

# Swapping: a natural bridge between named and indexed explicit substitution calculi

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This article is devoted to the presentation of  $\lambda_{\text{rex}}$ , an explicit substitution calculus with *de Bruijn* indexes and a simple notation. By being isomorphic to  $\lambda_{\text{ex}}$  – a recent formalism with variable names –,  $\lambda_{\text{rex}}$  accomplishes simulation of  $\beta$ -reduction (Sim), preservation of  $\beta$ -strong normalization (PSN) and meta-confluence (MC), among other desirable properties. Our calculus is based on a novel presentation of  $\lambda_{\text{dB}}$ , using a *swap* notion that was originally devised by de Bruijn. Besides  $\lambda_{\text{rex}}$ , two other indexed calculi isomorphic to  $\lambda_{\text{x}}$  and  $\lambda_{\text{xgc}}$  are presented, demonstrating the potential of our technique when applied to the design of indexed versions of known named calculi.

## 1 Introduction

This article is devoted to explicit substitutions (ES, for short), a formalism that has attracted attention since the appearance of  $\lambda\sigma$  [1] and, later, of Mellies' counterexample [17], showing the lack of the preservation of  $\beta$ -strong normalization property (PSN, for short) in  $\lambda\sigma$ . One of the main motivations behind the field of ES is studying how substitution behaves when internalized in the language it serves (in the classic  $\lambda$ -calculus, substitution is a meta-level operation). Several calculi have been proposed since the counterexample of Mellies, and few have been shown to have a whole set of desirable properties: simulation of  $\beta$ -reduction, PSN, meta-confluence, full composition, etc. For a detailed introduction to the ES field, we refer the reader to e.g. [16, 15, 20].

In 2008, D. Kesner proposed  $\lambda_{\text{ex}}$  [14, 15], a formalism with variable names that has the entire set of properties expected from an ES calculus. As Kesner points in [15], for implementation purposes a different approach to variable names should be taken, since bound variable renaming (*i.e.*, working modulo  $\alpha$ -equivalence) is known to be error-prone and computationally expensive. Among others, one of the ways this problem is tackled is by using *de Bruijn* notation [5], which is a technique that simply avoids the need of working modulo  $\alpha$ -equivalence. As far as we know, no ES calculus with *de Bruijn* indexes and the whole set of properties enjoyed by  $\lambda_{\text{ex}}$  exists to date. The main target of this article is the introduction of  $\lambda_{\text{rex}}$ , an ES calculus with *de Bruijn* indexes that, by being isomorphic to  $\lambda_{\text{ex}}$ , enjoys the same set of properties.  $\lambda_{\text{rex}}$  is based on  $\lambda_{\text{r}}$ , a novel *swapping*-based version of the classic  $\lambda_{\text{dB}}$  [5], that we also introduce here.

It is important to remark that the whole development was made on a staged basis: we first devised  $\lambda_{\text{r}}$ , and then made substitutions explicit orienting the definition for  $\lambda_{\text{r}}$ 's meta-substitution. At that point, we got a calculus we called  $\lambda_{\text{re}}$ , which turned out to be isomorphic to  $\lambda_{\text{x}}$  [4, 3]. Encouraged by this result, we added *Garbage Collection* to  $\lambda_{\text{re}}$ , obtaining a calculus isomorphic to  $\lambda_{\text{xgc}}$  [4]:  $\lambda_{\text{re}_{\text{gc}}}$ . Finally, we added composition of substitutions in the style of  $\lambda_{\text{ex}}$  to  $\lambda_{\text{re}_{\text{gc}}}$ , obtaining  $\lambda_{\text{rex}}$ . Thus, besides fulfilling our original aim, we introduced *swapping*, a technique that turns out to behave as a natural bridge between named and indexed formalisms. Furthermore, we didn't know any indexed isomorphic versions of  $\lambda_{\text{x}}$  nor  $\lambda_{\text{xgc}}$ .

The content of the article is as follows: in Section 2 we present  $\lambda r$ , an alternative version of  $\lambda_{dB}$ . Next, in Section 3, we introduce  $\lambda re$ ,  $\lambda re_{gc}$  and  $\lambda rex$ , the three ES calculi derived from  $\lambda r$  already mentioned. We show the isomorphism between these and the  $\lambda x$ ,  $\lambda xgc$  and  $\lambda ex$  calculi in Section 4. Last, in sections 5 and 6, we point out related work and present the conclusions, respectively. We refer the interested reader to [18] for complete proofs over the whole development.

## 2 A new presentation for $\lambda_{dB}$ : the $\lambda r$ -calculus

### 2.1 Intuition

The  $\lambda$ -calculus with *de Bruijn* indexes ( $\lambda_{dB}$ , for short) [5] accomplishes the elimination of  $\alpha$ -equivalence, since  $\alpha$ -equivalent  $\lambda$ -terms are syntactically identical under  $\lambda_{dB}$ . This greatly simplifies implementations, since caring about bound variable renaming is no longer necessary. One usually refers to a *de Bruijn* indexed calculus as a *nameless calculus*, for binding is positional – relative – instead of absolute (indexes are used in place of names for this purpose). We observe here that, even though this *nameless* notion makes sense in the classical  $\lambda_{dB}$ -calculus (because the substitution operator is located in the meta-level), it seems not to be the case in certain ES calculi derived from  $\lambda_{dB}$ , such as:  $\lambda s$  [11],  $\lambda s_e$  [12] or  $\lambda t$  [13]. These calculi have constructions of the form  $a[i := b]$  to denote ES (notations vary). Here, even though  $i$  is not a name *per se*, it plays a similar role:  $i$  indicates which free variable should be substituted; then, these calculi are not purely *nameless*, *i.e.*, binding is *mixed*: positional (relative) for abstractions and named (absolute) for closures.

In general, we observe that not a single ES calculus with *de Bruijn* indexes to date is completely nameless. This assertion rests on the following observation: in each and every case, the (Lamb) rule is of the form  $(\lambda a)[s] \rightarrow \lambda a[s']$ . Thus, since the term  $a$  is not altered, an “absolute binding technique” *must* be implemented inside  $s$  in order to indicate which free variable is to be substituted. To further support this *not-completely-nameless* assertion, we note that even though there is a known isomorphism between the classic  $\lambda$ -calculus and the  $\lambda_{dB}$ -calculus, when substitutions are made explicit in both calculi, the isomorphism does not hold just by adding the new ES case (which would be reasonable to expect). The problem is that  $\lambda_{dB}$ ’s classic definition is always – tacitly, at least – being used for the explicitation task, thus obtaining calculi with mixed binding approaches, as mentioned earlier. As shown throughout the rest of the paper, our (Lamb) rules will be of the form  $(\lambda a)[s] \rightarrow \lambda a'[s']$ , *i.e.*, altering *both a and s* to enforce a *completely nameless* approach.

In order to obtain a *completely nameless* notion for an explicit substitutions  $\lambda_{dB}$ , we start by eliminating the index  $i$  from the substitution operator. Then, we are left with terms of the form  $a[b]$ , and with a (Beta) reduction rule that changes from  $(\lambda a)b \rightarrow a[1 := b]$  to  $(\lambda a)b \rightarrow a[b]$ . The semantics of  $a[b]$  should be clear from the new (Beta) rule. The problem is, of course, how to define it. Two difficulties arise when a substitution crosses (goes into) an abstraction: first, the indexes of  $b$  should be incremented in order to reflect the new variable bindings; second – and the key to our technology –, some mechanism should be implemented in order to replace the need for indexes inside closures (since these should be incremented, too).

The first problem is solved easily: we just use an operator to progressively increment indexes with every abstraction crossing, in the style of  $\lambda t$  [13]. The second issue is a bit harder. Figure 1 will help us clarify what we do when a substitution crosses an abstraction, momentarily using  $\sigma^b a$  to denote  $a[b]$  in order to emphasize the binding character of the substitution (by writing the

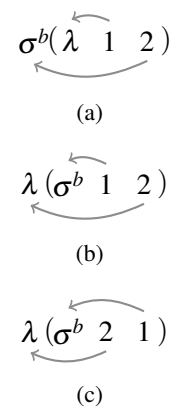


Figure 1: Bindings

substitution construction *before* the term and *annotating* it with the substituent – which does not actually affect binding –, it resembles the abstraction operation; thus, “reading” the term is much easier for those who are already familiar with *de Bruijn* notation). In this example we use the term  $\sigma^b(\lambda 1 2)$  (which stands for  $(\lambda 1 2)[b]$ ). Figure 1(a) shows the bindings in the original term; Figure 1(b) shows that bindings are inverted if we cross the abstraction and do not make any changes. Then, in order to get bindings “back on the road”, we just *swap* indexes 1 and 2! (Figure 1(c)). With this operation we recover, intuitively, the original semantics of the term. Summarizing, all that is needed when abstractions are crossed is: *swap* indexes 1 and 2 and, also, increment the indexes of the term carried in the substitution. That is exactly what  $\lambda r$  does, with substitutions in the meta-level.

