

SILTING REDUCTION AND CALABI–YAU REDUCTION OF TRIANGULATED CATEGORIES

OSAMU IYAMA AND DONG YANG

ABSTRACT. It is shown that the silting reduction $\mathcal{T}/\text{thick}\mathcal{P}$ of a triangulated category \mathcal{T} with respect to a presilting subcategory \mathcal{P} can be realised as a certain subfactor category of \mathcal{T} , and that there is a one-to-one correspondence between the set of (pre)silting subcategories of \mathcal{T} containing \mathcal{P} and the set of (pre)silting subcategories of $\mathcal{T}/\text{thick}\mathcal{P}$. This is analogous to a result for Calabi–Yau reduction. This result is applied to show that the Amiot–Guo–Keller construction of d -Calabi–Yau triangulated categories with d -cluster-tilting objects takes silting reduction to Calabi–Yau reduction, and conversely, Calabi–Yau reduction lifts to silting reduction.

CONTENTS

1. Introduction	2
2. Preliminaries	4
2.1. Some notation	4
2.2. Mutation pairs and cluster-tilting subcategories	6
2.3. Presilting and silting subcategories, t-structures and co-t-structures	8
2.4. From silting objects to t-structures	9
2.5. Derived categories of dg algebras	13
3. Silting reduction as subfactor category	13
3.1. The additive equivalence	13
3.2. The triangle equivalence	17
3.3. The correspondence between silting subcategories	18
3.4. A theorem of Buchweitz	20
3.5. Conjectures of Auslander–Reiten and Tachikawa	21
4. Silting reduction versus Calabi–Yau reduction	22
4.1. Calabi–Yau triples	23
4.2. The silting reduction of a Calabi–Yau triple	25
4.3. The Amiot–Guo–Keller cluster category of a Calabi–Yau triple	28

Date: August 13, 2014.

4.4. The main result	30
4.5. Lifting Calabi–Yau reduction	33
Appendix A. Proof of Theorem 4.7	35
References	38

1. INTRODUCTION

Two kinds of reduction process of triangulated categories were studied in representation theory. One is called *Calabi–Yau reduction*, introduced in [21]. This is defined for a rigid subcategory \mathcal{P} of a d -Calabi–Yau triangulated category \mathcal{T} as a certain subfactor category \mathcal{U} of \mathcal{T} . In this case \mathcal{U} is again a d -Calabi–Yau triangulated category, and there is a natural bijection between d -cluster-tilting subcategories of \mathcal{T} containing \mathcal{P} and d -cluster-tilting subcategories of \mathcal{U} .

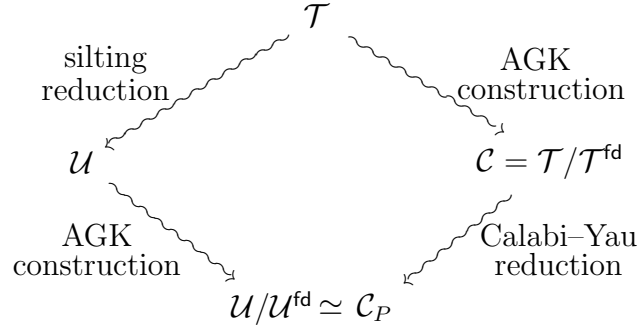
The other one is called *silting reduction*. This is defined for a presilting subcategory \mathcal{P} of a triangulated category \mathcal{T} as the triangle quotient $\mathcal{U} = \mathcal{T}/\text{thick}\mathcal{P}$. Our first main result justifies this terminology — we verify the analogy between silting reduction and Calabi–Yau reduction by showing that silting reduction can also be realised as a subfactor category of \mathcal{T} (Theorems 3.1 and 3.7). We recover, as a special case of this realisation, the well-known triangle equivalence due to Buchweitz[11]

$$\underline{\text{CMA}} \xrightarrow{\cong} \text{D}^b(\text{mod}A)/\text{K}^b(\text{proj}A)$$

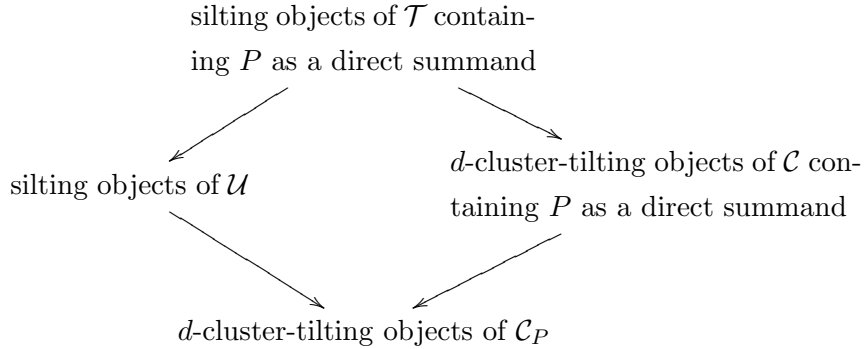
for an Iwanaga–Gorenstein ring A (Example 3.11). Moreover, there is a natural bijection between silting subcategories of \mathcal{T} containing \mathcal{P} and silting subcategories of \mathcal{U} (Theorem 3.8), which preserves a canonical partial order on the set of silting subcategories. A similar result was given in [2, Theorem 2.37] under the strong restriction that $\text{thick}\mathcal{P}$ is functorially finite in \mathcal{T} . We can drop this assumption thanks to the realisation of \mathcal{U} as a subfactor category of \mathcal{T} .

The second main result of this paper is to compare these two reduction processes using the Amiot–Guo construction [3, 16] (based on Keller’s work [25, 27]). Let \mathcal{T} be a triangulated category, $M \in \mathcal{T}$ an object and $\mathcal{T}^{\text{fd}} \subset \mathcal{T}$ a triangulated subcategory such that $(\mathcal{T}, \mathcal{T}^{\text{fd}}, M)$ is a $(d+1)$ -Calabi–Yau triple (see Section 4.1 for the precise definition). Let P be a direct summand of M . On the one hand, applying the Amiot–Guo–Keller construction, we obtain a d -Calabi–Yau triangulated category $\mathcal{C} = \mathcal{T}/\mathcal{T}^{\text{fd}}$ in which P becomes a d -rigid object, then we form the Calabi–Yau reduction \mathcal{C}_P of \mathcal{C} with respect to P , which is d -Calabi–Yau with d -cluster-tilting objects. On the other hand, we first

form the silting reduction $\mathcal{U} = \mathcal{T}/\text{thick}P$, which turns out to be part of a relative $(d+1)$ -Calabi–Yau triple $(\mathcal{U}, \mathcal{U}^{\text{fd}}, M)$, and then the Amiot–Guo–Keller construction yields a d -Calabi–Yau triangulated category $\mathcal{U}/\mathcal{U}^{\text{fd}}$ with d -cluster-tilting objects. We prove that the two resulting d -Calabi–Yau triangulated categories \mathcal{C}_P and $\mathcal{U}/\mathcal{U}^{\text{fd}}$ are triangle equivalent (Theorem 4.13). This can be illustrated by the following commutative diagram.



The case when \mathcal{T} is the perfect derived category of a Ginzburg dg algebra was studied by Keller in [27, Section 7]. The diagram above induces a commutative diagram of maps



where the two left-going maps are bijections due to properties of silting reduction and Calabi–Yau reduction and the two right-going maps are surjections when $d = 1$ and when $d = 2$ (due to Keller–Nicolás [29] in the algebraic setting) (Corollary 4.10).

Acknowledgement: The first-named author acknowledges financial support from JSPS Grant-in-Aid for Scientific Research (B) 24340004, (C) 23540045 and (S) 22224001. The second-named author acknowledges financial support from a JSPS postdoctoral fellowship program and from National Science Foundation in China No. 11371196 and No. 11301272. Both authors thank Martin Kalck for helpful comments. They are grateful to Xiao-Wu Chen for pointing out the application Corollary 3.12 and the fact that Theorems 3.1 and 3.7 generalise Buchweitz’ result.

2. PRELIMINARIES

In this section, we fix some notation. We recall the triangle structure of an additive quotient associated to a mutation pair. We recall the definitions of silting subcategories, silting reduction, cluster-tilting subcategories, Calabi–Yau reduction, t-structures and co-t-structures. We recall derived categories of dg algebras and Keller’s Morita theorem for triangulated categories.

2.1. Some notation. For a ring R , we denote by $\mathbf{mod}R$ the category of finitely generated right R -modules, by $\mathbf{proj}R$ the category of finitely generated projective right R -modules, by $\mathbf{D}^b(\mathbf{mod}R)$ the bounded derived category of $\mathbf{mod}R$ and by $\mathbf{K}^b(\mathbf{proj}R)$ the bounded homotopy category of $\mathbf{proj}R$.

Let \mathcal{T} be an additive category. For morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, we denote by $gf : X \rightarrow Z$ the composition. Let \mathcal{S} be a full subcategory of \mathcal{T} (for example, an object of \mathcal{T} will often be considered as a full subcategory with one object). For an object X of \mathcal{T} , we say that a morphism $f : S \rightarrow X$ is a *right \mathcal{S} -approximation* of X if $S \in \mathcal{S}$ and $\mathrm{Hom}_{\mathcal{T}}(S', f)$ is surjective for any $S' \in \mathcal{S}$. We say that \mathcal{S} is *contravariantly finite* if every object in \mathcal{T} has a right \mathcal{S} -approximation. Dually, we define *left \mathcal{S} -approximations* and *covariantly finite* subcategories. We say that \mathcal{S} is *functorially finite* if it is both contravariantly finite and covariantly finite [6].

Denote by $\mathbf{add}_{\mathcal{T}}\mathcal{S}$ (or simply $\mathbf{add}\mathcal{S}$) the smallest full subcategory of \mathcal{T} which contains \mathcal{S} and which is closed under taking isomorphisms, finite direct sums and direct summands. Denote by $[\mathcal{S}]$ the ideal of \mathcal{T} consisting of morphisms which factors through an object of \mathcal{S} and denote by $\frac{\mathcal{T}}{[\mathcal{S}]}$ the corresponding additive quotient of \mathcal{T} by \mathcal{S} . Define full subcategories

$$\begin{aligned} {}^{\perp}\mathcal{S} &:= \{X \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(X, \mathcal{S}) = 0\}, \\ \mathcal{S}^{\perp} &:= \{X \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(\mathcal{S}, X) = 0\}. \end{aligned}$$

When it does not cause confusion, we will simply write ${}^{\perp}\mathcal{S}$ and \mathcal{S}^{\perp} .

Let \mathcal{T} be a triangulated category (we will denote by $[1]$ the shift functor of any triangulated category unless otherwise stated). For two objects X and Y of \mathcal{T} and an integer n , by $\mathrm{Hom}_{\mathcal{T}}(X, Y[>n]) = 0$ (respectively, $\mathrm{Hom}_{\mathcal{T}}(X, Y[\geq n]) = 0$, $\mathrm{Hom}_{\mathcal{T}}(X, Y[<n]) = 0$, $\mathrm{Hom}_{\mathcal{T}}(X, Y[\leq n]) = 0$), we mean $\mathrm{Hom}_{\mathcal{T}}(X, Y[i]) = 0$ for all $i > n$ (respectively, for all $i \geq n$, $i < n$, $i \leq n$).

Let \mathcal{S} be a full subcategory of \mathcal{T} . Denote by $\mathbf{thick}_{\mathcal{T}}\mathcal{S}$ (or simply $\mathbf{thick}\mathcal{S}$) the smallest triangulated subcategory of \mathcal{T} which contains \mathcal{S} and which is closed under taking direct

summands. Denote by $\mathcal{T}/\text{thick}\mathcal{S}$ the triangle quotient of \mathcal{T} by $\text{thick}\mathcal{S}$. Let \mathcal{S}' be another full subcategory of \mathcal{T} . By $\text{Hom}_{\mathcal{T}}(\mathcal{S}, \mathcal{S}') = 0$, we mean $\text{Hom}_{\mathcal{T}}(S, S') = 0$ for all $S \in \mathcal{S}$ and $S' \in \mathcal{S}'$. Define

$$\mathcal{S} * \mathcal{S}' := \{X \in \mathcal{T} \mid \text{there is a triangle } S \rightarrow X \rightarrow S' \rightarrow S[1] \\ \text{with } S \in \mathcal{S} \text{ and } S' \in \mathcal{S}'\}.$$

We have the following easy observation.

Lemma 2.1. *If $X \in \text{add}(\mathcal{S} * \mathcal{S}')$ satisfies $\text{Hom}_{\mathcal{T}}(\mathcal{S}, X) = 0$, then $X \in \text{add}\mathcal{S}'$.*

Proof. There exist $Y \in \mathcal{T}$ and a triangle

$$S \xrightarrow{a} X \oplus Y \longrightarrow S' \longrightarrow S[1] \quad (2.1.1)$$

with $S \in \mathcal{S}$ and $S' \in \mathcal{S}'$. Since $\text{Hom}_{\mathcal{T}}(\mathcal{S}, X) = 0$, we can write $a = (0 \ b)$ for $b : S \rightarrow Y$. We extend b to a triangle $S \xrightarrow{b} Y \xrightarrow{c} Z \rightarrow S[1]$. Then we have a triangle

$$S \xrightarrow{a=(0 \ b)} X \oplus Y \xrightarrow{\begin{pmatrix} 1_X & 0 \\ 0 & c \end{pmatrix}} X \oplus Z \longrightarrow S[1].$$

Comparing this with (2.1.1), we have $S' \simeq X \oplus Z$. Thus $X \in \text{add}\mathcal{S}'$. \square

Even if $\mathcal{S} = \text{add}\mathcal{S}$ and $\mathcal{S}' = \text{add}\mathcal{S}'$ hold, we do not have $\mathcal{S} * \mathcal{S}' = \text{add}(\mathcal{S} * \mathcal{S}')$ in general. We have the following sufficient condition for this, which was known for Krull–Schmidt case in [21, Proposition 2.1].

Lemma 2.2. *Let \mathcal{S} and \mathcal{S}' be subcategories of \mathcal{T} satisfying $\mathcal{S} = \text{add}\mathcal{S}$ and $\mathcal{S}' = \text{add}\mathcal{S}'$. If $\text{Hom}_{\mathcal{T}}(\mathcal{S}, \mathcal{S}') = 0$ and $\mathcal{S}[1] \subset \mathcal{S}'$ hold, then $\mathcal{S} * \mathcal{S}' = \text{add}(\mathcal{S} * \mathcal{S}')$.*

Proof. Since \mathcal{S} and \mathcal{S}' are closed under direct sums, it follows easily from definition that $\mathcal{S} * \mathcal{S}'$ is also closed under direct sums. It remains to show that $\mathcal{S} * \mathcal{S}'$ is closed under direct summands. Assume that $X \oplus X' \in \mathcal{S} * \mathcal{S}'$, that is, there exists a triangle

$$S \xrightarrow{(a \ a')} X \oplus X' \xrightarrow{\begin{pmatrix} b \\ b' \end{pmatrix}} S' \rightarrow S[1] \quad (2.1.2)$$

with $S \in \mathcal{S}$ and $S' \in \mathcal{S}'$. Now we extend $a : S \rightarrow X$ to a triangle

$$S \xrightarrow{a} X \rightarrow Y \rightarrow S[1]. \quad (2.1.3)$$

Since $\text{Hom}_{\mathcal{T}}(\mathcal{S}, \mathcal{S}') = 0$, the map $\text{Hom}_{\mathcal{T}}(\mathcal{S}, S) \xrightarrow{(a \ a')} \text{Hom}_{\mathcal{T}}(\mathcal{S}, X \oplus X')$ is surjective by the triangle (2.1.2). In particular, the map $\text{Hom}_{\mathcal{T}}(\mathcal{S}, S) \xrightarrow{a} \text{Hom}_{\mathcal{T}}(\mathcal{S}, X)$ is also surjective. Thus we have $\text{Hom}_{\mathcal{T}}(\mathcal{S}, Y) = 0$ by the triangle (2.1.3) and our assumptions $\text{Hom}_{\mathcal{T}}(\mathcal{S}, \mathcal{S}') = 0$ and $\mathcal{S}[1] \subset \mathcal{S}'$.

On the other hand, $S' * S[1] \in \mathcal{S}'$ holds since any triangle $S' \rightarrow Z \rightarrow S[1] \rightarrow S'[1]$ splits by $\text{Hom}_{\mathcal{T}}(\mathcal{S}, \mathcal{S}') = 0$ and moreover $\mathcal{S}[1] \subset \mathcal{S}'$ holds. Thus we have

$$Y \in X * S[1] \subset \text{add}(S * S') * S[1] \subset \text{add}(S * S' * S[1]) = \text{add}(S * \mathcal{S}').$$

Now $\text{Hom}_{\mathcal{T}}(\mathcal{S}, Y) = 0$ and $Y \in \text{add}(S * \mathcal{S}')$ implies $Y \in \text{add}\mathcal{S}' = \mathcal{S}'$ by Lemma 2.1. \square

We need the following observation in the next subsection.

Proposition 2.3. *Let \mathcal{P} be a full subcategory of \mathcal{T} with $\mathcal{P} = \text{add}\mathcal{P}$ and $n \geq 0$. If $\text{Hom}_{\mathcal{T}}(\mathcal{P}, \mathcal{P}[i]) = 0$ for any i with $1 \leq i \leq n - 1$, then we have $\mathcal{P} * \mathcal{P}[1] * \cdots * \mathcal{P}[n] = \text{add}(\mathcal{P} * \mathcal{P}[1] * \cdots * \mathcal{P}[n])$.*

Proof. The assertion is clear for $n = 0$. Assume that it holds for $n - 1$. Then $\mathcal{S} := \mathcal{P}$ and $\mathcal{S}' := \mathcal{P}[1] * \mathcal{P}[2] * \cdots * \mathcal{P}[n - 1]$ satisfies the assumptions in Lemma 2.2. Thus $\mathcal{S} * \mathcal{S}' = \mathcal{P} * \mathcal{P}[1] * \cdots * \mathcal{P}[n]$ satisfies $\mathcal{S} * \mathcal{S}' = \text{add}(\mathcal{S} * \mathcal{S}')$. \square

2.2. Mutation pairs and cluster-tilting subcategories. Let \mathcal{T} be a triangulated category. Let \mathcal{P} be a full subcategory of \mathcal{T} such that $\text{Hom}_{\mathcal{T}}(\mathcal{P}, \mathcal{P}[1]) = 0$ and let \mathcal{Z} be an extension-closed full subcategory of \mathcal{T} which contains \mathcal{P} . Assume that $(\mathcal{Z}, \mathcal{Z})$ forms a \mathcal{P} -mutation pair in the sense of [21], i.e. the following conditions are satisfied:

- $\mathcal{P} \subset \mathcal{Z}$ and $\text{Hom}_{\mathcal{T}}(\mathcal{P}, \mathcal{Z}[1]) = 0 = \text{Hom}_{\mathcal{T}}(\mathcal{Z}, \mathcal{P}[1])$.
- For any $Z \in \mathcal{Z}$, there exists triangles $Z \rightarrow P' \rightarrow Z' \rightarrow Z[1]$ and $Z'' \rightarrow P'' \rightarrow Z \rightarrow Z''[1]$ with $P', P'' \in \mathcal{P}$ and $Z', Z'' \in \mathcal{Z}$.

