)* $\mathcal{D}_{m,\hat{\mu}}^{(a)} = \mathrm{qdim}_{\hat{\mu}} R_m^{(a)}$ *for* $0 \le m \le t_a k$,
- *(ii)* $\mathcal{D}_{m,\hat{\mu}}^{(a)} = \mathcal{D}_{t_a k}^{(a)} \mathcal{D}_{t_a k - m,\hat{\mu}}^{(a)*}$ *for* $0 \le m \le t_a k$,
- *(iii)* $\mathcal{D}_{t_a k,\hat{\mu}}^{(a)} = e^{-2\pi i (\tau_a \hat{\omega}_0 | \mu)}$,
- *(iv)* $\mathcal{D}_{t_a k+1,\hat{\mu}}^{(a)} = \mathcal{D}_{t_a k+2,\hat{\mu}}^{(a)} = \cdots = \mathcal{D}_{t_a(k+h^\vee)-1,\hat{\mu}}^{(a)} = 0$.

Proof. From the assumption that $\mathrm{qdim}_{\hat{\mu}}(R_m^{(a)}) = \mathcal{D}_{m,\hat{\mu}}^{(a)} \neq 0$ for all $0 \le m \le \lfloor \frac{t_a k}{2} \rfloor$, we have $\mathrm{qdim}_{\hat{\mu}}(R_m^{(a)}) \neq 0$ for all $0 \le m \le t_a k$ by Theorem 5.5 and Proposition 2.2. Thus we can conclude $\mathrm{qdim}_{\hat{\mu}}(R_m^{(a)})$ must be equal to $\mathcal{D}_{m,\hat{\mu}}^{(a)}$ for $0 \le m \le t_a k$ by Lemma 6.1. This proves (i), (ii) and (iii).

Since $\mathcal{D}^{(a)}_{t_ak-1,\hat{\mu}} \neq 0$ for each $a \in I$ by (ii) and $\left(\mathcal{D}^{(a)}_{t_ak,\hat{\mu}}\right)_{a\in I}$ is a solution of the boundary Q-system, we get

$$\mathcal{D}^{(a)}_{t_ak+1,\hat{\mu}} = \cdots = \mathcal{D}^{(a)}_{t_a(k+1),\hat{\mu}} = 0, \quad a \in I$$

by Lemma 4.6. Since X has a vertex listed in Table 7, there exists $b \in I$ such that

$$\mathcal{D}^{(b)}_{t_ak+1,\hat{\mu}} = \mathcal{D}^{(b)}_{t_ak+2,\hat{\mu}} = \cdots = \mathcal{D}^{(b)}_{t_a(k+h^\vee)-1,\hat{\mu}} = 0$$

by Theorem 3.10 and Theorem 3.4 (iv). Thus the assumptions of Lemma 4.7 are now all satisfied and we can conclude that

$$\mathcal{D}^{(a)}_{t_ak+1,\hat{\mu}} = \mathcal{D}^{(a)}_{t_ak+2,\hat{\mu}} = \cdots = \mathcal{D}^{(a)}_{t_a(k+h^\vee)-1,\hat{\mu}} = 0$$

for any $a \in I$. We thus have proved (iv). □

Now we have a proof of the KNS conjecture [KNS11, Conjecture 14.2] for all classical types.

Corollary 6.3. *Let X be a Dynkin diagram of types A, B, C and D. For each $a \in I$, the following properties hold :*

(i) $\mathcal{D}^{(a)}_m > 0$ *for* $0 \leq m \leq t_ak$,
(ii) $\mathcal{D}^{(a)}_m = \mathcal{D}^{(a)}_{t_ak-m}$ *for* $0 \leq m \leq t_ak$,
(iii) $\mathcal{D}^{(a)}_{t_ak} = 1$,
(iv) $\mathcal{D}^{(a)}_{t_ak+1} = \mathcal{D}^{(a)}_{t_ak+2} = \cdots = \mathcal{D}^{(a)}_{t_a(k+h^\vee)-1} = 0$,
(v) $\mathcal{D}^{(a)}_{m-1} < \mathcal{D}^{(a)}_m$ *for* $1 \leq m \leq \lfloor \frac{t_ak}{2} \rfloor$.

Proof. We know that $\mathcal{D}^{(a)}_m = \operatorname{qdim} R^{(a)}_m > 0$ for all $0 \leq m \leq \lfloor \frac{t_ak}{2} \rfloor$. Then by Theorem 6.2 (i), we have $\mathcal{D}^{(a)}_m = \operatorname{qdim} R^{(a)}_m > 0$ for all $0 \leq m \leq t_ak$. Now all the properties (i)-(iv) follow from Theorem 6.2 as a special case when $\hat{\mu} = k\hat{\omega}_0$.