In Section 2.2 we define both  $\lambda_{dB}$  and  $\lambda r$ ; in Section 2.3 we show that they are the same calculus.

## 2.2 Definitions

First of all, we define some operations on sets of natural numbers.

**Definition 1** (Operations on sets of natural numbers). For every  $N \subset \mathbb{N}$ ,  $k \in \mathbb{N}$ :

1.  $N + k = \{n + k : n \in N\}$
2.  $N - k = \{n - k : n \in N \wedge n > k\}$
3.  $N_{\oplus k} = \{n : n \in N \wedge n \oplus k\}$ , with  $\oplus \in \{=, <, \leq, >, \geq\}$

Terms for  $\lambda r$  are the same as those for  $\lambda_{dB}$ . That is:

**Definition 2** (Terms for  $\lambda_{dB}$  and  $\lambda r$ ). The set of terms for  $\lambda_{dB}$  and  $\lambda r$ , denoted  $\Lambda_{dB}$ , is given in BNF by:

$$a ::= n \mid aa \mid \lambda a \quad (n \in \mathbb{N}_{>0})$$

**Definition 3** (Free variables). The free variables of a term,  $FV : \Lambda_{dB} \rightarrow \mathcal{P}(\mathbb{N}_{>0})$ , is given by:

$$FV(n) = \{n\} \quad FV(aa) = FV(a) \cup FV(b) \quad FV(\lambda a) = FV(a) - 1$$

### Classical definitions

We recall the classical definitions for  $\lambda_{dB}$  (see e.g. [11] for a more detailed introduction).

**Definition 4** (Updating meta-operator for  $\lambda_{dB}$ ). For every  $k \in \mathbb{N}$ ,  $i \in \mathbb{N}_{>0}$ ,  $U_k^i : \Lambda_{dB} \rightarrow \Lambda_{dB}$  is given inductively by:

$$U_k^i(n) = \begin{cases} n & \text{if } n \leq k \\ n + i - 1 & \text{if } n > k \end{cases} \quad \begin{aligned} U_k^i(ab) &= U_k^i(a) U_k^i(b) \\ U_k^i(\lambda a) &= \lambda U_{k+1}^i(a) \end{aligned}$$

**Definition 5** (Meta-substitution for  $\lambda_{dB}$ ). For every  $a, b, c \in \Lambda_{dB}$ ,  $m, n \in \mathbb{N}_{>0}$ ,  $\bullet\{\bullet \leftarrow \bullet\} : \Lambda_{dB} \times \mathbb{N}_{>0} \times \Lambda_{dB} \rightarrow \Lambda_{dB}$  is given inductively by:

$$m\{\{n \leftarrow c\}\} = \begin{cases} m & \text{if } m < n \\ U_0^n(c) & \text{if } m = n \\ m - 1 & \text{if } m > n \end{cases} \quad \begin{aligned} (ab)\{\{n \leftarrow c\}\} &= a\{\{n \leftarrow c\}\} b\{\{n \leftarrow c\}\} \\ (\lambda a)\{\{n \leftarrow c\}\} &= \lambda a\{\{n + 1 \leftarrow c\}\} \end{aligned}$$

**Definition 6** ( $\lambda_{dB}$ -calculus). The  $\lambda_{dB}$ -calculus is the reduction system  $(\Lambda_{dB}, \beta_{dB})$ , where:

$$(\forall a, b \in \Lambda_{dB}) (a \rightarrow_{\beta_{dB}} b \iff (\exists C \text{ context; } c, d \in \Lambda_{dB}) (a = C[(\lambda c)d] \wedge b = C[c\{\{1 \leftarrow d\}\}]))$$

### New definitions

We now define the new meta-operators used to implement index increments and swaps.

**Definition 7** (*Increment operator* –  $\uparrow_i$ ). For every  $i \in \mathbb{N}$ ,  $\uparrow_i : \Lambda_{\text{dB}} \rightarrow \Lambda_{\text{dB}}$  is given inductively by:

$$\uparrow_i(n) = \begin{cases} n & \text{if } n \leq i \\ n+1 & \text{if } n > i \end{cases} \quad \begin{aligned} \uparrow_i(ab) &= \uparrow_i(a) \uparrow_i(b) \\ \uparrow_i(\lambda a) &= \lambda \uparrow_{i+1}(a) \end{aligned}$$

**Definition 8** (*Swap operator* –  $\Downarrow_i$ ). For every  $i \in \mathbb{N}_{>0}$ ,  $\Downarrow_i : \Lambda_{\text{dB}} \rightarrow \Lambda_{\text{dB}}$  is given inductively by:

$$\Downarrow_i(n) = \begin{cases} n & \text{if } n < i \vee n > i+1 \\ i+1 & \text{if } n = i \\ i & \text{if } n = i+1 \end{cases} \quad \begin{aligned} \Downarrow_i(ab) &= \Downarrow_i(a) \Downarrow_i(b) \\ \Downarrow_i(\lambda a) &= \lambda \Downarrow_{i+1}(a) \end{aligned}$$

Finally, we present the meta-level substitution definition for  $\lambda r$ , and then the  $\lambda r$ -calculus itself.

**Definition 9** (*Meta-substitution for  $\lambda r$* ). For every  $a, b, c \in \Lambda_{\text{dB}}$ ,  $n \in \mathbb{N}_{>0}$ ,  $\bullet\{\bullet\} : \Lambda_{\text{dB}} \times \Lambda_{\text{dB}} \rightarrow \Lambda_{\text{dB}}$  is given inductively by:

$$n\{c\} = \begin{cases} c & \text{if } n = 1 \\ n-1 & \text{if } n > 1 \end{cases} \quad \begin{aligned} (ab)\{c\} &= a\{c\}b\{c\} \\ (\lambda a)\{c\} &= \lambda \Downarrow_1(a)\{\uparrow_0(c)\} \end{aligned}$$

**Definition 10** ( $\lambda r$ -calculus). The  $\lambda r$ -calculus is the reduction system  $(\Lambda_{\text{dB}}, \beta_r)$ , where:

$$(\forall a, b \in \Lambda_{\text{dB}}) (a \rightarrow_{\beta_r} b \iff (\exists C \text{ context}; c, d \in \Lambda_{\text{dB}}) (a = C[(\lambda c)d] \wedge b = C[c\{d\}]))$$

### 2.3 $\lambda_{\text{dB}}$ and $\lambda r$ are the same calculus

We want to prove that  $\lambda r$  equals  $\lambda_{\text{dB}}$ . That is, we want to show that  $a\{\{1 \leftarrow b\}\} = a\{b\}$ . In order to do this, however, we should first prove the general case:  $a\{\{n \leftarrow b\}\} = a'\{b'\}$ , with  $a'$  and  $b'$  being the result of a series of *swaps* and *increments* over  $a$  and  $b$ , respectively. This comes from observing that, while  $\lambda_{\text{dB}}$  increments the index inside the substitution when going into an abstraction,  $\lambda r$  performs a *swap* over the affected term, and an index increment over the term carried in the substitution. Thus, comparing what happens after the “crossing” of  $n-1$  abstractions in  $(\underbrace{\lambda \cdots \lambda a}_{n-1})\{\{1 \leftarrow b\}\}$  and  $(\underbrace{\lambda \cdots \lambda a}_{n-1})\{b\}$ , we get to:

$$\underbrace{\lambda \cdots \lambda a}_{n-1} \{\{n \leftarrow b\}\} \quad \text{and} \quad \underbrace{\lambda \cdots \lambda}_{n-1} \Downarrow_1(\cdots \Downarrow_{n-1}(a) \cdots) \{\underbrace{\uparrow_0(\cdots \uparrow_0(b) \cdots)}_{n-1}\}$$

Therefore, the idea for the proof is showing that the above terms are equal for every  $n \in \mathbb{N}_{>0}$ . We formalize this idea by introducing two additional definitions: stacked swaps and stacked increments.

