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A MULTIPLICATION OF E-VARIETIES OF REGULAR E-SOLID SEMIGROUPS BY INVERSE SEMIGROUP VARIETIES

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Abstract. A multiplication of e-varieties of regular E-solid semigroups by inverse semigroup varieties is described both semantically and syntactically. The associativity of the multiplication is also proved.

1. Introduction

We investigate here an operator on the lattice of all e-varieties of regular semigroups. In [7] we defined semantically a partial multiplication on this lattice: $\mathcal{U} \square \mathcal{V}$ is defined if $\mathcal{U}$ is an e-variety of regular semigroups and $\mathcal{V}$ is an e-variety of inverse semigroups. The definition is based on a certain semidirect product of regular semigroups by inverse semigroups. In the case that $\mathcal{U}$ is an e-variety of orthodox semigroups we also described our multiplication syntactically in terms of biinvariant congruences for orthodox semigroups introduced in [5] by Kaďourek and Szendrei.

In this paper we present a syntactical description of our multiplication in the case that the first factor is an e-variety of regular E-solid semigroups. The description is essentially based on the notion of biinvariant congruences for regular E-solid semigroups given in [6] by Kaďourek and Szendrei. Moreover, we prove the associativity: $\mathcal{U} \square (\mathcal{V} \square \mathcal{W}) = (\mathcal{U} \square \mathcal{V}) \square \mathcal{W}$ for any e-variety $\mathcal{U}$ of regular E-solid semigroups and any inverse semigroup varieties $\mathcal{V}, \mathcal{W}$.

For basic notions in the theory of semigroups the reader is referred to [4].

2. Semantics

Let $S = (S, \cdot)$ be a semigroup. The set of all endomorphisms of S is denoted by $\mathrm{End}(S)$. Let $E(S)$ stand for the set of all idempotents of S. Denote by $C(S)$ the subsemigroup of S generated by $E(S)$ provided that $E(S) \neq \emptyset$. Clearly, for

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any element $a \in S$, there is at most one element $b \in S$ satisfying $aba = a, bab = b, ab = ba$. If such an element b really exists then we denote it by a^{-1}.

In [7] we used the following non-standard semidirect product of semigroups:

Let $T = (T, \cdot)$ be an inverse semigroup. For $a \in T$, the unique inverse of a is denoted by a'. Let $\varphi : (T, \cdot) \to (\mathrm{End}(S), \circ)$, where $\circ$ is the composition $(\alpha \circ \beta)(s) = \alpha(\beta(s))$ $(\alpha, \beta \in \mathrm{End}(S), s \in S)$, be a homomorphism.

Put $S \times_\varphi T = \{(s,t) \in S \times T |\, \varphi(tt')(s) = s\}$ and define

$$(s,t) \cdot (u,v) = (\varphi(tvv't')(s) \cdot \varphi(t)(u), t \cdot v)$$

for $(s,t), (u,v) \in S \times_\varphi T$.

2.1 Result. ([7], 2.1 Lemma, 2.2 Lemma)

(i) *$(S \times_\varphi T, \cdot)$ is a semigroup*
(ii) *If S is regular, then $S \times_\varphi T$ is also regular.*

Notice that this non-standard semidirect product of semigroups is in essence the so called λ -semidirect product of inverse semigroups introduced by Billhardt in [1].

2.2 Result. ([7], 2.3 Lemma) *Let $(s,t) \in S \times_\varphi T$. Then (s,t) is an idempotent in $S \times_\varphi T$ if and only if $s \in E(S)$ and $t \in E(T)$.*

2.3 Lemma. *Let $(s,t) \in S \times_\varphi T$. Then $(s,t) \in C(S \times_\varphi T)$ if and only if $s \in C(S)$ and $t \in E(T)$.*

Proof.