Theorem 2.4. ([21, Theorem 4.2]) *The category $\frac{\mathcal{Z}}{[\mathcal{P}]}$ has the structure of a triangulated category with respect to the following shift functor and triangles:*

- (a) *For $X \in \mathcal{Z}$, we take a triangle*

$$X \xrightarrow{\iota_X} P_X \rightarrow X\langle 1 \rangle \rightarrow X[1]$$

with a (fixed) left \mathcal{P} -approximation ι_X . Then $\langle 1 \rangle$ gives a well-defined auto-equivalence of $\frac{\mathcal{Z}}{[\mathcal{P}]}$, which is the shift functor of $\frac{\mathcal{Z}}{[\mathcal{P}]}$.

- (b) *For a triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ with $X, Y, Z \in \mathcal{Z}$, take the following commutative diagram of triangles:*

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\ \parallel & & \downarrow & & \downarrow a & & \parallel \\ X & \xrightarrow{\iota_X} & P_X & \longrightarrow & X\langle 1 \rangle & \longrightarrow & X[1] \end{array} \quad (2.2.1)$$

Then we have a complex $X \xrightarrow{\bar{f}} Y \xrightarrow{\bar{g}} Z \xrightarrow{\bar{a}} X\langle 1 \rangle$. We define triangles in $\frac{\mathcal{Z}}{[\mathcal{P}]}$ as the complexes which is isomorphic to a complex obtained in this way.

Let k be a field and \mathcal{T} be a k -linear triangulated category. Let $d \geq 1$ be an integer. \mathcal{T} is said to be d -Calabi–Yau if \mathcal{T} is Hom-finite, Krull–Schmidt and there is a bifunctorial isomorphism for any objects X and Y of \mathcal{T}

$$D \operatorname{Hom}_{\mathcal{T}}(X, Y) \simeq \operatorname{Hom}_{\mathcal{T}}(Y, X[d]).$$

Assume that \mathcal{T} is d -Calabi–Yau. A full subcategory \mathcal{P} of \mathcal{T} is d -rigid if $\operatorname{Hom}_{\mathcal{T}}(\mathcal{P}, \mathcal{P}[i]) = 0$ for all $1 \leq i \leq d-1$. It is d -cluster-tilting if \mathcal{P} is functorially finite and the following equivalence holds for $X \in \mathcal{T}$

$$\operatorname{Hom}_{\mathcal{T}}(\mathcal{P}, X[i]) = 0 \text{ for all } 1 \leq i \leq d-1 \iff X \in \operatorname{add} \mathcal{P}.$$

It is easy to check that a d -rigid subcategory \mathcal{P} of \mathcal{T} is d -cluster-tilting if and only if $\mathcal{T} = \mathcal{P} * \mathcal{P}[1] * \cdots * \mathcal{P}[d-1]$ holds. An object P of \mathcal{T} is d -rigid if $\operatorname{add} P$ is a d -rigid subcategory, and d -cluster-tilting if $\operatorname{add} P$ is a d -cluster-tilting subcategory. We point out that $\operatorname{add} P$ is always functorially finite.

Let \mathcal{P} be a functorially finite d -rigid subcategory of \mathcal{T} . Let

$$\mathcal{Z} := {}^{\perp \mathcal{T}}(\mathcal{P}[1] * \mathcal{P}[2] * \cdots * \mathcal{P}[d-1]) \quad \text{and} \quad \mathcal{T}_{\mathcal{P}} := \frac{\mathcal{Z}}{[\operatorname{add} \mathcal{P}]}.$$

Then the additive category $\mathcal{T}_{\mathcal{P}}$, called the *Calabi–Yau reduction* of \mathcal{T} with respect to \mathcal{P} in [21], carries a natural structure of a triangulated category, by Theorem 2.4. Moreover,

Theorem 2.5. ([21, Theorem 4.9]) *The projection functor $\mathcal{Z} \rightarrow \mathcal{T}_{\mathcal{P}}$ induces a one-to-one correspondence between the set of d -cluster-tilting subcategories of \mathcal{T} which contains \mathcal{P} and the set of d -cluster-tilting subcategories of $\mathcal{T}_{\mathcal{P}}$.*

We will use the following cluster-Beilinson criterion for triangle equivalence due to Keller–Reiten.

Proposition 2.6. ([31, Lemma 4.5]) *Let \mathcal{C} be another d -Calabi–Yau triangulated category and let $P \in \mathcal{T}$ and $P' \in \mathcal{C}$ be d -cluster-tilting objects and $F : \mathcal{T} \rightarrow \mathcal{C}$ be a triangle functor. If F induces an equivalence $\operatorname{add} P \rightarrow \operatorname{add} P'$, then F is a triangle equivalence.*

2.3. Presilting and silting subcategories, t-structures and co-t-structures. Let \mathcal{T} be a triangulated category.

A full subcategory \mathcal{P} of \mathcal{T} is *presilting* if $\mathrm{Hom}_{\mathcal{T}}(\mathcal{P}, \mathcal{P}[i]) = 0$ for any $i > 0$. It is *silting* if in addition $\mathcal{T} = \mathrm{thick}\mathcal{P}$. An object P of \mathcal{T} is *presilting* if $\mathrm{add}P$ is a presilting subcategory and *silting* if $\mathrm{add}P$ is a silting subcategory.

Clearly triangulated categories with silting subcategories satisfy the following property.

Lemma 2.7. ([2, Proposition 2.4]) *Let \mathcal{T} be a triangulated category with a silting subcategory. Then, for any $X, Y \in \mathcal{T}$, the Hom-group $\mathrm{Hom}_{\mathcal{T}}(X, Y[i])$ vanishes for $i \gg 0$.*

A *torsion pair* of \mathcal{T} is a pair $(\mathcal{X}, \mathcal{Y})$ of full subcategories of \mathcal{T} such that

- (T1) $\mathcal{X} = {}^{\perp}\mathcal{Y}$ and $\mathcal{Y} = \mathcal{X}^{\perp}$;
- (T2) $\mathcal{T} = \mathcal{X} * \mathcal{Y}$, namely, for each $M \in \mathcal{T}$ there is a triangle $X_M \rightarrow M \rightarrow Y_M \rightarrow X[1]$ in \mathcal{T} with $X_M \in \mathcal{X}$ and $Y_M \in \mathcal{Y}$.

It is elementary that the condition (T1) can be replaced by the following condition:

- (T1') $\mathrm{Hom}_{\mathcal{T}}(\mathcal{X}, \mathcal{Y}) = 0$, $\mathcal{X} = \mathrm{add}\mathcal{X}$ and $\mathcal{Y} = \mathrm{add}\mathcal{Y}$.

A *t-structure* on \mathcal{T} ([7]) is a pair $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ of full subcategories of \mathcal{T} such that $\mathcal{T}^{\geq 1} \subset \mathcal{T}^{\geq 0}$ and $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 1})$ is a torsion pair. Here for an integer n we denote $\mathcal{T}^{\leq n} = \mathcal{T}^{\leq 0}[-n]$ and $\mathcal{T}^{\geq n} = \mathcal{T}^{\geq 0}[-n]$. In this case, the triangle in the second condition above is unique up to a unique isomorphism, and the assignments $M \mapsto X_M$ and $M \mapsto Y_M$ define two functors $\sigma^{\leq 0} : \mathcal{T} \rightarrow \mathcal{T}^{\leq 0}$ and $\sigma^{\geq 1} : \mathcal{T} \rightarrow \mathcal{T}^{\geq 1}$, called the *truncation functors*. Clearly for an integer n the pair $(\mathcal{T}^{\leq n}, \mathcal{T}^{\geq n})$ is also a *t-structure* and we denote by $\sigma^{\leq n}$ and $\sigma^{\geq n+1}$ the associated truncation functors. The *heart* $\mathcal{H} := \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$ is always an abelian category. The t-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ is said to be *bounded* if

$$\bigcup_{n \in \mathbb{Z}} \mathcal{T}^{\leq n} = \mathcal{T} = \bigcup_{n \in \mathbb{Z}} \mathcal{T}^{\geq n},$$

equivalently, if $\mathcal{T} = \mathrm{thick}\mathcal{H}$.

A *co-t-structure* on \mathcal{T} ([39, 9]) is a pair $(\mathcal{T}_{\geq 0}, \mathcal{T}_{\leq 0})$ of full subcategories of \mathcal{T} such that $\mathcal{T}_{\geq 1} \subset \mathcal{T}_{\geq 0}$ and $(\mathcal{T}_{\geq 1}, \mathcal{T}_{\leq 0})$ is a torsion pair. Here for an integer n we denote $\mathcal{T}_{\geq n} = \mathcal{T}_{\geq 0}[-n]$ and $\mathcal{T}_{\leq n} = \mathcal{T}_{\leq 0}[-n]$. It is easy to see that the *co-heart* $\mathcal{P} := \mathcal{T}_{\geq 0} \cap \mathcal{T}_{\leq 0}$ is a presilting subcategory of \mathcal{T} , but it is usually not an abelian category. The co-t-structure $(\mathcal{T}_{\geq 0}, \mathcal{T}_{\leq 0})$ is said to be *bounded* if

$$\bigcup_{n \in \mathbb{Z}} \mathcal{T}_{\geq n} = \mathcal{T} = \bigcup_{n \in \mathbb{Z}} \mathcal{T}_{\leq n},$$

equivalently, if $\mathcal{T} = \mathrm{thick}\mathcal{P}$.

Now we show that any silting subcategory gives a co-t-structure on \mathcal{T} . The following proposition is well-known, which was proved as [35, Theorem 5.5], see also [2, Proposition 2.22], [9, proof of Theorem 4.3.2] and [28].

Proposition 2.8. *Let \mathcal{P} be a silting subcategory of \mathcal{T} with $\mathcal{P} = \text{add}\mathcal{P}$.*

(a) *Then $(\mathcal{T}_{\geq 0}, \mathcal{T}_{\leq 0})$ is a bounded co-t-structure on \mathcal{T} , where*

$$\mathcal{T}_{\geq 0} := \bigcup_{i \geq 0} \mathcal{P}[-i] * \cdots * \mathcal{P}[-1] * \mathcal{P} \quad \text{and} \quad \mathcal{T}_{\leq 0} := \bigcup_{i \geq 0} \mathcal{P} * \mathcal{P}[1] * \cdots * \mathcal{P}[i].$$

(b) *For any integers i and j , we have*

$$\mathcal{T}_{\geq i} \cap \mathcal{T}_{\leq j} = \begin{cases} \mathcal{P}[-j] * \mathcal{P}[1-j] * \cdots * \mathcal{P}[-i] & \text{if } i \leq j, \\ 0 & \text{if } i > j. \end{cases}$$

Proof. (a) For the convenience of the reader, we give a proof. Since $\text{Hom}_{\mathcal{T}}(\mathcal{P}, \mathcal{P}[>0]) = 0$, the condition (T1') holds by Proposition 2.3 and there is the following equality

$$\mathcal{T} = \bigcup_{i \geq 0} \text{add}(\mathcal{P}[-i] * \mathcal{P}[1-i] * \cdots * \mathcal{P}[i-1] * \mathcal{P}[i])$$

by [2, Lemma 2.15(b)]. Applying Proposition 2.3 again, we have $\mathcal{T} = \mathcal{T}_{\geq 0} * \mathcal{T}_{< 0}$.

(b) This can be shown easily by using Lemma 2.1. □

2.4. From silting objects to t-structures. Let \mathcal{T} be a triangulated category. In this section, we show that under certain conditions silting objects yield t-structures. We refer to [32, 34, 28, 10, 4, 40] for more on this subject. For a silting subcategory \mathcal{P} in \mathcal{T} , we consider subcategories of \mathcal{T} :

$$\begin{aligned} \mathcal{P}[<0]^{\perp \mathcal{T}} &= \{X \in \mathcal{T} \mid \text{Hom}_{\mathcal{T}}(\mathcal{P}[<0], X) = 0\}, \\ \mathcal{P}[>0]^{\perp \mathcal{T}} &= \{X \in \mathcal{T} \mid \text{Hom}_{\mathcal{T}}(\mathcal{P}[>0], X) = 0\}. \end{aligned}$$

We adopt the notation in Proposition 2.8 and we have the following immediate observation.

Lemma 2.9. *We have $\mathcal{P}[<0]^{\perp \mathcal{T}} = \mathcal{T}_{\leq 0}$ and $\text{Hom}_{\mathcal{T}}(\mathcal{P}[<0]^{\perp \mathcal{T}}, \mathcal{P}[>0]^{\perp \mathcal{T}}) = 0$.*

Proof. By Proposition 2.8 (a), we have $\mathcal{P}[<0]^{\perp \mathcal{T}} = \mathcal{T}_{\geq 0}^{\perp \mathcal{T}} = \mathcal{T}_{\leq 0}$. The latter assertion is a direct consequence. □

We are interested in when $(\mathcal{P}[<0]^{\perp \mathcal{T}}, \mathcal{P}[>0]^{\perp \mathcal{T}})$ forms a t-structure on \mathcal{T} . By Lemma 2.9, this is equivalent to that $\mathcal{T} = \mathcal{P}[<0]^{\perp \mathcal{T}} * \mathcal{P}[\geq 0]^{\perp \mathcal{T}}$ holds. We assume that \mathcal{T} satisfies the following finiteness condition:

(F) $\mathrm{Hom}_{\mathcal{T}}(X, Y)$ is finitely generated as an $\mathrm{End}_{\mathcal{T}}(X)$ -module and as an $\mathrm{End}_{\mathcal{T}}(Y)^{\mathrm{op}}$ -module.

This means that $\mathrm{add}X$ is a functorially finite subcategory of \mathcal{T} for any $X \in \mathcal{T}$.

Theorem 2.10. *Let \mathcal{T} be a triangulated category satisfying the condition (F). If there exists a silting object M of \mathcal{T} such that $(M[<0]^{\perp\tau}, M[>0]^{\perp\tau})$ forms a t-structure on \mathcal{T} , then $(N[<0]^{\perp\tau}, N[>0]^{\perp\tau})$ is a t-structure on \mathcal{T} for any silting object N in \mathcal{T} .*

Proof. We have $\mathrm{Hom}_{\mathcal{T}}(N[<0]^{\perp\tau}, N[>0]^{\perp\tau}[-1]) = 0$ by Lemma 2.9. Moreover, we have $(N[<0]^{\perp\tau})[1] = N[<1]^{\perp\tau} \subset N[<0]^{\perp\tau}$ and $(N[>0]^{\perp\tau})[-1] = N[>-1]^{\perp\tau} \subset N[>0]^{\perp\tau}$. Thus to show that $(N[<0]^{\perp\tau}, N[>0]^{\perp\tau})$ is a t-structure on \mathcal{T} , it remains to show $\mathcal{T} = (N[<0]^{\perp\tau}) * (N[\geq 0]^{\perp\tau})$.

Since N is a silting object of \mathcal{T} , it follows from Proposition 2.8 (a) that

$$N[<0]^{\perp\tau} = \bigcup_{i \geq 0} (\mathrm{add}N) * (\mathrm{add}N[1]) * \cdots * (\mathrm{add}N[i]). \quad (2.4.1)$$

We may assume, up to shift, that $N \in \mathcal{T}_{\leq 0}$. Because $(\mathcal{T}_{\geq 0}, \mathcal{T}_{\leq 0})$ is a bounded co-t-structure on \mathcal{T} , there exists an integer $n \geq 1$ such that $N \in \mathcal{T}_{\geq -n}$. Therefore, applying Proposition 2.8 (b) to $\mathcal{P} = \mathrm{add}M$, we obtain

$$N \in \mathcal{T}_{\geq -n} \cap \mathcal{T}_{\leq 0} = (\mathrm{add}M) * (\mathrm{add}M[1]) * \cdots * (\mathrm{add}M[n]). \quad (2.4.2)$$

It follows that $\mathrm{Hom}_{\mathcal{T}}(M, N[i]) = 0$ for any $i > 0$ and $\mathrm{Hom}_{\mathcal{T}}(N, M[i]) = 0$ for any $i > n+1$. Applying Proposition 2.8 (b) to $\mathcal{P} = \mathrm{add}N$, we have

$$M \in (\mathrm{add}N[-n]) * \cdots * (\mathrm{add}N[-1]) * (\mathrm{add}N). \quad (2.4.3)$$

With (2.4.1), (2.4.2) and (2.4.3), one can easily check that

$$N[<0]^{\perp\tau} = \bigcup_{i \geq n} (\mathrm{add}N) * \cdots * (\mathrm{add}N[n-1]) * (\mathrm{add}M[n]) * \cdots * (\mathrm{add}M[i]) \quad (2.4.4)$$

holds. Now we define subcategories \mathcal{X} and \mathcal{Y} of \mathcal{T} by

$$\mathcal{X} := (\mathrm{add}N) * (\mathrm{add}N[1]) * \cdots * (\mathrm{add}N[2n-2]) \quad \text{and} \quad \mathcal{Y} := \mathcal{X}^{\perp\tau}.$$

Then $\mathcal{X} = \mathrm{add}\mathcal{X}$ holds by Proposition 2.3. Since \mathcal{T} satisfies the condition (F), it follows from [21, Proposition 2.4] that $(\mathcal{X}, \mathcal{Y})$ is a torsion pair of \mathcal{T} . For any $Y \in \mathcal{Y}$, we take the triangle associated to the t-structure $(M[<0]^{\perp\tau}[-1+2n], M[>0]^{\perp\tau}[-1+2n])$

$$\sigma^{\leq 1-2n}Y \longrightarrow Y \longrightarrow \sigma^{\geq 2-2n}Y \longrightarrow (\sigma^{\leq 1-2n}Y)[1]. \quad (2.4.5)$$

We prove the following as a separate Lemma.

Lemma 2.11. *We have $\sigma^{\leq 1-2n}Y \in N[<0]^{\perp\tau}$ and $\sigma^{\geq 2-2n}Y \in N[\geq 0]^{\perp\tau}$.*

Proof. We have that $\sigma^{\leq 1-2n}Y$ belongs to $\mathcal{T}^{\leq 1-2n}$, which, by (2.4.4), is a subcategory of $N[<0]^{\perp\tau}$. The first assertion follows.

To prove the second assertion, we need to show $\text{Hom}_{\mathcal{T}}(N[<0]^{\perp\tau}, \sigma^{\geq 2-2n}Y) = 0$. By (2.4.4), it suffices to show the following:

- (i) $\text{Hom}_{\mathcal{T}}(M[i], \sigma^{\geq 2-2n}Y) = 0$ for any i with $2n - 1 \leq i$;
- (ii) $\text{Hom}_{\mathcal{T}}(M[i], \sigma^{\geq 2-2n}Y) = 0$ for any i with $n \leq i \leq 2n - 2$;
- (iii) $\text{Hom}_{\mathcal{T}}(N[i], \sigma^{\geq 2-2n}Y) = 0$ for any i with $0 \leq i \leq n - 1$.

The statement (i) is clear since $\sigma^{\geq 2-2n}Y \in \mathcal{T}^{\geq 2-2n}$.

We show (ii). Since $(\sigma^{\leq 1-2n}Y)[1] \in \mathcal{T}^{\leq -2n}$, we have $\text{Hom}_{\mathcal{T}}(M[i], (\sigma^{\leq 1-2n}Y)[1]) = 0$ for any $n \leq i \leq 2n - 2$. Since $Y \in \mathcal{Y}$ and $M[i] \in \mathcal{X} = (\text{add}N) * (\text{add}N[1]) * \cdots * (\text{add}N[2n-2])$ for any $n \leq i \leq 2n - 2$ by (2.4.3), we have $\text{Hom}_{\mathcal{T}}(M[i], Y) = 0$ for any $n \leq i \leq 2n - 2$. Thus the statement follows from the triangle (2.4.5).