Now we prove the inequality part (v). For $a, b \in I$, $1 \leq m \leq t_ak - 1$ and $1 \leq n \leq t_bk - 1$, or equivalently $(a,m),(b,n) \in \mathring{H}_k$, let

$$K^{m,n}_{a,b} = (\alpha_a|\alpha_b)\left(\min(t_bm, t_an) - \frac{mn}{k}\right).$$

Since $(K^{m,n}_{a,b})_{(a,m),(b,n)\in\mathring{H}_k}$ is a positive definite symmetric matrix, the system of equations

$$f^{(a)}_m = \prod_{(b,n)\in\mathring{H}_k}\left(1 - f^{(b)}_n\right)^{K^{m,n}_{a,b}}$$

has a unique positive solution $\left(f^{(a)}_m\right)_{(a,m)\in\mathring{H}_k}$ such that $0 < f^{(a)}_m < 1$ and it can be written as

$$1 - f^{(a)}_m = x^{(a)}_m = \frac{\mathcal{D}^{(a)}_{m-1}\mathcal{D}^{(a)}_{m+1}}{(\mathcal{D}^{(a)}_m)^2}, \quad (a,m) \in \mathring{H}_k$$

by (14.42),(14.45) and (14.46) of [KNS11].

Let $s = \lfloor \frac{t_ak}{2} \rfloor$ for the rest of the argument. If t_ak is odd, by the symmetry of solutions, we have $\mathcal{D}^{(a)}_{s+1} = \mathcal{D}^{(a)}_s$. Then from $x^{(a)}_s < 1$, we have

$$x^{(a)}_s = \frac{\mathcal{D}^{(a)}_{s-1}\mathcal{D}^{(a)}_{s+1}}{\mathcal{D}^{(a)}_s\mathcal{D}^{(a)}_s} = \frac{\mathcal{D}^{(a)}_{s-1}}{\mathcal{D}^{(a)}_s} < 1.$$

If $t_a k$ is even, again the symmetry condition implies $\mathcal{D}_{s+1}^{(a)} = \mathcal{D}_{s-1}^{(a)}$. Using this, we get

$$x_s^{(a)} = \frac{\mathcal{D}_{s-1}^{(a)}\mathcal{D}_{s+1}^{(a)}}{\mathcal{D}_s^{(a)}\mathcal{D}_s^{(a)}} = \frac{(\mathcal{D}_{s-1}^{(a)})^2}{(\mathcal{D}_s^{(a)})^2} = (\frac{\mathcal{D}_{s-1}^{(a)}}{\mathcal{D}_s^{(a)}})^2 < 1.$$

In both cases, we obtain the inequality $\frac{\mathcal{D}_{s-1}^{(a)}}{\mathcal{D}_s^{(a)}} < 1$.

Then the fact that

$$0 < x_m^{(a)} = \frac{\mathcal{D}_{m-1}^{(a)}\mathcal{D}_{m+1}^{(a)}}{(\mathcal{D}_m^{(a)})^2} < 1, \quad 1 \leq m \leq t_a k - 1$$

implies

$$\frac{\mathcal{D}_0^{(a)}}{\mathcal{D}_1^{(a)}} < \frac{\mathcal{D}_1^{(a)}}{\mathcal{D}_2^{(a)}} < \cdots < \frac{\mathcal{D}_{s-1}^{(a)}}{\mathcal{D}_s^{(a)}} < 1.$$

This proves the inequlity

$$\mathcal{D}_0^{(a)} < \mathcal{D}_1^{(a)} < \cdots < \mathcal{D}_s^{(a)}.$$

□

Corollary 6.4. *Let $\mathcal{A}_m^{(a)}$ be the admissibility matrix in Definition 5.7 for each $(a, m) \in H_k$. Under the same assumptions as in Theorem 6.2, $\mathcal{D}_{m,\hat{\mu}}^{(a)}$ is an eigenvalue of $\mathcal{A}_m^{(a)}$ and in particular, $\mathcal{D}_m^{(a)}$ is the Perron-Frobenius eigenvalue of it.*

Proof. It follows from Theorem 6.2 and Proposition 2.3. □

Appendix A. Proof of Proposition 3.8

For each $a \in I$ in Table 7, we will construct a sequence of positive roots $\beta_{h^\vee-1} = \theta, \beta_{h^\vee-2}, \cdots, \beta_1 = \alpha_a$ such that $(\omega_a|\beta_l) = 1$ and $(\rho|\beta_l) = l$.

Type A_r. We can use mathematical induction on r. The statement is true for A_1. Suppose that $r > 1$. Let $\beta_{h^\vee-1} = \theta$ be the highest root $\alpha_1 + \cdots + \alpha_r$. We have $(\omega_a|\beta_{h^\vee-1}) = 1$ and $(\rho|\beta_{h^\vee-1}) = h^\vee - 1$. We can choose a simple root $\alpha_j \neq \alpha_a$ such that $(\beta_{h^\vee-1}|\alpha_j) = 1$. Note that the only possible choices are $\alpha_j = \alpha_1$ or $\alpha_j = \alpha_r$. Then $\Pi - \{\alpha_j\}$ forms a simple system of type A_{r-1} with the highest root $\beta_{h^\vee-2} = \beta_{h^\vee-1} - \alpha_j$. Using induction hypothesis, we can construct a sequence of roots $\beta_{h^\vee-2}, \cdots, \beta_1 = \alpha_a$ satisfying the conditions $(\omega_a|\beta_l) = 1$ and $(\rho|\beta_l) = l$. Thus we have constructed a sequence of roots with desired properties.

