Bidimensional Regression: Assessing the Configural Similarity and Accuracy of Cognitive Maps and Other Two-Dimensional Data Sets

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Bidimensional regression is a method for comparing the degree of resemblance between 2 planar configurations of points and, more generally, for assessing the nature of the geometry (Euclidean and non-Euclidean) between 2-dimensional independent and dependent variables. For example, it can assess the similarity between location estimates from different tasks or participant groups, measure the fidelity between cognitive maps and actual locations, and provide parameters for psychological process models. The authors detail the formal similarity between uni- and bidimensional regression, provide computational methods and a new index of spatial distortion, outline the advantages of bidimensional regression over other techniques, and provide guidelines for its use. The authors conclude by describing substantive areas in psychology for which the method would be appropriate and uniquely illuminating.

Ever since Tolman (1948) introduced it, the construct of a cognitive map has played an important role in theorizing about psychological processes. A *cognitive map* is a representation of the elements of an environment and their spatial interrelations. The construct has appeared in literature as far ranging as adult and developmental human cognition, animal cognition, neuroscience, behavioral geography, behavioral ecology, and human factors in virtual reality (see Fagot, 2000; Kitchin & Blades, 2002; Kitchin & Fre-

undschuh, 2000; Newcombe & Huttenlocher, 2000, for comprehensive reviews). Though the nature and role of cognitive maps may take on subtly different meanings in each of these literatures, there is a general consensus that these representations play a functional role in many kinds of real-world spatial tasks for both animals and humans. For animals, spatial representations must underlie foraging and food caching, migration, some predatory behaviors, territorial patrolling, and so on. For humans, cognitive maps are believed to influence behaviors as diverse as navigating through familiar environments and learning novel ones, giving directions, and making decisions about where to live, work, shop, or spend a holiday. Thus, describing the nature of cognitive maps, and assessing their accuracy with respect to the real world, is essential to understanding how people and animals represent, reason about, and function in large- and small-scale spatial environments.

One of the main tools for assessing the configural relations between cognitive and actual maps is the *bidimensional regression methodology* introduced to the geography literature by Tobler (1965, 1966, 1994). Tobler developed bidimensional regression as a solution to the general problem of map comparison; the methodology is virtually unknown in the psychological literature (for examples from behavioral geography, see Kitchin & Blades, 2002; Lloyd, 1989; Lloyd & Heivly, 1987; Nakaya, 1997; and Wakabaya-

shi, 1994). The technique provides a unique means of analyzing the similarity between two or more configurations of points in a plane by postulating a regression-like relationship between coordinate pairs. In particular, whereas bivariate, unidimensional regression assesses the relation between independent and dependent variables that each measure a single dimension (e.g., actual and estimated distance in miles), bivariate, bidimensional regression assesses the relation between independent and dependent variables that are each two dimensional (2-D; e.g., actual and estimated locations in space). Bidimensional regression models are also inference tools for identifying the transformation rules between two planes. The analysis is thus appropriate for any data set in which the independent and dependent variables can each be described by two integral dimensions and for which it would be useful to understand the nature of the spatial geometry between them (e.g., comparisons between normal and brain damaged patients on spatial learning tasks, effects of different instructions on patterns of eye movements in scenes, or correlations between nesting locations from year to year in different species).

Furthermore, the method (a) is not limited to assessing Euclidean spaces; (b) yields parameters that can be analyzed in their own right (e.g., scale, angle of rotation, and shear) and that can form part of a psychological process model of either individual or group differences; (c) is well-suited for comparing two or more representations extracted through either empirical or statistical means (e.g., sketch maps and multidimensional scaling, or MDS), so it can be used as a measure of convergent validity among tasks; and (d) provides unique information as well as distinct advantages over other statistical techniques used to extract the dimensional nature of a data set, including MDS and canonical correlation. Thus, bidimensional regression is ideal for analyzing cognitive-map data as well as any data for which the 2-D geometric or spatial properties are important to evaluate and understand.

Because of the potential utility, but relative obscurity, of bidimensional regression in the psychological literature, our main intent in the present article is to introduce its concepts and computational methods and to demonstrate when and why it is to be preferred over several other analytical techniques. In addition, because an important feature of human cognitive maps is that they are often systematically distorted (e.g., Friedman & Brown, 2000a, 2000b; Friedman, Brown, & McGaffey, 2002; Friedman, Kerkman, & Brown, 2002; Glicksohn, 1994; Stevens & Coupe, 1981;

Tversky, 1981), we briefly describe a measure of distortion introduced by Waterman and Gordon (1984) that is an extension of bidimensional regression. However, the measure is somewhat flawed; we discuss why and advocate replacing it with a similar one that is more transparently related to the statistical constructs underlying bidimensional regression.

To accomplish these goals, we first briefly review the background of bidimensional regression and the index of distortion that was developed from it. Second, we provide for the first time, the detailed, formal correspondence between uni- and bidimensional regression. Doing so serves to make explicit the two fundamentally different ways in which bidimensional regression and indices of distortion in cognitive maps can be implemented. Third, we discuss the advantages of bidimensional regression compared with other methods. Fourth, we describe and critique Waterman and Gordon's (1984) distortion index (DI) and offer our alternative. Fifth, we demonstrate the two implementations of bidimensional regression with both a "toy world" and a real-world example, to illustrate how each implementation provides different information that may alter the interpretation of data. Finally, we provide some methodological guidelines and examples of substantive areas in psychology—both within and outside of cognitive mapping—in which bidimensional regression should provide unique and useful information.

Background

Tobler (1965, 1966, 1994) introduced bidimensional regression as a means of comparing the degree of resemblance between two or more representations of the same configuration of points, given a set of matching coordinates in each representation. He illustrated the method by comparing 37 locations identified on a 14th century map of the British Isles with their actual latitudes and longitudes, converted to planar xy coordinates. Tobler assigned the coordinates representing the modern latitudes and longitudes to the status of the independent variable in the bidimensional regression; locations on the 14th century map thus comprised the dependent variable. These assignments make sense because the variables are not symmetric in the statistical sense (e.g., the values of the independent variable are usually controlled or selected). For example, coordinates representing actual latitudes and longitudes are unchanging values against which the corresponding locations in all other maps,

both physical and cognitive, can be compared according to the degree to which they are similar. In both uni- and bidimensional regression, the terms *referent* and *variant* better capture how the two domains differ. However, not all treatments of bidimensional regression have implemented it with the referent map as the independent variable (e.g., Lloyd, 1989; Waterman & Gordon, 1984).

The Euclidean version of bidimensional regression shown in Equation 1 yields four parameters. When these parameters are applied to the coordinates of the referent map, they yield the "best fit" shape (i.e., set of $A'B'$ coordinate points) between the two maps, much as a regression line in the unidimensional case is the best-fit line between a single set of points.

$$\begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} + \begin{pmatrix} \beta_1 & -\beta_2 \\ \beta_2 & \beta_1 \end{pmatrix} \cdot \begin{pmatrix} X \\ Y \end{pmatrix} \qquad (1)$$

Two parameters capture the magnitude of the horizontal (α_1) and vertical (α_2) translation between the referent and the least squares solution. The remaining two parameters (β_1 and β_2) are used to derive the scale (ϕ) and angle (θ) values by which the original coordinates are transformed to derive the least squares fit. The *scale* transformation indicates the magnitude of contraction or expansion, and the *angle* determines how much and in which direction the predicted shape rotates with respect to the referent. Figure 1 shows four examples of the application of these transformations in a situation in which the bidimensional corre-
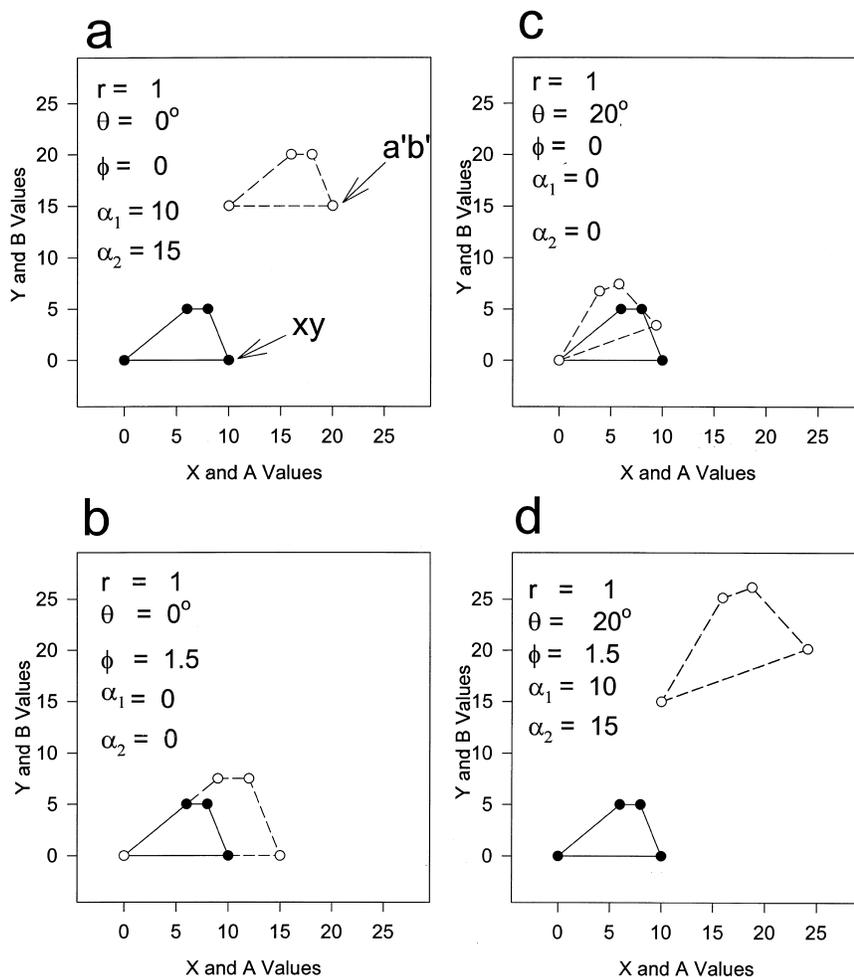

Figure 1. Bidimensional regression of a set of data in which the correlation between *XY* and *AB* is perfect. a: A simple translation of the *XY* coordinates. b: A scale increase. c: A rotation from the origin. d: All three transformations combined. Note that in this example, the *XY* coordinates serve as the independent variable in the regression.

lation between referent and variant is assumed to be perfect. It is clear that though a perfect correlation implies that the original and predicted configurations are identical, they can nevertheless be located in different places and be of different sizes and angles of rotation around the origin.