We show (iii). Since $Y \in \mathcal{Y}$, we have $\text{Hom}_{\mathcal{T}}(N[i], Y) = 0$ for any $0 \leq i \leq n - 1$. Since $(\sigma^{\leq 1-2n}Y)[1] \in \mathcal{T}^{\leq -2n} = \mathcal{T}_{\geq 1-2n}^{\perp\tau}$ and $N \in \mathcal{T}_{\geq -n}$, we have $\text{Hom}_{\mathcal{T}}(N[i], \sigma^{\leq 1-2n}Y) = 0$ for any $0 \leq i \leq n - 1$. Thus the statement follows from the triangle (2.4.5). \square

For any $Z \in \mathcal{T}$, we take a triangle

$$X \longrightarrow Z \longrightarrow Y \longrightarrow X[1] \quad (2.4.6)$$

with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. For Y , we take a triangle (2.4.5). Applying the octahedron axiom to triangles (2.4.6) and (2.4.5), we have a commutative diagram of triangles:

$$\begin{array}{ccccccc} X & \longrightarrow & W & \longrightarrow & \sigma^{\leq 1-2n}Y & \longrightarrow & X[1] \\ \parallel & & \downarrow & & \downarrow & & \parallel \\ X & \longrightarrow & Z & \longrightarrow & Y & \longrightarrow & X[1] \\ & & \downarrow & & \downarrow & & \\ & & \sigma^{\geq 2-2n}Y & \xlongequal{\quad} & \sigma^{\geq 2-2n}Y & & \\ & & \downarrow & & \downarrow & & \\ & & W[1] & \longrightarrow & (\sigma^{\leq 1-2n}Y)[1] & & \end{array}$$

Since $X \in \mathcal{X} \subset N[<0]^{\perp\tau}$ and $\sigma^{\leq 1-2n}Y \in N[<0]^{\perp\tau}$ by Lemma 2.11, the first row shows $W \in N[<0]^{\perp\tau}$. Since $\sigma^{\geq 2-2n}Y \in N[\geq 0]^{\perp\tau}$ by Lemma 2.11, the second column shows

$$Z \in W * (\sigma^{\geq 2-2n}Y) \subset (N[<0]^{\perp\tau}) * (N[\geq 0]^{\perp\tau}).$$

This completes the proof. \square

Next we have the following description of the heart of a t -structure which is “right adjacent” to a silting object (compare: [19, Theorem 1.3(c)] and [8, Theorem 3.4]).

Proposition 2.12. *Let \mathcal{T} be a Hom-finite triangulated category over a field k . Let M a silting object of \mathcal{T} such that $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}) := (M[<0]^{\perp\tau}, M[>0]^{\perp\tau})$ is a t -structure on \mathcal{T} and $\Lambda := \text{End}_{\mathcal{T}}(M)$. Then the functor $F := \text{Hom}_{\mathcal{T}}(M, -): \mathcal{T} \rightarrow \mathbf{mod}\Lambda$ restricts to an equivalence from the heart $\mathcal{H} = \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$ to $\mathbf{mod}\Lambda$.*

Proof. Consider the triangle

$$M_{\leq -1} \longrightarrow M \longrightarrow M_0 \longrightarrow M_{\leq -1}[1] \quad (M_{\leq -1} \in \mathcal{T}_{\leq -1} \text{ and } M_0 \in \mathcal{H}).$$

We have $\text{Hom}(M_{\leq -1}, \mathcal{H}) = 0$, and hence $F = \text{Hom}_{\mathcal{T}}(M, -)|_{\mathcal{H}}$ is isomorphic to $\text{Hom}_{\mathcal{T}}(M_0, -)|_{\mathcal{H}}$. Moreover, a direct diagram chasing shows that $\Lambda = \text{End}_{\mathcal{T}}(M)$ is isomorphic to $\text{End}_{\mathcal{T}}(M_0)$. So, by Morita’s theorem, it suffices to show that M_0 is a projective generator of \mathcal{H} .

Let

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

be a short exact sequence in \mathcal{H} . It comes from a triangle in \mathcal{T}

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1].$$

Because both $\text{Hom}_{\mathcal{T}}(M_0, X[-1])$ and $\text{Hom}_{\mathcal{T}}(M_0, X[1]) \cong \text{Hom}_{\mathcal{T}}(M, X[1])$ vanish, applying $\text{Hom}_{\mathcal{T}}(M_0, -)$ to the above triangle, we obtain an exact sequence

$$0 \longrightarrow \text{Hom}_{\mathcal{T}}(M_0, X) \longrightarrow \text{Hom}_{\mathcal{T}}(M_0, Y) \longrightarrow \text{Hom}_{\mathcal{T}}(M_0, Z) \longrightarrow 0,$$

showing that M_0 is projective in \mathcal{H} .

For $X \in \mathcal{H}$, take a right $\text{add}M$ -approximation $M^X \rightarrow X$ and form a triangle

$$N^X \longrightarrow M^X \longrightarrow X \longrightarrow N^X[1].$$

Applying $\text{Hom}_{\mathcal{T}}(M, -)$ to this triangle, we obtain long exact sequences

$$\text{Hom}_{\mathcal{T}}(M, M^X[i-1]) \rightarrow \text{Hom}_{\mathcal{T}}(M, X[i-1]) \rightarrow \text{Hom}_{\mathcal{T}}(M, N^X[i]) \rightarrow \text{Hom}_{\mathcal{T}}(M, M^X[i]).$$

We claim that $\text{Hom}_{\mathcal{T}}(M, N^X[i]) = 0$ for $i \geq 1$, hence $N^X \in \mathcal{T}^{\leq 0}$. Indeed, $\text{Hom}_{\mathcal{T}}(M, M^X[i])$ vanishes for all $i \geq 1$. If $i = 1$, then the left morphism is surjective; if $i > 1$, then $\text{Hom}_{\mathcal{T}}(M, X[i-1]) = 0$. The claim follows immediately. Now taking the 0th cohomology associated to the t -structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$, we obtain an exact sequence in \mathcal{H}

$$\begin{array}{ccccc} H^0(M^X) & \longrightarrow & H^0(X) & \longrightarrow & H^0(N^X[1]), \\ & & \parallel & & \parallel \\ & & X & & 0 \end{array}$$

showing that M_0 is a projective generator of \mathcal{H} , since $H^0(M^X) \in \text{add}H^0(M) = \text{add}M_0$. \square

2.5. Derived categories of dg algebras. We follow [24, 26].

Let k be a field and A be a dg (k -)algebra, that is, a graded algebra endowed with a compatible structure of a complex. A (right) dg A -module is a (right) graded A -module endowed with a compatible structure of a complex. Let $\mathbf{D}(A)$ denote the derived category of dg A -modules. This is a triangulated category whose shift functor is the shift of complexes. Let $\text{per}(A) = \text{thick}(A_A)$ and let $\mathbf{D}_{\text{fd}}(A)$ denote the full subcategory of $\mathbf{D}(A)$ consisting of dg A -modules whose total cohomology is finite-dimensional over k . These are two triangulated subcategories of $\mathbf{D}(A)$.

Let \mathcal{T} be an algebraic triangulated category (over k), that is, \mathcal{T} is triangle equivalent to the stable category of a Frobenius category. Assume that \mathcal{T} is idempotent complete and M is an object of \mathcal{T} such that $\mathcal{T} = \text{thick}(M)$. Then by [26, Theorem 3.8 b)], there is a dg algebra A together with a triangle equivalence $\mathcal{T} \rightarrow \text{per}(A)$ which takes M to A_A . We briefly describe the construction of A and refer to the proof of [24, Theorem 4.3] for more details. Let \mathcal{E} be a Frobenius category such that the stable category of \mathcal{E} is triangle equivalent to \mathcal{T} . Let $\text{proj}\mathcal{E}$ denote the full subcategory of projective objects of \mathcal{E} . Then $\mathbf{K}_{\text{ac}}(\text{proj}\mathcal{E})$, the homotopy category of acyclic complexes on $\text{proj}\mathcal{E}$, is triangle equivalent to \mathcal{T} . Let \widetilde{M} be a preimage of M under this equivalence and let A be the dg endomorphism algebra of \widetilde{M} . Then there is a natural triangle functor $\mathbf{K}_{\text{ac}}(\text{proj}\mathcal{E}) \rightarrow \text{per}(A)$ which turns out to be a triangle equivalence and which takes \widetilde{M} to A_A . Composing this equivalence with the equivalence $\mathbf{K}_{\text{ac}}(\text{proj}\mathcal{E}) \rightarrow \mathcal{T}$, we obtain a triangle equivalence $\mathcal{T} \rightarrow \text{per}(A)$ which takes M to A_A .

3. SILTING REDUCTION AS SUBFACTOR CATEGORY

In this section we show that under mild conditions, the silting reduction in the sense of [2] of a triangulated category \mathcal{T} by the thick triangulated category generated by a presilting subcategory is triangle equivalent a certain subfactor category of \mathcal{T} . We also discuss various applications of this result.

3.1. The additive equivalence. Let \mathcal{T} be a triangulated category. We fix a presilting subcategory \mathcal{P} of \mathcal{T} . Let

$$\mathcal{S} := \text{thick}_{\mathcal{T}}\mathcal{P} \quad \text{and} \quad \mathcal{U} := \mathcal{T}/\mathcal{S}.$$

We call \mathcal{U} the *silting reduction* of \mathcal{T} with respect to \mathcal{P} (see [2]). In the rest, we assume $\mathcal{P} = \text{add}\mathcal{P}$ for simplicity. Moreover we assume the following mild technical conditions:

(P1) \mathcal{P} is a functorially finite subcategory of \mathcal{T} .

(P2) For any $X \in \mathcal{T}$, we have $\mathrm{Hom}_{\mathcal{T}}(X, \mathcal{P}[\ell]) = 0 = \mathrm{Hom}_{\mathcal{T}}(\mathcal{P}, X[\ell])$ for $\ell \gg 0$.

For example, (P1) is satisfied when \mathcal{T} is Hom-finite over a field and $\mathcal{P} = \mathbf{add}(P)$ for a presilting object P ; By Lemma 2.7, (P2) is satisfied when \mathcal{T} admits a silting subcategory which contains \mathcal{P} .

For an integer ℓ , we define full subcategories of \mathcal{T} by

$$\begin{aligned} \mathcal{S}_{\geq \ell} = \mathcal{S}_{> \ell-1} &:= \bigcup_{i \geq 0} \mathcal{P}[-\ell - i] * \cdots * \mathcal{P}[-\ell - 1] * \mathcal{P}[-\ell], \\ \mathcal{S}_{\leq \ell} = \mathcal{S}_{< \ell+1} &:= \bigcup_{i \geq 0} \mathcal{P}[-\ell] * \mathcal{P}[-\ell + 1] * \cdots * \mathcal{P}[-\ell + i]. \end{aligned}$$

Then $\mathcal{S}_{\geq \ell} = \mathbf{add} \mathcal{S}_{\geq \ell}$ and $\mathcal{S}_{\leq \ell} = \mathbf{add} \mathcal{S}_{\leq \ell}$ by Proposition 2.3, cf. also [35, Remark 5.6]. We introduce a full subcategory \mathcal{Z} of \mathcal{T} by

$$\mathcal{Z} := ({}^{\perp \tau} \mathcal{S}_{< 0}) \cap (\mathcal{S}_{> 0}^{\perp \tau}) = ({}^{\perp \tau} \mathcal{P}[> 0]) \cap (\mathcal{P}[< 0]^{\perp \tau}).$$

The following result shows that we can realise the triangle quotient $\mathcal{U} = \mathcal{T}/\mathcal{S}$ as a subfactor category of \mathcal{T} . Let $\pi: \mathcal{T} \rightarrow \mathcal{U}$ be the canonical projection functor.

Theorem 3.1. *The composition $\mathcal{Z} \subset \mathcal{T} \xrightarrow{\pi} \mathcal{U}$ of natural functors induces an equivalence of additive categories:*

$$\bar{\pi}: \frac{\mathcal{Z}}{[\mathcal{P}]} \xrightarrow{\simeq} \mathcal{U}.$$

We will see in Example 3.11 that this result, together with Theorem 3.7, gives the famous triangle equivalence

$$\underline{\mathrm{CMA}} \simeq \mathrm{D}^b(\mathrm{mod} A) / \mathrm{K}^b(\mathrm{proj} A)$$

for an Iwanaga–Gorenstein ring A due to Buchweitz [11, Theorem 4.4.1(b)] by putting $\mathcal{T} := \mathrm{D}^b(\mathrm{mod} A)$ and $\mathcal{P} := \mathrm{proj} A$.

The rest of this subsection is devoted to the proof of Theorem 3.1. We start with the following useful observation, which generalises Proposition 2.8.

Proposition 3.2. *The two pairs $({}^{\perp \tau} \mathcal{S}_{< 0}, \mathcal{S}_{\leq 0})$ and $(\mathcal{S}_{\geq 0}, \mathcal{S}_{> 0}^{\perp \tau})$ are co-t-structures on \mathcal{T} with co-heart \mathcal{P} .*

Proof. We only prove that $({}^{\perp \tau} \mathcal{S}_{< 0}, \mathcal{S}_{\leq 0})$ is a co-t-structure on \mathcal{T} with co-heart \mathcal{P} since the other assertion can be shown similarly. We first show that $({}^{\perp \tau} \mathcal{S}_{< 0}, \mathcal{S}_{\leq 0})$ is a co-t-structure. This is equivalent to showing that $({}^{\perp \tau} \mathcal{S}_{< 0}, \mathcal{S}_{< 0})$ is a torsion pair. Since ${}^{\perp \tau} \mathcal{S}_{< 0} = \mathbf{add} {}^{\perp \tau} \mathcal{S}_{< 0}$

holds clearly and $\mathcal{S}_{\leq 0} = \text{add}\mathcal{S}_{\leq 0}$ holds by Proposition 2.3, it is enough to show that any object $X \in \mathcal{T}$ belongs to $({}^{\perp\tau}\mathcal{S}_{<0}) * \mathcal{S}_{<0}$. By our assumption (P2), there exists some integer ℓ such that $X \in {}^{\perp\tau}\mathcal{S}_{<-\ell}$. If $\ell \leq 0$, then the assertion is clear. Thus we assume $\ell > 0$ and induct on ℓ . By our assumption (P1), there exists a triangle

$$Y \longrightarrow X \xrightarrow{f} P[\ell] \longrightarrow Y[1]$$

with a left $\mathcal{P}[\ell]$ -approximation f of X . Applying $\text{Hom}_{\mathcal{T}}(-, \mathcal{S}_{<-\ell})$ and $\text{Hom}_{\mathcal{T}}(-, \mathcal{P}[\ell])$, we have $Y \in {}^{\perp\tau}\mathcal{S}_{\leq -\ell}$. By the induction hypothesis, we have $Y \in ({}^{\perp\tau}\mathcal{S}_{<0}) * \mathcal{S}_{<0}$. Thus $X \in Y * P[\ell] \in ({}^{\perp\tau}\mathcal{S}_{<0}) * (\mathcal{S}_{<0} * \mathcal{P}[\ell]) = ({}^{\perp\tau}\mathcal{S}_{<0}) * \mathcal{S}_{<0}$ holds since $\mathcal{S}_{\leq 0}$ is extension closed.

Next, we show that the co-heart of $({}^{\perp\tau}\mathcal{S}_{<0}, \mathcal{S}_{\leq 0})$ is \mathcal{P} . It is easy to see that \mathcal{P} is a subcategory of the co-heart. Conversely, let X be an object of the co-heart. Then $X \in \mathcal{S}_{\leq 0}$ and $\text{Hom}_{\mathcal{T}}(X, \mathcal{S}_{<0}) = 0$. Since $\mathcal{S}_{\leq 0} = \mathcal{P} * \mathcal{S}_{<0}$, there is a triangle

$$P \xrightarrow{a} X \xrightarrow{b} Y \longrightarrow P[1]$$

with $P \in \mathcal{P}$ and $Y \in \mathcal{S}_{<0}$. Then $b = 0$ holds since $\text{Hom}_{\mathcal{T}}(X, \mathcal{S}_{<0}) = 0$. Thus a is a split epimorphism and we have $X \in \mathcal{P}$. \square

We stress that to obtain Proposition 3.2 the condition (P2) is necessary:

Remark 3.3. For a presilting subcategory \mathcal{P} of \mathcal{T} , the condition (P2) is satisfied if $({}^{\perp\tau}\mathcal{S}_{<0}, \mathcal{S}_{\leq 0})$ and $(\mathcal{S}_{\geq 0}, \mathcal{S}_{>0}{}^{\perp\tau})$ are co-t-structures on \mathcal{T} .

Proof. Clearly $\text{Hom}({}^{\perp\tau}\mathcal{S}_{<0}, \mathcal{P}[>0]) = 0$ holds. For any X in $\mathcal{S}_{<0}$, $\text{Hom}(X, \mathcal{P}[\gg 0]) = 0$ holds. Since any X in \mathcal{T} belongs to $({}^{\perp\tau}\mathcal{S}_{<0}) * \mathcal{S}_{<0}$, we have $\text{Hom}(X, \mathcal{P}[\gg 0]) = 0$. Similarly, since any X in \mathcal{T} belongs to $\mathcal{S}_{>0} * (\mathcal{S}_{>0}{}^{\perp\tau})$, we have $\text{Hom}(\mathcal{P}, X[\gg 0]) = 0$. \square

Next we show that our functor in Theorem 3.1 is dense.

Lemma 3.4. *For any $X \in \mathcal{U}$, there exists $Y \in \mathcal{Z}$ satisfying $X \simeq Y$ in \mathcal{U} . As a consequence, the functor $\bar{\pi} : \frac{\mathcal{Z}}{[\mathcal{P}]} \rightarrow \mathcal{U}$ in Theorem 3.1 is dense.*

Proof. Let $X \in \mathcal{U}$. By Proposition 3.2, we have a triangle

$$X' \longrightarrow X \longrightarrow S \longrightarrow X'[1] \quad (X' \in {}^{\perp\tau}\mathcal{S}_{<0}, S \in \mathcal{S}_{<0}).$$

Then we have $X \simeq X'$ in \mathcal{U} . Again by Proposition 3.2, we have a triangle

$$S' \longrightarrow X' \longrightarrow X'' \longrightarrow S'[1] \quad (S' \in \mathcal{S}_{>0}, X'' \in \mathcal{S}_{>0}{}^{\perp\tau}).$$

Then we have $X \simeq X' \simeq X''$ in \mathcal{U} . Applying $\mathrm{Hom}_{\mathcal{T}}(-, \mathcal{S}_{<0})$, we see that $\mathrm{Hom}_{\mathcal{T}}(X'', \mathcal{S}_{<0}) \simeq \mathrm{Hom}_{\mathcal{T}}(X', \mathcal{S}_{<0})$ vanishes. Thus $Y := X''$ belongs to $({}^{\perp\tau}\mathcal{S}_{<0}) \cap (\mathcal{S}_{>0}{}^{\perp\tau}) = \mathcal{Z}$, and we have an isomorphism $X \simeq Y$ in \mathcal{U} . \square

Finally we show that our functor is fully faithful.

Lemma 3.5. *The functor $\mathcal{T} \rightarrow \mathcal{U}$ induces the following bijective maps for any $M \in {}^{\perp\tau}\mathcal{S}_{<0}$ and $N \in \mathcal{S}_{>0}{}^{\perp\tau}$*

$$\begin{aligned} \mathrm{Hom}_{\frac{\mathcal{T}}{[\mathcal{P}]}}(M, N) &\longrightarrow \mathrm{Hom}_{\mathcal{U}}(M, N), \\ \mathrm{Hom}_{\mathcal{T}}(M, N[\ell]) &\longrightarrow \mathrm{Hom}_{\mathcal{U}}(M, N[\ell]) \quad (\ell > 0). \end{aligned}$$

As a consequence, the functor $\bar{\pi} : \frac{\mathcal{Z}}{[\mathcal{P}]} \rightarrow \mathcal{U}$ in Theorem 3.1 is fully faithful.