Type B_r ($r \geq 2$). We have only one vertex $a = 1$. For $1 \leq l \leq r$, we define $\beta_{h^\vee - l}$ as follows :

$$\begin{aligned}
\beta_{h^\vee-1} &= \theta = \alpha_1 + 2\sum_{j=2}^{r} \alpha_j, \\
\beta_{h^\vee-2} &= s_2\beta_{h^\vee-1} = \alpha_1 + \alpha_2 + 2\sum_{j=3}^{r} \alpha_j, \\
\beta_{h^\vee-3} &= s_3\beta_{h^\vee-2} = \alpha_1 + \alpha_2 + \alpha_3 + 2\sum_{j=4}^{r} \alpha_j, \\
&\cdots, \\
\beta_{h^\vee-(r-1)} &= s_{r-1}\beta_{h^\vee-(r-2)} = \sum_{j=1}^{r-1} \alpha_j + 2\alpha_r, \\
\beta_{h^\vee-r} &= s_r\beta_{h^\vee-(r-1)} = \sum_{j=1}^{r-1} \alpha_j.
\end{aligned}$$

One can see that the conditions $(\omega_1|\beta_l) = 1$ and $(\rho|\beta_l) = l$ are satisfied for each $1 \leq l \leq r$. Note that $\Pi - \{\alpha_r\}$ forms a simple system of type A_{r-1} with the highest root $\beta_{h^\vee - r} = \beta_{r-1}$, we can now imitate the construction for type A_{r-1} to define the rest of the terms $\beta_{r-2}, \cdots, \beta_1 = \alpha_1$.

Type C_r ($r \geq 2$). Let $a = r$. For $1 \leq l \leq r$, we define $\beta_{h^\vee - l}$ as follows :

$$\begin{aligned}
\beta_{h^\vee-1} &= \theta = \alpha_r + 2\sum_{j=1}^{r-1} \alpha_j, \\
\beta_{h^\vee-2} &= s_1\beta_{h^\vee-1} = \alpha_r + 2\sum_{j=2}^{r-1} \alpha_j, \\
&\cdots, \\
\beta_{h^\vee-(r-1)} &= s_{r-2}\beta_{h^\vee-(r-2)} = \alpha_r + 2\alpha_{r-1}, \\
\beta_{h^\vee-r} &= s_{r-1}\beta_{h^\vee-(r-1)} = \alpha_r.
\end{aligned}$$

One can check that the conditions $(\omega_r|\beta_l) = 1$ and $(\rho|\beta_l) = l$ are satisfied for $1 \leq l \leq r = h^\vee - 1$.

Type D_r ($r \geq 4$). We assume that $a \in \{1, r-1, r\}$. For $1 \leq l \leq r-2$, we define $\beta_{h^\vee - l}$ as follows :

$$\begin{aligned}
\beta_{h^\vee-1} &= \theta = \alpha_1 + \alpha_{r-1} + \alpha_r + 2\sum_{j=2}^{r-2} \alpha_j, \\
\beta_{h^\vee-2} &= s_2 \beta_{h^\vee-1} = \alpha_1 + \alpha_2 + \alpha_{r-1} + \alpha_r + 2\sum_{j=3}^{r-2} \alpha_j, \\
&\cdots, \\
\beta_{h^\vee-(r-3)} &= s_{r-3}\beta_{h^\vee-(r-3)} = \sum_{j=1}^{r} \alpha_j + \alpha_{r-2}, \\
\beta_{h^\vee-(r-2)} &= s_{r-2}\beta_{h^\vee-(r-3)} = \sum_{j=1}^{r} \alpha_j.
\end{aligned}$$

Note that the conditions $(\omega_a|\beta_l) = 1$ and $(\rho|\beta_l) = l$ are satisfied for $1 \leq l \leq r-2$. To define the next term $\beta_{h^\vee-(r-1)} = \beta_{r-1}$, choose the vertex $j \in \{r-1, r\}$ such that $j \neq a$ and let $\beta_{r-1} = \beta_{h^\vee-(r-2)} - \alpha_j$. Since $\Pi - \{\alpha_j\}$ forms a simple system of type A_{r-1} with the highest root β_{r-1}, we can now use the construction for type A_{r-1} to define the rest of the terms $\beta_{r-2}, \cdots, \beta_1 = \alpha_a$.