**Definition 11** (*Stacked swap*). For every  $i \in \mathbb{N}_{>0}$ ,  $j \in \mathbb{N}$ ,  $\Downarrow_i^j : \Lambda_{\text{dB}} \rightarrow \Lambda_{\text{dB}}$  is given inductively by:

$$\Downarrow_i^j(a) = \begin{cases} a & \text{if } j = 0 \\ \Downarrow_i^{j-1}(\Downarrow_{i+j-1}(a)) & \text{if } j > 0 \end{cases}$$

The intuitive idea behind  $\Downarrow_i^j(a)$  is that of:  $\underbrace{\Downarrow_i(\Downarrow_{i+1}(\cdots \Downarrow_{i+j-1}(a) \cdots))}_{j \text{ swaps}}$

**Definition 12** (Stacked increment). For every  $i \in \mathbb{N}$ ,  $\uparrow^i: \Lambda_{\text{dB}} \rightarrow \Lambda_{\text{dB}}$  is given inductively by:

$$\uparrow^i(a) = \begin{cases} a & \text{if } i = 0 \\ \uparrow^{i-1}(\uparrow_0(a)) & \text{if } i > 0 \end{cases}$$

The intuitive idea behind  $\uparrow^i(a)$  is that of:  $\underbrace{\uparrow_0(\cdots \uparrow_0(a) \cdots)}_{i \text{ increments}}$

Based on this last two definitions, the next theorem states the relationship between  $\lambda r$  and  $\lambda_{\text{dB}}$  meta-substitution operators, having as an immediate corollary that  $\lambda r$  and  $\lambda_{\text{dB}}$  are the same calculus.

**Theorem 13** (Correspondence between  $\lambda_{\text{dB}}$  and  $\lambda r$  meta-substitution). For every  $a, b \in \Lambda_{\text{dB}}$ ,  $n \in \mathbb{N}_{>0}$ :

$$a\{\{n \leftarrow b\}\} = \Downarrow_1^{n-1}(a)\{\uparrow^{n-1}(b)\}$$

*Proof.* See Appendix A. □

**Corollary 14.** For every  $a, b \in \Lambda_{\text{dB}}$ :  $a\{\{1 \leftarrow b\}\} = a\{b\}$ . Therefore,  $\lambda_{\text{dB}}$  and  $\lambda r$  are the same calculus.

*Proof.* Use Theorem 13 with  $n = 1$ , and conclude the equality of both calculi by definition. This result was checked using the *Coq* theorem prover<sup>1</sup>. □

### 3 Devising the $\lambda re$ , $\lambda re_{\text{gc}}$ and $\lambda rex$ calculi

In order to derive an ES calculus from  $\lambda r$ , we first need to internalize substitutions in the language. Thus, we add the construction  $a[b]$  to  $\Lambda_{\text{dB}}$ , and call the resulting set of terms  $\Lambda re$ . The definition for the free variables of a term is extended to consider the ES case as follows:  $\text{FV}(a[b]) = (\text{FV}(a) - 1) \cup \text{FV}(b)$ . Also, and as a design decision, operators  $\uparrow_i$  and  $\Downarrow_i$  are left in the meta-level. Naturally, we must extend their definitions to the ES case, task that needs some lemmas over  $\lambda r$ 's meta-operators in order to ensure correctness. We use lemmas 26 and 27 in Appendix B for the extension of swap and increment meta-operators:

$$\uparrow_i(a[b]) = \uparrow_{i+1}(a)[\uparrow_i(b)] \quad \text{and} \quad \Downarrow_i(a[b]) = \Downarrow_{i+1}(a)[\Downarrow_i(b)]$$

Then, we just orient the equalities from the meta-substitution definition as expected and get a calculus we call  $\lambda re$  (that turns out to be isomorphic to  $\lambda x$  [4, 3], as we will later explain).

As a next step in our work, we add *Garbage Collection* to  $\lambda re$ . The goal is removing useless substitutions, *i.e.*, when the index 1 does not appear free in the term. When removing a substitution, free indexes of the term must be updated, decreasing them by 1. To accomplish this, we introduce a new meta-operator:  $\Downarrow_i$ . The operator is inspired in a similar one from [19]. We first define it for the set  $\Lambda_{\text{dB}}$ :

**Definition 15** (Decrement operator  $-\Downarrow_i$ ). For every  $i \in \mathbb{N}_{>0}$ ,  $\Downarrow_i: \Lambda_{\text{dB}} \rightarrow \Lambda_{\text{dB}}$  is given inductively by:

$$\Downarrow_i(n) = \begin{cases} n & \text{if } n < i \\ \text{undefined} & \text{if } n = i \\ n - 1 & \text{if } n > i \end{cases} \quad \begin{aligned} \Downarrow_i(ab) &= \Downarrow_i(a) \Downarrow_i(b) \\ \Downarrow_i(\lambda a) &= \lambda \Downarrow_{i+1}(a) \end{aligned}$$

**Note.** Notice that  $\Downarrow_i(a)$  is well-defined iff  $i \notin \text{FV}(a)$ .

<sup>1</sup>The proof can be downloaded from <http://www.mpi-sws.org/~beta/lambdar.v>

As for the  $\downarrow_i$  and  $\uparrow_i$  meta-operators, we need a few lemmas to ensure a correct definition for the extension of the  $\downarrow_i$  meta-operator to the ES case. Particularly, Lemma 28 (see Appendix B) is used for this purpose. The extension resembles those of the  $\downarrow_i$  and  $\uparrow_i$  meta-operators:

$$\downarrow_i(a[b]) = \downarrow_{i+1}(a)[\downarrow_i(b)]$$

The *Garbage Collection* rule added to  $\lambda re$  (GC) can be seen in Figure 2, and the resulting calculus is called  $\lambda re_{gc}$  (which, as we will see, is isomorphic to  $\lambda xgc$  [4]).

Finally, in order to mimic the behavior of  $\lambda ex$  [15], an analogue method for the composition of substitutions must be devised. In  $\lambda ex$ , composition is handled by one rule and one equation:

$$\begin{array}{lcl} t[x := u][y := v] & \rightarrow_{(\text{Comp})} & t[y := v][x := u[y := v]] \quad \text{if } y \in \text{FV}(u) \\ t[x := u][y := v] & =_{\text{C}} & t[y := v][x := u] \quad \text{if } y \notin \text{FV}(u) \wedge x \notin \text{FV}(v) \end{array}$$

The rule (Comp) is used when substitutions are *dependent*, and reasoning modulo C-equation is needed for *independent* substitutions. Since in  $\lambda r$ -derived calculi there is no simple way of implementing an ordering of substitutions (remember: no indexes inside closures!), and thus no trivial path for the elimination of equation C exists, we need an analogue equation.

Let us start with the composition rule: in a term of the form  $a[b][c]$ , substitutions  $[b]$  and  $[c]$  are *dependent* iff  $1 \in \text{FV}(b)$ . In such a term, indexes 1 and 2 in  $a$  are being affected by  $[b]$  and  $[c]$ , respectively. Consequently, if we were to reduce to a term of the form  $a'[c][b]$ , a *swap* should be performed over  $a$ . Moreover, as substitution  $[c]$  crosses the binder  $[b]$ , an index increment should also be done. Finally, since substitutions are dependent – that is,  $[c]$  affects  $b$  –,  $b'$  should be  $b[c]$ . Then, we are left with the term  $\downarrow_1(a)[\uparrow_0(c)][b[c]]$ .

For the equation, let us suppose we negate the composition condition (*i.e.*,  $1 \notin \text{FV}(b)$ ). Using *Garbage Collection* in the last term, we have  $\downarrow_1(a)[\uparrow_0(c)][b[c]] \rightarrow_{(\text{GC})} \downarrow_1(a)[\uparrow_0(c)][\downarrow_1(b)]$ . It is important to notice that the condition in rule (Comp) is essential; that is: we cannot leave (Comp) unconditional and let (GC) do its magic: we would immediately generate infinite reductions, losing PSN. Thus, our composition rule and equation are:

$$\begin{array}{lcl} a[b][c] & \rightarrow_{(\text{Comp})} & \downarrow_1(a)[\uparrow_0(c)][b[c]] \quad \text{if } 1 \in \text{FV}(b) \\ a[b][c] & =_{\text{D}} & \downarrow_1(a)[\uparrow_0(c)][\downarrow_1(b)] \quad \text{if } 1 \notin \text{FV}(b) \end{array}$$

Rules for the  $\lambda rex$ -calculus can be seen in Figure 2. The relation  $rex_p$  is generated by the set of rules (App), (Lamb), (Var), (GC) and (Comp);  $\lambda rex_p$  by (Beta) +  $rex_p$ . D-equivalence is the least equivalence and compatible relation generated by (EqD). Relations  $\lambda rex$  (resp.  $rex$ ) are obtained from  $\lambda rex_p$  (resp.  $rex_p$ ) modulo D-equivalence (thus specifying rewriting on D-equivalence classes). That is,

$$\forall a, a' \in \Lambda re : a \rightarrow_{(\lambda)rex} a' \iff \left( \exists b, b' \in \Lambda re : a =_{\text{D}} b \rightarrow_{(\lambda)rex_p} b' =_{\text{D}} a' \right)$$

We define  $\lambda rex$  as the reduction system  $(\Lambda re, \lambda rex)$ . We shall define  $\lambda re$  and  $\lambda re_{gc}$  next. Since the rule (VarR) does not belong to  $\lambda rex$ , but only to  $\lambda re$  and  $\lambda re_{gc}$ , we present it here:

$$(\text{VarR}) \quad (n+1)[c] \rightarrow n$$

The relation  $re$  is generated by (App), (Lamb), (Var) and (VarR);  $\lambda re$  by (Beta) +  $re$ ; the relation  $re_{gc}$  by  $re$  + (GC); and  $\lambda re_{gc}$  by (Beta) +  $re_{gc}$ . Finally, the  $\lambda re$  and  $\lambda re_{gc}$  calculi are the reduction systems  $(\Lambda re, \lambda re)$  and  $(\Lambda re, \lambda re_{gc})$ , respectively.