Proof. (i) We show the surjectivity.

Let $\ell \geq 0$. Any morphism $\mathrm{Hom}_{\mathcal{U}}(M, N[\ell])$ has a representative of the form $M \xrightarrow{f} X \xleftarrow{s} N[\ell]$, where $f \in \mathrm{Hom}_{\mathcal{T}}(M, X)$ and $s \in \mathrm{Hom}_{\mathcal{T}}(N[\ell], X)$, such that the cone of s is in \mathcal{S} . Take a triangle

$$N[\ell] \xrightarrow{s} X \longrightarrow S \xrightarrow{a} N[\ell + 1] \quad (S \in \mathcal{S}).$$

By Proposition 3.2, we can take a triangle

$$S_{\geq 0} \xrightarrow{b} S \longrightarrow S_{<0} \longrightarrow S_{\geq 0}[1] \quad (S_{\geq 0} \in \mathcal{S}_{\geq 0}, S_{<0} \in \mathcal{S}_{<0}).$$

Since $ba = 0$ by $S_{\geq 0} \in \mathcal{S}_{\geq 0}$ and $N[\ell + 1] \in \mathcal{S}_{>-\ell-1}{}^{\perp\tau}$, we have the following commutative diagram by the octahedral axiom.

$$\begin{array}{ccccccc} & & S_{\geq 0} & \xlongequal{\quad} & S_{\geq 0} & & \\ & & \downarrow & & \downarrow b & & \\ N[\ell] & \xrightarrow{s} & X & \longrightarrow & S & \xrightarrow{a} & N[\ell + 1] \\ & \parallel & \downarrow c & & \downarrow & & \parallel \\ N[\ell] & \xrightarrow{sc} & X' & \xrightarrow{d} & S_{<0} & \longrightarrow & N[\ell + 1] \\ & & \downarrow & & \downarrow & & \\ & & S_{\geq 0}[1] & \xlongequal{\quad} & S_{\geq 0}[1] & & \end{array}$$

Then we have $gcd = 0$ by $M \in {}^{\perp\tau}\mathcal{S}_{<0}$ and $S_{<0} \in \mathcal{S}_{<0}$. Thus there exists $e \in \mathrm{Hom}_{\mathcal{T}}(M, N[\ell])$ such that $fc = esc$. Now $(f - es)c = 0$ implies that $f - es$ factors through $S_{\geq 0} \in \mathcal{S}$. Thus $f = es$ and $fs^{-1} = e$ hold in \mathcal{U} , and we have the assertion.

(ii) We show the injectivity.

Let $\ell \geq 0$. Assume that a morphism $f \in \text{Hom}_{\mathcal{T}}(M, N[\ell])$ is zero in \mathcal{U} . Then it factors through \mathcal{S} (by, for example, [37, Lemma 2.1.26]), that is, there exist $S \in \mathcal{S}$, $g \in \text{Hom}_{\mathcal{T}}(M, S)$ and $a \in \text{Hom}_{\mathcal{T}}(S, N[\ell])$ such that $f = ga$. Take a triangle

$$S_{>-\ell} \xrightarrow{b} S \xrightarrow{c} S_{\leq-\ell} \longrightarrow S_{>-\ell}[1] \quad (S_{>-\ell} \in \mathcal{S}_{>-\ell}, S_{\leq-\ell} \in \mathcal{S}_{\leq-\ell}).$$

Since $ba = 0$ by $S_{>-\ell} \in \mathcal{S}_{>-\ell}$ and $N[\ell] \in \mathcal{S}_{>-\ell}^{\perp\tau}$, there exists $d \in \text{Hom}_{\mathcal{T}}(S_{\leq-\ell}, N[\ell])$ such that $a = cd$.

$$\begin{array}{ccccc} S_{>-\ell} & \xrightarrow{b} & S & \xrightarrow{c} & S_{\leq-\ell} \\ & \nearrow g & & \searrow a & \downarrow d \\ M & \xrightarrow{f} & & & N[\ell] \end{array}$$

First we assume $\ell > 0$. Then $gc = 0$ because $M \in {}^{\perp\tau}\mathcal{S}_{<0}$ and $S_{\leq-\ell} \in \mathcal{S}_{\leq-\ell} \subset \mathcal{S}_{<0}$. Thus we have $f = gcd = 0$.

Next we assume $\ell = 0$. Take a triangle

$$P \longrightarrow S_{\leq 0} \xrightarrow{e} S_{<0} \longrightarrow P[1] \quad (P \in \mathcal{P}, S_{<0} \in \mathcal{S}_{<0}).$$

Then we have $gce = 0$ by $M \in {}^{\perp\tau}\mathcal{S}_{<0}$ and $S_{<0} \in \mathcal{S}_{<0}$. Thus gc factors through P , and $f = gcd = 0$ in $\frac{\mathcal{T}}{[\mathcal{P}]}$. \square

3.2. The triangle equivalence. Let \mathcal{T} be a triangulated category and \mathcal{P} a presilting subcategory of \mathcal{T} satisfying (P1) and (P2). Keep the notation in Section 3.1. The aim of this subsection is to show that the additive category $\frac{\mathcal{Z}}{[\mathcal{P}]}$ has the structure of a triangulated category, and that the equivalence given in Theorem 3.1 is a triangle equivalence.

Lemma 3.6. *The pair $(\mathcal{Z}, \mathcal{Z})$ forms a \mathcal{P} -mutation pair (see Section 2.2). More precisely, for $T \in \mathcal{T}$, the following conditions are equivalent.*

- (a) $T \in \mathcal{Z}$.
- (b) *There exists a triangle $X \xrightarrow{a} P \rightarrow T \rightarrow X[1]$ with $X \in \mathcal{Z}$ and a left \mathcal{P} -approximation a .*
- (c) *There exists a triangle $T \rightarrow P' \xrightarrow{b} Y \rightarrow T[1]$ with $Y \in \mathcal{Z}$ and a right \mathcal{P} -approximation b .*

Proof. We only show the equivalence of (a) and (b) since the equivalence of (a) and (c) can be shown dually.

(b) \Rightarrow (a) It is easy to check, by applying $\mathrm{Hom}_{\mathcal{T}}(\mathcal{P}, -)$ to the triangle, that $\mathrm{Hom}_{\mathcal{T}}(\mathcal{P}, T[>0]) = 0$ holds. Similarly by applying $\mathrm{Hom}_{\mathcal{T}}(-, \mathcal{P})$ to the triangle, it is easy to check that $\mathrm{Hom}_{\mathcal{T}}(T, \mathcal{P}[>0]) = 0$ holds. Therefore $T \in \mathcal{Z}$.

(a) \Rightarrow (b) We take a triangle $X \xrightarrow{a} P \xrightarrow{b} T \rightarrow X[1]$ with a right \mathcal{P} -approximation b . It is easy to check, by applying $\mathrm{Hom}_{\mathcal{T}}(\mathcal{P}, -)$ to the triangle, that $\mathrm{Hom}_{\mathcal{T}}(\mathcal{P}, X[>0]) = 0$ holds. Similarly by applying $\mathrm{Hom}_{\mathcal{T}}(-, \mathcal{P})$ to the triangle, it is easy to check that $\mathrm{Hom}_{\mathcal{T}}(X, \mathcal{P}[>0]) = 0$ holds and that a is a left \mathcal{P} -approximation. Therefore $X \in \mathcal{Z}$. \square

As a consequence of this lemma, the category $\frac{\mathcal{Z}}{[\mathcal{P}]}$ has a natural structure of triangulated category, according to Theorem 2.4. Now we prove the following result.

Theorem 3.7. *The category $\frac{\mathcal{Z}}{[\mathcal{P}]}$ has a structure of a triangulated category given in Theorem 2.4 such that the functor $\bar{\pi}: \frac{\mathcal{Z}}{[\mathcal{P}]} \rightarrow \mathcal{U}$ in Theorem 3.1 is a triangle equivalence.*

Proof. We need to show that the equivalence $\bar{\pi}: \frac{\mathcal{Z}}{[\mathcal{P}]} \rightarrow \mathcal{U}$ is a triangle functor.

Applying the triangle functor π to the triangle $X \rightarrow P_X \rightarrow X\langle 1 \rangle \rightarrow X[1]$ in Theorem 2.4(a), we have an isomorphism $X\langle 1 \rangle \rightarrow X[1]$ in \mathcal{U} , which defines a natural isomorphism $\bar{\pi} \circ \langle 1 \rangle \simeq [1] \circ \bar{\pi}$.

Let

$$X \xrightarrow{\bar{f}} Y \xrightarrow{\bar{g}} Z \xrightarrow{\bar{a}} X\langle 1 \rangle \quad (3.2.1)$$

be a triangle given in Theorem 2.4(b). Applying the triangle functor $\mathcal{T} \rightarrow \mathcal{U}$ to (2.2.1), we have a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\ \parallel & & \downarrow & & \downarrow a & & \parallel \\ X & \longrightarrow & 0 & \longrightarrow & X\langle 1 \rangle & \xrightarrow{\sim} & X[1] \end{array}$$

of triangles in \mathcal{U} . Thus the image of (3.2.1) by the functor $\frac{\mathcal{Z}}{[\mathcal{P}]} \rightarrow \mathcal{U}$ is a triangle. \square

3.3. The correspondence between silting subcategories. Let \mathcal{T} be a triangulated category. We denote by $\mathrm{silt} \mathcal{T}$ (respectively, $\mathrm{presilt} \mathcal{T}$) the class of silting (respectively, presilting) subcategories of \mathcal{T} . As usual we identify two (pre)silting subcategories \mathcal{M} and \mathcal{N} of \mathcal{T} when $\mathrm{add} \mathcal{M} = \mathrm{add} \mathcal{N}$.

Fix a presilting subcategory \mathcal{P} of \mathcal{T} and denote by $\mathrm{silt}_{\mathcal{P}} \mathcal{T}$ (respectively, $\mathrm{presilt}_{\mathcal{P}} \mathcal{T}$) the class of silting (respectively, presilting) subcategories of \mathcal{T} containing \mathcal{P} . Assume further that the conditions (P1) and (P2) are satisfied. Keep the notation in Section 3.1.

Theorem 3.8. *The natural functor $\pi: \mathcal{T} \rightarrow \mathcal{U}$ induces bijections $\text{silt}_{\mathcal{P}} \mathcal{T} \rightarrow \text{silt} \mathcal{U}$ and $\text{presilt}_{\mathcal{P}} \mathcal{T} \rightarrow \text{presilt} \mathcal{U}$.*

Proof. (i) We will show that π induces a map $\text{presilt}_{\mathcal{P}} \mathcal{T} \rightarrow \text{presilt} \mathcal{U}$.

Let \mathcal{M} be a silting subcategory of \mathcal{T} containing \mathcal{P} . Clearly we have $\mathcal{M} \subset {}^{\perp} \mathcal{Z}$. By Lemma 3.5, we have

$$\text{Hom}_{\mathcal{U}}(\mathcal{M}, \mathcal{M}[>0]) = \text{Hom}_{\mathcal{T}}(\mathcal{M}, \mathcal{M}[>0]) = 0.$$

Thus $\pi \mathcal{M}$ is a presilting subcategory of \mathcal{U} .

(ii) We will show that the map $\text{presilt}_{\mathcal{P}} \mathcal{T} \rightarrow \text{presilt} \mathcal{U}$ is bijective.

Since π induces an equivalence $\frac{\mathcal{Z}}{[\mathcal{P}]} \simeq \mathcal{U}$, the correspondence $\text{presilt}_{\mathcal{P}} \mathcal{T} \rightarrow \text{presilt} \mathcal{U}$ is injective. We will show the surjectivity. For a silting subcategory \mathcal{N} of \mathcal{U} , we define a subcategory \mathcal{M} of \mathcal{T} by

$$\mathcal{M} := \{X \in \mathcal{Z} \mid \pi X \in \mathcal{N}\}.$$

Then $\pi \mathcal{M} = \mathcal{N}$ holds. Moreover, by Lemma 3.5, we have

$$\text{Hom}_{\mathcal{T}}(\mathcal{M}, \mathcal{M}[>0]) = \text{Hom}_{\mathcal{U}}(\mathcal{N}, \mathcal{N}[>0]) = 0.$$

Thus the assertion follows.

(iii) We will show that π induces a bijective map $\text{silt}_{\mathcal{P}} \mathcal{T} \rightarrow \text{silt} \mathcal{U}$.

Let \mathcal{M} be a presilting subcategory of \mathcal{T} containing \mathcal{P} and $\mathcal{N} := \pi \mathcal{M}$ the corresponding presilting subcategory of \mathcal{U} . By (ii), it is enough to show that $\text{thick}_{\mathcal{T}} \mathcal{M} = \mathcal{T}$ holds if and only if $\text{thick}_{\mathcal{U}} \mathcal{N} = \mathcal{U}$ holds. This follows from the fact that π induces a bijection between thick subcategories of \mathcal{T} containing \mathcal{P} and thick subcategories of \mathcal{U} . \square

For $\mathcal{M}, \mathcal{N} \in \text{silt} \mathcal{T}$, we write $\mathcal{M} \geq \mathcal{N}$ if $\text{Hom}_{\mathcal{T}}(\mathcal{M}, \mathcal{N}[>0]) = 0$. This gives a partial order \geq on $\text{silt} \mathcal{T}$, see [2, Theorem 2.11].

Corollary 3.9. *The bijections $\text{silt}_{\mathcal{P}} \mathcal{T} \rightarrow \text{silt} \mathcal{U}$ and $\text{presilt}_{\mathcal{P}} \mathcal{T} \rightarrow \text{presilt} \mathcal{U}$ in Theorem 3.8 are isomorphisms of partially ordered sets.*

Proof. Let \mathcal{M} and \mathcal{N} be silting subcategories of \mathcal{T} containing \mathcal{P} . By Lemma 3.5, we have

$$\text{Hom}_{\mathcal{T}}(\mathcal{M}, \mathcal{N}[>0]) \simeq \text{Hom}_{\mathcal{U}}(\mathcal{M}, \mathcal{N}[>0]).$$

Thus $\mathcal{M} \geq \mathcal{N}$ if and only if $\pi \mathcal{M} \geq \pi \mathcal{N}$. \square

Next we discuss the completion of ‘‘almost complete’’ presilting subcategories.

Let X be an object in \mathcal{T} such that $\mathcal{M} = \text{add}\{\mathcal{P}, X\}$ is a silting subcategory of \mathcal{T} . Let

$$X \xrightarrow{f} P' \longrightarrow Y \longrightarrow X[1] \quad \text{and} \quad Z \longrightarrow P'' \xrightarrow{g} X \longrightarrow Z[1]$$

be triangles in \mathcal{T} with a left \mathcal{P} -approximation f of X and a right \mathcal{P} -approximation g of X . It was shown in [2, Theorem 2.31] that $\mu_X^-(\mathcal{M}) := \text{add}\{\mathcal{P}, Y\}$ and $\mu_X^+(\mathcal{M}) := \text{add}\{\mathcal{P}, Z\}$ are again silting subcategories of \mathcal{T} , which we call *the left mutation and the right mutation of \mathcal{M} at X* , respectively.

The following result was shown in [2, Theorem 2.44] under the strong restriction that $\text{thick}\mathcal{P}$ is functorially finite in \mathcal{T} .

Corollary 3.10. *Assume further that \mathcal{T} is Krull–Schmidt. If there exists an indecomposable object $X \in \mathcal{T}$ such that $X \notin \mathcal{P}$ and $\mathcal{M} = \text{add}\{\mathcal{P}, X\}$ is a silting subcategory of \mathcal{T} , then we have*

$$\text{silt}_{\mathcal{P}}\mathcal{T} = \{\mathcal{M}_i = \text{add}\{\mathcal{P}, X_i\} \mid i \in \mathbb{Z}\},$$

where $\mathcal{M}_{i\pm 1} = \mu_{X_i}^{\mp}(\mathcal{M}_i)$ for all $i \in \mathbb{Z}$.

Proof. By Theorem 3.8, we have a bijection $\text{silt}_{\mathcal{P}}\mathcal{T} \rightarrow \text{silt}\mathcal{U}$. In particular \mathcal{U} has an indecomposable silting object X . By [2, Theorem 2.26], we have $\text{silt}\mathcal{U} = \{X\langle i \rangle \mid i \in \mathbb{Z}\}$. Since $X_i = X\langle i \rangle$ by our construction, we have the assertion. \square

3.4. A theorem of Buchweitz. Recall that a noetherian ring A is called *Iwanaga–Gorenstein* if A has finite injective dimension as an A -module and also as an A^{op} -module (see e.g. [14]). In this case, we define the category of *Cohen–Macaulay A -modules* (also often called *modules of Gorenstein dimension zero*, *Gorenstein projective modules*, or *totally reflexive modules*) by

$$\text{CMA} := \{X \in \text{mod}A \mid \text{Ext}_A^i(X, A) = 0 \text{ for any } i > 0\}.$$

This has a natural structure of a Frobenius category whose projective-injective objects are exactly the projective A -modules, and we denote by $\underline{\text{CMA}}$ its stable category.

Example 3.11. Let A be an Iwanaga–Gorenstein ring, $\mathcal{T} := \text{D}^b(\text{mod}A)$ and $\mathcal{P} := \text{proj}A$. Then $\mathcal{Z} = ({}^{\perp\tau}\mathcal{S}_{<0}) \cap (\mathcal{S}_{>0}{}^{\perp\tau})$ is given by CMA . Therefore the silting reduction $\frac{\mathcal{Z}}{[\mathcal{P}]}$ is the stable category $\underline{\text{CMA}}$, and we have a triangle equivalence

$$\underline{\text{CMA}} \xrightarrow{\sim} \text{D}^b(\text{mod}A)/\text{K}^b(\text{proj}A),$$

which is a classical result [11, Theorem 4.4.1(b)] due to Buchweitz.

Proof. Let $(\mathcal{D}^{\leq 0}(\mathbf{mod}B), \mathcal{D}^{\geq 0}(\mathbf{mod}B))$ be the standard t-structure on $\mathcal{D}^b(\mathbf{mod}B)$ for $B = A$ or A^{op} . Let $\mathcal{T}' := \mathcal{D}^b(\mathbf{mod}A^{\text{op}})$, $\mathcal{P}' := \text{proj}A^{\text{op}}$ and $\mathcal{S}'_{>0} := \bigcup_{i \geq 0} \mathcal{P}'[-i] * \cdots * \mathcal{P}'[-2] * \mathcal{P}'[-1]$. Then we have

$$\mathcal{S}_{>0}^{\perp \tau} = \mathcal{D}^{\leq 0}(\mathbf{mod}A) \quad \text{and} \quad \mathcal{S}'_{>0}{}^{\perp \tau'} = \mathcal{D}^{\leq 0}(\mathbf{mod}A^{\text{op}}). \quad (3.4.1)$$

In particular, we have $\mathbf{mod}A \subset \mathcal{S}_{>0}^{\perp \tau}$ and

$$\mathbf{mod}A \cap \mathcal{Z} = \mathbf{mod}A \cap {}^{\perp \tau} \mathcal{S}_{<0} = \text{CMA}.$$

It is enough to show $\mathcal{Z} \subset \mathbf{mod}A$. Since A is Iwanaga–Gorenstein, we have a duality

$$(-)^* := \mathbf{RHom}_A(-, A): \mathcal{T} \rightarrow \mathcal{T}'$$

(e.g. [36, Corollary 2.11]) as in the commutative case [18]. Since $\mathcal{S}_{<0} = (\mathcal{S}'_{>0})^*$ holds clearly, we have

$${}^{\perp \tau} \mathcal{S}_{<0} = (\mathcal{S}'_{>0}{}^{\perp \tau'})^* \stackrel{(3.4.1)}{=} (\mathcal{D}^{\leq 0}(\mathbf{mod}A^{\text{op}}))^* \subset \mathcal{D}^{\geq 0}(\mathbf{mod}A).$$

Therefore $\mathcal{Z} = ({}^{\perp \tau} \mathcal{S}_{<0}) \cap (\mathcal{S}_{>0}{}^{\perp \tau}) \subset \mathcal{D}^{\leq 0}(\mathbf{mod}A) \cap \mathcal{D}^{\geq 0}(\mathbf{mod}A) = \mathbf{mod}A$ holds. \square

Another application of Theorem 3.1 is the following.

