Infinite Series Expansion of Subdivision Surfaces

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Abstract. Here is the abstract

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1 Introduction

1.1 Previous Work

1.2 Overview

2 Basic Idea

For simplicity, we use $I(X)$ to represent the interpolation surface of mesh X, $S(X)$ to represent the limit surface of X and $L(X)$ to represent all the limit points of X. For a given control mesh M, we need to find a smooth surface $I(M)$ that interpolates M. Suppose $S(M)$ is the limit surface of M by using some subdivision scheme, say Catmull-Clark subdivision scheme. If we can find a surface T_1 or K_1, such that

$$T_1 + S(M) = I(M)$$

or

$$K_1 * S(M) = I(M)$$

then the interpolation problem is solved. Here T_1 (or K_1) can be regarded as an offset (scaling) surface which moves (scales) $S(M)$ to $I(M)$ evrywhere. We believe T_1 and K_1 are silimar to construct, hence it is sufficient to present one of them.

The difference of $I(M)$ and $S(M)$ at the vertices of M can be calculated as follows.

$$M_1 = M - L(M).$$

Therefore $T_1 = I(M_1)$, i.e., $I(M_1)$ interpolates all the difference between $I(M)$ and $S(M)$. M_1 has the same topology as M, hence $I(M_1)$ and $I(M)$ are equally difficult to construct. However, the above process can be repeated to find a series of meshes M_i ($1 \leq i \leq \infty$) such that

$$I(M_{i+1}) + S(M_i) = I(M_i),$$

and

$$M_{i+1} = M_i - L(M_i) \quad (1)$$

Let $M = M_0$, from the above series we have

$$I(M) = \sum_{i=0}^{n} S(M_i) + I(M_{n+1}). \quad (2)$$

From eq. (1), we can get M_i easily as follows.

$$M_i = (E - A)^i M_0, \quad (3)$$

where E is the identity matrix and A is the matrix that calculates all the limit points of the given mesh M. It is easy to see (the proof is shown in the appendix) that

$$\lim_{n\to\infty} I(M_{n+1}) = \mathbf{0}.$$

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Because A is invertable (see the appendix), it also is easy to get

$$\sum_{i=0}^{n} S(M_i) = S(\sum_{i=0}^{n} M_i) = S(A^{-1}(E-(E-A)^{n+1})M_0)$$

Combining the above two equations, we have

$$I(M) = S(\sum_{i=0}^{\infty} M_i) = S(A^{-1}M_0). \quad (4)$$

If we define

$$\sum_{i=0}^{\infty} M_i = \hat{M},$$

then $\hat{M} = A^{-1}M_0$ holds as well. Hence I(M) is also a subdivision surface and $\hat{M}$ is the mesh whose limit surface interpolates the given M. Traditionally, people try to directly find $A^{-1}M_0$ by solving an linear system

[5, 12]. Hence it is difficult to deal with meshes of large number of vertices. However, with Eq. (4), $\hat{M}$ can be obtained by interatively applying eq. (1) until some given tolerance. Hence there is no problem to deal with large meshes. More importantly, just like Fourier transformation, any subdivision surface now can be represented by a summation of an infinite series of subdivision surfaces. For example, for any given mesh M, $S(M)$ can be represented with an infinite series of subdivision surfaces as follows.

$$S(M) = I(L(M)) = S(\sum_{i=0}^{\infty} L(M_i)).$$

Similar to Fourier transformation, we believe this good property can be used for a lot of applications in computer graphics and modelling, like fairing, smoothing, sharpening, lowpass or high pass filtering etc.

3 Test Results

The proposed techniques have been implemented in *C++* using *OpenGL* as the supporting graphics system on the Windows platform. Quite a few examples have been tested with the techniques described here. All the examples have extra-ordinary vertices. Some of the tested results are shown in Figures **??**, **??** and **??**. From these examples we can see smooth and visually pleasant interpolation shapes can be obtained.

4 Summary

Here is the Summary

5 Appendix

5.1 Proof of convergence of $(E - A)^i$

To prove this, we just need to show all the eigen values λ_i of A are $0 < \lambda_i \leq 1$. Here we present the proof using generalized Catmull-Clark subdivision scheme. Other shcemes can be proven similarily. For generalized Catmull-Clark subdivision scheme, new face points and new edge points are calcuated the same way as they are in a standard Catmull-Clark subdivision scheme, but the new vertex points are calculated differently using the following formula.

$$V' = \frac{n-2}{n} V + \frac{1}{n^2} \sum (\alpha V + (1-\alpha) E_j) + \frac{1}{n^2} \sum F_j,$$

where $0 \leq \alpha \leq 1$. When $\alpha = 0$, it becomes the standard Catmull-Clark subdivision scheme. The limit point of the vertex V_i of degree n_i can be calculated as follows.