(EqD)	$a[b][c]$	=	$\Downarrow_1(a)[\uparrow_0(c)][\downarrow_1(b)]$	$(1 \notin \text{FV}(b))$
(Beta)	$(\lambda a) b$	$\rightarrow$	$a[b]$	
(App)	$(ab)[c]$	$\rightarrow$	$a[c] b[c]$	
(Lamb)	$(\lambda a)[c]$	$\rightarrow$	$\lambda \Downarrow_1(a)[\uparrow_0(c)]$	
(Var)	$1[c]$	$\rightarrow$	$c$	
(GC)	$a[c]$	$\rightarrow$	$\downarrow_1(a)$	$(1 \notin \text{FV}(a))$
(Comp)	$a[b][c]$	$\rightarrow$	$\Downarrow_1(a)[\uparrow_0(c)][b[c]]$	$(1 \in \text{FV}(b))$

Figure 2: Equations and rules for the  $\lambda$ rex-calculus

## 4 The isomorphisms

For the isomorphism between  $\lambda$ ex and  $\lambda$ rex (and also between  $\lambda$ x and  $\lambda$ re; and between  $\lambda$ xgc and  $\lambda$ re<sub>gc</sub>), we must first give a translation from the set  $\Lambda x$  (*i.e.*, the set of terms for  $\lambda$ x,  $\lambda$ xgc and  $\lambda$ ex; see e.g. [15] for the expected definition) to  $\Lambda$ re, and vice versa. It is important to notice that our translations depend on a list of variables, which will determine the indexes of the free variables. All this work is inspired in a similar proof that shows the isomorphism between the  $\lambda$  and  $\lambda_{\text{dB}}$  calculi, found in [13].

**Definition 16** (Translation from  $\Lambda x$  to  $\Lambda$ re). For every  $t \in \Lambda x$ ,  $n \in \mathbb{N}$ , such that  $\text{FV}(t) \subseteq \{x_1, \dots, x_n\}$ ,  $w_{[x_1, \dots, x_n]} : \Lambda x \rightarrow \Lambda$ re is given inductively by:

$$\begin{aligned} w_{[x_1, \dots, x_n]}(x) &= \min \{j : x_j = x\} & w_{[x_1, \dots, x_n]}(\lambda x.t) &= \lambda w_{[x, x_1, \dots, x_n]}(t) \\ w_{[x_1, \dots, x_n]}(tu) &= w_{[x_1, \dots, x_n]}(t) w_{[x_1, \dots, x_n]}(u) & w_{[x_1, \dots, x_n]}(t[x := u]) &= w_{[x, x_1, \dots, x_n]}(t) [w_{[x_1, \dots, x_n]}(u)] \end{aligned}$$

**Definition 17** (Translation from  $\Lambda$ re to  $\Lambda x$ ). For every  $a \in \Lambda$ re,  $n \in \mathbb{N}$ , such that  $\text{FV}(a) \subseteq \{1, \dots, n\}$ ,  $u_{[x_1, \dots, x_n]} : \Lambda$ re  $\rightarrow \Lambda x$ , with  $\{x_1, \dots, x_n\}$  different variables, is given inductively by:

$$\begin{aligned} u_{[x_1, \dots, x_n]}(j) &= x_j & u_{[x_1, \dots, x_n]}(\lambda a) &= \lambda x. u_{[x, x_1, \dots, x_n]}(a) \\ u_{[x_1, \dots, x_n]}(ab) &= u_{[x_1, \dots, x_n]}(a) u_{[x_1, \dots, x_n]}(b) & u_{[x_1, \dots, x_n]}(a[b]) &= u_{[x, x_1, \dots, x_n]}(a) [x := u_{[x_1, \dots, x_n]}(b)] \end{aligned}$$

with  $x \notin \{x_1, \dots, x_n\}$  in the cases of abstraction and closure.

Translations are correct w.r.t.  $\alpha$ -equivalence. That is,  $\alpha$ -equivalent  $\Lambda x$  terms have the same image under  $w_{[x_1, \dots, x_n]}$ , and identical  $\Lambda$ re terms have  $\alpha$ -equivalent images under different choices of  $x$  for  $u_{[x_1, \dots, x_n]}$ . Besides, adding variables at the end of translation lists does not affect the result; thus, uniform translations  $w$  and  $u$  can be defined straightforwardly, depending only on a preset ordering of variables. See Appendix C for details.

We now state the isomorphisms:

**Theorem 18** ( $\lambda$ ex  $\cong$   $\lambda$ rex,  $\lambda$ x  $\cong$   $\lambda$ re and  $\lambda$ xgc  $\cong$   $\lambda$ re<sub>gc</sub>). The  $\lambda$ ex (resp.  $\lambda$ x,  $\lambda$ xgc) and  $\lambda$ rex (resp.  $\lambda$ re,  $\lambda$ re<sub>gc</sub>) calculi are isomorphic. That is,

- A.  $w \circ u = \text{Id}_{\Lambda\text{re}} \wedge u \circ w = \text{Id}_{\Lambda x}$
- B.  $\forall t, u \in \Lambda x : t \rightarrow_{\lambda\text{ex}(\lambda x, \lambda\text{xgc})} u \implies w(t) \rightarrow_{\lambda\text{rex}(\lambda\text{re}, \lambda\text{re}_{\text{gc}})} w(u)$
- C.  $\forall a, b \in \Lambda\text{re} : a \rightarrow_{\lambda\text{rex}(\lambda\text{re}, \lambda\text{re}_{\text{gc}})} b \implies u(a) \rightarrow_{\lambda\text{ex}(\lambda x, \lambda\text{xgc})} u(b)$

*Proof.* This is actually a three-in-one theorem. Proofs require many auxiliary lemmas that assert the interaction between translations and meta-operators. See Appendix D for details.  $\square$

Finally, in order to show meta-confluence (MC) for  $\lambda_{\text{rex}}$ , meta-variables are added to the set of terms, and hence, functions and meta-operators are extended accordingly. Particularly, each metavariable is decorated with a set  $\Delta$  of available free variables. This, in order to achieve an isomorphism with  $\lambda_{\text{ex}}$ 's corresponding extension (c.f. [15]). Extensions are as follows:

1. Set of terms  $\Lambda_{\text{rex}}: a ::= n \mid X_{\Delta} \mid aa \mid \lambda a \mid a[a] \quad (n \in \mathbb{N}_{>0}, X \in \{X, Y, Z, \dots\}, \Delta \in \mathcal{P}(\mathbb{N}_{>0}))$
2. Free variables of a metavariable:  $\text{FV}(X_{\Delta}) = \Delta$
3. Swap over a metavariable:  $\downarrow_i(X_{\Delta}) = X_{\Delta'}$  with  $\Delta' = \Delta_{<i} \cup \Delta_{>i+1} \cup (\Delta_{=i} + 1) \cup (\Delta_{=i+1} - 1)$
4. Increment over a metavariable:  $\uparrow_i(X_{\Delta}) = X_{\Delta'}$  with  $\Delta' = \Delta_{\leq i} \cup (\Delta_{>i} + 1)$
5. Decrement over a metavariable:  $\downarrow_i(X_{\Delta}) = \begin{cases} X_{\Delta'} & \text{with } \Delta' = \Delta_{<i} \cup (\Delta_{>i} - 1) \text{ if } i \notin \Delta \\ \text{undefined} & \text{if } i \in \Delta \end{cases}$
6. Translation from  $\Lambda_{\text{xop}}$  to  $\Lambda_{\text{rex}}: w_{[x_1, \dots, x_n]}(X_{\Delta}) = X_{\Delta'}$  with  $\Delta' = \{w_{[x_1, \dots, x_n]}(x) : x \in \Delta\}$
7. Translation from  $\Lambda_{\text{rex}}$  to  $\Lambda_{\text{xop}}: u_{[x_1, \dots, x_n]}(X_{\Delta}) = X_{\Delta'}$  with  $\Delta' = \{u_{[x_1, \dots, x_n]}(j) : j \in \Delta\}$

**Theorem 19.** The  $\lambda_{\text{rex}}$  and  $\lambda_{\text{ex}}$  calculi on open terms are isomorphic.

*Proof.* This is proved as an extension of the proof for Theorem 18, considering the new case. A few simple lemmas about how meta-operators alter the set of free variables are needed. We refer the reader to [18], chapter 6, section 3 for details (space constraints disallow further technicality here).  $\square$

As a direct consequence of theorems 18 and 19, we have:

**Corollary 20** (Preservation of properties). The  $\lambda_{\text{ex}}$  (resp.  $\lambda_{\text{x}}$ ,  $\lambda_{\text{xgc}}$ ) and  $\lambda_{\text{rex}}$  (resp.  $\lambda_{\text{re}}$ ,  $\lambda_{\text{re}_{\text{gc}}}$ ) have the same properties. In particular, this implies  $\lambda_{\text{rex}}$  has, among other properties, Sim, PSN and MC.