Tobler proposed a family of four bidimensional regression models (Tobler, 1966, 1994); thus far, only the Euclidean and affine models have provided useful descriptions of psychological data (Lloyd, 1989; Nakaya, 1997). The Euclidean model is a rigid transformation: The original *XY* coordinates are scaled, rotated, and translated by the same values so the overall configuration retains the same shape. In the affine transformation, *X* and *Y* can be scaled independently, and the entire configuration of points can exhibit shear (e.g., when a square becomes a parallelogram). In the projective transformation, the size, shape, and orientation of a configuration can change as a function of viewpoint. For example, the 2-D projection of a cube can become a quadrilateral shape in which no three points are collinear. The Euclidean, affine, and projective transformations form a hierarchy, and each provides a linear mapping between the independent and dependent coordinates: Straight lines in the original space are straight lines in the transformed space. In addition, parallel lines remain parallel in the Euclidean and affine transformations, but they may not in the projective transformation.

Tobler's (1994) fourth model is curvilinear and can take many forms. Because it can always fit the observations exactly, given a sufficiently high order (see also Nakaya, 1997), it is likely to be too general to be of practical use and will not be considered further here. We illustrate the points in the present article with the Euclidean model and elaborate the affine and its computational methods in the Appendix. We provide a Microsoft Excel spreadsheet to compute the Euclidean bidimensional regression on the Web at dx.doi.org/10.1037/1082.989X.8.4.468.supp.

There is a complete analogy between uni- and bidimensional regression. Nakaya (1997) capitalized on the analogy to derive inferential statistics for testing differences among parameter estimates in the bidimensional case. In the current context, one of the most important aspects of the analogy derives from the fact that in the unidimensional case one can regress either *A* on *X* or *X* on *A* (Figure 2). The two regression lines intersect at the same point as the means for the independent and dependent variables (see Hays, 1994, pp. 616–618).
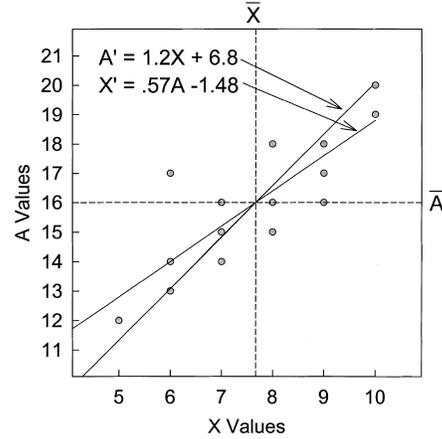

Figure 2. A set of unidimensional data (gray dots) showing the two possible regression lines: one when *X* is used as the independent variable and the other when *A* is used as the independent variable.

Analogously, it is possible in principle to regress a variant on the referent map or the referent on the variant, as shown in Figures 3 and 4. A comparison of these figures makes it clear that in the Euclidean case, even with an imperfect correlation the best-fit shape is formally similar to whichever shape's coordinates are used as the independent variable. If, for example, the *XY* coordinates are the referent, then the best fit looks like the referent has been mapped into the variant's "space" (Figure 3).

For both uni- and bidimensional regression, it makes no difference in principle which variables take on the role of the independent variable. However, the parameter estimates necessarily change when the variables exchange roles; this affects the interpretation of the results. That is, the two regression equations and their parameters in both the uni- and bidimensional cases are not simply inverses. We elaborate the consequences of this below.

Waterman and Gordon (1984) extended bidimensional regression to devise a measure that permitted comparisons of distortions among different cognitive maps (or other variants). They proposed computing a distortion distance (*D*) using each variant's coordinates as the independent variable in the bidimensional regression. The true map's coordinates were then used to compute the denominator of a distortion index (*DI = D/Dmax*) for each cognitive map.[1] However, this

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[1] Waterman and Gordon (1984) multiplied *DI* by 100 to be able to discuss the percentage of distortion. In this article
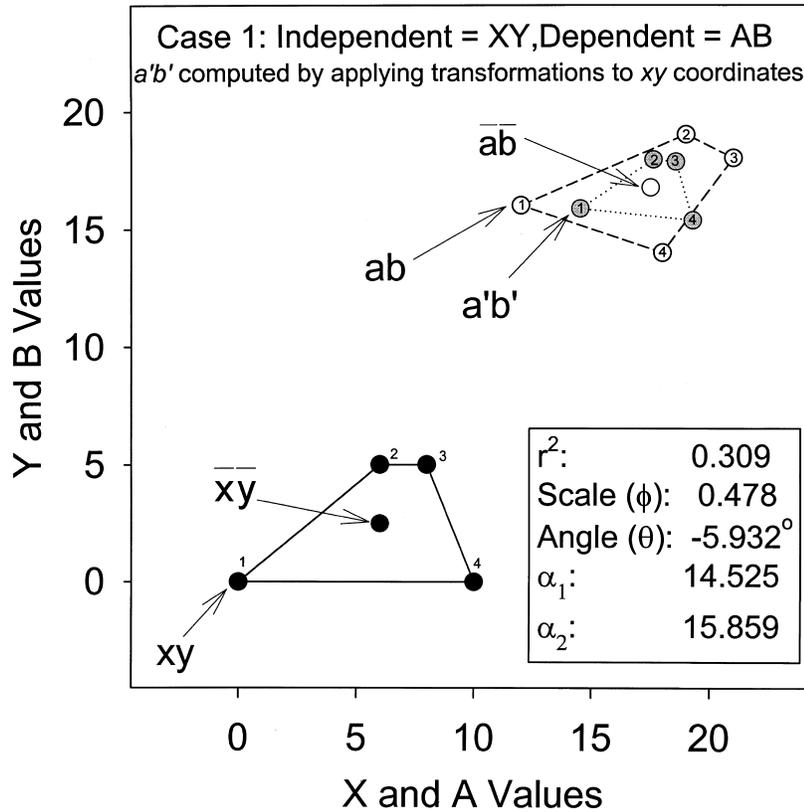
Figure 3. The toy world: The *XY* (true map) coordinates (black dots) were the independent variable, and the *AB* (cognitive map) coordinates (white dots) were the dependent variable. The gray dots represent the least squares solution. The numbers beside or inside the dots indicate corresponding coordinates.

definition of *Dmax* disrupts the strict analogy between uni- and bidimensional regression when the true map's coordinates are used as the independent variable, which is the more conventional case. Thus, we describe and advocate an alternative measure that we believe is both warranted and elegant because it reinstates the analogy.

The Formal Analogy Between Unidimensional and Bidimensional Regression

In this section, we highlight the correspondences between uni- and bidimensional regression. For ease of exposition and to facilitate the comparison, Table 1 shows the equations relevant to the two implementations of unidimensional regression, and Table 2 shows their analogs in bidimensional regression.

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we want to emphasize *DI* as a proportion of unexplained variance, so we do not multiply it by 100.

Unidimensional Regression

In the default notation for the unidimensional case, *X* refers to values of the independent variable, *A* refers to values of the dependent variable, and *A′* refers to the predicted values of the dependent variable that fall on the regression line, given a set of *X* values.

In ordinary bivariate regression, the regression line in Equation 1.1 is specified by a slope, β (Equation 1.2), and an intercept, α (Equation 1.3). Each value of the independent variable is scaled by the slope and shifted by the intercept to achieve the corresponding value on the regression line. The degree of association, or correlation, between the independent and dependent variables is given by Equations 1.4 and 1.5. The correlation is necessarily identical whether *X* is regressed on *A* or *A* is regressed on *X,* but the values of the parameters (α and β) necessarily change; that is, they are not inverses. This is because if *X* is regressed on *A,* then Equations 1.1–1.3 in Table 1 become Equations 1.7–1.9.
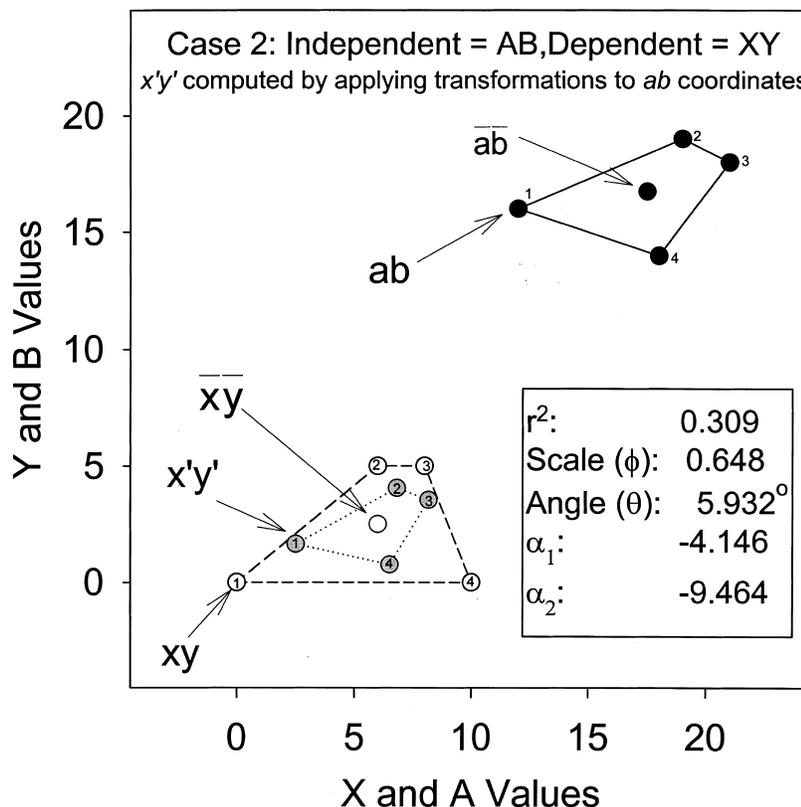
Figure 4. The toy world: The *AB* (cognitive map) coordinates (black dots) were the independent variable, and the *XY* (true map) coordinates (white dots) were the dependent variable. The gray dots represent the least squares solution. The numbers beside or inside the dots indicate corresponding coordinates.

Squaring Equation 1.5 expresses the proportion of the total variability of the scores that is a function of either the explained variability of the predicted values around the mean of the scores or the difference between a perfect correlation and the residual (unexplained) variability. The relation among these three types of sums of squares is given by Equation 1.6.

Bidimensional Regression

To move from the 1- to the 2-D case, the independent variable *X,* dependent variable *A,* and predicted value of the dependent variable *A′,* are each represented by a point in a 2-D space: *XY, AB,* and *A′B′,* respectively. Vectors represent the slope and intercept. The bidimensional regression equation is thus $(A'B') = \alpha + \beta \cdot (XY)$, where β is given by Equation 2.2 and α by Equation 2.3.

Representing the slope and intercept as vectors emphasizes that they each comprise two components. For the intercept, each component represents a trans-

lation relative to the origin of the referent's axis system: left or right, denoted by α_1; and up or down, denoted by α_2. For the slope, one component indicates whether and by how much the variant's points have expanded or contracted with respect to the referent (the magnitude or scale transformation ϕ). The interpretation of this parameter is similar to that of the slope in the unidimensional case: $\phi < 1$ indicates a contraction, and $\phi > 1$ indicates an expansion relative to the referent. The second component of the slope (the *angle,* θ) indicates whether and by how much the variant's points (the *AB* plane) have been rotated with respect to the referent's (the *XY* plane): counterclockwise if θ is positive or clockwise if θ is negative.