Corollary 3.12. *Let k be a field and A be a finite-dimensional k -algebra. Assume that P is a finitely generated projective A -module which has finite injective dimension. Then the triangle quotient $\mathcal{D}^b(\mathbf{mod}A)/\text{thick}P$ is Hom-finite and Krull–Schmidt.*

Proof. Let $\mathcal{P} = \text{add}P$ and define the full subcategory \mathcal{Z} of $\mathcal{D}^b(\mathbf{mod}A)$ as in Section 3. Then \mathcal{Z} is closed under direct summands. Thus it is Hom-finite and Krull–Schmidt, so is the additive quotient $\frac{\mathcal{Z}}{[\mathcal{P}]} \cong \mathcal{D}^b(\mathbf{mod}A)/\text{thick}P$. \square

3.5. Conjectures of Auslander–Reiten and Tachikawa. In this subsection, we discuss the relationship between silting theory and the conjecture of Tachikawa and that of Auslander–Reiten.

Let k be a field and A be a finite-dimensional k -algebra and let n be the number of pairwise non-isomorphic simple A -modules. Motivated by Tachikawa’s study [41] on the famous Nakayama conjecture, Auslander and Reiten proposed the following conjecture:

The Auslander–Reiten Conjecture ([5]) *If $X \in \mathbf{mod}A$ satisfies $\text{Ext}_A^i(X, X \oplus A) = 0$ for all $i > 0$, then X is a projective A -module.*

Now we pose the following conjectures in the context of silting theory.

Conjecture 3.13. $D^b(\text{mod}A)$ has no presilting object X such that $\text{add}X$ contains $\text{proj}A$ as a proper subcategory.

Conjecture 3.14. There does not exist a thick subcategory \mathcal{T} of $D^b(\text{mod}A)$ containing $K^b(\text{proj}A)$ such that the Grothendieck group $K_0(\mathcal{T})$ is a free abelian group with rank strictly bigger than n .

We have the following observation (see also Section 4 of [17]).

Theorem 3.15. *Conjecture 3.14 \Rightarrow Conjecture 3.13 \Rightarrow the Auslander–Reiten Conjecture.*

Proof. To prove the first implication, assume that a non-projective A -module X satisfies $\text{Ext}_A^i(X, X \oplus A) = 0$ for all $i > 0$. Then $\mathcal{T} := \text{thick}(X \oplus A)$ is a thick subcategory of $D^b(\text{mod}A)$ containing $K^b(\text{proj}A)$. Clearly $X \oplus A$ is a silting object in \mathcal{T} . It is shown in [2, Theorem 2.27] that the Grothendieck group $K_0(\mathcal{T})$ is a free abelian group and the rank is equal to the number of non-isomorphic indecomposable direct summands of $X \oplus A$. Thus the assertion follows.

The second one is clear: if $X \in \text{mod}A$ is not projective and satisfies $\text{Ext}_A^i(X, X \oplus A) = 0$ for all $i > 0$, then $X \oplus A$ is a presilting object of $D^b(\text{mod}A)$ such that $\text{add}(X \oplus A)$ contains $\text{proj}A$ as a proper subcategory. \square

When A is self-injective, Auslander–Reiten Conjecture takes the following form due to Tachikawa.

The Tachikawa Conjecture ([41]) *Assume that A is self-injective. If $X \in \text{mod}A$ satisfies $\text{Ext}_A^i(X, X) = 0$ for all $i > 0$, then X is a projective module.*

Clearly this is equivalent to the following conjecture stated in terms of presilting objects.

Conjecture 3.16. *Assume that A is self-injective. Then the stable category $\underline{\text{mod}}A$ has no non-trivial presilting objects.*

What we know is the following.

Proposition 3.17. ([2, Example 2.5]) *Assume that A is self-injective. Then the stable category $\underline{\text{mod}}A$ has no silting objects unless A is semi-simple.*

4. SILTING REDUCTION VERSUS CALABI–YAU REDUCTION

In Theorems 3.1 and 3.7, we realise silting reduction as subfactor categories. This is analogous to the Calabi–Yau reduction introduced by Yoshino and the first author in

[21]. In this section, we relate these two constructions, using the results in the preceding section. We will show that silting reduction of Calabi–Yau triangulated categories induces Calabi–Yau reduction (Theorem 4.13), and conversely, Calabi–Yau reduction lifts to silting reduction (Theorem 4.18).

Throughout this section, let k be a field and let $D = \text{Hom}_k(-, k)$ denote the k -dual. Let $d \geq 1$ be an integer.

4.1. Calabi–Yau triples. Let \mathcal{T} be k -linear triangulated category, M an object of \mathcal{T} and \mathcal{T}^{fd} a triangulated subcategory of \mathcal{T} . We say that $(\mathcal{T}, \mathcal{T}^{\text{fd}}, M)$ is a $(d+1)$ -Calabi–Yau triple if the following conditions are satisfied.

(CY0) The category \mathcal{T} is Hom-finite and Krull–Schmidt.

(CY1) The object M is a silting object in \mathcal{T} .

(CY2) The pair $(\mathcal{T}, \mathcal{T}^{\text{fd}})$ is *relative* $(d+1)$ -Calabi–Yau in the sense that there exists a bifunctorial isomorphism for any $X \in \mathcal{T}^{\text{fd}}$ and $Y \in \mathcal{T}$:

$$D \text{Hom}_{\mathcal{T}}(X, Y) \simeq \text{Hom}_{\mathcal{T}}(Y, X[d+1]).$$

(CY3) The pair $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}) := (M[<0]^{\perp\tau}, M[>0]^{\perp\tau})$ forms a t-structure in \mathcal{T} such that $\mathcal{T}^{\geq 0}$ is contained in \mathcal{T}^{fd} .

It follows from Theorem 2.10 that the condition (CY3) is independent of the choice of a silting object M , namely, if $(\mathcal{T}, \mathcal{T}^{\text{fd}}, M)$ is a $(d+1)$ -Calabi–Yau triple, then $(\mathcal{T}, \mathcal{T}^{\text{fd}}, N)$ is $(d+1)$ -Calabi–Yau triple for any silting object N of \mathcal{T} .

In the rest of this subsection, let $(\mathcal{T}, \mathcal{T}^{\text{fd}}, M)$ be a $(d+1)$ -Calabi–Yau triple. Put

$$\begin{aligned} \mathcal{T}_{\leq 0} &:= \bigcup_{i \geq 0} (\text{add}M) * (\text{add}M[1]) * \cdots * (\text{add}M[i]), \\ \mathcal{T}_{\geq 0} &:= \bigcup_{i \geq 0} (\text{add}M[-i]) * \cdots * (\text{add}M[-1]) * (\text{add}M). \end{aligned}$$

Then $(\mathcal{T}_{\geq 0}, \mathcal{T}_{\leq 0})$ is a bounded co-t-structure on \mathcal{T} with co-heart $\text{add}M$, by Proposition 3.2. As a consequence, $\mathcal{T}_{\leq 0} = (\mathcal{T}_{\geq 0}[-1])^{\perp} = M[<0]^{\perp\tau} = \mathcal{T}^{\leq 0}$. Thus the co-t-structure $(\mathcal{T}_{\geq 0}, \mathcal{T}_{\leq 0})$ is left adjacent to the t-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ in the sense of Bondarko [9]. Moreover, since \mathcal{T}^{fd} is closed under shifts, we have $\mathcal{T}^{\leq i} \subset \mathcal{T}^{\text{fd}}$ for any $i \in \mathbb{Z}$.

The subcategory \mathcal{T}^{fd} of \mathcal{T} has the following characterisation, which justifies our notation for \mathcal{T}^{fd} .

Lemma 4.1. *Let X be an object of \mathcal{T} . The following statements are equivalent:*

- (a) X belongs to \mathcal{T}^{fd} ;

- (b) for any $Y \in \mathcal{T}$ the space $\mathrm{Hom}_{\mathcal{T}}(Y, X[i])$ vanishes for almost all $i \in \mathbb{Z}$;
(c) the space $\mathrm{Hom}_{\mathcal{T}}(M, X[i])$ vanishes for almost all $i \in \mathbb{Z}$.

Proof. (a) \Rightarrow (b): Let $X \in \mathcal{T}^{\mathrm{fd}}$ and $Y \in \mathcal{T}$. Then we have $\mathrm{Hom}_{\mathcal{T}}(Y, X[i]) = 0$ and $\mathrm{Hom}_{\mathcal{T}}(Y, X[-i]) \cong D \mathrm{Hom}_{\mathcal{T}}(X, Y[d+i]) = 0$ for $i \gg 0$ by Lemma 2.7.

(b) \Rightarrow (c): This is clear.

(c) \Rightarrow (a): If (c) holds, then there exists an integer i such that $\mathrm{Hom}_{\mathcal{T}}(M, X[<i]) = 0$, i.e. $X \in M[>-i]^{\perp\tau}$. Since $M[>-i]^{\perp\tau} = \mathcal{T}^{\geq i}$ is contained in $\mathcal{T}^{\mathrm{fd}}$, it follows that X belongs to $\mathcal{T}^{\mathrm{fd}}$. \square

Now we show that the t-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ restricts to a t-structure on $\mathcal{T}^{\mathrm{fd}}$.

Lemma 4.2. *The pair $(\mathcal{T}^{\mathrm{fd}} \cap \mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ is a bounded t-structure on $\mathcal{T}^{\mathrm{fd}}$. Its has the same heart \mathcal{H} as $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$. Consequently, $\mathcal{T}^{\mathrm{fd}}$ is the smallest triangulated subcategory of \mathcal{T} containing \mathcal{H} .*

Proof. For $X \in \mathcal{T}^{\mathrm{fd}}$, there is a triangle

$$\sigma^{\leq 0} X \longrightarrow X \longrightarrow \sigma^{\geq 1} X \longrightarrow (\sigma^{\leq 0} X)[1].$$

Since both X and $\sigma^{\geq 1} X$ belong to the triangulated subcategory $\mathcal{T}^{\mathrm{fd}}$ of \mathcal{T} , it follows that $\sigma^{\leq 0} X$ belongs to $\mathcal{T}^{\mathrm{fd}}$ and hence to $\mathcal{T}^{\mathrm{fd}} \cap \mathcal{T}^{\leq 0}$. This shows that $(\mathcal{T}^{\mathrm{fd}} \cap \mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ is a t-structure on $\mathcal{T}^{\mathrm{fd}}$.

Let X be any object of $\mathcal{T}^{\mathrm{fd}}$. By Lemma 4.1, there exist integers $i \leq j$ such that $\mathrm{Hom}_{\mathcal{T}}(M, X[<i]) = 0$ and $\mathrm{Hom}_{\mathcal{T}}(M, X[>j]) = 0$. Namely, X belongs to $\mathcal{T}^{\mathrm{fd}} \cap \mathcal{T}^{\leq j} \cap \mathcal{T}^{\geq i}$. By definition the t-structure $(\mathcal{T}^{\mathrm{fd}} \cap \mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ is bounded.

The second statement is clear, as $\mathcal{T}^{\geq 0} \subset \mathcal{T}^{\mathrm{fd}}$. \square

Remark 4.3. Assume further that \mathcal{T} is algebraic. Then there is a dg algebra A such that there is a triangle equivalence $\mathcal{T} \rightarrow \mathrm{per}(A)$ which takes M to A , see Section 2.5. It follows that $H^i(A) \cong \mathrm{Hom}_{\mathrm{per}(A)}(A, A[i]) \cong \mathrm{Hom}_{\mathcal{T}}(M, M[i]) = 0$ for $i > 0$ and $H^0(A) \cong \mathrm{End}_{\mathrm{per}(A)}(A) \cong \mathrm{End}_{\mathcal{T}}(M)$ is finite-dimensional over k . Let

$$\mathcal{H} := \{X \in \mathrm{per}(A) \mid H^i(X) = 0 \text{ for any } i \neq 0\}.$$

By Lemma 2.12, we have an equivalence

$$H^0 = \mathrm{Hom}_{\mathrm{per}(A)}(A, -): \mathcal{H} \rightarrow \mathrm{mod}H^0(A).$$

Therefore we have an equality

$$\mathcal{H} = \{X \in \mathrm{D}(A) \mid H^i(X) = 0 \text{ for any } i \neq 0, H^0(X) \in \mathrm{mod}H^0(A)\},$$

which implies $\text{per}(A) \supset \mathbf{D}_{\text{fd}}(A)$, since $\mathbf{D}_{\text{fd}}(A)$ is the smallest triangulated subcategory of $\mathbf{D}(A)$ containing \mathcal{H} , see for example [22, Proposition 2.5(b)]. Comparing this with Lemma 4.2, we obtain that the equivalence $\mathcal{T} \rightarrow \text{per}(A)$ restricts to a triangle equivalence $\mathcal{T}^{\text{fd}} \rightarrow \mathbf{D}_{\text{fd}}(A)$. Thus the dg algebra A satisfies the following conditions

- (1) $H^i(A) = 0$ for $i > 0$.
- (2) $H^0(A)$ is finite-dimensional over k .
- (3) $\text{per}(A) \supset \mathbf{D}_{\text{fd}}(A)$.
- (4) there is a bifunctorial isomorphism for $X \in \mathbf{D}_{\text{fd}}(A)$ and $Y \in \text{per}(A)$

$$D \text{Hom}_{\text{per}(A)}(X, Y) \simeq \text{Hom}_{\text{per}(A)}(Y, X[d]).$$

This is very close to the original setting of Amiot in [3, Section 2] and of Guo in [16, Section 1].

4.2. The silting reduction of a Calabi–Yau triple. Let $(\mathcal{T}, \mathcal{T}^{\text{fd}}, M)$ be a $(d+1)$ -Calabi–Yau triple, as in Section 4.1. Let P be a direct summand of M . Then P is a presilting object in \mathcal{T} . Let

$$\mathcal{S} := \text{thick}P, \quad \mathcal{U} := \mathcal{T}/\mathcal{S}.$$

By the relative $(d+1)$ -Calabi–Yau property (CY2), we have $\mathcal{T}^{\text{fd}} \cap \mathcal{S}^{\perp\tau} = \mathcal{T}^{\text{fd}} \cap {}^{\perp\tau}\mathcal{S}$, which will be denoted by \mathcal{U}^{fd} , i.e.

$$\mathcal{U}^{\text{fd}} := \mathcal{T}^{\text{fd}} \cap \mathcal{S}^{\perp\tau} = \mathcal{T}^{\text{fd}} \cap {}^{\perp\tau}\mathcal{S}.$$

This category can be considered as a full subcategory of \mathcal{U} (by, for example, [37, Lemma 9.1.5]).

Theorem 4.4. *The triple $(\mathcal{U}, \mathcal{U}^{\text{fd}}, M)$ is a $(d+1)$ -Calabi–Yau triple. Namely,*

- (a) \mathcal{U} is Hom-finite and Krull–Schmidt.
- (b) $M \in \mathcal{U}$ is a silting object.
- (c) The pair $(\mathcal{U}, \mathcal{U}^{\text{fd}})$ is relative $(d+1)$ -Calabi–Yau.
- (d) The pair $(M[<0]^{\perp u}, M[>0]^{\perp u})$ forms a t -structure on \mathcal{U} such that $M[>0]^{\perp u} \subset \mathcal{U}^{\text{fd}}$.

In the proof of this proposition a crucial role is played by the following description of \mathcal{U} obtained in the preceding section: Let

$$\mathcal{P} = \text{add}_{\mathcal{T}}P \quad \text{and} \quad \mathcal{Z} := ({}^{\perp\tau}\mathcal{S}_{<0}) \cap (\mathcal{S}_{>0}{}^{\perp\tau}), \tag{4.2.1}$$

then we have a triangle equivalence (Theorems 3.1 and 3.7)

$$G: \frac{\mathcal{Z}}{[\mathcal{P}]} \xrightarrow{\simeq} \mathcal{U}.$$

Our strategy is to show that under G the triple $(\mathcal{U}, \mathcal{U}^{\text{fd}}, M)$ is equivalent to $(\frac{\mathcal{Z}}{[\mathcal{P}]}, \mathcal{T}^{\text{fd}} \cap \mathcal{Z}, M)$ and then prove Theorem 4.4 for $(\frac{\mathcal{Z}}{[\mathcal{P}]}, \mathcal{T}^{\text{fd}} \cap \mathcal{Z}, M)$. We need some further preparation.

Lemma 4.5. *We have an equality $\mathcal{U}^{\text{fd}} = \mathcal{T}^{\text{fd}} \cap \mathcal{Z}$ of subcategories of \mathcal{T} .*

Proof. Let $X \in \mathcal{T}^{\text{fd}}$. Then $X \in \mathcal{Z}$ if and only if $\text{Hom}_{\mathcal{T}}(X, \mathcal{S}_{<0}) = 0$ and $\text{Hom}_{\mathcal{T}}(\mathcal{S}_{>0}, X) = 0$. By the relative $(d+1)$ -Calabi–Yau property, this amounts to $\text{Hom}_{\mathcal{T}}(\mathcal{S}_{<d+1}, X) = 0$ and $\text{Hom}_{\mathcal{T}}(\mathcal{S}_{>0}, X) = 0$, which, by Proposition 3.2, is equivalent to $X \in \mathcal{S}^{\perp\tau}$. \square

For $X \in \mathcal{T}$, we have a triangle

$$\sigma^{\leq 0} X \xrightarrow{a_X} X \xrightarrow{b_X} \sigma^{\geq 1} X \xrightarrow{c_X} \sigma^{\leq 0} X[1] \quad (4.2.2)$$

in \mathcal{T} such that $\sigma^{\leq 0} X \in \mathcal{T}^{\leq 0}$ and $\sigma^{\geq 1} X \in \mathcal{T}^{\geq 1} \subset \mathcal{T}^{\text{fd}}$.

Lemma 4.6. *Let $X \in \mathcal{Z}$. Then $\sigma^{\geq 1} X \in \mathcal{T}^{\text{fd}} \cap \mathcal{Z}$ and $\sigma^{\leq 0} X \in \mathcal{Z}$.*

Proof. Since $P \in \text{add}_{\mathcal{T}}(M)$, we have by the definition of $\mathcal{T}^{\geq 1}$ that

$$\text{Hom}_{\mathcal{T}}(P, \sigma^{\geq 1} X[i]) = 0 \quad \text{for any } i \leq 0, \quad (4.2.3)$$

and by the definition of $\mathcal{T}^{\leq 0}$ that

$$\text{Hom}_{\mathcal{T}}(P, \sigma^{\leq 0} X[i]) = 0 \quad \text{for any } i \geq 1. \quad (4.2.4)$$

Applying $\text{Hom}_{\mathcal{T}}(P, -)$ to the triangle (4.2.2), we obtain an exact sequence

$$\text{Hom}_{\mathcal{T}}(P, X[i]) \rightarrow \text{Hom}_{\mathcal{T}}(P, \sigma^{\geq 1} X[i]) \rightarrow \text{Hom}_{\mathcal{T}}(P, \sigma^{\leq 0} X[i+1]).$$

Assume $i \geq 1$. Then the left term vanishes because $X \in \mathcal{Z}$ and the right term vanishes due to (4.2.4). Thus we have $\text{Hom}_{\mathcal{T}}(P, \sigma^{\geq 1} X[i]) = 0$ for any $i \geq 1$. Combined with (4.2.3), this yields $\sigma^{\geq 1} X \in \mathcal{T}^{\text{fd}} \cap \mathcal{S}^{\perp\tau} = \mathcal{U}^{\text{fd}}$. By Lemma 4.5, $\sigma^{\geq 1} X \in \mathcal{T}^{\text{fd}} \cap \mathcal{Z}$.