Type E_6. Let $a \in \{1, 5\}$. For $1 \leq l \leq 4$, we define $\beta_{h^\vee-l}$ as follows :

$$\begin{aligned}
\beta_{h^\vee-1} &= \theta = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6, \\
\beta_{h^\vee-2} &= s_6\beta_{h^\vee-1} = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \\
\beta_{h^\vee-3} &= s_3\beta_{h^\vee-2} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \\
\beta_{h^\vee-4} &= \begin{cases} s_4\beta_{h^\vee-3} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 & \text{if } a = 1 \\ s_2\beta_{h^\vee-3} = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 & \text{if } a = 5 \end{cases}
\end{aligned}$$

Note that the conditions $(\omega_a|\beta_l) = 1$ and $(\rho|\beta_l) = l$ are satisfied for $1 \leq l \leq 4$. In order to define the next term $\beta_{h^\vee-5} = \beta_7$, choose the vertex $j \in \{1, 5\}$ such that $j \neq a$ and let

$$\beta_{h^\vee-5} = s_j\beta_{h^\vee-4} = \begin{cases} \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_6 & \text{if } a = 1 \\ \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 & \text{if } a = 5 \end{cases}.$$

Since $\Pi - \{\alpha_j\}$ forms a simple system of type D_5 with the highest root $\beta_{h^\vee-5} = \beta_7$, we can now use the construction for type D_5 to define the rest of the sequence $\beta_6, \cdots, \beta_1 = \alpha_a$.

Type E_7. We have only one vertex $a = 6$. For $1 \le l \le r$, we define $\beta_{h^\vee - l}$ by

$$\begin{aligned}
\beta_{h^\vee-1} &= \theta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + 2\alpha_7,\\
\beta_{h^\vee-2} &= s_1\beta_{h^\vee-1} = \alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + 2\alpha_7,\\
\beta_{h^\vee-3} &= s_2\beta_{h^\vee-2} = \alpha_1 + 2\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + 2\alpha_7,\\
\beta_{h^\vee-4} &= s_3\beta_{h^\vee-3} = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + 2\alpha_7,\\
\beta_{h^\vee-5} &= s_4\beta_{h^\vee-4} = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + 2\alpha_7,\\
\beta_{h^\vee-6} &= s_7\beta_{h^\vee-5} = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7,\\
\beta_{h^\vee-7} &= s_3\beta_{h^\vee-6} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7,\\
\beta_{h^\vee-8} &= s_2\beta_{h^\vee-7} = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7,\\
\beta_{h^\vee-9} &= s_1\beta_{h^\vee-8} = \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7.
\end{aligned}$$

One can see that the conditions $(\omega_1|\beta_l) = 1$ and $(\rho|\beta_l) = l$ are satisfied for $1 \le l \le r$. Since $\Pi - \{\alpha_1\}$ forms a simple system of type D_6 with the highest root $\beta_{h^\vee-9} = \beta_9$, we now imitate the construction of type D_6 to define the rest of the sequence $\beta_8, \cdots, \beta_1 = \alpha_a$.

Appendix B. Solutions of level k restricted Q-systems

Let $(a, m) \in H$. We describe the decomposition (2.15) of $W_m^{(a)}$ for all classical types. See [HKO+99, Appendix] for a reference. In these cases, $Z(a, m, \omega)$ is 0 or 1 for each $\omega \in P_+$. Hence it is enough to describe the set $\Omega_m^{(a)}$ of weights defined by

$$\Omega_m^{(a)} := \{\omega \in P_+ | Z(a, m, \omega) = 1\}.$$

We will give it in a form by which we can determine the coefficient of $\hat{\omega}_0$ easily when we extend $\omega \in \Omega_m^{(a)}$ to an affine weight of level k.

Type A_r. For $1 \le a \le r$,

$$\Omega_m^{(a)} = \{m\omega_a\}. \tag{B.1}$$

For $\omega \in \Omega_m^{(a)}$ in (B.1), its level k affinization $\hat{\omega}$ is given by

$$\hat{\omega} = (k - m)\hat{\omega}_0 + m\hat{\omega}_a$$

and it is clear that $\hat{\omega} \in \hat{P}_+^k$ when $(a, m) \in H_k$.

Type B_r. For $a \in I$ even,

$$\omega \in \Omega_m^{(a)} \iff \begin{cases} \omega = k_a\omega_a + k_{a-2}\omega_{a-2} + \cdots + k_2\omega_2 \\ k_a + t_a(k_{a-2} + \cdots + k_2 + k_0) = m \\ k_a, k_{a-2}, \cdots, k_2, k_0 \in \mathbb{Z}_{\ge 0} \end{cases}. \tag{B.2}$$

For $\omega \in \Omega_m^{(a)}$ in (B.2), its level k affinization $\hat{\omega}$ is given by

$$\hat{\omega} = k_a\hat{\omega}_a + k_{a-2}\hat{\omega}_{a-2} + \cdots + k_2\hat{\omega}_2 + \hat{k}_0\hat{\omega}_0$$

where

$$\hat{k}_0 = \begin{cases} k - 2m + 2k_0 & \text{if } 1 \le a \le r-1 \\ k - m + 2k_0 & \text{if } a = r \end{cases}. \tag{B.3}$$

For $a \in I$ odd,

$$\omega \in \Omega_m^{(a)} \iff \begin{cases} \omega = k_a\omega_a + k_{a-2}\omega_{a-2} + \cdots + k_1\omega_1 \\ k_a + t_a(k_{a-2} + \cdots + k_1) = m \\ k_a, k_{a-2}, \cdots, k_1 \in \mathbb{Z}_{\geq 0} \end{cases} . \tag{B.4}$$