In the definitional equation for the bidimensional correlation (Equation 2.4), complex numbers represent the *XY* and *AB* coordinates; the second term in the numerator is the complex conjugate of the independent variable. The complex conjugate of a complex number is given by changing the sign of the imaginary

Table 1
Equations for Unidimensional Regression When X or A Is the Independent Variable

Equation	Variable assignment
	X independent, *A* dependent
1.1	$A' = \alpha + \beta \cdot X$
1.2	$\beta = \dfrac{\text{cov } AX}{\text{var } X} = \dfrac{\sum(A - M_A) \cdot (X - M_X)}{\sum(X - M_X)^2}$
1.3	$\alpha = M_A - \beta \cdot M_X$
1.4	$r = \dfrac{\text{cov } AX}{\sqrt{\text{var } X \cdot \text{var } A}} = \dfrac{\sum(A - M_A) \cdot (X - M_X)}{\sqrt{\sum(X - M_X)^2 \cdot \sum(A - M_A)^2}}$
1.5	$r = \sqrt{\dfrac{\sum(A' - M_A)^2}{\sum(A - M_A)^2}} = \sqrt{1 - \dfrac{\sum(A - A')^2}{\sum(A - M_A)^2}}$
1.6	$\sum(A - M_A)^2 = \sum(A' - M_A)^2 + \sum(A - A')^2$ Total = Explained + Unexplained
	A independent, *X* dependent
1.7	$X' = \alpha + \beta \cdot A$
1.8	$\beta = \dfrac{\text{cov } XA}{\text{var } A} = \dfrac{\sum(X - M_X) \cdot (A - M_A)}{\sum(A - M_A)^2}$
1.9	$\alpha = M_X - \beta \cdot M_A$
1.10	$r = \dfrac{\text{cov } XA}{\sqrt{\text{var } A \cdot \text{var } X}} = \dfrac{\sum(X - M_X) \cdot (A - M_A)}{\sqrt{\sum(A - M_A)^2 \cdot \sum(X - M_X)^2}}$
1.11	$r = \sqrt{\dfrac{\sum(X' - M_X)^2}{\sum(X - M_X)^2}} = \sqrt{1 - \dfrac{\sum(X - X')^2}{\sum(X - M_X)^2}}$
1.12	$\sum(X - M_X)^2 = \sum(X' - M_X)^2 + \sum(X - X')^2$ Total = Explained + Unexplained

part (e.g., the conjugate of $z = a + ib$ is $z^* = a - ib$). That the complex conjugate of the independent variable is used in the bidimensional covariance is an important point: In the unidimensional case, $\Sigma(A - M_A) \cdot (X - M_X)$ and $\Sigma(X - M_X) \cdot (A - M_A)$ return the same result, but in the bidimensional case, the conjugated variable provides the base from which the other variable is rotated by θ degrees. Thus, the order of terms is relevant, and the coordinates representing the reference plane must become the complex conjugate.

The degree of association between the *XY* and *AB* coordinate pairs is given by the magnitude of the vector in Equation 2.4 and by Equation 2.5. The angle components of **r**, **β**, and cov *AB, XY* have the identical value. An interesting consequence of the fact that the bidimensional correlation is the magnitude of a vector is that it cannot be negative.

As in the unidimensional case, r^2 expresses the ratio of either the explained, or one minus the unexplained variance to the total variance among the dependent variable's scores. Thus, extending the analogy from the unidimensional case, the relation among the three sums of squares in Equation 2.5 is given by Equation 2.6.

Table 2
Bidimensional Regression Equations When XY or AB Is the Independent Variable

Equation	Variable assignment
	XY independent, *AB* dependent
2.1	$(A'B') = \boldsymbol{\alpha} + \boldsymbol{\beta} \cdot (XY)$, or $\begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} + \begin{pmatrix} \beta_1 & -\beta_2 \\ \beta_2 & \beta_1 \end{pmatrix} \cdot \begin{pmatrix} X \\ Y \end{pmatrix}$ in matrix form.
2.2	$\boldsymbol{\beta} = \beta_1 + i\beta_2 = \dfrac{\text{cov } AB, XY}{\text{var } X + \text{var } Y} = \dfrac{\sum [(A - M_A) + i(B - M_B)] \cdot [(X - M_X) + i(Y - M_Y)]^*}{\sum [(X - M_X)^2 + (Y - M_Y)^2]}$
2.3	$\boldsymbol{\alpha} = \alpha_1 + i\alpha_2 = (M_A + iM_B) - \boldsymbol{\beta} \cdot (M_X + iM_Y)$
2.4	$\mathbf{r} = \dfrac{\text{cov } AB, XY}{\sqrt{(\text{var } X + \text{var } Y) \cdot (\text{var } A + \text{var } B)}} = \dfrac{\sum [(A - M_A) + i(B - M_B)] \cdot [(X - M_X) + i(Y - M_Y)]^*}{\sqrt{\sum [(X - M_X)^2 + (Y - M_Y)^2] \cdot \sum [(A - M_A)^2 + (B - M_B)^2]}}$
2.5	$r = \sqrt{\dfrac{\sum [(A' - M_A)^2 + (B' - M_B)^2]}{\sum [(A - M_A)^2 + (B - M_B)^2]}} = \sqrt{1 - \dfrac{\sum [(A - A')^2 + (B - B')^2]}{\sum [(A - M_A)^2 + (B - M_B)^2]}}$
2.6	$\sum [(A - M_A)^2 + (B - M_B)^2] = \sum [(A' - M_A)^2 + (B' - M_B)^2] + \sum [(A - A')^2 + (B - B')^2]$ Total = Explained + Unexplained
	AB independent, *XY* dependent
2.7	$(X'Y') = \boldsymbol{\alpha} + \boldsymbol{\beta} \cdot (AB)$, or $\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} + \begin{pmatrix} \beta_1 & -\beta_2 \\ \beta_2 & \beta_1 \end{pmatrix} \cdot \begin{pmatrix} A \\ B \end{pmatrix}$ in matrix form.
2.8	$\boldsymbol{\beta} = \beta_1 + i\beta_2 = \dfrac{\text{cov } XY, AB}{\text{var } A + \text{var } B} = \dfrac{\sum [(X - M_X) + i(Y - M_Y)] \cdot [(A - M_A) + i(B - M_B)]^*}{\sum [(A - M_A)^2 + (B - M_B)^2]}$
2.9	$\boldsymbol{\alpha} = \alpha_1 + i\alpha_2 = (M_X + iM_Y) - \boldsymbol{\beta} \cdot (M_A + iM_B)$
2.10	$\mathbf{r} = \dfrac{\text{cov } XY, AB}{\sqrt{(\text{var } A + \text{var } B) \cdot (\text{var } X + \text{var } Y)}} = \dfrac{\sum [(X - M_X) + i(Y - M_Y)] \cdot [(A - M_A) + i(B - M_B)]^*}{\sqrt{\sum [(A - M_A)^2 + (B - M_B)^2] \cdot \sum [(X - M_X)^2 + (Y - M_Y)^2]}}$
2.11	$r = \sqrt{\dfrac{\sum [(X' - M_X)^2 + (Y' - M_Y)^2]}{\sum [(X - M_X)^2 + (Y - M_Y)^2]}} = \sqrt{1 - \dfrac{\sum [(X - X')^2 + (Y - Y')^2]}{\sum [(X - M_X)^2 + (Y - M_Y)^2]}}$
2.12	$\sum [(X - M_X)^2 + (Y - M_Y)^2] = \sum [(X' - M_X)^2 + (Y' - M_Y)^2] + \sum [(X - X')^2 + (Y - Y')^2]$ Total = Explained + Unexplained

Note. The asterisks in Equations 2.2, 2.4, 2.8, and 2.10 denote the term in the numerator that is the complex conjugate.

Once again, though the value of r does not change when the variables exchange roles, the values of the parameters ($\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$) do. Furthermore, as in unidimensional regression, as r decreases from 1 to 0, the scale factor (or slope, ϕ) necessarily decreases, until the projected referent's shape becomes a point (i.e., the mean of the dependent variable's coordinates; see Figures 3 and 4). Of course, this does not preclude scale values greater than 1. For example, in Figure 1d, the scale value begins at 1.5 when $r = 1$ and remains greater than 1 until $r < .70$.

Solving for the Parameters: (XY Independent)

Tobler (1994) provided a means of solving for the bidimensional regression parameters using matrices.

$$
\begin{bmatrix}
N & 0 & \sum X & -\sum Y \\
0 & N & \sum Y & \sum X \\
\sum X & \sum Y & \sum (X^2 + Y^2) & 0 \\
-\sum Y & \sum X & 0 & \sum (X^2 + Y^2)
\end{bmatrix}
\cdot
\begin{bmatrix}
\alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2
\end{bmatrix}
=
\begin{bmatrix}
\sum A \\ \sum B \\ \sum (AX + BY) \\ \sum (BX - AY)
\end{bmatrix}
\tag{3}
$$

For the Euclidean model in Equation 1, the system of equations that yields the regression parameters when XY is the independent variable is given above; N is the number of coordinate pairs.[2]

Given a set of coordinate pairs, XY and AB, the entries in the first and third matrices in Equation 3 can be computed and the parameters obtained by taking the inverse of the first matrix and postmultiplying it by the third matrix. Solving this system of equations for the four parameters yields the following relations, which can be used to find the parameters directly:

$$
\beta_1 = \frac{\text{cov } AX + \text{cov } BY}{\text{var } X + \text{var } Y},
$$
$$
\beta_2 = \frac{\text{cov } BX - \text{cov } AY}{\text{var } X + \text{var } Y},
$$

and

$$
\alpha_1 = M_A - \beta_1 \cdot M_X + \beta_2 \cdot M_Y,
$$
$$
\alpha_2 = M_B - \beta_2 \cdot M_X - \beta_1 \cdot M_Y.
$$

The form of the numerator in the equations for β_1 and β_2 is a direct result of expressing the bidimensional covariance as a complex number; that is, $AX + BY$ is the real part of the bidimensional covariance, and $BX - AY$ is the imaginary part.