1. Let $(s,t) \in C(S \times_\varphi T)$. Then $(s,t) = (e_1, f_1) \dots (e_k, f_k)$ for some $(e_1, f_1), \dots, (e_k, f_k) \in E(S \times_\varphi T)$. We know that $e_1, \dots, e_k \in E(S)$ and $f_1, \dots, f_k \in E(T)$ (see 2.2). Put $(u_i, v_i) = (e_1, f_1) \dots (e_i, f_i)$ $(i = 1, \dots, k)$. We will show that $u_i \in C(S)$ and $v_i \in E(T)$ $(i = 1, \dots, k)$. Clearly, $e_1 \in C(S), f_1 \in E(T)$. Let $1 < i \leq k$ and $u_{i-1} \in C(S), v_{i-1} \in E(T)$. We have $\varphi(v_{i-1} f_i f_i' v_{i-1}')(u_{i-1}) \in C(S)$, since $\varphi(v_{i-1} f_i f_i' v_{i-1}') \in \mathrm{End}(S)$. Further, $\varphi(v_{i-1})(e_i) \in E(S)$. We see that $u_i = \varphi(v_{i-1} f_i f_i' v_{i-1}')(u_{i-1}) \cdot \varphi(v_{i-1})(e_i) \in C(S)$. Finally, $v_i = v_{i-1} \cdot f_i \in E(T)$.
2. Let $s \in C(S)$ and $t \in E(T)$. Then $s = e_1 \dots e_k$ for some $e_1, \dots, e_k \in E(S)$. Put $f_i = \varphi(tt')(e_i)$ $(i = 1, \dots, k)$. Clearly, $f_i \in E(S)$ $(i = 1, \dots, k)$ and $s = f_1 \dots f_k$. Further, $(f_i, t) \in S \times_\varphi T$, since $\varphi(tt')(f_i) = f_i$ $(i = 1, \dots, k)$. Using 2.2 we obtain $(f_i, t) \in E(S \times_\varphi T)$ $(i = 1, \dots, k)$. We will prove that $(f_1, t) \dots (f_i, t) = (f_1 \dots f_i, t)$ $(i = 1, \dots, k)$. Let $1 < i \leq k$. Then

$$\begin{aligned}(f_1,t)\dots(f_{i-1},t)(f_i,t) &= (f_1 \dots f_{i-1}, t)(f_i, t) \\ &= (\varphi(ttt't')(f_1 \dots f_{i-1}) \cdot \varphi(t)(f_i), t^2) \\ &= (\varphi(tt')(f_1 \dots f_{i-1}) \cdot \varphi(tt')(f_i), t) \\ &= (f_1 \dots f_i, t).\end{aligned}$$

For $i = k$ we get $(s,t) = (f_1 \dots f_k, t) = (f_1, t) \dots (f_k, t) \in C(S \times_\varphi T)$. □

A semigroup S is called regular E-solid if it is regular and $C(S)$ is completely regular.

2.4 Lemma. *If S is regular E-solid, then $S \times_\varphi T$ is also regular E-solid.*

Proof. We know that $S \times_\varphi T$ is regular (see 2.1(ii)). We have to show that $C(S \times_\varphi T)$ is completely regular. Let $(s,t) \in C(S \times_\varphi T)$. Then $s \in C(S), t \in E(T)$, by 2.3. Since S is regular E-solid, there exists $a \in C(S)$ such that $sas = s, asa = a, sa = as$. Put $b = \varphi(tt')(a)$. Then $b \in C(S)$ and $sbs = s, bsb = b, sb = bs$. Clearly, $(b,t) \in S \times_\varphi T$. Using 2.3 we obtain $(b,t) \in C(S \times_\varphi T)$. Further,

$$\begin{aligned}(s,t)(b,t)(s,t) &= (\varphi(ttt't')(s) \cdot \varphi(t)(b), t^2)(s,t) = (sb,t)(s,t) \\ &= (\varphi(ttt't')(sb) \cdot \varphi(t)(s), t^2) = (sbs,t) = (s,t).\end{aligned}$$

Similarly, $(b,t)(s,t)(b,t) = (b,t)$ and $(s,t)(b,t) = (b,t)(s,t)$. □

For any class $\mathcal{V}$ of regular semigroups, we will denote by $H(\mathcal{V}), S_r(\mathcal{V})$ and $P(\mathcal{V})$, respectively, the classes of all homomorphic images, regular subsemigroups and direct products of semigroups in $\mathcal{V}$.

We adopt the following notations for classes of regular semigroups:
R — the class of all regular semigroups;
ES — the class of all regular E-solid semigroups;
I — the class of all inverse semigroups.

A class $\mathcal{V} \subseteq \mathbf{R}$ satisfying $H(\mathcal{V}) \subseteq \mathcal{V}, S_r(\mathcal{V}) \subseteq \mathcal{V}$ and $P(\mathcal{V}) \subseteq \mathcal{V}$ is called an e-variety. The classes **R, ES, I** are examples of e-varieties. The concept of e-variety was introduced by Hall in [3]. Simultaneously and independently Kaďourek and Szendrei in [5] have considered e-varieties of orthodox semigroups, which they called bivarieties of orthodox semigroups.