*Proof sketch for e.g. PSN in  $\lambda_{\text{rex}}$ .* Assume PSN does not hold in  $\lambda_{\text{rex}}$ . Then, there exists  $a \in \text{SN}_{\lambda_{\text{dB}}}$  s.t.  $a \notin \text{SN}_{\lambda_{\text{rex}}}$ . Besides,  $a \in \text{SN}_{\lambda_{\text{dB}}}$  implies  $u(a) \in \text{SN}_{\lambda}$ . Therefore, by PSN of  $\lambda_{\text{ex}}$  [15],  $u(a) \in \text{SN}_{\lambda_{\text{ex}}}$ . Now, since  $a \notin \text{SN}_{\lambda_{\text{rex}}}$ , there exists an infinite reduction  $a \rightarrow_{\lambda_{\text{rex}}} a_1 \rightarrow_{\lambda_{\text{rex}}} a_2 \rightarrow_{\lambda_{\text{rex}}} \dots$ . Thus, by Theorem 18, we have  $u(a) \rightarrow_{\lambda_{\text{ex}}} u(a_1) \rightarrow_{\lambda_{\text{ex}}} u(a_2) \rightarrow_{\lambda_{\text{ex}}} \dots$ , contradicting the fact that  $u(a) \in \text{SN}_{\lambda_{\text{ex}}}$ .  $\square$

## 5 Related work

It is important to mention that, even though independently discovered, the swapping mechanism introduced in this article was first depicted by de Bruijn for his ES calculus  $C\lambda\xi\phi$  [6], and, later, updated w.r.t. notation –  $\lambda\xi\phi$  – and compared to  $\lambda v$  in [2]. We will now briefly discuss the main differences between these calculi and our swapping-based approach.

Firstly, neither  $C\lambda\xi\phi$  nor  $\lambda\xi\phi$  have composition of substitutions nor Garbage Collection, two keys for the accomplishment of meta-confluence. In that sense, these two calculi only resemble closely our first  $\lambda_{\text{r}}$ -based ES calculus:  $\lambda_{\text{re}}$ . Thus, both  $\lambda_{\text{re}_{\text{gc}}}$  and  $\lambda_{\text{rex}}$  represent a relevant innovation for swapping-based formalisms, specially considering the fact that, as far as we know, no direct successor of  $C\lambda\xi\phi$  nor  $\lambda\xi\phi$  was found to satisfy PSN and MC.

As a second fundamental difference, both  $C\lambda\xi\phi$  and  $\lambda\xi\phi$  are entirely explicit formalisms. In the end, internalizing meta-operations is desirable, both theoretically and practically; nevertheless, the presence of meta-operations in  $\lambda_{\text{re}}$ ,  $\lambda_{\text{re}_{\text{gc}}}$  and  $\lambda_{\text{rex}}$  are mandatory for the accomplishment of isomorphisms w.r.t.  $\lambda_{\text{x}}$ ,  $\lambda_{\text{xgc}}$  and  $\lambda_{\text{ex}}$ , respectively. Particularly, the isomorphism between  $\lambda_{\text{ex}}$  and  $\lambda_{\text{rex}}$  represents a step forward in the explicit substitutions area. Moreover, these isomorphisms – impossible in the case



of  $C\lambda\xi\phi$  and  $\lambda\xi\phi$  – allow simple and straightforward proofs for every single property enjoyed by the calculi.

Last but not least, in  $C\lambda\xi\phi$  as well as in  $\lambda\xi\phi$ , swap and increment operations are implemented by means of a special sort of substitution that only operates on indexes (c.f. [2]). Even though undoubtedly a very clever setting for these operations – specially compared to ours, much more conservative –, the fact is that we still use meta-operations. With this in mind, it may be the case that de Bruijn’s formulation for both the swap and increment operations, if taken to the meta-level, would lead to *exactly the same* functional relations between terms than those defined by our method. Consequently, this difference loses importance in the presence of meta-operations. Nevertheless, if swap and increment meta-operations were to be made explicit, a deep comparison between our approach and de Bruijn’s should be carried out before deciding for the use of either.

## 6 Conclusions and further work

We have presented  $\lambda_{\text{rex}}$ , an ES calculus with *de Bruijn* indexes that is isomorphic to  $\lambda_{\text{ex}}$ , a formalism with variable names that fulfills a whole set of interesting properties. As a consequence of the isomorphism,  $\lambda_{\text{rex}}$  inherits all of  $\lambda_{\text{ex}}$ ’s properties. This, together with a simple notation makes it, as far as we know, the first calculus of its kind. Besides, the  $\lambda_{\text{re}}$  and  $\lambda_{\text{re}_{\text{gc}}}$  calculi (isomorphic to  $\lambda_{\text{x}}$  and  $\lambda_{\text{xgc}}$ , respectively) were also introduced. The development was based on a novel presentation of the classical  $\lambda_{\text{dB}}$ . Given the homogeneity of definitions and proofs, not only for  $\lambda_{\text{r}}$  and  $\lambda_{\text{rex}}$ , but also for  $\lambda_{\text{re}}$  and  $\lambda_{\text{re}_{\text{gc}}}$ , we think we found a truly *natural* bridge between named and indexed formalisms. We believe this opens a new set of possibilities in the area: either by translating and studying existing calculi with good properties; or by rethinking old calculi from a different perspective (*i.e.*, with  $\lambda_{\text{r}}$ ’s concept in mind).

Work is yet to be done in order to get a more suitable theoretical tool for implementation purposes, for unary closures and equations still make such a task hard. In this direction, a mix of ideas from  $\lambda_{\text{rex}}$  and calculi with  $n$ -ary substitutions (*i.e.*,  $\lambda_{\sigma}$ -styled calculi) may lead to the solution of both issues. Particularly, a swap-based  $\lambda_{\sigma_{\uparrow}}$  [7] could be an option. This comes from the following observation: in  $\lambda_{\sigma_{\uparrow}}$ , the (Lamb) rule is:

$$\text{(Lamb)} \quad (\lambda a)[s] \rightarrow \lambda a[\uparrow(s)]$$

where the intuitive semantics of  $\uparrow(s)$  is:  $1 \cdot (s \circ \uparrow)$ . We observe here that *this is not nameless!* The reason is that, even though there are no explicit indexes inside closures, this lift operation resembles closely the classic definition of the  $\lambda_{\text{dB}}$  calculus (particularly, leaving lower indexes untouched). Thus, we propose replacing this rule by one of the form:

$$\text{(Lamb)} \quad (\lambda a)[s] \rightarrow \lambda \Downarrow(a)[\uparrow(s)]$$

with the semantics of  $\uparrow(s)$  being  $s \circ \uparrow$ , and that of  $\Downarrow(a)$  being *swapping*  $a$ ’s indexes in concordance with the substitution  $s$ , therefore mimicking  $\lambda_{\text{r}}$ ’s behavior. This approach is still in its early days, but we feel it is quite promising.

In a different line of work, the explicitation of meta-operators may also come to mind: we think this is not a priority, because the main merit of  $\lambda_{\text{rex}}$  is evidencing the accessory nature of index updates.

From a different perspective, an attempt to use  $\lambda_{\text{rex}}$  in proof assistants or higher order unification [8] implementations may be taken into account. In such a case, a typed version of  $\lambda_{\text{rex}}$  should be developed as well. Also, adding an  $\eta$  rule to  $\lambda_{\text{rex}}$  should be fairly simple using the decrement meta-operator. Finally, studying the possible relation between these *swapping-based* formalisms and nominal logic or nominal rewriting (see e.g. [10, 9]) could be an interesting approach in gathering a deeper understanding of  $\lambda_{\text{r}}$ ’s underlying logic.