β_1 and β_2 are used to compute the scale and angle values, according to the equations

$$
\text{Scale} = \phi = \sqrt{\beta_1^2 + \beta_2^2}
$$
$$
\text{Angle} = \theta = \tan^{-1}\left(\frac{\beta_2}{\beta_1}\right).
$$

Because the arctangent function ($\tan^{-1}$) covers only the range between $-90°$ and $90°$ (i.e., two quadrants) it is necessary to add $180°$ to θ if $\beta_1 < 0$. This extends the range of the $\tan^{-1}$ function from $-90°$ to $270°$, covering all four quadrants.[3]

Each predicted pair of coordinates can be computed using either the regression parameters or the scale and angle transformations. This is illustrated in Equations 4 and 5 for the situation in which XY is the indepen-

dent variable and in Equations 6 and 7 for the situation in which AB is the independent variable. Note that the values of α_1, α_2, β_1, and β_2 in Equations 4 and 5 are different from those in Equations 6 and 7:

$$
A' = \alpha_1 + \beta_1(X) - \beta_2(Y)
$$
$$
B' = \alpha_2 + \beta_2(X) + \beta_1(Y),
\tag{4}
$$

$$
A' = \alpha_1 + \phi(X\cos\theta - Y\sin\theta)
$$
$$
B' = \alpha_2 + \phi(X\sin\theta + Y\cos\theta),
\tag{5}
$$

$$
X' = \alpha_1 + \beta_1(A) - \beta_2(B)
$$
$$
Y' = \alpha_2 + \beta_2(A) + \beta_1(B),
\tag{6}
$$

$$
X' = \alpha_1 + \phi(A\cos\theta - B\sin\theta)
$$
$$
Y' = \alpha_2 + \phi(A\sin\theta + B\cos\theta).
\tag{7}
$$

Inferential statements about the relation between two configurations can be supported in at least two ways. First, the bidimensional correlation coefficient can be tested to see whether it is significantly different from zero:

$$
F = \frac{2N - p}{p - 2} \cdot \frac{r^2}{1 - r^2}, \ df = (p - 2), (2N - p),
$$

(Nakaya, 1997, Equation 50) where N is the number of coordinate pairs, and p is the number of parameters in the model being tested (Euclidean, affine, projective, or curvilinear). Second, the difference between

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[2] There is a slight error in Tobler's (1994) original matrix, which was supposed to provide the parameters for the case when the true map was the independent variable; the result is that θ has the wrong sign. We have adapted Tobler's matrix equation so that it gives the same result as Equation 2.4 in Table 2.

[3] Tan^{-1} is the arctangent, or inverse tangent, for the point XY relative to the origin (0,0). It returns the counterclockwise angle from the X-axis in the range from $-90°$ to $90°$. In Microsoft Excel, the ATAN2 function automatically adjusts for the situation when $\beta_1 < 0$.

two nested models in the amount of variance accounted for can be tested:

$$F_{12} = \frac{2N - p_2}{p_2 - p_1} \cdot \frac{r_2^2 - r_1^2}{1 - r_2^2}, df = (p_2 - p_1), (2N - p_2),$$

(Nakaya, 1997, Equation 58) where Model 1 has fewer parameters than Model 2.

Advantages of Bidimensional Regression

The first, and perhaps most important, advantage of bidimensional regression over other uni- or multivariate methods of analyzing 2-D data is that it is sensitive to, and provides measures of, the geometry of the spatial relations between two 2-D variables that (a) form a plane and (b) can be identified with the same locations.[4] Thus, pragmatically, bidimensional regression is a statistical method of comparing any two or more sets of planar coordinates, however they may have been generated, so long as they can be digitized. For example, a configuration of landmarks from a set of sketch maps can be compared with one that is inferred from a multidimensional scaling (MDS) solution generated from a list of all possible pairs of distances between the landmarks, and the accuracy of the maps produced by these two methods can be compared with the actual map from which they were presumably derived. This means that the bidimensional r^2 can be used to measure the convergent validity between various tasks (e.g., Kitchin, 1996).

A second advantage of using bidimensional regression is that its parameters provide the basis for computing the transformations required to perform the mapping between the two planes under consideration. Thus, the parameters derived from the bidimensional regression (α_1, α_2, β_1, β_2) permit the computation of transformations (scale and rotation) that can be compared across individuals and groups; the transformations could form part of a psychological process model that specifically addresses how geometric distortions arise in spatial representations. For example, Lloyd (1989) used bidimensional regression analytically to compute differences in the patterns of distance and directional errors between groups of participants who had learned a city primarily through navigation or by studying a map. He used the bidimensional regression parameters to compute and then make inferences about different underlying alignment and rotation heuristics (transformations) used to scale the cognitive maps in the two groups.

Third, as pointed out by Nakaya (1997, p. 174), bidimensional regression is unique in supposing a one-to-one mapping between two point distributions. Yet, assuming independent normal error and constant variance, orthodox computational procedures for regression analysis can be used for parameter estimation as well as for computing confidence intervals and significance tests. Equally, the strengths and limitations of unidimensional regression are applicable to bidimensional regression. Thus, they should be relatively familiar to psychologists.

What About Other Methods?

Because bidimensional regression presupposes that the dependent and independent variables are planar coordinates, the issue of how many dimensions best describe the data is moot—it is always two. Thus, methods designed to discover how many dimensions describe a set of data, such as factor analysis or canonical correlation, are not necessarily germane to data appropriate for bidimensional regression. In addition, because of the bidimensional, as opposed to multivariate, nature of data appropriate for bidimensional regression, other common methods that might be entertained for its analysis are not appropriate. We discuss two of these below, and then turn to a comparison between bidimensional regression and MDS, because though the MDS methodology is appropriate (in principle) for analyzing cognitive mapping and other 2-D data, it may not be ideal.

Individual regressions for each dimension. It should be clear that separately correlating the *X* and *A* (horizontal) and the *Y* and *B* (vertical) values misses the essential point that *XY* and *AB* are integral coordinates that represent a single location in space; that is, location is a 2-D variable. Separate correlations on each dimension do not yield configural information and are also insensitive to relative stretching. Furthermore, separate slope and intercept parameters computed for the regression of *X* on *A* (or *Y* on *B*) cannot be used to derive scale or angle parameters that transform the configuration between planes.

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[4] In the present article we have focused on situations in which there is one independent variable and one dependent variable. Tobler (1994) extended the formal analysis to cases where there is more than one dependent variable; for example, the features of more than one child's face can be compared with those of a parent, provided corresponding locations on each are converted to planar coordinates.

Canonical correlation. Canonical correlation is among the most general of the multivariate techniques; its goal is to analyze the relationships between two or more sets of variables (Tabachnick & Fidell, 2001). Thus, canonical correlation at first seems appropriate for analyzing the relation between, for example, 2-D location estimates and actual locations. However, in canonical correlation, each participant is typically measured on two sets of variables and the analysis determines how the sets are related to each other (e.g., one set of variables might be scores on various measures of scholastic ability, and the other set might be measures of success in school). Thus, a data set appropriate for this kind of analysis should have participants measured on a minimum of four variables (two independent and two dependent). Because the *XY* and *AB* coordinate pairs are inextricably linked together, they do not qualify. Again, *X* and *Y* are two dimensions of a location marker (mental or physical), not two different variables.

Multidimensional scaling (MDS). Unlike separate unidimensional correlations or canonical analysis, metric and nonmetric MDS are techniques that can be used to recover the 2-D structure embedded within a matrix of proximities (for example, participants might provide similarity or difference ratings for all possible pairs of stimuli). As such, MDS is an appropriate candidate for assessing cognitive-mapping data and has been used in this manner in the geography literature (e.g., Buttenfield, 1986; Gatrell, 1983; Kitchin, 1996; Magaña, Evans, & Romney, 1981). MDS takes a matrix of proximities and displays it in Euclidean space such that there is a minimal degree of distortion between the distances in the matrix and the distances in the MDS solution.

Stress measures how well any given MDS solution fits the original matrix from which it was generated. Thus, stress is not a measure of accuracy of the solution "map" with respect to the real world (or any other referent) but only with respect to the original proximity matrix. This means that MDS by itself cannot provide a measure of accuracy between the experimental data generated (by whatever technique) and the actual distances in the geometric environment. By contrast, bidimensional regression provides exactly this kind of accuracy measure. For example, Kitchin (1996) used bidimensional regression to compare the cognitive maps inferred from using MDS procedures (e.g., filling in a matrix of all possible distance estimates) with those obtained using several other methods, including drawing, spatial cuing, and cloze procedures. He

found that the maps produced by subjecting the distance estimates to MDS were substantially less accurate than the maps produced by the other methods. Additional drawbacks are that MDS (a) assumes that participants are equally familiar with all the locations, which limits its empirical use; (b) might introduce geometric distortion into the data (Buttenfield, 1986); and (c) might not be a valid technique to infer a latent 2-D configuration. This is because people might not know where places are in relation to each other in a map-like sense yet still be able to generate a distance estimate, perhaps using travel times as a proxy (Kitchin and Blades, 2002, p. 133).

Furthermore, the similarity judgments required to generate an MDS solution typically constitute a large amount of information that is time consuming to collect. The method is not suitable, therefore, for certain participant populations (e.g., young children or animals). Even if one were to infer the proximity matrix by, say, computing all possible distances between points generated on a sketch map of landmarks, the MDS solution would still need to be compared with the real-world configuration for accuracy using bidimensional regression or some other technique (for a further critique of MDS as a method for investigating the mental representation of space, see Hunt & Waller, 1999; Kitchin, 1996; and Waller & Haun, in press). In summary, MDS is not in any sense substitutable for bidimensional regression.

Distortion Distance and *DI*

Throughout the remainder of this article, we always refer to the *XY* coordinates as locations on an actual map and the *AB* coordinates as the participants' location estimates, either singly or in aggregate. Either set of coordinates can play the role of dependent or independent variable. As noted previously, Waterman and Gordon's (1984) proposed distortion distance, *D*, was unconventional both conceptually and statistically. In particular, to compute *D*, they assigned the cognitive coordinates (*AB*) to the role of independent variable and the true map's coordinates (*XY*) to the role of dependent variable. Thus, Waterman and Gordon explicitly chose to have "the coordinates of the mental map undergo a transformation so that the sum of squares of the distances from the 'true' or given points to the 'transformed' points is minimal" (p. 327). The assignment of coordinates to fixed roles meant that "The best-fit solution always brings the mean center of the mental map to the mean center of

the true map" (p. 328). Their index thus reflects the situation depicted in Figure 4 and does not allow for the case in which the participants' coordinates are the dependent variable.

In contrast to Waterman and Gordon (1984), because there are two possible implementations of bidimensional regression, we believe D should always be defined in terms of whichever coordinates are to be the dependent variable; on this view, when AB is dependent, as in Figure 3,

$$D_{AB} = \sqrt{\sum[(A - A')^2 + (B - B')^2]}, \qquad (8)$$

and when XY is dependent, as in Figure 4,

$$D_{XY} = \sqrt{\sum[(X - X')^2 + (Y - Y')^2]}. \qquad (9)$$

It should be noted that Equation 9 is identical to Waterman and Gordon's (1984) definition yet is consistent with the notation we have been using throughout this article.[5] It is also important to note, from Equation 2.6, that D_{AB}^2 is simply the unexplained variance between the coordinates of the cognitive map and their predicted values. Analogously, from Equation 2.12, D_{XY}^2 is the unexplained variance between the coordinates of the true map and their predicted values. Thus, allowing D to be defined in terms of the dependent variable's coordinates brings its interpretation meaningfully into the bidimensional regression analysis.