Applying $\text{Hom}_{\mathcal{T}}(-, P)$ to (4.2.2), we obtain an exact sequence

$$\text{Hom}_{\mathcal{T}}(X, P[i]) \rightarrow \text{Hom}_{\mathcal{T}}(\sigma^{\leq 0} X, P[i]) \rightarrow \text{Hom}_{\mathcal{T}}(\sigma^{\geq 1} X, P[i+1]).$$

Assume $i \geq 1$. Then the two outer terms vanish because both X and $\sigma^{\geq 1} X$ belong to \mathcal{Z} . Thus we have $\text{Hom}_{\mathcal{T}}(\sigma^{\leq 0} X, P[i]) = 0$ for any $i \geq 1$. Combined with (4.2.4), this yields $\sigma^{\leq 0} X \in \mathcal{Z}$. \square

Proof of Theorem 4.4. By Lemma 4.5, the category $\mathcal{T}^{\text{fd}} \cap \mathcal{Z}$ is left and right orthogonal to \mathcal{P} , thus it can be viewed as a full subcategory of $\frac{\mathcal{Z}}{[\mathcal{P}]}$. Moreover, it follows from Lemma 4.5 that on $\mathcal{T}^{\text{fd}} \cap \mathcal{Z}$ there is a natural isomorphism $\langle 1 \rangle \simeq [1]$. Therefore $\mathcal{T}^{\text{fd}} \cap \mathcal{Z}$ is naturally a

triangulated subcategory of $\frac{\mathcal{Z}}{[\mathcal{P}]}$. Thanks to the equivalence G , to prove the proposition it suffices to show that the statements (a), (b), (c) and (d) hold for the triple $(\frac{\mathcal{Z}}{[\mathcal{P}]}, \mathcal{T}^{\text{fd}} \cap \mathcal{Z}, M)$.

(a) The category \mathcal{Z} is a full subcategory of \mathcal{T} which is closed under direct summands. Thus it is a Hom-finite and Krull–Schmidt, so is the additive quotient $\frac{\mathcal{Z}}{[\mathcal{P}]}$.

(b) By Theorem 3.8, $M \in \frac{\mathcal{Z}}{[\mathcal{P}]}$ is a sifting object.

(c) Since on $\mathcal{T}^{\text{fd}} \cap \mathcal{Z}$ there is a natural isomorphism $\langle 1 \rangle \simeq [1]$, it follows that for $X \in \mathcal{T}^{\text{fd}} \cap \mathcal{Z}$ and $Y \in \frac{\mathcal{Z}}{[\mathcal{P}]}$ we have $\text{Hom}_{\mathcal{T}}(\mathcal{P}, X) \simeq D \text{Hom}_{\mathcal{T}}(X, \mathcal{P}[d+1]) = 0$ and $\text{Hom}_{\mathcal{T}}(\mathcal{P}, X[d+1]) = 0$. Therefore we have bifunctorial isomorphisms

$$\begin{aligned} D \text{Hom}_{\frac{\mathcal{Z}}{[\mathcal{P}]}}(X, Y) &= D \text{Hom}_{\mathcal{Z}}(X, Y) \simeq \text{Hom}_{\mathcal{Z}}(Y, X[d+1]) = \text{Hom}_{\frac{\mathcal{Z}}{[\mathcal{P}]}}(Y, X[d+1]) \\ &\simeq \text{Hom}_{\frac{\mathcal{Z}}{[\mathcal{P}]}}(Y, X\langle d+1 \rangle). \end{aligned}$$

(d) By (b) and Proposition 2.8(a), we have $M\langle < 0 \rangle^{\perp \frac{\mathcal{Z}}{[\mathcal{P}]}} = \bigcup_{i \geq 0} (\text{add} M) * \cdots * (\text{add} M[i])$. Thus $\text{Hom}_{\frac{\mathcal{Z}}{[\mathcal{P}]}}(M\langle < 0 \rangle^{\perp \frac{\mathcal{Z}}{[\mathcal{P}]}}, M\langle \geq 0 \rangle^{\perp \frac{\mathcal{Z}}{[\mathcal{P}]}}) = 0$ holds.

It remains to prove $\frac{\mathcal{Z}}{[\mathcal{P}]} = (M\langle < 0 \rangle^{\perp \frac{\mathcal{Z}}{[\mathcal{P}]}}) * (M\langle \geq 0 \rangle^{\perp \frac{\mathcal{Z}}{[\mathcal{P}]}})$. Let $X \in \mathcal{Z}$. By Theorem 2.4(b), the triangle (4.2.2) induces a triangle in $\frac{\mathcal{Z}}{[\mathcal{P}]}$

$$\sigma^{\leq 0} X \xrightarrow{a_X} X \xrightarrow{b_X} \sigma^{\geq 1} X \longrightarrow \sigma^{\leq 0} X\langle 1 \rangle. \quad (4.2.5)$$

We only have to show that $\sigma^{\leq 0} X \in M\langle < 0 \rangle^{\perp \frac{\mathcal{Z}}{[\mathcal{P}]}}$ and $\sigma^{\geq 1} X \in M\langle \geq 0 \rangle^{\perp \frac{\mathcal{Z}}{[\mathcal{P}]}}$. We know that $\sigma^{\geq 1} X \in \mathcal{T}^{\text{fd}} \cap \mathcal{Z}$ and $\sigma^{\leq 0} X \in \mathcal{Z}$ hold by Lemma 4.6.

Fix $i \geq 0$. Then we have $M\langle i \rangle \in \mathcal{P} * \cdots * \mathcal{P}[i-1] * M[i]$ by the construction of $\langle i \rangle$. This implies $\text{Hom}_{\mathcal{T}}(M\langle i \rangle, \mathcal{T}^{\geq 1}) = 0$. Hence $\mathcal{T}^{\geq 1} \cap \mathcal{Z} \ni \sigma^{\geq 1} X$ is contained in $M\langle \geq 0 \rangle^{\perp \frac{\mathcal{Z}}{[\mathcal{P}]}}$.

Fix $i > 0$. Then we have $M\langle 1-i \rangle \in M[1-i] * \mathcal{P}[2-i] * \cdots * \mathcal{P}$ by the construction of $\langle 1-i \rangle$. This implies $\text{Hom}_{\mathcal{T}}(M\langle 1-i \rangle[-1], \mathcal{T}^{\leq 0}) = 0$. Further, we have a triangle

$$M\langle 1-i \rangle[-1] \longrightarrow M\langle -i \rangle \xrightarrow{b} P' \xrightarrow{a} M\langle 1-i \rangle$$

with a right \mathcal{P} -approximation a . Applying $\text{Hom}_{\mathcal{T}}(-, \mathcal{T}^{\leq 0})$ to this triangle, we have that the map $\text{Hom}_{\mathcal{T}}(P', \mathcal{T}^{\leq 0}) \rightarrow \text{Hom}_{\mathcal{T}}(M\langle -i \rangle, \mathcal{T}^{\leq 0})$ is surjective. Hence $\text{Hom}_{\frac{\mathcal{Z}}{[\mathcal{P}]}}(M\langle -i \rangle, \mathcal{T}^{\leq 0} \cap \mathcal{Z}) = 0$, and $\mathcal{T}^{\leq 0} \cap \mathcal{Z} \ni \sigma^{\leq 0} X$ is contained in $M\langle < 0 \rangle^{\perp \frac{\mathcal{Z}}{[\mathcal{P}]}}$.

Consequently, $(M\langle < 0 \rangle^{\perp \frac{\mathcal{Z}}{[\mathcal{P}]}}, M\langle > 0 \rangle^{\perp \frac{\mathcal{Z}}{[\mathcal{P}]}})$ forms a t-structure on $\frac{\mathcal{Z}}{[\mathcal{P}]}$. Finally, if $X \in M\langle > 0 \rangle^{\perp \frac{\mathcal{Z}}{[\mathcal{P}]}}$, we have an isomorphism $X \simeq \sigma^{\geq 1} X \in \mathcal{T}^{\text{fd}} \cap \mathcal{Z}$ in $\frac{\mathcal{Z}}{[\mathcal{P}]}$. This completes the proof. \square

4.3. The Amiot–Guo–Keller cluster category of a Calabi–Yau triple. Assume that $(\mathcal{T}, \mathcal{T}^{\text{fd}}, M)$ is a $(d+1)$ -Calabi–Yau triple. We keep the notation in Section 4.1. Consider the triangle quotient

$$\mathcal{C} := \mathcal{T}/\mathcal{T}^{\text{fd}},$$

which we call *Amiot–Guo–Keller cluster category* of \mathcal{T} . Let $\pi: \mathcal{T} \rightarrow \mathcal{C}$ denote the canonical projection functor. We define a full subcategory \mathcal{F} of \mathcal{T} by

$$\mathcal{F} := \mathcal{T}_{\geq 1-d} \cap \mathcal{T}_{\leq 0} \stackrel{\text{Prop. 2.8(b)}}{=} (\text{add}M) * (\text{add}M[1]) * \cdots * (\text{add}M[d-1]).$$

Now we give the following generalisation of fundamental results due to Amiot and Guo [3, 16] to our setting of $(d+1)$ -Calabi–Yau triples. In particular, the statement (b) says that \mathcal{F} is a fundamental domain of \mathcal{C} in \mathcal{T} . We give a detailed proof in Appendix A, which improves the proofs of [3, 16].

Theorem 4.7. (a) *The category \mathcal{C} is a d -Calabi–Yau triangulated category.*
 (b) *The functor $\pi: \mathcal{T} \rightarrow \mathcal{C}$ restricts to an equivalence $\mathcal{F} \rightarrow \mathcal{C}$ of additive categories.*
 (c) *$\pi(M)$ is a d -cluster-tilting object of \mathcal{C} whose endomorphism algebra is isomorphic to $\text{End}_{\mathcal{T}}(M)$.*

The following proposition will play an important role in the proof of Theorem 4.7 and Theorem 4.13.

Proposition 4.8. *The functor $\pi: \mathcal{T} \rightarrow \mathcal{C}$ induces a bijection (respectively, injection) $\text{Hom}_{\mathcal{T}}(U, V) \rightarrow \text{Hom}_{\mathcal{C}}(U, V)$ for any $U \in \mathcal{T}_{\leq 0}$ and $V \in \mathcal{T}_{\geq 1-d}$ (respectively, $V \in \mathcal{T}_{\geq -d}$). Consequently, it restricts to a fully faithful functor $\mathcal{F} \rightarrow \mathcal{C}$.*

To prove this proposition we need the following lemma.

Lemma 4.9. *Let $X \in \mathcal{T}_{\leq 0}$ and $Y \in \mathcal{T}$. Then any morphism in $\text{Hom}_{\mathcal{C}}(X, Y)$ has a representative of the form $X \xleftarrow{s} Z \xrightarrow{f} Y$ such that the cone of s belongs to $\mathcal{T}_{\leq 0} \cap \mathcal{T}^{\text{fd}}$.*

Proof. Any morphism $X \rightarrow Y$ in \mathcal{C} can be written as $X \xleftarrow{s} Z \xrightarrow{f} Y$ such that there exists a triangle

$$Z \xrightarrow{s} X \xrightarrow{t} W \longrightarrow Z[1]$$

with $W \in \mathcal{T}^{\text{fd}}$. Recall that $\mathcal{T}_{\leq 0} = \mathcal{T}^{\leq 0}$. Thus any morphism $g \in \text{Hom}_{\mathcal{T}}(X, W)$ with $W \in \mathcal{T}^{\text{fd}}$ factors through $\sigma^{\leq 0}W \rightarrow W$ since $\text{Hom}_{\mathcal{T}}(X, \sigma^{\geq 1}W) = 0$. We obtain a commutative

diagram of triangles:

$$\begin{array}{ccccccc}
& & & & \sigma^{\geq 1}W & & \\
& & & & \uparrow & & \\
Z & \xrightarrow{s} & X & \xrightarrow{t} & W & \longrightarrow & Z[1] \\
\uparrow h & & \parallel & & \uparrow & & \uparrow \\
Z' & \xrightarrow{s'} & X & \longrightarrow & \sigma^{\leq 0}W & \longrightarrow & Z'[1]
\end{array}$$

Because the cone $\sigma^{\geq 1}W$ of s' belongs to $\mathcal{T}_{\leq 0} \cap \mathcal{T}^{\text{fd}}$ by Lemma 4.2, the morphism $X \xleftarrow{s} Z \xrightarrow{f} Y$ is equivalent to $X \xleftarrow{s'} Z' \xrightarrow{fh} Y$, so the assertion follows. \square

Proof of Proposition 4.8. Let $U \in \mathcal{T}_{\leq 0}$ and $V \in \mathcal{T}_{\geq -d}$.

First we show that $\text{Hom}_{\mathcal{T}}(U, V) \rightarrow \text{Hom}_{\mathcal{C}}(U, V)$ is injective. Assume that $f \in \text{Hom}_{\mathcal{T}}(U, V)$ becomes zero in \mathcal{C} . Then it factors through some $W \in \mathcal{T}^{\text{fd}}$ (by, for example, [37, Lemma 2.1.26]), and further through $\sigma^{\leq 0}W$ because $U \in \mathcal{T}_{\leq 0}$. By the relative $(d+1)$ -Calabi–Yau property, we have

$$\text{Hom}_{\mathcal{T}}(\sigma^{\leq 0}W, V) \simeq D \text{Hom}_{\mathcal{T}}(V, (\sigma^{\leq 0}W)[d+1]) = 0$$

as $V \in \mathcal{T}_{\geq -d}$. Thus f must be zero.

Next we show that $\text{Hom}_{\mathcal{T}}(U, V) \rightarrow \text{Hom}_{\mathcal{C}}(U, V)$ is surjective if $V \in \mathcal{T}_{\geq 1-d}$. By Lemma 4.9, a morphism in $\text{Hom}_{\mathcal{C}}(U, V)$ has a representative of the form $U \xleftarrow{s} Y \xrightarrow{f} V$ such that the cone W of s belongs to $\mathcal{T}_{\leq 0} \cap \mathcal{T}^{\text{fd}}$. We have an exact sequence

$$\text{Hom}_{\mathcal{T}}(U, V) \xrightarrow{s} \text{Hom}_{\mathcal{T}}(Y, V) \rightarrow \text{Hom}_{\mathcal{T}}(W[-1], V).$$

As $W[-1] \in \mathcal{T}^{\text{fd}}$, we can apply the relative $(d+1)$ -Calabi–Yau property to obtain

$$\text{Hom}_{\mathcal{T}}(W[-1], V) \simeq D \text{Hom}_{\mathcal{T}}(V, W[d]) = 0.$$

The last equality holds because $V \in \mathcal{T}_{\geq 1-d}$ and $W[d] \in \mathcal{T}_{\leq -d}$. So there exists $g \in \text{Hom}_{\mathcal{T}}(U, V)$ such that $f = gs$, and hence $U \xleftarrow{s} Y \xrightarrow{f} V$ is equivalent to $U \xrightarrow{g} V$. It follows that $\text{Hom}_{\mathcal{T}}(U, V) \rightarrow \text{Hom}_{\mathcal{C}}(U, V)$ is surjective. \square

We end this subsection with the following observation, where the $d=2$ case of part (b) is due to Keller and Nicolás [29] in the algebraic case, see also [10, Theorem 4.5].

Corollary 4.10. *Let $\text{silt } \mathcal{T}$ be the set of sifting objects of \mathcal{T} and $d\text{-ctilt } \mathcal{C}$ be the set of d -cluster-tilting objects of \mathcal{C} .*

- (a) *The functor $\pi: \mathcal{T} \rightarrow \mathcal{C}$ gives a map $\pi: \text{silt } \mathcal{T} \rightarrow d\text{-ctilt } \mathcal{C}$.*

- (b) *This restricts to an injection $\pi: \mathcal{F} \cap \text{silt } \mathcal{T} \rightarrow d\text{-ctilt } \mathcal{C}$, which is a bijection if $d = 1$ or $d = 2$.*

Proof. For any $N \in \text{silt } \mathcal{T}$, it follows from Theorem 2.10 that $(\mathcal{T}, \mathcal{T}^{\text{fd}}, N)$ is a $(d + 1)$ -Calabi–Yau triple. Thus, by Theorem 4.7, πN is a d -cluster-tilting object of \mathcal{C} . In this way, we obtain a map $\pi: \text{silt } \mathcal{T} \rightarrow d\text{-ctilt } \mathcal{C}$. Since $\pi: \mathcal{F} \rightarrow \mathcal{C}$ is fully faithful by Proposition 4.8, the induced map $\pi: \mathcal{F} \cap \text{silt } \mathcal{T} \rightarrow d\text{-ctilt } \mathcal{C}$ is injective.

We show that it is surjective for $d = 1$ and $d = 2$. For $d = 1$ this is clear, since, up to multiplicities, any object of $\mathcal{F} \cap \text{silt } \mathcal{T}$ is M and any object of $d\text{-ctilt } \mathcal{C}$ is $\pi(M)$. Next assume $d = 2$. For $X \in \mathcal{F}$, assume that $\pi X \in \mathcal{C}$ is 2-cluster-tilting. Since $\text{Hom}_{\mathcal{T}}(X, X[1]) \rightarrow \text{Hom}_{\mathcal{C}}(X, X[1])$ is injective by Proposition 4.8, $X \in \mathcal{T}$ is presilting. Since $|X| = |\pi X| = |\pi M| = |M|$ holds, X is silting by [20, Proposition 4.3]. Here for an object Y of \mathcal{T} , we denote by $|Y|$ the number of pairwise non-isomorphic indecomposable direct summands of Y . \square

Remark 4.11. Assume $d = 2$ and let $\Lambda := \text{End}_{\mathcal{T}}(M)$. It is shown in [1] that we have a bijection $2\text{-silt } \Lambda \rightarrow 2\text{-ctilt } \mathcal{C}$, where $2\text{-silt } \Lambda$ denotes the set of 2-term silting objects in $K^b(\text{proj } \Lambda)$. Thus there is a bijective map $\mathcal{F} \cap \text{silt } \mathcal{T} \rightarrow 2\text{-silt } \Lambda$ making the following diagram of bijective maps commutative

$$\begin{array}{ccc} \mathcal{F} \cap \text{silt } \mathcal{T} & \xrightarrow{\quad} & 2\text{-silt } \Lambda \\ & \searrow \pi & \swarrow \\ & 2\text{-ctilt } \mathcal{C} & \end{array}$$

In the algebraic setting this is given in [10]. Note, however, that in the algebraic setting there is a triangle functor $\mathcal{T} \rightarrow K^b(\text{proj } \Lambda)$, which induces a bijective map $\mathcal{F} \cap \text{silt } \mathcal{T} \rightarrow 2\text{-silt } \Lambda$ making the above diagram commutative. In the general setting the triangle functor $\mathcal{T} \rightarrow K^b(\text{proj } \Lambda)$ and the direct definition of the map $\mathcal{F} \cap \text{silt } \mathcal{T} \rightarrow 2\text{-silt } \Lambda$ are not available.