For $\omega \in \Omega_m^{(a)}$ in (B.4), we get

$$\hat{\omega} = k_a\hat{\omega}_a + k_{a-2}\hat{\omega}_{a-2} + \cdots + k_1\hat{\omega}_1 + \hat{k}_0\hat{\omega}_0$$

where

$$\hat{k}_0 = \begin{cases} k - 2m + k_1 & \text{if } 1 \leq a \leq r-1 \\ k - m + k_1 & \text{if } a = r \end{cases} . \tag{B.5}$$

(B.3) and (B.5) show that $\hat{\omega} \in \hat{P}_+^k$ for any $a \in I$ and $0 \leq m \leq \lfloor \frac{t_a k}{2} \rfloor$.

type C_r. For $1 \leq a \leq r-1$,

$$\omega \in \Omega_m^{(a)} \iff \begin{cases} \omega = k_a\omega_a + k_{a-1}\omega_{a-1} + \cdots + k_1\omega_1 \\ k_a + k_{a-1} + \cdots + k_1 + k_0 = m \\ k_b \equiv m\delta_{a,b} \pmod 2 \\ k_a, k_{a-1}, \cdots, k_1, k_0 \in \mathbb{Z}_{\geq 0} \end{cases} . \tag{B.6}$$

For $\omega \in \Omega_m^{(a)}$ in (B.6), the corresponding affine weight is

$$\hat{\omega} = k_a\hat{\omega}_a + k_{a-1}\hat{\omega}_{a-1} + \cdots + k_1\hat{\omega}_1 + \hat{k}_0\hat{\omega}_0$$

where

$$\hat{k}_0 = k - m + k_0. \tag{B.7}$$

For $a = r$,

$$\Omega_m^{(a)} = \{m\omega_a\}$$

Again, we can observe that $\hat{\omega} \in \hat{P}_+^k$ for any $a \in I$ and $0 \leq m \leq \lfloor \frac{t_a k}{2} \rfloor$.

Type D_r. For even a such that $2 \leq a \leq r-2$,

$$\omega \in \Omega_m^{(a)} \iff \begin{cases} \omega = k_a\omega_a + k_{a-2}\omega_{a-2} + \cdots + k_2\omega_2 \\ k_a + k_{a-2} + \cdots + k_2 + k_0 = m \\ k_a, k_{a-2}, \cdots, k_2, k_0 \in \mathbb{Z}_{\geq 0} \end{cases} . \tag{B.8}$$

For $\omega \in \Omega_m^{(a)}$ in (B.8), the level k affinization is

$$\hat{\omega} = k_a\hat{\omega}_a + k_{a-2}\hat{\omega}_{a-2} + \cdots + k_1\hat{\omega}_1 + \hat{k}_0\hat{\omega}_0$$

where

$$\hat{k}_0 = k - 2m + 2k_0.$$

For odd a such that $1 \leq a \leq r-2$,

$$\omega \in \Omega_m^{(a)} \iff \begin{cases} \omega = k_a\omega_a + k_{a-2}\omega_{a-2} + \cdots + k_1\omega_1 \\ k_a + k_{a-2} + \cdots + k_1 + k_0 = m \\ k_a, k_{a-2}, \cdots, k_1, k_0 \in \mathbb{Z}_{\geq 0} \end{cases} . \tag{B.9}$$

For $\omega \in \Omega_m^{(a)}$ in (B.9), we get

$$\hat{\omega} = k_a\hat{\omega}_a + k_{a-2}\hat{\omega}_{a-2} + \cdots + k_1\hat{\omega}_1 + \hat{k}_0\hat{\omega}_0$$

where

$$\hat{k}_0 = k - 2m + k_0 + k_1.$$

For $a = r-1$ and r, we have $\Omega_m^{(a)} = \{m\omega_a\}$.

Note that $\hat{\omega} \in \hat{P}_+^k$ for any $a \in I$ and $0 \le m \le \lfloor \frac{k}{2} \rfloor$.

Appendix C. Proof of Lemma 5.3 for type B_r

Let $\hat{\Omega}_m^{(a)} = \{\hat{\omega} \in \hat{P}^k | \omega \in \Omega_m^{(a)}\}$. For the rest of this section, we will denote the element $k_a\hat{\omega}_a + k_{a-2}\hat{\omega}_{a-2} + \cdots + k_2\hat{\omega}_2 + \hat{k}_0\hat{\omega}_0 \in \hat{P}^k$ by $(k_a, k_{a-2}, \cdots, k_2, \hat{k}_0)$ when a is even and $k_a\hat{\omega}_a + k_{a-2}\hat{\omega}_{a-2} + \cdots + k_1\hat{\omega}_1 + \hat{k}_0\hat{\omega}_0 \in \hat{P}^k$ by $(k_a, k_{a-2}, \cdots, k_1, \hat{k}_0)$ when a is odd.

The case of the vertex $a = r$.