Waterman and Gordon (1984) realized that D had limited utility because its value does not indicate whether a given amount of distortion is large or small. To standardize the amount of distortion among different cognitive maps of the same place, they proposed that the true map's coordinates be used to compute $Dmax$, which they defined as "the maximum value that D can achieve . . . obtained when all the points on the mental map coincide in a single point" (p. 328; see also their Appendix). That is, they proposed that

$$Dmax = \sqrt{\sum[(X - M_X)^2 + (Y - M_Y)^2]}. \quad (10)$$

They then proposed that $DI = D/Dmax$ be computed for each individual (or for the means across individuals). Because D for Waterman and Gordon (1984) is always computed using the cognitive map as the referent, DI thus reinstated the true map's coordinates as a kind of referent. That is, on their analysis, DI is a dimensionless value whose magnitude indicates the amount of distortion in a mental map relative to the true map regardless of scale.

However, from Equation 2.12, $Dmax^2$ as defined by Waterman and Gordon (1984) is identical to the total variance among the coordinates of the true map. Thus, DI^2 defined by their method is simply the proportion of variance in the bidimensional regression that is unexplained. However, this relation between r^2 and DI^2 holds only when the estimated locations (AB) are the independent variable, and the true coordinates (XY) are the dependent variable. If the estimated coordinates are used as the dependent variable but $Dmax$ is still computed using the coordinates of the true map, the correspondence between the bidimensional regression and DI necessarily breaks down. In particular, though D_{AB}^2 is still equivalent to the unexplained variance (now between the AB and $A'B'$ coordinates), $Dmax$ defined in terms of the true map has no meaning in this implementation of the regression.

In contrast, if D and $Dmax$ are both defined as a function of whichever coordinates are used as the dependent variable (DV), then DI_{DV}^2 becomes the proportion of total variance in the dependent variable scores that remains unexplained by the bidimensional regression. Furthermore, if DI is defined this way, then this residual value remains equivalent irrespective of which set of coordinates (XY or AB) is used as the independent variable, and the relation between the bidimensional correlation coefficient and DI becomes

$$DI_{DV}^2 = \frac{D_{DV}^2}{Dmax_{DV}^2} = 1 - r^2. \qquad (12)$$

Assessing the Configural Similarity Between Sets of 2-D Coordinates

The Toy World

As noted earlier, Figures 3 and 4 depict the situation when either the true map's coordinates (XY) or the participants' judgments (AB), respectively, serve as the independent variable. Table 3 shows the data

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[5] Waterman and Gordon (1984) used the notation $A'B'$ to represent the predicted values, even though they used the AB (data) coordinates as the independent variable in the computations (see their Appendix). We believe it is less confusing and more consistent with the regression analysis to use the notation $X'Y'$ to refer to the least squares solution when the AB coordinates are independent and the XY coordinates are dependent, and that is what we do in the present article.

Table 3

Data, Predicted Values, Regression Parameters, and Distortion Parameters for the Toy Worlds Depicted in Figures 3 and 4

Case 1: *XY* independent							
Independent		Dependent		Predicted		Parameters	
X	*Y*	*A*	*B*	*A'*	*B'*	Regression	Distortion
0	0	12	16	14.5247	15.8580	$\beta_1 = .4753$	$D_{AB}^2 = 41.25$
10	0	19	19	19.2778	15.3642	$\beta_2 = -.0494$	$\boldsymbol{Dmax_{AB}^2 = 59.75}$
8	5	21	18	18.5741	17.8395	$\alpha_1 = 14.5247$	$\boldsymbol{Dmax_{XY}^2 = 81.00}$
6	5	18	14	17.6235	17.9383	$\alpha_2 = 15.8580$	$DI_{AB}^2 = .6903$
6.00	*2.50*	*17.50*	*16.75*	*17.50*	*16.75*	$\theta = -5.9315°$	$DI_{XY}^2 = .5088$
						$\phi = .4779$	
						$\boldsymbol{r^2 = .3095}$	

Case 2: *AB* independent							
Independent		Dependent		Predicted		Parameters	
A	*B*	*X*	*Y*	*X'*	*Y'*	Regression	Distortion
12	16	0	0	2.5063	1.6485	$\beta_1 = .6444$	$D_{XY}^2 = 55.92$
19	19	10	0	6.8159	4.0502	$\beta_2 = .0669$	$\boldsymbol{Dmax_{AB}^2 = 59.75}$
21	18	8	5	8.1715	3.5397	$\alpha_1 = -4.1458$	$\boldsymbol{Dmax_{XY}^2 = 81.00}$
18	14	6	5	6.5063	0.7615	$\alpha_2 = -9.4644$	$DI_{AB}^2 = .9359$
17.50	*16.75*	*6.00*	*2.50*	*6.00*	*2.50*	$\theta = 5.9315°$	$\boldsymbol{DI_{XY}^2 = .6903}$
						$\phi = .6478$	
						$\boldsymbol{r^2 = .3095}$	

Note. Boldface indicates values that agree in both analyses. Numbers in italics are means for each column. *D* = distortion distance; *Dmax* = maximum value of *D*; *DI* = distortion index.

that were used in the figures, the regression parameter estimates, the predicted values based on those parameters, and the distortion parameters. We have highlighted in the table those values that are identical in the two analyses. It should be noted that these analyses are functionally equivalent to item (aggregate) analyses because there was no averaging conducted to obtain the *AB* values.

Several things are notable about Table 3. For the regression parameters, only the value of r^2 is identical between the two analyses, though θ is the same except for the sign. Because the parameter estimates differ, the description of the data and nature of the inferences one might make also differs. For instance, when the true map's coordinates (*XY*) are the independent variable, transforming them to the predicted values for the cognitive map (*A'B'*) requires shifting the *XY* coordinates to the north and east by approximately the same amount, scaling them by about half, and rotating them clockwise by about 6°. In contrast, when the *AB* coordinates are the independent variable, transforming them to the predicted values for the true map (*X'Y'*) requires a westerly

translation that was about half as much as the necessary southerly translation, a scale difference of about 65%, and a 6° rotation counterclockwise. Clearly, the two descriptions are not symmetric, reflecting the fact that the parameters are not inverses. Furthermore, even if the true and cognitive maps were centered on the origin prior to conducting the regression, though the translation parameters become zero in both analyses, the scale and magnitude values still differ.

Though it seems counterintuitive that both scale values are contractions, this is the result of conducting two regressions with different independent variables, even though the data are identical. Normally, only one set of coordinates (true or cognitive) is selected to function as the independent variable, and the psychological interpretation of the single computed scale value can be made in terms of either the value itself or its inverse. For example, in the unidimensional case, once *X* is chosen as the independent variable, there is only one relevant regression line, specified by *XA'* coordinates. If any *X* must be multiplied by 0.5 to find the corresponding predicted value of the

dependent variable on the regression line (i.e., A'), then conversely, any value of A' must be multiplied by 2 to find the corresponding value of X on that same regression line. Given the same original data, if A were chosen as the independent variable the slope (and its inverse) would be different because the regression line would be different. Similarly, for the bidimensional case, if XY is independent and the computed scale value is 0.5 (as it is in the toy world example), it is correct to say that to transform the predicted $A'B'$ points back to the original XY referent map requires multiplying the predicted values by 2. However, different values are obtained for the scale and its inverse when AB is independent (indeed, in the toy world example those values are 0.65 and 1.5, respectively). This is why the selection of which map's coordinates are to provide the independent variable is fundamentally so important.

Because of the different parameter estimates and because it is a reasonable assumption that cognitive maps are formed from participants' experience with actual maps (among many other things; Friedman & Brown, 2000a, 2000b; Friedman, Brown, & Mc-Gaffey, 2002; Friedman, Kerkman, & Brown, 2002), it would normally (though not always) make more sense to describe the parameters that show how the

true map has been transformed into the cognitive map rather than the other way around.

D can be computed only one way in each analysis (Equations 8 or 9), but we computed both $Dmax$ and DI assuming that either XY or AB was the dependent variable (the squared values are shown in Table 3). Only when $Dmax$ is based on the dependent variable's coordinates does the value of DI remain the same across the analyses. This symmetry is the main reason we advocate replacing Waterman and Gordon's (1984) method with our own. In contrast, if $Dmax$ were always computed from the true map's coordinates, DI necessarily changes. This is illustrated in Table 4, which shows the values for each sum of squares in each analysis. These values are related according to the equation below:

$$r^2 = \left[\frac{38.71^2}{(81.00) \cdot (59.75)} \right] = \left[\frac{18.50}{59.75} \right] = \left[1 - \frac{41.25}{59.75} \right] = \left[\frac{25.08}{81.00} \right] = \left[1 - \frac{55.92}{81.00} \right] = .309.$$

In summary, if the measure of distortion is to maintain a transparent relation to the bidimensional regression, then D must be the root of the unexplained variance, and $Dmax$ must be the root of the total variance

Table 4

Sums of Squares and Corresponding Terms in Regression and Distortion Analyses for the Toy Worlds Depicted in Figures 3 and 4

Sum of squares	Value	Regression	Distortion index
Case 1: *XY* independent			
$\Sigma[(A - M_A) + i(B - M_B)] \cdot [(X - M_X) + i(Y - M_Y)]^*$	38.71	Cov *AB, XY*	
$\Sigma[(X - M_X)^2 + (Y - M_Y)^2]$	81.00	Total SS (IV)	$Dmax^2_{XY}$ (true map)
$\Sigma[(A - M_A)^2 + (B - M_B)^2]$	59.75	Total SS (DV)	$Dmax^2_{AB}$ (cognitive map)
$\Sigma[(A' - M_A)^2 + (B' - M_B)^2]$	18.50	Explained SS	
$\Sigma[(A - A')^2 + (B - B')^2]$	41.25	Unexplained SS	D^2_{AB}
$\Sigma[(X - A')^2 + (Y - B')^2]$	1,363.75		
$\Sigma[(X - A)^2 + (Y - B)^2]$	1,405.00		
Case 2: *AB* independent			
$\Sigma[(X - M_X) + i(Y - M_Y)] \cdot [(A - M_A) + i(B - M_B)]^*$	38.71	Cov *XY, AB*	
$\Sigma[(X - M_X)^2 + (Y - M_Y)^2]$	81.00	Total SS (DV)	$Dmax^2_{XY}$ (true map)
$\Sigma[(A - M_A)^2 + (B - M_B)^2]$	59.75	Total SS (IV)	$Dmax^2_{AB}$ (cognitive map)
$\Sigma[(X' - M_X)^2 + (Y' - M_Y)^2]$	25.08	Explained SS	
$\Sigma[(X - X')^2 + (Y - Y')^2]$	55.92	Unexplained SS	D^2_{XY}
$\Sigma[(A - X')^2 + (B - Y')^2]$	1,349.08		
$\Sigma[(A - X)^2 + (B - Y)^2]$	1,405.00		

Note. The asterisks in the first equation in each section indicate that the second term is a complex conjugate. SS = sum of squares; IV = independent variable; DV = dependent variable; D = distortion distance; $Dmax$ = maximum value of D.

among the dependent variable's coordinates. The value of *DI* will then be the same irrespective of which coordinates are independent, and which analysis one chooses depends on a variety of factors.