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## A Proofs for the $\lambda_{\text{dB}} = \lambda \mathbf{r}$ assertion

We first show the auxiliary lemmas that allow us to prove the main theorem of Subsection 2.3.

**Lemma 21.** For every  $a \in \Lambda_{\text{dB}}$ ,  $n \in \mathbb{N}_{>0}$ ,  $\uparrow^{n-1}(a) = U_0^n(a)$

*Proof.* Easy induction on  $n$ , using that  $l \leq k < l + j \implies U_k^i(U_l^j(a)) = U_l^{j+i-1}(a)$  (c.f. [12], lemma 6), and the fact that  $\uparrow_0(a) = U_0^2(a)$ .  $\square$

**Lemma 22.** For every  $m, i \in \mathbb{N}_{>0}$ ,  $n \in \mathbb{N}$ :

1.  $m > n + i \implies \Downarrow_i^n(m) = m$
2.  $i \leq m < n + i \implies \Downarrow_i^n(m) = m + 1$
3.  $\Downarrow_i^n(n + i) = i$

*Proof.* Easy inductions on  $n$ .  $\square$

**Lemma 23.** For every  $a, b \in \Lambda_{\text{dB}}$ ,  $n \in \mathbb{N}$ ,  $i \in \mathbb{N}_{>0}$ :

1.  $\Downarrow_i^n(ab) = \Downarrow_i^n(a) \Downarrow_i^n(b)$
2.  $\Downarrow_i^n(\lambda a) = \lambda \Downarrow_{i+1}^n(a)$
3.  $(\lambda \Downarrow_2^n(a))\{\uparrow^n(b)\} = \lambda \Downarrow_1^{n+1}(a)\{\uparrow^{n+1}(b)\}$

*Proof.* Easy inductions on  $n$ .  $\square$

We now restate and prove the main theorem:

**Theorem (13).** For every  $a, b \in \Lambda_{\text{dB}}$ ,  $n \in \mathbb{N}_{>0}$ , we have that  $a\{n \leftarrow b\} = \Downarrow_1^{n-1}(a)\{\uparrow^{n-1}(b)\}$ .

*Proof.* Induction on  $a$ .

- $a = m \in \mathbb{N}_{>0}$ . Then,  $a\{n \leftarrow b\} = m\{n \leftarrow b\} = \begin{cases} m-1 & \text{if } m > n \\ U_0^n(b) & \text{if } m = n \\ m & \text{if } m < n \end{cases}$

We consider each case separately:

1.  $m > n \implies \Downarrow_1^{n-1}(m)\{\uparrow^{n-1}(b)\} \stackrel{\text{L.22.1}}{=} m\{\uparrow^{n-1}(b)\} \stackrel{m > n \geq 1}{=} m-1$
2.  $m = n \implies \Downarrow_1^{n-1}(n)\{\uparrow^{n-1}(b)\} \stackrel{\text{L.22.3}}{=} 1\{\uparrow^{n-1}(b)\} \stackrel{\text{def}}{=} \uparrow^{n-1}(b) \stackrel{\text{L.21}}{=} U_0^n(b)$
3.  $m < n \implies \Downarrow_1^{n-1}(m)\{\uparrow^{n-1}(b)\} \stackrel{\text{L.22.2}}{=} m+1\{\uparrow^{n-1}(b)\} \stackrel{m+1 > 1}{=} m$

Then,

$$m\{n \leftarrow b\} = \Downarrow_1^{n-1}(m)\{\uparrow^{n-1}(b)\}$$

- $a = cd$ ,  $c, d \in \Lambda_{\text{dB}}$ . Use inductive hypothesis and Lemma 23.1.
- $a = \lambda c$ ,  $c \in \Lambda_{\text{dB}}$ . Then,

$$\begin{aligned} a\{n \leftarrow b\} &= (\lambda c)\{n \leftarrow b\} \stackrel{\text{def}}{=} \lambda c\{n+1 \leftarrow b\} \stackrel{\text{HI}}{=} \lambda \Downarrow_1^n(c)\{\uparrow^n(b)\} \stackrel{\text{L.23.3}}{=} \\ &(\lambda \Downarrow_2^{n-1}(c))\{\uparrow^{n-1}(b)\} \stackrel{\text{L.23.2}}{=} \Downarrow_1^{n-1}(\lambda c)\{\uparrow^{n-1}(b)\} = \Downarrow_1^{n-1}(a)\{\uparrow^{n-1}(b)\} \end{aligned}$$

$\square$

## B Extension lemmas for the $\downarrow_i$ , $\uparrow_i$ and $\downarrow_i$ meta-operators

**Lemma 24.** For every  $a \in \Lambda_{\text{dB}}$ ,  $i, j \in \mathbb{N}_{>0}$ ,  $k \in \mathbb{N}$ :

1.  $k < i \implies \downarrow_{i+1}(\uparrow_k(a)) = \uparrow_k(\downarrow_i(a))$
2.  $j \geq 2 \implies \downarrow_{i+j}(\downarrow_i(a)) = \downarrow_i(\downarrow_{i+j}(a))$

*Proof.* Easy induction on  $a$ . □

**Lemma 25.** For every  $a \in \Lambda_{\text{dB}}$ ,  $i \in \mathbb{N}_{>0}$ ,  $j \in \mathbb{N}$ :

1.  $i \geq j + 2 \wedge i - 1 \notin \text{FV}(a) \implies \downarrow_i(\uparrow_j(a)) = \uparrow_j(\downarrow_{i-1}(a))$
2.  $j \geq 2 \wedge i + j \notin \text{FV}(a) \implies \downarrow_{i+j}(\downarrow_i(a)) = \downarrow_i(\downarrow_{i+j}(a))$

*Proof.* Easy induction on  $a$ . □

**Lemma 26.** For every  $a, b \in \Lambda_{\text{dB}}$ ,  $i \in \mathbb{N}_{>0}$ ,  $\downarrow_i(a\{b\}) = \downarrow_{i+1}(a)\{\downarrow_i(b)\}$

*Proof.* Easy induction on  $a$ , using Lemma 24. □

**Lemma 27.** For every  $a, b \in \Lambda_{\text{dB}}$ ,  $i \in \mathbb{N}$ ,  $\uparrow_i(a\{b\}) = \uparrow_{i+1}(a)\{\uparrow_i(b)\}$

*Proof.* Use that  $U_k^i(a\{1 \leftarrow b\}) = U_{k+1}^i(a)\{1 \leftarrow U_k^i(b)\}$  (c.f. [12], Lemma 10 with  $n = 1$ ), the fact that  $\uparrow_i(a) = U_i^2(a)$  and Corollary 14. □

**Lemma 28.** For every  $a, b \in \Lambda_{\text{dB}}$ ,  $i \in \mathbb{N}_{>0}$ ,  $i + 1 \notin \text{FV}(a) \wedge i \notin \text{FV}(b)$ , we have that  $\downarrow_i(a\{b\}) = \downarrow_{i+1}(a)\{\downarrow_i(b)\}$ .

*Proof.* Easy induction on  $a$ , using Lemma 25. □

## C Correction proofs for translations

We show the lemmas necessary to prove that the translations given (*i.e.*,  $w_{[x_1, \dots, x_n]}$  and  $u_{[x_1, \dots, x_n]}$ ) are correct w.r.t.  $\alpha$ -equivalence.

**Lemma 29.** For every  $t \in \Lambda x$ ,  $n \in \mathbb{N}$  such that  $\text{FV}(t) \subseteq \{x_1, \dots, x_n\}$ , we have that  $\forall y \notin \{x_1, \dots, x_n\}$ ,  $z \in \{x_1, \dots, x_n\}$ ,  $w_{[x_1, \dots, x_n]}(t) = w_{[x_1, \dots, x_{k-1}, y, x_{k+1}, \dots, x_n]}(t\{z := y\})$ , with  $k = \min \{j : x_j = z\}$ .

*Proof.* Easy induction on  $t$ , but using the non-Barendregt-variable-convention definition for the meta-substitution operation (otherwise, we would be assuming that  $t =_\alpha u \implies w_{[x_1, \dots, x_n]}(t) = w_{[x_1, \dots, x_n]}(u)$ , which is what we ultimately want to prove). See e.g. [3] for an expected definition. □

**Lemma 30.** For every  $t, u \in \Lambda x$ ,  $n \in \mathbb{N}$  such that  $\text{FV}(t) \subseteq \{x_1, \dots, x_n\}$ , we have that  $t =_\alpha u \implies w_{[x_1, \dots, x_n]}(t) = w_{[x_1, \dots, x_n]}(u)$ . Notice that  $w_{[x_1, \dots, x_n]}(u)$  is well-defined, since  $t =_\alpha u \implies \text{FV}(t) = \text{FV}(u)$ .