Proposition C.1. *Let r be even and $a = r$. Then $W_{k+1}^{(a)} = \tau_a(W_{k-1}^{(a)*}) = W_{k-1}^{(a)}$.*

Proof. Note that $\hat{\Omega}_{k-1}^{(a)} \subseteq \hat{\Omega}_{k+1}^{(a)}$ and

$$\hat{\Omega}_{k+1}^{(a)} \setminus \hat{\Omega}_{k-1}^{(a)} = \left\{ (k_a, k_{a-2}, \cdots, k_2, k_0) \in \hat{P}^k \mid \begin{array}{l} k_a + 2(k_{a-2} + \cdots + k_2) = k+1 \\ k_a, k_{a-2}, \cdots, k_2 \in \mathbb{Z}_{\ge 0} \end{array} \right\}.$$

If $\hat{\omega} = (k_a, k_{a-2}, \cdots, k_2, \hat{k}_0) \in \hat{\Omega}_{k+1}^{(a)} \setminus \hat{\Omega}_{k-1}^{(a)}$, then $\hat{k}_0 = -1$ by (B.3) and thus $V_{\hat{\omega}} = 0$ since $s_0 \cdot \hat{\omega} = \hat{\omega}$. This proves that $W_{k+1}^{(a)} = W_{k-1}^{(a)}$. □

Proposition C.2. *Let r be odd and $a = r$. Then, $W_{k+1}^{(a)} = \tau_a(W_{k-1}^{(a)*})$.*

Proof. For any $\omega = (k_a, k_{a-2}, \cdots, k_1, k_0) \in \hat{\Omega}_{k+1}^{(a)}$ with $k_1 = 0$, we get $\hat{k}_0 = -1$ by (B.5) and thus $V_{\hat{\omega}} = 0$.

Let

$$(\hat{\Omega}_{k+1}^{(a)})' = \{(k_a, k_{a-2}, \cdots, k_1, \hat{k}_0) \in \hat{\Omega}_{k+1}^{(a)} \mid k_1 \ge 1\}.$$

Then we can write $W_{k+1}^{(a)} = \sum_{\omega \in (\hat{\Omega}_{k+1}^{(a)})'} V_{\hat{\omega}}$.

Let us define a map from $(\hat{\Omega}_{k+1}^{(a)})'$ to $\hat{\Omega}_{k-1}^{(a)}$ by

$$(k_a, k_{a-2}, \cdots, k_1, \hat{k}_0) \mapsto (k_a, k_{a-2}, \cdots, \hat{k}_0, k_1). \tag{C.1}$$

For $(k_a, k_{a-2}, \cdots, k_1, \hat{k}_0) \in (\hat{\Omega}_{k+1}^{(a)})'$, we get $\hat{k}_0 = k_1 - 1 \ge 0$ by (B.5). This shows that $(k_a, k_{a-2}, \cdots, \hat{k}_0, k_1) \in \hat{\Omega}_{k-1}^{(a)}$ and thus the map (C.1) is well-defined. It is clear that this is injective.

Conversely, any element $(k_a, k_{a-2}, \cdots, k_1, \hat{k}_0) \in \hat{\Omega}_{k-1}^{(a)}$ satisfies $\hat{k}_0 = k_1 + 1 \ge 1$ again by (B.5) and it proves that (C.1) is surjective and thus bijective. This proves our proposition. □

The case of the vertices $1 \le a \le r-1$ when k is odd. Let $s = \frac{k-1}{2}$.

Proposition C.3. *If a is even and $1 \le a \le r-1$, then $W_s^{(a)} = W_{s+1}^{(a)}$.*

Proof. Note that $\hat{\Omega}_s^{(a)} \subseteq \hat{\Omega}_{s+1}^{(a)}$ and

$$\hat{\Omega}_{s+1}^{(a)} \setminus \hat{\Omega}_s^{(a)} = \left\{ (k_a, k_{a-2}, \cdots, k_2, \hat{k}_0) \in \hat{P}^k \mid \begin{array}{l} k_a + k_{a-2} + \cdots + k_2 = s+1 \\ k_a, k_{a-2}, \cdots, k_2 \in \mathbb{Z}_{\ge 0} \end{array} \right\}.$$

If $\hat{\omega} = (k_a, k_{a-2}, \cdots, k_2, \hat{k}_0) \in \hat{\Omega}_{s+1}^{(a)} \setminus \hat{\Omega}_s^{(a)}$, then $\hat{k}_0 = -1$. So for any $\hat{\omega} \in \hat{\Omega}_{s+1}^{(a)} \setminus \hat{\Omega}_s^{(a)}$, $V_{\hat{\omega}} = 0$ since $s_0 \cdot \hat{\omega} = \hat{\omega}$. Thus $W_{s+1}^{(a)} = W_s^{(a)}$. □

Proposition C.4. *If a is odd and $1 \le a \le r-1$, then $W^{(a)}_{s+1} = \tau_a(W^{(a)*}_s)$.*

Proof. Let $\hat{\omega} = (k_a, k_{a-2}, \cdots, k_1, \hat{k}_0) \in \hat{\Omega}^{(a)}_{s+1}$. If $k_1 = 0$, then $\hat{k}_0 = -1$ by (B.5) and thus $V_{\hat{\omega}} = 0$ since $s_0 \cdot \hat{\omega} = \hat{\omega}$.