In the toy world example, $r^2 = .309$ for the Euclidean model and .885 for the affine model (see Appendix). There are only four sets of points in the configurations, so neither of these correlations were significantly different from zero, $F(2, 4) = 0.89$, $p < 1$, and $F(4, 2) = 3.87$, $p < 1$, respectively. The difference between the Euclidean and affine models was also not significant, $F(2, 2) = 5.03$, $p < 1$.

The Real World

For unidimensional regression, when a constant set of values is used as the independent variable for all participants, then the slope and intercept averaged across individuals is identical to the slope and intercept calculated from the item means. Similarly, for bidimensional regression, if only one referent map's coordinates are used as the independent variable then the average of the participants' individual regression parameters (α_1, α_2, β_1, and β_2) have identical values to those obtained by averaging location estimates over individuals for each place and computing the regression parameters from the analysis of the item means. This is because the denominator of β is the same constant for both the item and participant analyses, and all four parameter estimates are derived from this vector. In contrast, when the cognitive map's coordinates are used as the independent variable, the parameter estimates from the participant and item means differ. This disparity is illustrated with actual location estimates.

We obtained location estimates from 32 participants who had lived in Edmonton, Alberta, Canada for a minimum of 5 years ($M = 16.40$ years, $SE = 1.00$). An important geographic feature of the city of Edmonton is the North Saskatchewan river, which meanders from the southwest to the northeast (see Figure 5). In addition, Edmonton uses a numeric street and avenue grid system in which the streets and avenues are orthogonal. The centre of downtown is located at 100th Street and 100th Avenue; the streets run from 1st Street at the eastern border to approximately 200th Street at the west; and the avenues run from 1st Avenue at the southern border to about 200th Avenue at the north. This numbering scheme allowed us to obtain absolute location estimates (i.e., street and avenue numbers), rather than having participants estimate locations relative to reference points. This
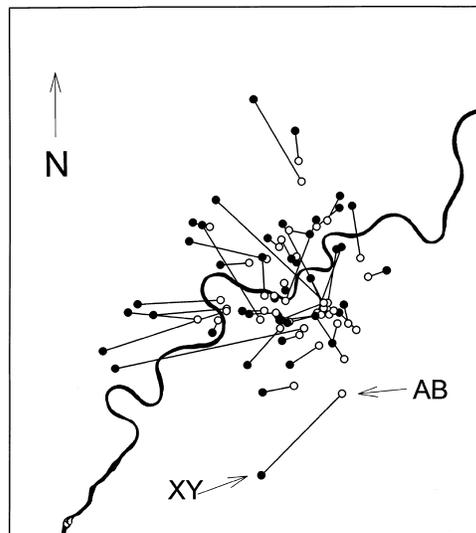

Figure 5. Actual (black circles) and mean estimated (white circles) locations of 44 places in the city of Edmonton, Alberta, Canada. The actual and estimated locations for each place are connected by a line. They were converted to pixel coordinates by hand and superimposed on a digitized map of Edmonton. The solid line running through the graph represents the actual location of the North Saskatchewan river.

was advantageous because landmarks and other reference points often distort cognitive maps (Holyoak & Mah, 1982; McNamara & Didwakar, 1997). However, numeric estimates per se are not required for bidimensional regression; any data that can be converted to *xy* coordinates can be used.

The participants estimated the street and avenue location nearest to the centroid of 44 public places. They were tested individually; place names appeared one at a time on a computer screen; half the participants entered the value of the street before the avenue, and the remainder did the reverse. The places estimated included shopping malls, theaters and auditoriums, museums, sports centers, high schools, and popular restaurants and bars. Most of the locations spanned at least one city block, so the centroid estimate was prima facie reasonable. The locations were selected to represent places that would reflect a range of familiarity; that this was achieved was corroborated by knowledge ratings: On a 0 (*no knowledge*) to 9 (*a lot of knowledge*) scale, the mean rating for the 44 places was 4.30 ($SD = 0.26$). The knowledge ratings were collected before the location estimates, using a similar procedure. Because of the variation in rated

knowledge, the selected locations would not be appropriate for an MDS analysis.

The mean estimated and actual locations are shown in Figure 5. When averaged across participants, the bidimensional correlation based on the Euclidean model was .487 [r^2 (explained variance) = .237; DI^2 (unexplained variance) = .763], whereas for the item means it was .839 (r^2 = .703, DI^2 = .297). Both of these values were significantly different from zero, $F(2, 84) = 13.05$, $p < .01$, and $F(2, 84) = 99.41$, $p < .01$, respectively. The difference in r^2 values between the participant and item analysis reflects the well-known point that item means usually overestimate the strength of the association between variables, compared with averaging r^2 over participants. Conversely, DI will be underestimated with item means.

Table 5 shows the bidimensional regression and distortion parameters computed from the average of the participants' estimates as well as from the item means. In addition, the table shows the parameter values obtained when either the true map or the cognitive map coordinates were treated as independent. The parameter values computed from the item means were used to compute the predicted values displayed in Figures 6 and 7, so that these figures are completely analogous to Figures 3 and 4, respectively. Thus, notably, the regularity apparent between the connected points in Figures 6 and 7, compared with Figure 5, reflects the fact that the configurations of predicted values (gray dots) in Figures 6 and 7 are formally similar to the referent configurations from which they were transformed (black dots). That is, in the Euclid-
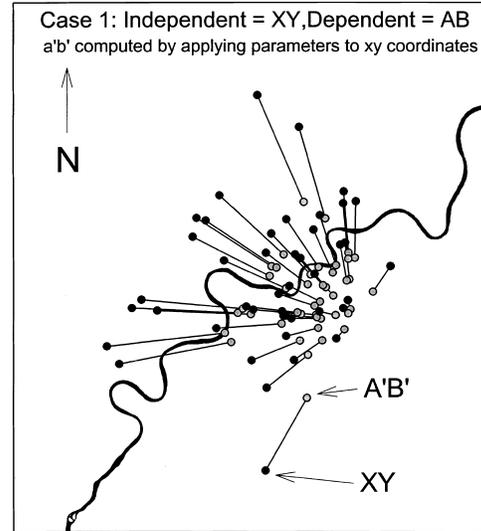

Figure 6. Actual (black circles) and predicted (gray circles) locations of 44 places in the city of Edmonton, Alberta, Canada. The parameter estimates were obtained by assuming that the *XY* (true map) coordinates were the independent variable. The actual and predicted locations for each place are connected by a line. The actual street and avenue locations were converted to pixel coordinates by hand, and the scale and angle transformations from Table 5, Column 1, were applied to the pixel values. The translation parameters were found by converting the mean actual and estimated street and avenue values to pixel units by hand and using the results in Equation 2.3. The final predicted values were superimposed on a digitized map of Edmonton. The solid line running through the graph represents the actual location of the North Saskatchewan river.

ean model, the predicted values and their predictors have identical global configurations.

The transformation from the actual locations to the predicted values on the cognitive map (*XY* independent; Figure 6) required a scale shift of 0.52 (with the inverse direction for that implementation requiring a shift of 1.92), whereas the transformation from the cognitive map to predicted values on the actual map (*AB* independent; Figure 7) entailed an expansion of 1.35 (with the inverse being 0.75). Again, these are very different descriptions. In contrast to the toy-world example, there was almost no rotation observed between the actual and predicted configurations ($\theta = 0.25°$ or $-0.25°$); thus, our participants functionally preserved the known canonical orientation of the streets and avenues of Edmonton in their estimates.

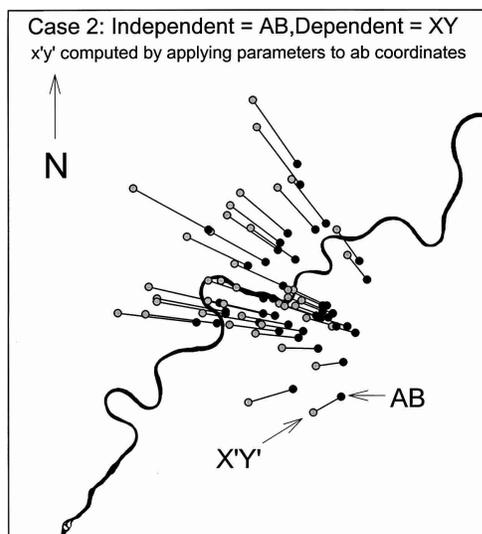From the statistical point of view, it is clear from the two leftmost columns of Table 5 that when the

Table 5
Regression and Distortion Parameter Estimates for Participant and Item Analyses of Experiment 1

| Parameter | XY independent | | AB independent | |
	Item means	Participant means	Item means	Participant means
β_1	.5221	.5221	1.3472	.4161
β_2	.0023	.0023	−.0059	−.0012
α_1	40.6262	40.6262	−22.2823	66.3934
α_2	38.9062	38.9062	−24.5402	54.6260
θ	.2506°	−10.6258°	−.2506°	10.6258°
ϕ	.5221	.5542	1.3474	.4374
r^2	.7034	.2375	.7034	.2375
D	80.96	231.996	130.01	198.60
$Dmax$	148.70	276.2243	238.8322	238.8322
DI	54.46	83.1544	54.46	83.1544

Note. D = distortion distance; *Dmax* = maximum value of *D; DI* = distortion index.

Figure 7. Mean estimated (black circles) and predicted (gray circles) locations of 44 places in the city of Edmonton, Alberta, Canada. The parameter estimates were obtained by assuming that the *AB* (cognitive map) coordinates were the independent variable. The estimated and predicted locations for each place are connected by a line. The estimated street and avenue locations were converted to pixel coordinates by hand, and the scale and angle transformations from Table 5, Column 3, were applied to the pixel values. The translation parameters were found by converting the mean actual and estimated street and avenue values to pixel units by hand and using the results in Equation 2.9. The final predicted values were superimposed on a digitized map of Edmonton. The solid line running through the graph represents the actual location of the North Saskatchewan river.

referent map's coordinates (*XY*) are the independent variable, α_1, α_2, β_1, and β_2 are identical for both items and participants; ϕ and θ are not. Recall that ϕ and θ are the magnitude and angle of $\boldsymbol{\beta}$, respectively; when these transformations are computed separately for each participant and averaged over participants, as we have done in the second column of Table 5, it is functionally the same as averaging the magnitudes and angles of vectors, which is not usually meaningful. Because β_1 and β_2 are identical across participants and items when *XY* is independent, we believe the values of ϕ and θ computed from these parameters are the ones that should be reported and tested (via Nakaya's [1997] methods).