We do not know if the map $\pi: \mathcal{F} \cap \text{silt } \mathcal{T} \rightarrow d\text{-ctilt } \mathcal{C}$ in Corollary 4.10 (b) is bijective for $d > 2$. We conjecture that this is the case.

Conjecture 4.12. *The map $\pi: \mathcal{F} \cap \text{silt } \mathcal{T} \rightarrow d\text{-ctilt } \mathcal{C}$ in Corollary 4.10 (b) is bijective for all $d \geq 1$.*

4.4. The main result. Let $(\mathcal{T}, \mathcal{T}^{\text{fd}}, M)$ be a $(d + 1)$ -Calabi–Yau triple, as in Section 4.1. Let P be a direct summand of M .

By Theorem 4.7, $\mathcal{C} = \mathcal{T}/\mathcal{T}^{\text{fd}}$ is a d -Calabi–Yau triangulated category and $\pi(M)$ is a d -cluster-tilting object of \mathcal{C} . In particular, $\pi(P)$ is d -rigid. Here $\pi : \mathcal{T} \rightarrow \mathcal{C}$ is the canonical projection functor. By abuse of notation, we will write M and P for $\pi(M)$ and $\pi(P)$.

Analogous to (4.2.1), we define subcategories of \mathcal{C} by

$$\mathcal{P}' := \text{add}_{\mathcal{C}} P \quad \text{and} \quad \mathcal{Z}' := {}^{\perp c}(\mathcal{P}'[1] * \mathcal{P}'[2] * \cdots * \mathcal{P}'[d-1]).$$

Thus, we can form the Calabi–Yau reduction as explained in Section 2.2:

$$\mathcal{C}_P := \frac{\mathcal{Z}'}{[\mathcal{P}']}.$$

By Theorem 2.5, the object M in \mathcal{C}_P is d -cluster-tilting. Its endomorphism algebra is

$$\text{End}_{\mathcal{C}_P}(M) = \text{End}_{\mathcal{C}}(M)/[\mathcal{P}'](M, M) \stackrel{\text{Prop. 4.8}}{\simeq} \text{End}_{\mathcal{T}}(M)/[\mathcal{P}](M, M). \quad (4.4.1)$$

On the other hand, let $\mathcal{S} := \text{thick} P$ and $\mathcal{U} := \mathcal{T}/\mathcal{S}$. We consider $\mathcal{U}^{\text{fd}} := \mathcal{T}^{\text{fd}} \cap \mathcal{S}^{\perp \tau}$ as a full subcategory of \mathcal{U} . Then $(\mathcal{U}, \mathcal{U}^{\text{fd}}, M)$ is a relative $(d+1)$ -Calabi–Yau triple by Theorem 4.4, and the triangle quotient

$$\mathcal{U}/\mathcal{U}^{\text{fd}}$$

is a d -Calabi–Yau triangulated category by Theorem 4.7. Moreover, the object M in $\mathcal{U}/\mathcal{U}^{\text{fd}}$ is d -cluster-tilting and its endomorphism algebra is

$$\text{End}_{\mathcal{U}/\mathcal{U}^{\text{fd}}}(M) \stackrel{\text{Prop. 4.8}}{\simeq} \text{End}_{\mathcal{U}}(M) \stackrel{\text{Thm. 3.1}}{=} \text{End}_{\frac{\mathcal{Z}}{[\mathcal{P}]}}(M) = \text{End}_{\mathcal{T}}(M)/[\mathcal{P}](M, M). \quad (4.4.2)$$

Therefore, we obtain two $(d+1)$ -Calabi–Yau triangulated categories, \mathcal{C}_P and $\mathcal{U}/\mathcal{U}^{\text{fd}}$, which have d -cluster-tilting objects. These two d -cluster-tilting objects have isomorphic endomorphism algebras. In fact, the following our main result asserts that these two triangulated categories are equivalent.

Theorem 4.13. *The two categories \mathcal{C}_P and $\mathcal{U}/\mathcal{U}^{\text{fd}}$ are triangle equivalent.*

In this sense, we say that the AGK cluster category construction Theorem 4.7 takes the silting reduction of \mathcal{T} with respect to P to the Calabi–Yau reduction of \mathcal{C} with respect to $\pi(P)$.

Remark 4.14. Let (Q, W) be a quiver with potential and $\Gamma = \Gamma(Q, W)$ be its complete Ginzburg dg algebra, see [12, 15, 33]. Assume that $H^0(\Gamma)$ is finite-dimensional. Then the triple $(\text{per}(\Gamma), \text{D}_{\text{fd}}(\Gamma), \Gamma)$ is a 3-Calabi–Yau triple. The triangle quotient

$$\mathcal{C}(Q, W) = \text{per}(\Gamma)/\text{D}_{\text{fd}}(\Gamma)$$

is called the *cluster category* of (Q, W) . Let i be a vertex of Q , $e = e_i$ be the trivial path at i , and (Q', W') be the quiver with potential obtained from (Q, W) by deleting the vertex i . Then it follows from [27, Corollary 7.3] and Theorem 4.13 that there is a triangle equivalence between the Calabi–Yau reduction of $\mathcal{C}(Q, W)$ with respect to $e_i\Gamma$ and the cluster category $\mathcal{C}(Q', W')$ of (Q', W') . This provides an alternative approach to [27, Theorem 7.4] .

We start the proof of Theorem 4.13 with two lemmas.

Lemma 4.15. *For any $X \in \mathcal{Z}$ and for $i \leq d - 1$, the map*

$$\mathrm{Hom}_{\mathcal{T}}(X, P[i]) \rightarrow \mathrm{Hom}_{\mathcal{C}}(X, P[i]) \quad (4.4.3)$$

is bijective. In particular, $\mathrm{Hom}_{\mathcal{C}}(X, P[i]) = 0$ for $1 \leq i \leq d - 1$.

Proof. Consider the triangle (4.2.2), which induces a commutative diagram for $i \leq d - 1$

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{T}}(X, P[i]) & \xrightarrow{a_X} & \mathrm{Hom}_{\mathcal{T}}(\sigma^{\leq 0} X, P[i]) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{C}}(X, P[i]) & \xrightarrow{a_X} & \mathrm{Hom}_{\mathcal{C}}(\sigma^{\leq 0} X, P[i]). \end{array}$$

The upper map is bijective since $\sigma^{\geq 1} X \in \mathcal{U}^{\mathrm{fd}} \subset {}^{\perp \mathcal{T}} \mathcal{S}$ holds by Lemma 4.6 and Lemma 4.5, and the lower map is bijective since $a_X : \sigma^{\leq 0} X \rightarrow X$ becomes an isomorphism in \mathcal{C} . Further, since $\sigma^{\leq 0} X \in \mathcal{T}^{\leq 0} = \mathcal{T}_{\leq 0}$ and $P[i] \in \mathcal{T}_{\geq 1-d}$, the right map is bijective by Proposition 4.8. The bijectivity of the left map follows immediately.

As $X \in \mathcal{Z}$, we have $\mathrm{Hom}_{\mathcal{T}}(X, P[>0]) = 0$. In conjunction with the first statement, this implies the second statement. \square

Lemma 4.16. *The functor $\pi : \mathcal{T} \rightarrow \mathcal{C}$ induces a dense functor $\mathcal{Z} \rightarrow \mathcal{Z}'$.*

Proof. By Lemma 4.15, π gives a functor $\mathcal{Z} \rightarrow \mathcal{Z}'$. We need to show that this is dense.

Fix any $Y \in \mathcal{Z}'$. By Theorem 4.7 (b), there exists $X \in \mathcal{F} = \mathcal{T}_{\geq 1-d} \cap \mathcal{T}_{\leq 0}$ such that $\pi(X) \simeq Y$. Since $P \in \mathbf{add} M$, we have $\mathrm{Hom}_{\mathcal{T}}(P, X[\geq 1]) = 0$ and $\mathrm{Hom}_{\mathcal{T}}(X, P[\geq d]) = 0$. By Lemma 4.15, we have $\mathrm{Hom}_{\mathcal{T}}(X, P[i]) \simeq \mathrm{Hom}_{\mathcal{C}}(Y, P[i]) = 0$ for $1 \leq i \leq d - 1$. Thus $X \in \mathcal{Z}$ and the assertion follows. \square

Therefore the functor $\pi : \mathcal{T} \rightarrow \mathcal{C}$ induces additive functors $\mathcal{Z} \rightarrow \mathcal{Z}'$ and $\mathcal{P} \rightarrow \mathcal{P}'$, and further induces an additive functor

$$\tilde{\pi} : \mathcal{U} \simeq \frac{\mathcal{Z}}{[\mathcal{P}]} \longrightarrow \mathcal{C}_P = \frac{\mathcal{Z}'}{[\mathcal{P}']}. \quad (4.4.4)$$

We observed in Sections 3.2 and 2.2 that both categories $\frac{\mathcal{Z}}{[\mathcal{P}]}$ and $\frac{\mathcal{Z}'}{[\mathcal{P}]'}$ has structures of triangulated categories. Now we show the following.

Proposition 4.17. *The functor $\tilde{\pi}: \mathcal{U} \rightarrow \mathcal{C}_P$ is a triangle functor which is dense.*

Proof. By Lemma 4.15, the image of a left \mathcal{P} -approximation in \mathcal{T} gives a left \mathcal{P}' -approximation in \mathcal{C} . Thus the functor commutes with shifts.

Next we show that the functor sends triangles to triangles. The triangles in $\frac{\mathcal{Z}}{[\mathcal{P}]}$ are defined by the commutative diagram (2.2.1) in Theorem 2.4. The image of (2.2.1) in \mathcal{C} is also a commutative diagram of triangles with a left \mathcal{P}' -approximation ι_X by Lemma 4.15. Thus $X \xrightarrow{\bar{f}} Y \xrightarrow{\bar{g}} Z \xrightarrow{\bar{a}} X\langle 1 \rangle$ is a triangle in $\frac{\mathcal{Z}'}{[\mathcal{P}]'}$. Thus the assertion follows.

That the functor $\tilde{\pi}: \mathcal{U} \rightarrow \mathcal{C}_P$ is dense by Lemma 4.16. \square

Now we are ready to prove Theorem 4.13.

Proof of Theorem 4.13. Since $\pi(\mathcal{T}^{\text{fd}}) = 0$ and $\mathcal{U}^{\text{fd}} \subset \mathcal{T}^{\text{fd}}$, we have $\tilde{\pi}(\mathcal{U}^{\text{fd}}) = 0$. Therefore $\tilde{\pi}$ induces a triangle functor $\pi': \mathcal{U}/\mathcal{U}^{\text{fd}} \rightarrow \mathcal{C}_P$. It remains to show that π' is an equivalence. Tracing the construction of π' , we see that π' sends the d -cluster-tilting object M of $\mathcal{U}/\mathcal{U}^{\text{fd}}$ to the d -cluster-tilting object M of \mathcal{C}_P . Moreover, we have isomorphisms of algebras

$$\text{End}_{\mathcal{U}/\mathcal{U}^{\text{fd}}}(M) \stackrel{(4.4.2)}{\simeq} \text{End}_{\mathcal{T}}(M)/[\mathcal{P}](M, M) \stackrel{(4.4.1)}{\simeq} \text{End}_{\mathcal{C}_P}(M).$$

which compose to the homomorphism induced by π' . Thus the triangle functor $\pi': \mathcal{U}/\mathcal{U}^{\text{fd}} \rightarrow \mathcal{C}_P$ is an equivalence by Proposition 2.6. \square

4.5. Lifting Calabi–Yau reduction. In this subsection, we show a result which is in some sense a converse to Theorem 4.13.

Let \mathcal{C} be a Hom-finite Krull–Schmidt 2-Calabi–Yau algebraic triangulated category with a cluster-tilting object T .

Theorem 4.18. *Any Calabi–Yau reduction of \mathcal{C} lifts to a sifting reduction. Precisely, there is a $(d+1)$ -Calabi–Yau triple $(\mathcal{T}, \mathcal{T}^{\text{fd}}, M)$ such that $\mathcal{T}/\mathcal{T}^{\text{fd}}$ is triangle equivalent to \mathcal{C} and this equivalence takes M to T ; further, for any direct summand T' of T , there is a direct summand P of M such that the AGK construction takes the sifting reduction of \mathcal{T} with respect to P to the Calabi–Yau reduction of \mathcal{C} with respect to T' .*

Proof. The existence of a desired Calabi–Yau triple is essentially established in [23]. We give the construction for the convenience of the reader.

Let \mathcal{E} be a Frobenius category such that there is a triangle equivalence between \mathcal{C} and the stable category of \mathcal{E} , which we identify. Let \mathcal{M} be the preimage of $\underline{\mathcal{M}} = \text{add}_{\mathcal{C}} M$

under the projection $\mathcal{E} \rightarrow \mathcal{C}$. Let $\mathbf{D}(\mathcal{M})$ be the derived category of modules over \mathcal{M} (=contravariant functors from \mathcal{M} to $\mathbf{Mod}k$), and let $\mathbf{K}_{\underline{\mathcal{M}}}^b(\mathcal{M})$ be the homotopy category of bounded complexes of objects of \mathcal{M} whose cohomologies are supported on $\underline{\mathcal{M}}$. By [30, Proposition 4(c)], there is a bifunctorial isomorphism

$$D \operatorname{Hom}_{\mathbf{D}(\mathcal{M})}(X, Y) \simeq \operatorname{Hom}_{\mathbf{D}(\mathcal{M})}(Y, X[d+1]) \quad (4.5.1)$$

for $X \in \mathbf{K}_{\underline{\mathcal{M}}}^b(\mathcal{M})$ and $Y \in \mathbf{D}(\mathcal{M})$.

Let $\mathbf{D}_{\underline{\mathcal{M}}}(\mathcal{M})$ denote the full subcategory of $\mathbf{D}(\mathcal{M})$ consisting of complexes of modules over \mathcal{M} whose cohomologies are supported on $\underline{\mathcal{M}}$.

Proposition 4.19. *There is a dg algebra A with $H^i(A) = 0$ ($i > 0$) together with a triangle equivalence $\mathbf{D}(A) \rightarrow \mathbf{D}_{\underline{\mathcal{M}}}(\mathcal{M})$ which restricts to a triangle equivalence $\mathbf{D}_{\text{fd}}(A) \rightarrow \mathbf{K}_{\underline{\mathcal{M}}}^b(\mathcal{M})$.*

Sketch. We sketch the construction of A . We take the dg quotient \mathcal{N} [13] of \mathcal{M} by the category $\mathbf{proj}\mathcal{E}$ of projective objects of \mathcal{E} . By construction, the morphism complexes of \mathcal{N} have cohomologies concentrated in non-positive degrees. Further, objects of $\mathbf{proj}\mathcal{E}$ are homotopic to zero in \mathcal{N} . Thus, if N denotes a preimage of M under the projection $\mathcal{M} \rightarrow \underline{\mathcal{M}}$, then the inclusion $\mathcal{A} = \mathbf{add}_{\mathcal{N}}(N) \hookrightarrow \mathcal{N}$ is a dg functor which induces an equivalence $H^0(\mathcal{A}) \rightarrow H^0(\mathcal{N})$. We take $A = \operatorname{End}_{\mathcal{A}}(N)$. \square

The isomorphism (4.5.1) translates to a bifunctorial isomorphism

$$D \operatorname{Hom}_{\mathbf{D}(A)}(X, Y) \simeq \operatorname{Hom}_{\mathbf{D}(A)}(Y, X[d+1])$$

for $X \in \mathbf{D}_{\text{fd}}(A)$ and $Y \in \mathbf{D}(A)$. Moreover, $\mathbf{per}(A) \supset \mathbf{D}_{\text{fd}}(A)$. It follows that the triple $(\mathbf{per}(A), \mathbf{D}_{\text{fd}}(A), A_A)$ is a $(d+1)$ -Calabi–Yau triple. It turns out that there is a triangle equivalence

$$\mathbf{per}(A)/\mathbf{D}_{\text{fd}}(A) \xrightarrow{\sim} \mathcal{C}$$

which takes A_A to T . This can be obtained by translating [38, Proposition 3] (some work is necessary) or applying a result of [23] (which explains the relationship between the “relative singularity categories” and the “singularity category”). The triple $(\mathcal{T}, \mathcal{T}^{\text{fd}}, M) = (\mathbf{per}(A), \mathbf{D}_{\text{fd}}(A), A_A)$ is as desired.

Now let T' be a direct summand of T . Since, by Theorem 4.7 (b), the functor $\mathcal{T} \rightarrow \mathcal{C}$ induces an additive equivalence $\mathbf{add}_{\mathcal{T}}(M) \simeq \mathbf{add}_{\mathcal{C}}(T)$, there is a direct summand P of M such that P is sent to T' by this equivalence. By Theorem 4.13, the AGK cluster category construction takes the silting reduction of \mathcal{T} with respect to P to the Calabi–Yau reduction of \mathcal{C} with respect to T' . \square

APPENDIX A. PROOF OF THEOREM 4.7

In this appendix, we give a proof of Theorem 4.7. We postpone the proof of the part (b) and first give the proof of the parts (a) and (c).

Proof of Theorem 4.7(a) and (c). (a) First, by (b), the category \mathcal{C} is Hom-finite.

Secondly, we show that \mathcal{C} is d -Calabi–Yau. Let X and Y be objects of \mathcal{T} . Recall that $(\mathcal{T}_{\geq 0}, \mathcal{T}_{\leq 0})$ is a bounded co-t-structure on \mathcal{T} . It follows that there exists an integer i such that Y belongs to $\mathcal{T}_{\geq i}$. Now consider the triangle

$$\sigma^{\leq i-1} X \longrightarrow X \longrightarrow \sigma^{\geq i} X \longrightarrow \sigma^{\leq i-1} X[1].$$

Because $\sigma^{\leq i-1} X \in \mathcal{T}^{\leq i-1} = \mathcal{T}_{\leq i-1}$, we have $\mathrm{Hom}_{\mathcal{T}}(Y, \sigma^{\leq i-1} X) = 0$. It follows that the induced homomorphism $\mathrm{Hom}_{\mathcal{T}}(Y, X) \rightarrow \mathrm{Hom}_{\mathcal{T}}(Y, \sigma^{\geq i} X)$ is injective. So the morphism $X \rightarrow \sigma^{\geq i} X$ is a local $\mathcal{T}^{\mathrm{fd}}$ -envelope of X relative to Y in the sense of [3, Definition 1.2]. Therefore by [3, Lemma 1.1, Theorem 1.3 and Proposition 1.4] we see that \mathcal{C} is d -Calabi–Yau.

(c) As all $M[i]$, $0 \leq i \leq d-1$ belong to \mathcal{F} , we have by Proposition 4.8 that $\mathrm{End}_{\mathcal{C}}(M) \simeq \mathrm{End}_{\mathcal{T}}(M)$, and $\mathrm{Hom}_{\mathcal{C}}(M, M[i]) \simeq \mathrm{Hom}_{\mathcal{T}}(M, M[i]) = 0$ for $1 \leq i \leq d-1$, i.e. M is d -rigid. Since $\mathcal{F} = (\mathrm{add} M) * (\mathrm{add} M[1]) * \cdots * (\mathrm{add} M[d-1])$ by definition and $\pi: \mathcal{F} \rightarrow \mathcal{C}$ is dense, we have $\mathcal{C} = (\mathrm{add} M) * (\mathrm{add} M[1]) * \cdots * (\mathrm{add} M[d-1])$. Thus $M \in \mathcal{C}$ is d -cluster-tilting. \square

In the rest of this appendix, we prove Theorem 4.7 (b). We prepare several auxiliary results.