Let

$$(\hat{\Omega}^{(a)}_{s+1})' = \{(k_a, k_{a-2}, \cdots, k_1, \hat{k}_0) \in \hat{\Omega}^{(a)}_{s+1} \mid k_1 \ge 1\}.$$

Let us construct a bijection between $(\hat{\Omega}^{(a)}_{s+1})'$ and $\hat{\Omega}^{(a)}_s$. Define a map from $(\hat{\Omega}^{(a)}_{s+1})'$ to $\hat{\Omega}^{(a)}_s$ by

$$(k_a, k_{a-2}, \cdots, k_1, \hat{k}_0) \mapsto (k_a, k_{a-2}, \cdots, \hat{k}_0, k_1). \tag{C.2}$$

To see that the map is well-defined, note that if $(k_a, k_{a-2}, \cdots, k_1, \hat{k}_0) \in (\hat{\Omega}^{(a)}_{s+1})'$, then $\hat{k}_0 = k_1 - 1 \ge 0$ by (B.5). Since

$$k_a + k_{a-2} + \cdots + k_3 + \hat{k}_0 = k_a + k_{a-2} + \cdots + k_3 + (k_1 - 1) = s,$$

we have $(k_a, k_{a-2}, \cdots, \hat{k}_0, k_1) \in \hat{\Omega}^{(a)}_s$. The map (C.2) is clearly injective.

Conversely, any element $(k_a, k_{a-2}, \cdots, k_1, \hat{k}_0) \in \hat{\Omega}^{(a)}_s$ satisfies the condition $\hat{k}_0 = k_1 + 1 \ge 1$ which shows that (C.2) is surjective. We thus have proved that (C.2) is a bijection between $(\hat{\Omega}^{(a)}_{s+1})'$ and $\hat{\Omega}^{(a)}_s$. This proves our assertion. □

The case of the vertices $1 \le a \le r-1$ when k is even. Let $s = \frac{k}{2}$.

Lemma C.5. *Let a be even and $1 \le a \le r-1$. If $\hat{\omega} = (k_a, k_{a-2}, \cdots, k_2, \hat{k}_0) \in \hat{P}^k$ satisfies $k_2 = 0$ and $\hat{k}_0 = -2$, then $V_{\hat{\omega}} = 0$.*

Proof. From $(s_0 s_2 s_0) \cdot (k_a, k_{a-2}, \cdots, 0, -2) = (k_a, k_{a-2}, \cdots, 0, -2)$, we can deduce that $V_{\hat{\omega}} = 0$. □

Proposition C.6. *If a is even and $1 \le a \le r-1$, then $W^{(a)}_{s+1} = W^{(a)}_{s-1}$.*

Proof. Recall that

$$\hat{\Omega}^{(a)}_{s-1} = \left\{ (k_a, k_{a-2}, \cdots, k_2, \hat{k}_0) \in \hat{P}^k \mid \begin{array}{l} k_a + k_{a-2} + \cdots + k_2 \le s-1 \\ k_a, k_{a-2}, \cdots, k_2 \in \mathbb{Z}_{\ge 0} \end{array} \right\}$$

and

$$\hat{\Omega}^{(a)}_{s+1} = \left\{ (k_a, k_{a-2}, \cdots, k_2, \hat{k}_0) \in \hat{P}^k \mid \begin{array}{l} k_a + k_{a-2} + \cdots + k_2 \le s+1 \\ k_a, k_{a-2}, \cdots, k_2 \in \mathbb{Z}_{\ge 0} \end{array} \right\}.$$

Let us define three disjoint subsets R, S and T of $\hat{\Omega}^{(a)}_{s+1}$ by

$$R = \{(k_a, k_{a-2}, \cdots, k_2, \hat{k}_0) \in \hat{\Omega}^{(a)}_{s+1} | k_a + k_{a-2} + \cdots + k_2 = s+1, k_2 = 0\},$$
$$S = \{(k_a, k_{a-2}, \cdots, k_2, \hat{k}_0) \in \hat{\Omega}^{(a)}_{s+1} | k_a + k_{a-2} + \cdots + k_2 = s+1, k_2 \ge 1\},$$
$$T = \{(k_a, k_{a-2}, \cdots, k_2, \hat{k}_0) \in \hat{\Omega}^{(a)}_{s+1} | k_a + k_{a-2} + \cdots + k_2 = s\}.$$

For $\hat{\omega} \in R$, we get $\hat{k}_0 = -2$. Thus $V_{\hat{\omega}} = 0$ by Lemma C.5 and so $\sum_{\hat{\omega} \in R} V_{\hat{\omega}} = 0$. We now want to prove $\sum_{\hat{\omega} \in S \cup T} V_{\hat{\omega}} = 0$. By (B.3), if $(k_a, k_{a-2}, \cdots, k_2, \hat{k}_0) \in S$, then $\hat{k}_0 = -2$ and if $(k_a, k_{a-2}, \cdots, k_2, \hat{k}_0) \in T$, then $\hat{k}_0 = 0$. We have a bijection between S and T since

$$s_0 \cdot (k_a, k_{a-2}, \cdots, k_2, -2) = (k_a, k_{a-2}, \cdots, k_2 - 1, 0).$$