The two rightmost columns in Table 5 show the regression and distortion parameters when participants' judgments were the independent variable. Here, none of the four regression parameters (α_1, α_2, β_1, and β_2) correspond across item and participant

analyses, which is potentially problematic. For example, it is not clear which set of regression parameters to use to compute ϕ and θ; however, because averaging angles and magnitudes of vectors is not meaningful it is probably reasonable to use β_1, and β_2 computed from the item means.

Statistical Issues and Guidelines

The Fisher-transformed bidimensional correlation coefficient, regression parameters, and *DI* can be computed for each individual and treated as dependent measures in analysis of variance or multivariate analysis of variance to test individual or group differences. In addition, Nakaya (1997) has a thorough treatment of significance testing for the bidimensional regression parameters, transformation values, and goodness-of-fit comparisons between models (e.g., Euclidean and affine).

Which Model?

In general, the Euclidean model, with the fewest parameters (four) and simplest assumptions, should be tested first, followed by the affine (six), and the projective (eight). Because shear in the affine alters the angles of intersections of lines relative to the original image, this model is particularly useful for testing hypotheses about cognitive distortions arising from "alignment and rotation heuristics" (Glicksohn, 1994; Nakaya, 1997; Tversky, 1981; Wakabayashi, 1994). In addition, because the projective model takes account of viewpoint, it should be useful for testing hypotheses in which the subjective distance between locations changes with viewpoint (Holyoak & Mah, 1982).

Which Implementation?

The most frequent use of bidimensional regression in cognitive mapping has been to compare cognitive maps with actual maps for accuracy. In this context, when participants' judgments are the dependent variable, the bidimensional parameters and the transforms derived from them presumably reflect the psychological processes (perceptual and memorial) that were applied to a set of inputs (e.g., actual maps) to produce a given individual's or group's cognitive map(s). That is, the parameters reflect how the cognitive map was derived from the actual map. Conversely, when the actual map's coordinates form the dependent variable, the parameters and transforms reflect how an indi-

vidual or group's cognitive map has to be transformed to "fit" back into the actual map. The parameters thus reflect how the actual map can be derived from transforms to an individual or group's cognitive map.

Using the actual map as the independent variable and participants' judgments as the dependent variable is conventional and generally preferable. First, actual locations are typically sampled a priori, not randomly (e.g., Golledge, Rivizzigno, & Spector, 1976; Glicksohn, 1994). Second, as shown in the previous section, it is only when a single set of coordinates serves as the independent variable that the parameter estimates for participants and items are identical (see Table 5). Third, *DI* is the same regardless of which coordinates are used as the independent variable, so if one is interested in comparing indices of distortion across individuals it is not necessary to "force" individual participant's maps into the actual map's space to do so.

Nevertheless, there are certain circumstances for which it makes sense to select a particular variant to play the role of the independent variable, rather than using accurate real-world coordinates. For example, if one were comparing sketch maps of an environment across age groups, the adults' mean estimates might be used as the referent coordinates, irrespective of their accuracy with respect to the real world. Note that in this case there is still a single set of referent coordinates to which all the other variants are compared. Thus, this case is completely analogous to the case in which the actual maps' coordinates are the independent variable.

There are also some circumstances in which using the objective coordinates as the dependent variable is warranted. For example, suppose an archaeologist wanted to determine the actual locations of sites on an ancient map that were not identifiable with respect to present day knowledge. In this case one would use the ancient map as the independent variable, to predict the modern coordinates of the unknown sites.[6] In a psychological version of this example, adults might require larger transformations than children require to fit their cognitive maps to an actual map because adults' maps are more distorted (Kerkman, Friedman, Brown, Stea, & Carmichael, 2003). Thus, the kind of question being addressed should dictate which set of coordinates plays the role of independent and dependent variable. However, the caveats with respect to *DI* still hold: *D* and *Dmax* should be computed with respect to whichever coordinates are used as the dependent variable.

Other Issues

Strictly speaking, the interpretation of the bidimensional correlation coefficient is analogous to the interpretation of a unidimensional correlation: When the true map provides coordinates for the independent variable, the bidimensional correlation reflects how much of the variability in the estimated locations can be accounted for by the true map's configuration. And, like the unidimensional case, there are many reasons why a correlation coefficient might be low, including the possibility that the method has been used with inappropriate data. Thus, just as one should not use Pearson *r* if the relation between independent and dependent unidimensional variables is not linear, one should not make inferences based on the Euclidean implementation of bidimensional regression if the data are not well described by a Euclidean metric. Conversely, computing the bidimensional correlation using the Euclidean model can certainly indicate how well a Euclidean metric fits a set of data if that were unknown a priori. For instance, Wakabayashi (1994) had participants draw sketch maps of the locations of 19 places in Kanazawa City, Japan. This city is characterized by two major channels of the Sai and Asano rivers running from the southeast to the northwest. Importantly, all the streets deviate from the cardinal directions in a manner that parallels these channels (i.e., by 10° and 30°), which makes Kanazawa City quite different from Edmonton in this respect. Nakaya (1997) reanalyzed Wakabayashi's data using bidimensional regression. Because the cognitive maps were globally skewed relative to the actual maps, the affine model accounted for significantly more of the variance in the location estimates than did the Euclidean model. Nevertheless, the Euclidean model accounted for a significant portion of the variance in the estimated locations. Note that here, the bidimensional regression methodology was still appropriate for analyzing the data but only when implemented by testing successive models.

In general, though 2-D data are in principle appropriate for examining with bidimensional regression, if the data exhibit significant local nonlinearities (e.g., Huttenlocher, Hedges, & Duncan, 1991), none of the first three bidimensional models (Euclidean, affine, and projective) would be warranted. In this case, the

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[6] We are grateful to an anonymous reviewer for suggesting this example.

curvilinear model might be appropriate, but it would then be important to have an a priori rationale for selecting the order of the exponent.

Finally, because bidimensional regression is completely analogous to unidimensional regression, researchers must be vigilant about the same sorts of issues. For instance, missing data is problematic in a repeated measures design (e.g., when participants make a number of location estimates, which is most often the case). One may decide to delete participants who have missing data from consideration, which lowers statistical power and restricts the population to which the results may be generalized. Alternatively, one may decide to use modern missing data procedures, such as multiple imputation, which make maximal use of the available data and produce unbiased estimates and standard errors when data are missing at random (Schafer & Graham, 2002). Additionally, issues arise in combining separate parameter estimates that are computed individually for each participant. Some participants might have relatively little variability in their judgments and others might have a lot; the groups might differ in either their knowledge or in the care with which they made their judgments. In either case one might consider some sort of weighting scheme for different participants (e.g., empirical Bayes methods; Kreft & De Leeuw, 1998, p. 132).

Psychological Issues Amenable to Bidimensional Regression

In this final section we provide examples from extant psychological literature that are appropriate for and would benefit from bidimensional regression analysis. In some instances, bidimensional regression would provide completely new information than what is currently reported or available and in other instances, the bidimensional correlation coefficient and transformations would augment extant analyses.

Animal Cognition and Behavioral Ecology

For animals, survival in an environment often depends on processing spatial information; spatial ability, and specifically that required for localization, is required for activities such as establishing shelter, attaining food, avoiding or engaging in predation, and migratory behavior. As long as these behaviors can be observed (either in the natural environment or in the laboratory), quite a few issues in animal spatial cognition are amenable to examination using bidimensional regression techniques. For example, what is the

accuracy of location memory in species that store food (e.g., Shettleworth, 2002)? What is the accuracy of a predator's striking distance from its prey (Snyder, 2001)? For species that are mobile, are there correlations between predator and prey locations or between nesting locations from year to year (Bélisle & St. Clair, 2001; St. Clair, McLean, Murie, Phillipson, & Studholme, 1999)? For animals that hunt or forage, is there evidence that they use landmark-based information for food or prey localization (Cheng, 1999; Snyder, 2001)?; related to this, do bees and other foragers visit the same areas across successive forays (Cheng, 2000)? Is there regularity in the patrolled sites within an animal's home territory? For several of these issues (e.g., the last two questions), one would use time of day as a temporal marker for setting the spatial coordinates/locations to be compared. Individuals could be observed (or tracked with global positioning system collars) at specific locations at particular times either within or across days.

Developmental Spatial Cognition

Spatial abilities comprise a core set of intellectual abilities; a large portion of the literature on the development of spatial cognition (see Newcombe & Huttenlocher, 2000, for review) addresses issues that are amenable to analysis with bidimensional regression. For example, how do young children's sketch maps of a classroom (neighborhood; city) compare with those of older children and adults? How accurate are each of these groups' maps with respect to an actual map? How is location memory affected by barriers (Newcombe & Libben, 1982)? How do children develop location coding? This type of study examines the accuracy of children's placement of, or memory for, targets relative to landmarks or in the absence of landmarks (e.g., Herman & Siegal, 1978; Huttenlocher & Newcombe, 1984; Huttenlocher, Newcombe, & Sandberg 1994; Newcombe, Huttenlocher, Drummey, & Wiley, 1998). The r^2 between the children's placements and the absolute locations of objects on a map or grid could be compared to the r^2 between the children's placements and environmental landmarks. Finally, which method of eliciting location estimates from young children provides the most accurate data with respect to the objective coordinates?

Adult Spatial Cognition

As with children, there is quite a lot of research on adult spatial cognition, and many of the questions

addressed could be examined with bidimensional regression. For example,

1. Do adults' estimates of locations on a city and/or global scale provide evidence that they have "normalized" the representation? Here, besides comparing digitized location estimates to an actual map for accuracy (r^2), one would examine the nature of the transformations required to change the actual map into the cognitive map (or vice versa). Cognitive maps of the same locations obtained from people who lived in different places could be compared for differences in their accuracy and patterns of distortion (see Friedman, Kerkman, & Brown, 2002, for a global-level example and Lloyd, 1989, for an example at the urban level). In addition, one could compare the r^2 computed from the Euclidean model with that computed from the affine model to determine which of these better fit the data (see Nakaya, 1997, for an urban example). If the affine model fit better, one would then interpret how the scale, angle, and shear characteristics differed among the groups (see Appendix).

2. Is learning a layout via virtual reality as accurate as learning it from a map or from the actual environment? Are their similarities in the distortions displayed after each kind of learning? How does visual fidelity affect accuracy of location memory (Waller, Knapp, & Hunt, 2001)?