Lemma A.1. *For any integer i , we have $\sigma^{\leq -i}(\mathcal{T}_{\geq 1-d-i}) \subset \mathcal{F}[i]$.*

Proof. It is enough to show the case $i = 0$. Fix $X \in \mathcal{T}_{\geq 1-d}$. We need to show $\sigma^{\leq 0}(X) \in \mathcal{T}_{\geq 1-d}$, that is, $\mathrm{Hom}_{\mathcal{T}}(\sigma^{\leq 0}(X), M[\geq d]) = 0$. Consider the triangle

$$\sigma^{\leq 0}(X) \longrightarrow X \longrightarrow \sigma^{\geq 1}(X) \longrightarrow \sigma^{\leq 0}(X)[1].$$

Applying $\mathrm{Hom}_{\mathcal{T}}(-, M[\geq d])$, we have an exact sequence

$$\mathrm{Hom}_{\mathcal{T}}(X, M[\geq d]) \rightarrow \mathrm{Hom}_{\mathcal{T}}(\sigma^{\leq 0}(X), M[\geq d]) \rightarrow \mathrm{Hom}_{\mathcal{T}}(\sigma^{\geq 1}(X)[-1], M[\geq d]).$$

Since $X \in \mathcal{T}_{\geq 1-d}$, we have $\mathrm{Hom}_{\mathcal{T}}(X, M[\geq d]) = 0$. Moreover $\mathrm{Hom}_{\mathcal{T}}(\sigma^{\geq 1}(X)[-1], M[\geq d]) \simeq D \mathrm{Hom}_{\mathcal{T}}(M, \sigma^{\geq 1}(X)[\leq 0]) = 0$. Thus the assertion follows. \square

Proposition A.2. *For any $i \geq 0$, the functor $\sigma^{\leq -i}: \mathcal{T} \rightarrow \mathcal{T}^{\leq -i}$ induces a dense functor $\mathcal{F} \rightarrow \mathcal{F}[i]$.*

Proof. By Lemma A.1, we have $\sigma^{\leq -i}(\mathcal{F}) \subset \mathcal{F}[i]$. We need to show that this is dense. It is enough to show that $\sigma^{\leq -1}: \mathcal{F} \rightarrow \mathcal{F}[1]$ is dense since then $\sigma^{\leq -i}: \mathcal{F}[i-1] \rightarrow \mathcal{F}[i]$ is also dense and the composition

$$\sigma^{\leq -i} = \left(\mathcal{F} \xrightarrow{\sigma^{\leq -1}} \mathcal{F}[1] \xrightarrow{\sigma^{\leq -2}} \mathcal{F}[2] \xrightarrow{\sigma^{\leq -3}} \dots \xrightarrow{\sigma^{\leq -i}} \mathcal{F}[i] \right)$$

is dense.

Fix $X \in \mathcal{F}$. We show that there exists $Y \in \mathcal{F}$ such that $\sigma^{\leq -1}(Y) \simeq X[1]$. Take a triangle

$$X_{\geq 3-d} \longrightarrow X \longrightarrow X_{\leq 2-d} \longrightarrow X_{\geq 3-d}[1] \quad (\text{A.0.2})$$

with $X_{\geq 3-d} \in \mathcal{T}_{\geq 3-d}$ and $X_{\leq 2-d} \in \mathcal{T}_{\leq 2-d}$. Since $X \in \mathcal{F} = \mathcal{T}_{\geq 1-d} \cap \mathcal{T}_{\leq 0}$, we have $X_{\leq 2-d} \in \mathcal{T}_{\geq 1-d} \cap \mathcal{T}_{\leq 2-d}$. Thus $W := X_{\leq 2-d}[2-d]$ belongs to $\mathcal{T}_{\geq -1} \cap \mathcal{T}_{\leq 0} = (\text{add}M) * (\text{add}M[1])$, and there exists a triangle

$$M_1 \xrightarrow{f} M_0 \longrightarrow W \longrightarrow M_1[1] \quad (\text{A.0.3})$$

with $M_0, M_1 \in \text{add}M$. Now let $\Lambda := \text{End}_{\mathcal{T}}(M)$ and we denote by

$$\nu := D \text{Hom}_{\Lambda}(-, \Lambda): \text{proj} \Lambda \rightarrow \text{inj} \Lambda$$

the Nakayama functor, which is an equivalence. By Lemma 2.12, the functor $F := \text{Hom}_{\mathcal{T}}(M, -): \mathcal{T} \rightarrow \text{mod} \Lambda$ induces an equivalence $F: \mathcal{H} \rightarrow \text{mod} \Lambda$. We define $L \in \mathcal{H}$ by the exact sequence

$$0 \longrightarrow F(L) \longrightarrow \nu F(M_1) \xrightarrow{\nu F(f)} \nu F(M_0) . \quad (\text{A.0.4})$$

Lemma A.3. *There exists a morphism $g \in \text{Hom}_{\mathcal{T}}(L, X[2])$ which induces a functorial isomorphism for $U \in \mathcal{T}^{\leq 0}$:*

$$\text{Hom}_{\mathcal{T}}(g, -): \text{Hom}_{\mathcal{T}}(X[2], U[d+1]) \xrightarrow{\simeq} \text{Hom}_{\mathcal{T}}(L, U[d+1]).$$

Proof. We first show that there are the following functorial isomorphisms:

- (i) $\text{Hom}_{\mathcal{T}}(X[2], U[d+1]) \simeq \text{Hom}_{\mathcal{T}}(W, U[1])$;
- (ii) $\text{Hom}_{\mathcal{T}}(W, U[1]) \simeq D \text{Hom}_{\Lambda}(F(U), F(L))$;
- (iii) $D \text{Hom}_{\Lambda}(F(U), F(L)) \simeq \text{Hom}_{\mathcal{T}}(L, U[d+1])$.

By the triangle (A.0.2), we have an exact sequence

$$\text{Hom}_{\mathcal{T}}(X_{\geq 3-d}[3], -) \rightarrow \text{Hom}_{\mathcal{T}}(W[d], -) \rightarrow \text{Hom}_{\mathcal{T}}(X[2], -) \rightarrow \text{Hom}_{\mathcal{T}}(X_{\geq 3-d}[2], -).$$

Evaluated at $U[d+1]$, this gives the functorial isomorphism (i), since $\text{Hom}_{\mathcal{T}}(X_{\geq 3-d}[\leq 3], U[d+1]) = 0$.

The triangle (A.0.3) and the exact sequence (A.0.4) yield a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & D \operatorname{Hom}_{\mathcal{T}}(W, U[1]) & \longrightarrow & D \operatorname{Hom}_{\mathcal{T}}(M_1, U) & \xrightarrow{D \operatorname{Hom}(f, U)} & D \operatorname{Hom}_{\mathcal{T}}(M_0, U) \\
& & & & \downarrow & & \downarrow \\
0 & \longrightarrow & \operatorname{Hom}_{\Lambda}(F(U), F(L)) & \longrightarrow & \operatorname{Hom}_{\Lambda}(F(U), \nu F(M_1)) & \xrightarrow{\operatorname{Hom}(F(U), \nu F(f))} & \operatorname{Hom}_{\Lambda}(F(U), \nu F(M_0)).
\end{array}$$

Here we used the vanishing of $D \operatorname{Hom}_{\mathcal{T}}(M_0, U[1])$. The vertical arrows are obtained by composing the functorial isomorphisms for $M' \in \mathbf{add} M$

$$\operatorname{Hom}_{\mathcal{T}}(M', U) \simeq \operatorname{Hom}_{\Lambda}(F(M'), F(U)) \simeq D \operatorname{Hom}_{\Lambda}(F(U), \nu F(M')).$$

As a consequence, the diagram gives us the functorial isomorphism (ii).

Since $\sigma^{\geq 0}U \in \mathcal{H}$ and $F(U) \simeq F(\sigma^{\geq 0}U)$, we have functorial isomorphisms

$$\operatorname{Hom}_{\Lambda}(F(U), F(L)) \simeq \operatorname{Hom}_{\Lambda}(F(\sigma^{\geq 0}U), F(L)) \simeq \operatorname{Hom}_{\mathcal{T}}(\sigma^{\geq 0}U, L) \simeq \operatorname{Hom}_{\mathcal{T}}(U, L).$$

Using the relative $(d+1)$ -Calabi-Yau property, we obtain the functorial isomorphism (iii).

Consequently, we have a functorial isomorphism

$$\operatorname{Hom}_{\mathcal{T}}(X[2], U[d+1]) \simeq \operatorname{Hom}_{\mathcal{T}}(L, U[d+1])$$

for $U \in \mathcal{T}^{\leq 0}$. Using the relative $(d+1)$ -Calabi-Yau property, we have a functorial isomorphism

$$\operatorname{Hom}_{\mathcal{T}}(-, L) \simeq \operatorname{Hom}_{\mathcal{T}}(-, X[2])$$

on \mathcal{H} . By Yoneda's Lemma, this is induced by a morphism $g \in \operatorname{Hom}_{\mathcal{T}}(L, X[2])$. \square

We extend the morphism g given in Lemma A.3 to a triangle

$$X[1] \longrightarrow Y \longrightarrow L \xrightarrow{g} X[2]. \quad (\text{A.0.5})$$

Since $\sigma^{\leq -1}(L[i]) = 0$ for $i = 0, -1$, we have an isomorphism $X[1] = \sigma^{\leq -1}(X[1]) \simeq \sigma^{\leq -1}(Y)$. Therefore it suffices to prove $Y \in \mathcal{F}$. Since both $X[1]$ and L belongs to $\mathcal{T}_{\leq 0}$, so does Y . It remains to show that $Y \in \mathcal{T}_{1-d}$. Since $\operatorname{Hom}_{\mathcal{T}}(X[1], M[\geq d+1]) = 0$ by $M \in \mathcal{T}_{\geq 1-d}$ and $\operatorname{Hom}_{\mathcal{T}}(L, M[\neq d+1]) = D \operatorname{Hom}_{\mathcal{T}}(M, L[\neq 0]) = 0$ by $L \in \mathcal{H}$, it follows that $\operatorname{Hom}_{\mathcal{T}}(Y, M[\geq d+2]) = 0$. Moreover, we have an exact sequence

$$\begin{aligned}
0 = \operatorname{Hom}_{\mathcal{T}}(L, M[d]) &\rightarrow \operatorname{Hom}_{\mathcal{T}}(Y, M[d]) \rightarrow \operatorname{Hom}_{\mathcal{T}}(X[2], M[d+1]) \xrightarrow{g} \operatorname{Hom}_{\mathcal{T}}(L, M[d+1]) \\
&\rightarrow \operatorname{Hom}_{\mathcal{T}}(Y, M[d+1]) \rightarrow \operatorname{Hom}_{\mathcal{T}}(X[1], M[d+1]) = 0.
\end{aligned}$$

By Lemma A.3, the map g is bijective, and hence $\operatorname{Hom}_{\mathcal{T}}(Y, M[d]) = 0 = \operatorname{Hom}_{\mathcal{T}}(Y, M[d+1])$. So $Y \in \mathcal{T}_{\geq 1-d}$ and the proof is complete. \square

Now we are ready to prove Theorem 4.7 (b).

Proof of Theorem 4.7 (b). The functor $\mathcal{F} \rightarrow \mathcal{C}$ is fully faithful by Proposition 4.8. It remains to show that it is dense. Let X be any object of \mathcal{C} and view it as an object of \mathcal{T} . Since $(\mathcal{T}_{\geq 0}, \mathcal{T}_{\leq 0})$ is a bounded co-t-structure on \mathcal{T} , there is an integer $i \gg 0$ such that $X \in \mathcal{T}_{\geq 1-d-i}$. We have $\sigma^{\leq -i}X \in \mathcal{F}[i]$ by Lemma A.1, and $\sigma^{\leq -i}X \simeq X$ in \mathcal{C} by $\sigma^{\geq 1-i}X \in \mathcal{T}^{\text{fd}}$. By Proposition A.2, there exists $Y \in \mathcal{F}$ such that $\sigma^{\leq -i}X \simeq \sigma^{\leq -i}Y$. Then we have $X \simeq \sigma^{\leq -i}X \simeq \sigma^{\leq -i}Y \simeq Y$ in \mathcal{C} . Thus the assertion follows. \square

REFERENCES

- [1] Takahide Adachi, Osamu Iyama and Idun Reiten, τ -tilting theory, *Compos. Math.* 150 (2014), no. 3, 415–452.
- [2] Takuma Aihara and Osamu Iyama, *Silting mutation in triangulated categories*, *J. Lond. Math. Soc.* (2) **85** (2012), no. 3, 633–668.
- [3] Claire Amiot, *Cluster categories for algebras of global dimension 2 and quivers with potential*, *Ann. Inst. Fourier (Grenoble)* **59** (2009), no. 6, 2525–2590.
- [4] Lidia Angeleri Hügel, Frederik Marks, Jorge Vitória, *Silting modules*, arXiv:1405.2531.
- [5] Maurice Auslander and Idun Reiten, *On a generalized version of the Nakayama conjecture*, *Proc. Amer. Math. Soc.* **52** (1975), 69–74.
- [6] Maurice Auslander and Sverre O. Smalø, *Almost split sequences in subcategories*, *J. Algebra* 69 (1981), no. 2, 426–454.
- [7] Alexander A. Beilinson, Joseph Bernstein, and Pierre Deligne, *Faisceaux pervers*, *Astérisque*, vol. 100, Soc. Math. France, 1982 (French).
- [8] Apostolos Beligiannis and Idun Reiten, *Homological and homotopical aspects of torsion theories*, *Mem. Amer. Math. Soc.* 188 (2007), no. 883.
- [9] Mikhail V. Bondarko, *Weight structures vs. t-structures; weight filtrations, spectral sequences, and complexes (for motives and in general)*, *J. K-Theory* **6** (2010), no. 3, 387–504.
- [10] Thomas Brüstle and Dong Yang, *Ordered exchange graphs*, *Advances in Representation Theory of Algebras (ICRA Bielefeld 2012)*, 135–193. arXiv:1302.6045.
- [11] Ragnar-Olaf Buchweitz, *Maximal Cohen-Macaulay modules and Tate-Cohomology over Gorenstein rings*, preprint 1987.
- [12] Harm Derksen, Jerzy Weyman, and Andrei Zelevinsky, *Quivers with potentials and their representations. I. Mutations*, *Selecta Math. (N.S.)* **14** (2008), no. 1, 59–119.
- [13] Vladimir Drinfeld, *DG quotients of DG categories*, *J. Algebra* **272** (2004), no. 2, 643–691.
- [14] Edgar E. Enochs, Overtoun M. G. Jenda, *Relative homological algebra*, de Gruyter Expositions in Mathematics, 30. Walter de Gruyter & Co., Berlin, 2000.
- [15] Victor Ginzburg, *Calabi–Yau algebras*, arXiv:math/0612139v3 [math.AG].
- [16] Lingyan Guo, *Cluster tilting objects in generalized higher cluster categories*, *J. Pure Appl. Algebra* **215** (2011), no. 9, 2055–2071.

- [17] Dieter Happel, *Reduction techniques for homological conjectures*, Tsukuba J. Math. **17** (1993), no. 1, 115–130.
- [18] Robin Hartshorne, *Residues and duality*, Lecture Notes in Mathematics, No. 20 Springer-Verlag, Berlin-New York 1966.
- [19] Mitsuo Hoshino, Yoshiaki Kato and Jun-ichi Miyachi, *On t -structures and torsion theories induced by compact objects*, J. Pure Appl. Algebra **167** (2002), no. 1, 15–35.
- [20] Osamu Iyama, Peter Jørgensen and Dong Yang, *Intermediate co- t -structures, two-term sifting objects, τ -tilting modules, and torsion classes*, arXiv:1311.4891.
- [21] Osamu Iyama and Yuji Yoshino, *Mutations in triangulated categories and rigid Cohen-Macaulay modules*, Inv. Math. **172** (2008), 117–168.
- [22] Martin Kalck and Dong Yang, *Relative singularity categories I: Auslander resolutions*, arXiv:1205.1008v2.
- [23] ———, *Relative singularity categories III: cluster resolutions*, in preparation.
- [24] Bernhard Keller, *Deriving DG categories*, Ann. Sci. École Norm. Sup. (4) **27** (1994), no. 1, 63–102.
- [25] ———, *On triangulated orbit categories*, Doc. Math. **10** (2005), 551–581.
- [26] ———, *On differential graded categories*, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006, pp. 151–190.
- [27] ———, *Deformed Calabi-Yau completions*, J. Reine Angew. Math. **654** (2011), 125–180, With an appendix by Michel Van den Bergh.
- [28] Bernhard Keller and Pedro Nicolás, *Weight structures and simple dg modules for positive dg algebras*, Int. Math. Res. Not. IMRN 2013, no. 5, 1028–1078.
- [29] ———, *Cluster hearts and cluster tilting objects*, work in preparation.
- [30] Bernhard Keller and Idun Reiten, *Cluster-tilted algebras are Gorenstein and stably Calabi-Yau*, Adv. Math. **211** (2007), 123–151.
- [31] ———, *Acyclic Calabi-Yau categories*, Compos. Math. **144** (2008), no. 5, 1332–1348, With an appendix by Michel Van den Bergh.
- [32] Bernhard Keller and Dieter Vossieck, *Aisles in derived categories*, Bull. Soc. Math. Belg. Sér. A **40** (1988), no. 2, 239–253.
- [33] Bernhard Keller and Dong Yang, *Derived equivalences from mutations of quivers with potential*, Adv. Math. **226** (2011), no. 3, 2118–2168.
- [34] Steffen Koenig and Dong Yang, *Sifting objects, simple-minded collections, t -structures and co- t -structures for finite dimensional algebras*, Doc. Math. **19** (2014), 403–438.
- [35] Octavio Mendoza, Edith C. Sáenz, Valente Santiago, and María José Souto Salorio, *Auslander-Buchweitz context and co- t -structures*, Appl. Categor. Struct. **21** (2013), 417–440.
- [36] Jun-ichi Miyachi, *Duality for derived categories and cotilting bimodules*, J. Algebra **185** (1996), no. 2, 583–603.
- [37] Amnon Neeman, *Triangulated categories*, Annals of Mathematics Studies, vol. 148, Princeton University Press, 2001.
- [38] Yann Palu, *Grothendieck group and generalized mutation rule for 2-Calabi-Yau triangulated categories*, J. Pure Appl. Algebra **213** (2009), no. 7, 1438–1449.

- [39] David Pauksztello, *Compact corigid objects in triangulated categories and co-t-structures*, Cent. Eur. J. Math. **6** (2008), no. 1, 25–42.
- [40] Chrysostomos Psaroudakis and Jorge Vitória, work in preparation.
- [41] Hiroyuki Tachikawa, *Quasi-Frobenius rings and generalizations. QF-3 and QF-1 rings*, Notes by Claus Michael Ringel. Lecture Notes in Mathematics, Vol. 351. Springer-Verlag, Berlin-New York, 1973.

O. IYAMA: GRADUATE SCHOOL OF MATHEMATICS, NAGOYA UNIVERSITY, CHIKUSA-KU, NAGOYA, 464-8602 JAPAN

E-mail address: iyama@math.nagoya-u.ac.jp

D. YANG: DEPARTMENT OF MATHEMATICS, NANJING UNIVERSITY, 22 HANKOU ROAD, NANJING 210093, P. R. CHINA

E-mail address: dongyang2002@gmail.com