By (2.11), $\sum_{\hat{\omega}\omega\in S\cup T} V_{\hat{\omega}} = 0$. Consequently,

$$W_{s+1}^{(a)} = \sum_{\hat{\omega}\in\hat{\Omega}_{s+1}^{(a)}} V_{\hat{\omega}} = \sum_{\hat{\omega}\in\hat{\Omega}_{s+1}^{(a)}\setminus(R\cup S\cup T)} V_{\hat{\omega}}.$$

From $\hat{\Omega}_{s-1}^{(a)} = \hat{\Omega}_{s+1}^{(a)} \setminus (R\cup S\cup T)$, we obtain $W_{s+1}^{(a)} = W_{s-1}^{(a)}$. □

Lemma C.7. *Let a be odd and $1 \le a \le r-1$. If $\hat{\omega} = (k_a, k_{a-2}, \cdots, 1, -1) \in \hat{P}^k$ or $\hat{\omega} = (k_a, k_{a-2}, \cdots, 0, -2) \in \hat{P}^k$, then $V_{\hat{\omega}} = 0$.*

Proof. It suffices to check that

$$s_0 \cdot (k_a, k_{a-2}, \cdots, 1, -1) = (k_a, k_{a-2}, \cdots, 1, -1)$$

and

$$(s_0 s_2 s_0) \cdot (k_a, k_{a-2}, \cdots, 0, -2) = (k_a, k_{a-2}, \cdots, 0, -2).$$

□

Proposition C.8. *If a is odd and $1 \le a \le r-1$, then $W_{s+1}^{(a)} = \tau_a(W_{s-1}^{(a)*})$.*

Proof. For any $\hat{\omega} = (k_a, k_{a-2}, \cdots, k_1, \hat{k}_0) \in \hat{\Omega}_{s+1}^{(a)}$ with $k_1 = 0$ or $k_1 = 1$, $V_{\hat{\omega}} = 0$ by Lemma C.7. Let $(\hat{\Omega}_{s+1}^{(a)})' = \{(k_a, k_{a-2}, \cdots, k_1, \hat{k}_0) \in \hat{\Omega}_{s+1}^{(a)} \mid k_1 \ge 2\}$. Then we can write $z_{s+1}^{(a)} = \sum_{\hat{\omega}\in(\hat{\Omega}_{s+1}^{(a)})'} V_{\hat{\omega}}$.

Let us define a map from $(\hat{\Omega}_{s+1}^{(a)})'$ to $\hat{\Omega}_{s-1}^{(a)}$ by

$$(k_a, k_{a-2}, \cdots, k_1, \hat{k}_0) \mapsto (k_a, k_{a-2}, \cdots, \hat{k}_0, k_1). \tag{C.3}$$

For $(k_a, k_{a-2}, \cdots, k_1, \hat{k}_0) \in (\hat{\Omega}_{s+1}^{(a)})'$, $\hat{k}_0 = k_1 - 2 \ge 0$. It shows that

$$(k_a, k_{a-2}, \cdots, \hat{k}_0, k_1) \in \hat{\Omega}_{s-1}^{(a)}$$

and thus the map (C.3) is well-defined. It is clear that this is injective.

Conversely, any element $(k_a, k_{a-2}, \cdots, k_1, \hat{k}_0) \in \hat{\Omega}_{s-1}^{(a)}$ satisfies $\hat{k}_0 = k_1 + 2 \ge 2$. It proves that (C.3) is surjective and thus bijective. This finishes our proof of the proposition. □

Appendix D. Proof of Lemma 5.3 for type C_r

The case of the vertices $1 \le a \le r-1$.

Proposition D.1. *Let $1 \le a \le r-1$. For $k \ge 2$, $W_{k-1}^{(a)} = W_{k+1}^{(a)}$*

Proof. Note that $\hat{\Omega}_{k-1}^{(a)} \subseteq \hat{\Omega}_{k+1}^{(a)}$ and

$$\hat{\Omega}_{k+1}^{(a)} \setminus \hat{\Omega}_{k-1}^{(a)} = \left\{ (k_a, k_{a-1}, \cdots, k_1, \hat{k}_0) \in \hat{P}^k \;\middle|\; \begin{array}{l} k_a + k_{a-1} + \cdots + k_1 = k+1 \\ k_a, k_{a-1}, \cdots, k_1 \in \mathbb{Z}_{\ge 0} \end{array} \right\}.$$

If $\hat{\omega} = (k_a, k_{a-1}, \cdots, k_1, \hat{k}_0) \in \hat{\Omega}_{k+1}^{(a)} \setminus \hat{\Omega}_{k-1}^{(a)}$, then $\hat{k}_0 = -1$ by (B.7). So for any $\hat{\omega} \in \hat{\Omega}_{k+1}^{(a)} \setminus \hat{\Omega}_{k-1}^{(a)}$, $V_{\hat{\omega}} = 0$ since $s_0 \cdot \hat{\omega} = \hat{\omega}$. Thus $W_{k+1}^{(a)} = W_{k-1}^{(a)}$. □

The case of the vertex $a = r$. It follows from Theorem 3.10.

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