3. Providing participants with a few well chosen "seed facts" (i.e., actual latitudes and longitudes of a few cities; Friedman & Brown, 2000a, 2000b) dramatically alters their location estimates. Bidimensional correlations could be used to compare pre- and postseeding cognitive maps to actual maps, as well as the same participants' pre- and postseeding cognitive maps to each other. The latter would be a measure of the consistency of the configural structure of item knowledge within a region, which we have been previously unable to quantify.

Neuropsychology

There is a large literature that examines the effects of brain damage on spatial learning and memory for both animals and humans. For example, what is the effect of hippocampal lesions on memory for location (Hampton & Shettleworth, 1996a, 1996b)? Bidimensional regression could be used to evaluate differences in the effects of lesions among individuals or within the same individuals over time.

Other Possibilities

Bidimensional regression should be useful in any domain in which the geometry of the configuration is of interest in its own right. For example, it provides a means of comparing patterns of individuals' or groups' eye movements as they examined scenes under different instructions; these patterns have been previously described only qualitatively (e.g., Buswell, 1935). In the semantic domain, one could compare individuals or groups on the similarity of their MDS solutions for particular semantic spaces (e.g., Rips, Shoben, & Smith, 1973).

Bidimensional regression should also prove useful for examining direct similarity scaling of multidimensional stimuli. For example, in Garner's (1974) analysis, a Euclidean metric is appropriate for describing integral dimensions (e.g., value and chroma; Handel and Imai, 1972) whereas a city-block metric is more appropriate for separable dimensions (e.g., lightness and size; Handel & Imai, 1972; see also Hyman & Well, 1967, 1968).

There are likely to be many potential uses for bidimensional regression besides the ones we have described, including those that are not limited to a Euclidean metric. Thus, the analysis should be useful for and adaptable to many psychological issues.

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Appendix

Computations for the Affine Transformation (*XY* Independent)

The affine transformation consists of a series of separate operations that translate, rotate, scale, and shear points from the referent's space into the variant's space (or vice versa). Like the Euclidean case, scale values greater or less than one represent an expansion or contraction, respectively. Unlike the Euclidean case, in the affine transformation the scaling is performed independently for each dimension, and scale values can be negative. Negative scale values produce configurations that are mirror reflections of those produced by positive scale values. The angle transform functions identically to the Euclidean case and reflects how much the variant's coordinates are rotated with respect to the referent's plane. The effect of a shear transformation (γ) is to "push" a 2-D configuration in a direction parallel to one of the coordinate axes. For example, one can "push" in the *X* direction, positive or negative, and keep the *Y* coordinates unchanged, or vice versa. However, even though shear can occur along the *X*- or *Y*-axis, when rotation is taken into account, one of the axes becomes redundant. Consequently, we have elected to illustrate shear along the *X*-axis.

We illustrate the affine transformation assuming that *XY* is the independent variable. Like all the models, the parameter estimates change when *AB* is independent.

The affine model has six parameters: Two are for trans-lation, and the remaining four are combined to form the scale, angle, and shear transformations. The matrix version of the model is

$$\begin{bmatrix} A' \\ B' \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} + \begin{bmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \end{bmatrix}.$$

Eliminating α allows the system of equations to be rewritten as

$$\begin{bmatrix} A' \\ B' \end{bmatrix} - \begin{bmatrix} M_A \\ M_B \end{bmatrix} = \begin{bmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{bmatrix} \cdot \left[\begin{pmatrix} X \\ Y \end{pmatrix} - \begin{pmatrix} M_X \\ M_Y \end{pmatrix} \right],$$

which can be solved as follows:

$$\begin{bmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{bmatrix} = AB^T \cdot XY \cdot (XY^T \cdot XY)^{-1},$$

where

$$AB = \begin{bmatrix} A' \\ B' \end{bmatrix} - \begin{bmatrix} M_A \\ M_B \end{bmatrix}$$

and

$$XY = \begin{bmatrix} X \\ Y \end{bmatrix} - \begin{bmatrix} M_X \\ M_Y \end{bmatrix},$$

organized as one coordinate pair per row (i.e., an $N \times 2$ matrix). Solving this system of equations for the four β

(Appendix continues)

values yields the following relations, which can be used to compute the parameters directly:

$$\beta_1 = \frac{\text{cov } AX \cdot \text{var } Y - \text{cov } AY \cdot \text{cov } XY}{\text{var } X \cdot \text{var } Y - (\text{cov } XY)^2}, \qquad (13)$$

$$\beta_2 = \frac{\text{cov } AY \cdot \text{var } X - \text{cov } AX \cdot \text{cov } XY}{\text{var } X \cdot \text{var } Y - (\text{cov } XY)^2}, \qquad (14)$$

$$\beta_3 = \frac{\text{cov } BX \cdot \text{var } Y - \text{cov } BY \cdot \text{cov } XY}{\text{var } X \cdot \text{var } Y - (\text{cov } XY)^2}, \qquad (15)$$

$$\beta_4 = \frac{\text{cov } BY \cdot \text{var } X - \text{cov } BX \cdot \text{cov } XY}{\text{var } X \cdot \text{var } Y - (\text{cov } XY)^2}. \qquad (16)$$

The translation parameters are found by

$$\alpha_1 = M_A - \beta_1 \cdot M_X - \beta_2 \cdot M_Y$$

and

$$\alpha_2 = M_B - \beta_3 \cdot M_X - \beta_4 \cdot M_Y,$$

and each predicted pair of coordinates can then be computed as follows:

$$A' = \alpha_1 + \beta_1 \cdot X + \beta_2 \cdot Y$$

and

$$B' = \alpha_2 + \beta_3 \cdot X + \beta_4 \cdot Y.$$

Once the predicted values have been obtained, they can be used in Equation 2.5 to find the bidimensional correlation coefficient. For the toy world values in Table 3, Case 1, the values of α_1 and α_2 are 11.846 and 15.673, respectively; the values of β_1 to β_4 are .7308, .5077, .3654, and −.4462; and $r^2 = .885$. Note that this is much higher than r^2 for the Euclidean case (.309), but of course, involves two additional parameters.

Obtaining Values for the Transformations

Each of the transformations except translation can be expressed by its own matrix:

$$\text{Rotation} \qquad \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

$$\text{Shear} \qquad \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix},$$

and

$$\text{Scale} \qquad \begin{bmatrix} \phi_X & 0 \\ 0 & \phi_Y \end{bmatrix}.$$

The matrices are multiplied together in the order in which the operations are applied to the referent's coordinates. For example, for the order rotation, shear, and scale, the β matrix becomes

$$\begin{bmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \phi_X & 0 \\ 0 & \phi_Y \end{bmatrix}.$$

However, given a particular β matrix computed from Equations 13–16, the values of θ, γ, ϕ_X, and ϕ_Y and the equations to compute them depend on the order in which the transformations are to be applied. Conversely, if the values of θ, γ, ϕ_X, and ϕ_Y, are known, then the values of β_1–β_4 will depend on whether, for example, the coordinates are first rotated, sheared, and then scaled, or whether they are first sheared, scaled, and then rotated. Thus, whether one is theorizing about the values of the βs or of the transforms, it is important to have a rationale for why the transforms are applied in a particular order and to take care in their interpretation. However, regardless of the order of transforms and values of θ, γ, ϕ_X, and ϕ_Y, the same values of $A'B'$ points are produced for the same set of XY points (as well as the same r^2 and DI).

Excluding translation, there are six possible orders of the θ, γ, and ϕ transforms; we illustrate the system of equations for two of these. In these examples we have always applied the translation operation first, to ensure that the origin $(0, 0)$ in the referent space maps simply to the point (α_1, α_2) in the variant's space. Otherwise, the order in which the translation is applied with the other transformations also makes a difference. Furthermore, even when the same order of operations is used, there are several ways to describe the same transformed coordinates. For example, mirror reflections about the x-axis are identical to the combination of a 180° rotation followed by a mirror reflection about the y-axis. Thus, it is necessary to restrict the solutions to some of the equations below.

Given the four β values, if the XY coordinates are to be rotated, sheared, and then scaled, the values for θ, γ, ϕ_X, and ϕ_Y are computed as follows:

$$\text{Angle } \theta = \tan^{-1}\left(\frac{\beta_3}{\beta_1}\right) \text{ (add 180° if } \beta_1 < 0\text{)},$$

$$\text{Shear } \gamma = \frac{\dfrac{\beta_4}{\beta_2} \cdot \sin\theta + \cos\theta}{\dfrac{\beta_4}{\beta_2} \cdot \cos\theta - \sin\theta},$$

$$\text{Scale } \phi_X = \sqrt{\beta_1^2 + \beta_3^2} \text{ (the negative solution is redundant)},$$

$$\text{Scale } \phi_Y = \pm\sqrt{\frac{\beta_2^2 + \beta_4^2}{\gamma^2 + 1}} \text{ or}$$

$$\frac{\beta_2}{\gamma \cdot \cos\theta - \sin\theta} \text{ or } \frac{\beta_4}{\gamma \cdot \sin\theta + \cos\theta}.$$

The values of θ, γ, ϕ_X, and ϕ_Y for the toy world in Table 3, Case 1, are 26.565°, −.4066, .8170, and −.6261, respectively. In contrast, by applying shear, scale, and then rotation, the equations become

$$\text{Angle } \theta = \tan^{-1}\frac{\beta_3}{\beta_4} \text{ (add } 180° \text{ if } \beta_4 < 0),$$

$$\text{Shear } \gamma = \frac{\beta_1 \cdot \sin\theta + \beta_2 \cdot \cos\theta}{\beta_3 \cdot \sin\theta + \beta_4 \cdot \cos\theta},$$

$$\text{Scale } \phi_X = \pm\sqrt{\beta_1^2 + \beta_2^2 - \gamma^2 \cdot (\beta_3^2 + \beta_4^2)} \text{ or } \frac{\beta_1 - \gamma \cdot \beta_3}{\cos\theta} \text{ or}$$

$$-\frac{\beta_2 - \gamma \cdot \beta_4}{\sin\theta},$$

and

$$\text{Scale } \phi_Y = \sqrt{\beta_3^2 + \beta_4^2} \text{ (the negative solution is redundant)}.$$

The values of θ, γ, ϕ_X, and ϕ_Y in this case are 140.684°, .1218, −.8870, and .5767.

Note that in the first case, the first equation for ϕ_Y yields its magnitude without concern about division by zero, and the remaining equations indicate whether ϕ_Y is positive or negative. We chose to restrict the solutions for ϕ_X to positive values and to allow both positive and negative solutions for ϕ_Y. In the second case, the first equation for ϕ_X provides the value without concern about division by zero, and the remaining equations provide its sign. Here, we restricted the solutions for ϕ_Y to positive values. Alternately, it is possible to restrict the range of θ and allow both positive and negative values for ϕ_X and ϕ_Y.