$$V_i^{\infty} = \frac{1}{n_i(n_i+5)} (a_i V_i + \sum_j b_{ij} E_j + \sum_j c_{ij} F_j),$$

where

$$\begin{aligned} a_i &= (n_i - 1)n_i + n_i \alpha + \sum \tfrac{4}{d_{ij}} \\ b_{ij} &= 2 - \alpha + \tfrac{4}{d_{ij}} + \tfrac{4}{d_{ji}} \\ c_{ij} &= 4/d_{ij} \end{aligned}$$

Note that the above formula is used for a vertex whose surrounding faces might not be four-sided. Hence in the above formula, E_j are the edge points, but F_j are all the generalized faces points. d_{ij} are the number of sides of the face, of which (V_i, V_j) is an edge or a diagonal line (see figure 1). Note that d_{ij} might not equal to d_{ji} because the two faces adjacent to the edge (V_i, V_j) could be of different side. But if (V_i, V_j) is a diagonal line of a face, $d_{ij} = d_{ji}$.

It is easy to see $A_{ij} \geq 0$ and for each row, $\sum_i A_{ij} = 0$, hence $\lambda_i \leq 1$; A common coefficient $1/n_i/(n_i+5)$ can be factored out for each row of A, where n_i is the valance of vertex i in the given mesh M. As a result, A can be represented as $A = diag(1/n_i/(n_i+5)) * B$. As defined above, B then is a matrix that satisfies:
1. $B_{ii} = a_i$,
2. $B_{ij} = b_{ij}$ if (V_i, V_j) is an edge of a face,
3. $B_{ij} = c_{ij}$ if (V_i, V_j) is a diagonal line of a face,
4. $B_{ij} = 0$ if (V_i, V_j) is not in a common face,
5. B is a symetric matrix. Hence λ_i are real numbers.

To finish the proof, we just need to show the eigen values of A or B are bigger than 0, which is equavilant to prove B is positive definite. This can be achieved by proving $X^T B X > 0$ for any vector $X \neq 0$. It is easy to see this if we expand $X^T B X$ as follows.

$$X^T B X = \sum_{all\ edges} 2 b_{ij} x_i x_j + \sum_{all\ diagonals} 2 c_{ij} x_i x_j + \sum a_i x_i^2$$

$$= \sum_{all\ edges} (b_{ij} - c_{ij} - c_{ji})(x_i + x_j)^2 + \sum_{all\ faces} c_{ij} (x_i + x_j + \cdots + x_p)^2 +$$

$$\sum_i (a_i - \sum_{(V_i,V_j)\ is\ an\ edge} (b_{ij} - 2c_{ij}) - \sum_{(V_i,V_j) is\ an\ edge} c_{ij}) x_i^2$$

Let

$$\rho_i = a_i - \sum_{(V_i,V_j)\ is\ an\ edge} (b_{ij} - 2c_{ij}) - \sum_{(V_i,V_j) is\ a\ edge} c_{ij}.$$

Becasue $b_{ij} > 0$, $c_{ij} > 0$ and $b_{ij} - c_{ij} - c_{ji} = 2 - \alpha > 0$, we just need to show $\rho_i > 0$. By plugging in a_i, b_{ij} and c_{ij}, we have

$$\rho_i = n_i^2 - 3 * n_i + 2 n_i \alpha$$

Obviously, because $n_i \geq 3$ and $0 \leq \alpha \leq 1$, we have $\rho_i \geq 0$. Therefore we cannot conclude that B is positive definite yet. However if there exists at least one vertex V_i such that $\rho_i > 0$, then $X^TBX > 0$. This can be proven by contradiction. Suppose this is not the case, then there exists an $X \neq 0$ such that $X^TBX = 0$. It is easy to see that $x_i = 0$ otherwise $X^TBX \geq \rho_i x_i > 0$. In addition, all x_j where (x_i, x_j) is an edge or a diagonal line must be 0 as well otherwise $X^TBX \geq (b_{ij} - 2c_{ij})(x_i + x_j)^2 = (b_{ij} - 2c_{ij})x_j^2 > 0$. Similarly, all x_k directly or indirectly connecting to X_i are all equal to 0. Because M is a connected mesh, all x_i are 0, which contradicts $X \neq 0$. Hence our claim holds. We call the mesh, whose A is positive definite, *interpolatable* using our method. Hence to make a mesh interpolatable, we just need to choose a proper α such that all $\rho_i \geq 0$ and at least one $\rho_i > 0$.

It is easy to see for any $\alpha \in [0, 1]$, $\rho \geq 0$, and when $n_i > 3$, for any $\alpha \in [0, 1]$, $\rho > 0$. Therefore for any given mesh M with at least one vertex of degree bigger than 3, it is guaranteed that M is interpolatable using our method. Hence now the question for interpolatability only is left for meshes with all vertices being degree of three. Note that when $\alpha > 0$, $\rho > 0$. Therefore, for any mesh non-interpolatable using our method, by changing the value of α, we can make it interpolatable. For example, for a mesh M with topology of a cube, it is not interpolatable when $\alpha = 0$. But if we change the value of α, such that $\alpha > 0$, say $\alpha = 0.5$, then, M is interpolatable. Furthermore, the bigger α is, the faster the convergence is.

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