*Proof.* Easy induction on  $t$ , using Lemma 29. Once again, the non-Barendregt-variable-convention definition for the meta-substitution operation must be used here. □

**Lemma 31.** For every  $a \in \Lambda \text{re}$ ,  $n \in \mathbb{N}$ ,  $\{x_1, \dots, x_n\}$  distinct variables such that  $\text{FV}(a) \subseteq \{1, \dots, n\}$ , we have that  $\forall y \notin \{x_1, \dots, x_n\}$ ,  $1 \leq k \leq n : u_{[x_1, \dots, x_n]}(a)\{x_k := y\} =_\alpha u_{[x_1, \dots, x_{k-1}, y, x_{k+1}, \dots, x_n]}(a)$ .

*Proof.* Easy induction on  $a$ . □

**Lemma 32.** For every  $a, b \in \text{Are}$ ,  $n \in \mathbb{N}$ ,  $\{x_1, \dots, x_n\}$  distinct variables,  $x, y \notin \{x_1, \dots, x_n\}$  such that  $\text{FV}(a) \subseteq \{1, \dots, n\}$ , we have that:

1.  $\lambda x. \mathbf{u}_{[x, x_1, \dots, x_n]}(a) =_\alpha \lambda y. \mathbf{u}_{[y, x_1, \dots, x_n]}(a)$
2.  $\mathbf{u}_{[x, x_1, \dots, x_n]}(a)[x := \mathbf{u}_{[x_1, \dots, x_n]}(b)] =_\alpha \mathbf{u}_{[y, x_1, \dots, x_n]}(a)[y := \mathbf{u}_{[x_1, \dots, x_n]}(b)]$

*Proof.* Direct in both cases, using the  $\alpha$ -equivalence definition and Lemma 31.  $\square$

Last, we show two lemmas that assert that adding variables at the end of translation lists does not affect the result of the translation and, thus, gives the possibility of defining uniform translations that depend only on a preset ordering of variables.

**Lemma 33.** For every  $t \in \Lambda x$  such that  $\text{FV}(t) \subseteq \{x_1, \dots, x_n\}$ , and for every  $\{y_1, \dots, y_m\} \subset \mathbb{V}$ , we have that  $\mathbf{w}_{[x_1, \dots, x_n]}(t) = \mathbf{w}_{[x_1, \dots, x_n, y_1, \dots, y_m]}(t)$ .

*Proof.* Easy induction on  $t$ .  $\square$

**Lemma 34.** For every  $a \in \text{Are}$ ,  $\{x_1, \dots, x_n\}$  distinct variables such that  $\text{FV}(a) \subseteq \{1, \dots, n\}$ , and for every  $\{y_1, \dots, y_m\}$  distinct variables such that  $\{x_1, \dots, x_n\} \cap \{y_1, \dots, y_m\} = \emptyset$ , we have that  $\mathbf{u}_{[x_1, \dots, x_n]}(a) =_\alpha \mathbf{u}_{[x_1, \dots, x_n, y_1, \dots, y_m]}(a)$ .

*Proof.* Easy induction on  $a$ .  $\square$

Last, we show the definitions for uniform translations.

**Definition 35** (Uniform translation from  $\Lambda x$  to  $\text{Are}$ ). Given an enumeration  $[v_1, v_2, \dots]$  of  $\mathbb{V}$ , for every  $t \in \Lambda x$ ,  $n \in \mathbb{N}$  such that  $\text{FV}(t) \subseteq \{v_1, \dots, v_n\}$ , we define  $\mathbf{w} : \Lambda x \rightarrow \text{Are}$  as:  $\mathbf{w}(t) = \mathbf{w}_{[v_1, \dots, v_n]}(t)$ .

**Definition 36** (Uniform translation from  $\text{Are}$  to  $\Lambda x$ ). Given an enumeration  $[v_1, v_2, \dots]$  of  $\mathbb{V}$ , for every  $a \in \text{Are}$ ,  $n \in \mathbb{N}$  such that  $\text{FV}(a) \subseteq \{1, \dots, n\}$ , we define  $\mathbf{u} : \text{Are} \rightarrow \Lambda x$  as:  $\mathbf{u}(a) = \mathbf{u}_{[v_1, \dots, v_n]}(a)$ .

## D Isomorphisms proofs

In order to prove Theorem 18, we must show:

- A.  $\mathbf{w} \circ \mathbf{u} = \text{Id}_{\text{Are}} \wedge \mathbf{u} \circ \mathbf{w} = \text{Id}_{\Lambda x}$
- B.  $\forall t, u \in \Lambda x : t \rightarrow_{\lambda \text{ex}(\lambda x, \lambda \text{xgc})} u \implies \mathbf{w}(t) \rightarrow_{\lambda \text{rex}(\lambda \text{re}, \lambda \text{re}_{\text{gc}})} \mathbf{w}(u)$
- C.  $\forall a, b \in \text{Are} : a \rightarrow_{\lambda \text{rex}(\lambda \text{re}, \lambda \text{re}_{\text{gc}})} b \implies \mathbf{u}(a) \rightarrow_{\lambda \text{ex}(\lambda x, \lambda \text{xgc})} \mathbf{u}(b)$

For Part A, the following two lemmas are needed.

**Lemma 37.** For every  $t \in \Lambda x$ ,  $a \in \text{Are}$ ,  $\{x_1, \dots, x_n\}$  variables,  $\{y_1, \dots, y_m\}$  distinct variables:

1.  $\text{FV}(t) \subseteq \{x_1, \dots, x_n\} \implies \text{FV}(\mathbf{w}_{[x_1, \dots, x_n]}(t)) \subseteq \{1, \dots, n\}$
2.  $\text{FV}(a) \subseteq \{1, \dots, m\} \implies \text{FV}(\mathbf{u}_{[y_1, \dots, y_m]}(a)) \subseteq \{y_1, \dots, y_m\}$

*Proof.* Easy inductions on  $t$  and  $a$ , respectively.  $\square$

**Lemma 38.** For every  $a \in \text{Are}$ ,  $t \in \Lambda x$ :

1.  $\mathbf{w}(\mathbf{u}(a)) = a$
2.  $\mathbf{u}(\mathbf{w}(t)) =_\alpha t$

*Proof.* Easy inductions on  $a$  and  $t$ , respectively, using Lemma 37.  $\square$

Next, to prove Part B of the theorem, we need several auxiliary lemmas, that we now state.

**Lemma 39.** For every  $\{x_1, \dots, x_n\}$ ,  $y \in \{x_1, \dots, x_n\}$ ,  $x \notin \{x_1, \dots, x_n\}$ ,  $w_{[x, x_1, \dots, x_n]}(y) = w_{[x_1, \dots, x_n]}(y) + 1$ .

*Proof.* Direct, using  $w$ 's definition.  $\square$

**Lemma 40.** For every  $t \in \Lambda x$ ,  $i \in \mathbb{N}_{>0}$  such that  $FV(t) \subseteq \{x_1, \dots, x_n\} \wedge i < n \wedge x_i \neq x_{i+1}$ ,

$$\uparrow_i(w_{[x_1, \dots, x_i, x_{i+1}, \dots, x_n]}(t)) = w_{[x_1, \dots, x_{i+1}, x_i, \dots, x_n]}(t).$$

*Proof.* Easy induction on  $t$ .  $\square$

**Lemma 41.** For every  $t \in \Lambda x$ ,  $m \in \mathbb{N}$ ,  $x \in \mathbb{V}$  such that  $FV(t) \subseteq \{x_1, \dots, x_n\} \wedge m \leq n \wedge x \notin \{x_1, \dots, x_n\}$ ,

$$w_{[x_1, \dots, x_m, x, x_{m+1}, \dots, x_n]}(t) = \uparrow_m(w_{[x_1, \dots, x_n]}(t)).$$

*Proof.* Easy induction on  $t$ .  $\square$

**Lemma 42.** For every  $t \in \Lambda x$ ,  $m \in \mathbb{N}$ ,  $x \in \mathbb{V}$  such that  $FV(t) \subseteq \{x_1, \dots, x_n\} \wedge 1 \leq m \leq n + 1$ :

1.  $x \notin \{x_1, \dots, x_n\} \implies m \notin FV(w_{[x_1, \dots, x_{m-1}, x, x_m, \dots, x_n]}(t))$
2.  $x \notin \{x_1, \dots, x_{m-1}\} \wedge x \in FV(t) \implies m \in FV(w_{[x_1, \dots, x_{m-1}, x, x_m, \dots, x_n]}(t))$
3.  $x \notin \{x_1, \dots, x_n\} \implies w_{[x_1, \dots, x_n]}(t) = \downarrow_m(w_{[x_1, \dots, x_{m-1}, x, x_m, \dots, x_n]}(t))$

*Proof.* Easy inductions on  $t$ .  $\square$

Given the auxiliary lemmas, we proceed to prove Part B of the isomorphism theorem. Item 1 of the next lemma is enough to prove the reduction preservation under translation  $w$  for the  $\lambda_{re}$  and  $\lambda_{re_{gc}}$  calculi. For  $\lambda_{rex}$ , Item 2 – showing the preservation of the equivalence relations under translation  $w$  – is also needed. Then, preservation for  $\lambda_{rex}$  follows immediately from the definition of reduction modulo an equivalence relation.