Denote by $\langle \mathcal{V} \rangle$ the least e-variety of regular semigroups containing the class $\mathcal{V} \subseteq \mathbf{R}$.

Let $\mathcal{U} \subseteq \mathbf{R}$ and $\mathcal{V} \subseteq \mathbf{I}$ be e-varieties. In [7] we defined a multiplication $\Box$ in the following way:

$$\mathcal{U} \Box \mathcal{V} = \langle \{S \times_\varphi T \mid S \in \mathcal{U}, T \in \mathcal{V}, \varphi : (T, \cdot) \to (\mathrm{End}(S), \circ) \text{ is a homomorphism}\} \rangle .$$

2.5 Result. ([7], 2.5 Lemma) *Let $I \neq \emptyset$. Let S_i be a semigroup for $i \in I$. Let T_i be an inverse semigroup for $i \in I$. Finally, let $\varphi_i : (T_i, \cdot) \to (\mathrm{End}(S_i), \circ)$ be a homomorphism for $i \in I$. Then*

$$\prod_{i\in I}(S_i \times_{\varphi_i} T_i) \cong \prod_{i\in I} S_i \times_\varphi \prod_{i \in I} T_i,$$

where the homomorphism

$$\varphi : (\prod_{i\in I} T_i, \cdot) \to (\mathrm{End}(\prod_{i\in I} S_i), \circ)$$

is given by

$$\varphi((t_i)_{i\in I})((s_i)_{i\in I}) = (\varphi_i(t_i)(s_i))_{i\in I} .$$

The isomorphism is given by

$$((s_i, t_i))_{i\in I} \mapsto ((s_i)_{i\in I}, (t_i)_{i\in I}) .$$

2.6 Result. ([6], Proposition 2.3) *Let* $\mathcal{V} \subseteq \mathbf{ES}$. *Then*

(i) $S_r H(\mathcal{V}) \subseteq HS_r(\mathcal{V})$
(ii) $\langle \mathcal{V} \rangle = HS_r P(\mathcal{V})$.

2.7 Lemma. *Let* $\mathcal{U} \subseteq \mathbf{ES}$ *and* $\mathcal{V} \subseteq \mathbf{I}$ *be e-varieties. Then*

$$\mathcal{U} \,\Box\, \mathcal{V} = HS_r(\{S \times_\varphi T \mid S \in \mathcal{U}, T \in \mathcal{V},\ \varphi : (T, \cdot) \to (\mathrm{End}\ (S), \circ) \text{ is a homomorphism}\}).$$

Proof. Put $\mathcal{W} = \{S \times_\varphi T \mid S \in \mathcal{U}, T \in \mathcal{V}, \varphi : (T, \cdot) \to (\mathrm{End}(S), \circ)$ is a homomorphism$\}$. It follows from 2.4 that $\mathcal{W} \subseteq \mathbf{ES}$. Then $\mathcal{U} \,\Box\, \mathcal{V} = HS_r P(\mathcal{W})$ by 2.6(ii). It is clear that $HS_r(\mathcal{W}) \subseteq HS_r P(\mathcal{W})$. Further, $P(\mathcal{W}) \subseteq H(\mathcal{W})$, by 2.5. Then $HS_r P(\mathcal{W}) \subseteq HS_r H(\mathcal{W})$. This together with 2.6(i) gives $HS_r P(\mathcal{W}) \subseteq HS_r(\mathcal{W})$. □

3. Syntax

Recall the notions of biidentities and biinvariant congruences in the class of regular E-solid semigroups introduced by Kaďourek and Szendrei in [6].

A unary semigroup is an algebra $S = (S, \cdot, ')$ with an associative multiplication and with a unary operation $'$.

Let Y be a non–empty set. We add new symbols (and $)'$ to the set Y and obtain a set $Y_0 = Y \cup \{(,)'\}$. Let us denote the free semigroup on the alphabet A by A^+. Let $U(Y)$ be the smallest one among the subsets T in Y_0^+ which satisfy

(i) $Y \subseteq T$
(ii) $u, v \in T$ implies $uv \in T$
(iii) $u \in T$ implies $(u)' \in T$.