**Lemma 43.** For every  $t, u \in \Lambda x$ :

1.  $t \rightarrow_{Bx(\lambda x, \lambda_{xgc})} u \implies w(t) \rightarrow_{\lambda_{rex_p}(\lambda_{re}, \lambda_{re_{gc}})} w(u)$
2.  $t =_C u \implies w(t) =_D w(u)$

*Proof. Part 1.* Induction on  $t$ . The only interesting cases are those of the explicit substitution when the reduction takes place at the root. The rest of the cases are either trivial or easily shown by using the inductive hypothesis. We will show the explicit substitution case in which reduction is done by using the (Comp) rule. The other two relevant cases, (Lamb) and (GC), omitted here for a matter of space, are proved in a similar fashion. Since we are working in the explicit substitution case,  $t$  is of the form  $t_1[x := t_2]$ . Now, as the (Comp) rule is used, we have that:

$$t_1[x := t_2] = t_3[y := t_4][x := t_2] \xrightarrow[\text{(Comp)}]{Bx} t_3[x := t_2][y := t_4[x := t_2]] = u$$

with  $x \in FV(t_4)$ . By the variable convention, we assume  $x \neq y \wedge y \notin \{x_1, \dots, x_n\}$ . Thus,

$$\begin{aligned} w_{[x_1, \dots, x_n]}(t_3[y := t_4][x := t_2]) & \stackrel{\text{def}}{=} w_{[x, x_1, \dots, x_n]}(t_3[y := t_4]) [w_{[x_1, \dots, x_n]}(t_2)] \stackrel{\text{def}}{=} \\ & w_{[y, x, x_1, \dots, x_n]}(t_3) [w_{[x, x_1, \dots, x_n]}(t_4)] [w_{[x_1, \dots, x_n]}(t_2)] \xrightarrow[\text{L.42.2, } x \in FV(t_4), \text{(Comp)}]{\lambda_{rex_p}} \\ & \uparrow_1(w_{[y, x, x_1, \dots, x_n]}(t_3)) [\uparrow_0(w_{[x_1, \dots, x_n]}(t_2))] [w_{[x, x_1, \dots, x_n]}(t_4)] [w_{[x_1, \dots, x_n]}(t_2)] \stackrel{\text{def}}{=} \end{aligned}$$

$$\begin{aligned}
& \mathbf{w}_{[x,y,x_1,\dots,x_n]}(t_3) \left[ \uparrow_0(\mathbf{w}_{[x_1,\dots,x_n]}(t_2)) \right] \left[ \mathbf{w}_{[x,x_1,\dots,x_n]}(t_4) \left[ \mathbf{w}_{[x_1,\dots,x_n]}(t_2) \right] \right] \stackrel{L.41, y \notin \text{FV}(t_2)}{=} \\
& \mathbf{w}_{[x,y,x_1,\dots,x_n]}(t_3) \left[ \mathbf{w}_{[y,x_1,\dots,x_n]}(t_2) \right] \left[ \mathbf{w}_{[x,x_1,\dots,x_n]}(t_4) \left[ \mathbf{w}_{[x_1,\dots,x_n]}(t_2) \right] \right] \stackrel{\text{def}}{=} \\
& \mathbf{w}_{[x,y,x_1,\dots,x_n]}(t_3) \left[ \mathbf{w}_{[y,x_1,\dots,x_n]}(t_2) \right] \left[ \mathbf{w}_{[x_1,\dots,x_n]}(t_4[x := t_2]) \right] \stackrel{\text{def}}{=} \\
& \mathbf{w}_{[y,x_1,\dots,x_n]}(t_3[x := t_2]) \left[ \mathbf{w}_{[x_1,\dots,x_n]}(t_4[x := t_2]) \right] \stackrel{\text{def}}{=} \mathbf{w}_{[x_1,\dots,x_n]}(t_3[x := t_2][y := t_4[x := t_2]])
\end{aligned}$$

**Part 2.** Induction on the inference of  $t =_C u$ . The only interesting case is when the actual equation is used. Then,  $t = t_1[y := t_2][x := t_3] =_C t_1[x := t_3][y := t_2] = u$ , with  $x \neq y \wedge x \notin \text{FV}(t_2) \wedge y \notin \text{FV}(t_3)$ . By the variable convention, assume that  $\{x, y\} \cap \{x_1, \dots, x_n\} = \emptyset$ . Proceed in a similar way than that of the proof of Part 1 in the (Comp) case.  $\square$

Finally, to prove Part C of the isomorphism theorem, we also need several auxiliary lemmas analogue to those used for Part B. We will now state them.

**Lemma 44.** For every  $a \in \text{Are}$ ,  $i \in \mathbb{N}_{>0}$ ,  $\{x_1, \dots, x_n\}$  distinct variables such that  $\text{FV}(a) \subseteq \{1, \dots, n\} \wedge i < n$ , we have that  $\mathbf{u}_{[x_1, \dots, x_i, x_{i+1}, \dots, x_n]}(a) = \alpha \mathbf{u}_{[x_1, \dots, x_{i+1}, x_i, \dots, x_n]}(\downarrow_i(a))$ .

*Proof.* Easy induction on  $a$ .  $\square$

**Lemma 45.** For every  $a \in \text{Are}$ ,  $m \in \mathbb{N}$ ,  $\{x_1, \dots, x_n\}$  distinct variables,  $x \notin \{x_1, \dots, x_n\}$  such that  $\text{FV}(a) \subseteq \{1, \dots, n\} \wedge m \leq n$ , we have that  $\mathbf{u}_{[x_1, \dots, x_m, x, x_{m+1}, \dots, x_n]}(\uparrow_m(a)) = \alpha \mathbf{u}_{[x_1, \dots, x_n]}(a)$ .

*Proof.* Easy induction on  $a$ .  $\square$

**Lemma 46.** For every  $a \in \text{Are}$ ,  $m \in \mathbb{N}$ ,  $x \in \mathbb{V}$ ,  $\{x_1, \dots, x_n\}$  distinct variables such that  $\text{FV}(a) \subseteq \{1, \dots, n\} \wedge 1 \leq m \leq n+1 \wedge x \notin \{x_1, \dots, x_n\}$ :

1.  $m \notin \text{FV}(a) \implies x \notin \text{FV}(\mathbf{u}_{[x_1, \dots, x_{m-1}, x, x_m, \dots, x_n]}(a))$
2.  $m \in \text{FV}(a) \implies x \in \text{FV}(\mathbf{u}_{[x_1, \dots, x_{m-1}, x, x_m, \dots, x_n]}(a))$
3.  $m \notin \text{FV}(a) \implies \mathbf{u}_{[x_1, \dots, x_{m-1}, x, x_m, \dots, x_n]}(a) = \alpha \mathbf{u}_{[x_1, \dots, x_n]}(\downarrow_m(a))$

*Proof.* Easy inductions on  $a$ .  $\square$

Given, once again, the auxiliary lemmas, we will now state Part C of the isomorphism theorem. As for Part B, Item 1 of Lemma 47 will be enough to prove preservation for the  $\lambda x$  and  $\lambda xgc$  calculi, whereas Item 2 will also be needed for the case of  $\lambda ex$ , concluding preservation by definition of reduction modulo an equation.

**Lemma 47.** For every  $a, b \in \text{Are}$ :

1.  $a \rightarrow_{\lambda \text{rex}_p(\lambda \text{re}, \lambda \text{re}_{gc})} b \implies \mathbf{u}(a) \rightarrow_{\text{Bx}(\lambda x, \lambda xgc)} \mathbf{u}(b)$
2.  $a =_D b \implies \mathbf{u}(a) =_C \mathbf{u}(b)$

*Proof.* For part 1, perform induction on  $a$  analogue to that of lemma 43.1. For part 2, perform induction on the inference of  $a =_D b$ , analogue to that of lemma 43.2. In both cases, use auxiliary lemmas 44, 45 and 46.  $\square$