The set $U(Y)$ will be often considered as a unary semigroup with a binary operation given by the concatenation of words and with a unary operation $' : U(Y) \to U(Y)$ given by $u \mapsto (u)'$. The unary semigroup $U(Y)$ is the free unary semigroup on the set Y.

In order to reduce the number of brackets in formulas, we will omit them if it causes no confusion. For example, we will often write u' instead of $(u)'$.

Consider a set Y' disjoint from Y and a bijection $' : Y \to Y', y \mapsto y'$. The union $Y \cup Y'$ will be denoted by $\overline{Y}$. For any $y \in Y$, we will identify $(y)'$ with y', and so $\overline{Y}$ becomes a subset in $U(Y)$.

If S is an inverse semigroup and $a \in S$ then the unique inverse of a is denoted by a'. In this way a unary operation $'$ on S is given and the inverse semigroup $S = (S, \cdot)$ can be considered as a unary semigroup $S = (S, \cdot, ')$. Moreover, this unary semigroup satisfies the identities

(id 1) $(x')' = x$
(id 2) $(xy)' = y'x'$
(id 3) $xx'x = x$
(id 4) $xx'yy' = yy'xx'$.

Conversely, if $S = (S, \cdot, ')$ is a unary semigroup satisfying the identities (id 1) — (id 4) then $S = (S, \cdot)$ is an inverse semigroup and the unique inverse of $a \in S$ is the element a'. Speaking about varieties of inverse semigroups we have in mind varieties of unary semigroups satisfying the identities (id 1) — (id 4).

In fact, the e-varieties contained in **I** are precisely the varieties of inverse semigroups. We can use the terms 'variety' and 'e-variety' interchangeably in this context.

Given an infinite set Y, we will denote by $\gamma(Y)$ and $\iota(Y)$, respectively, the fully invariant congruences on $U(Y)$ corresponding to the varieties of all groups and all inverse semigroups. $1(Y)$ stands for the identity element of the group $U(Y)/\gamma(Y)$.

3.1 Lemma. *Let $u \in U(Y)$. Then $u\gamma(Y) = 1(Y)$ if and only if $u^2 \,\iota(Y)\, u$.*

Proof.

1. Let $u\gamma(Y) = 1(Y)$. In view of the well–known solution of the word problem for free groups it suffices to show the following facts:
   (a) $(vw)^2 \,\iota(Y)\, vw \Rightarrow (vzz'w)^2 \,\iota(Y)\, vzz'w\,, \qquad (v, w \in U(Y), z \in \overline{Y})$;
   (b) $v^2 \,\iota(Y)\, v \Rightarrow (vzz')^2 \,\iota(Y)\, vzz'\,, \qquad (v \in U(Y), z \in \overline{Y})$;
   (c) $w^2 \,\iota(Y)\, w \Rightarrow (zz'w)^2 \,\iota(Y)\, zz'w\,, \qquad (w \in U(Y), z \in \overline{Y})$;
   Now the proofs follow:
   (a)
   $$\begin{aligned} \underline{v}zz'wvzz'\underline{w} \;&\iota(Y)\; v\underline{v'vzz'}wv\underline{zz'ww'}w \\ &\iota(Y)\; vzz'v'\underline{vwvw}w'zz'w \\ &\iota(Y)\; v\underline{zz'v'v}\,\underline{ww'zz'}w \\ &\iota(Y)\; \underline{vv'v}\,\underline{zz'zz'}\,\underline{ww'w} \\ &\iota(Y)\; vzz'w; \end{aligned}$$
   (b)
   $$\begin{aligned} \underline{v}zz'vzz' \;&\iota(Y)\; v\underline{v'vzz'}vzz' \\ &\iota(Y)\; vzz'v'\underline{vv}zz' \\ &\iota(Y)\; v\underline{zz'v'v}zz' \\ &\iota(Y)\; \underline{vv'v}\,\underline{zz'zz'} \\ &\iota(Y)\; vzz'\,. \end{aligned}$$
   (c) It is similar to the case (b).
2. Let $u^2 \,\iota(Y)\, u$. It is clear that $u\gamma(Y) = 1(Y)$. □

Let $U_r(Y)$ be the smallest one among the subsets T in $U(Y)$ which satisfy

(i) $\overline{Y} \subseteq T$

(ii) $u, v \in T$ implies $uv \in T$

(iii) $u \in T$ and $u\gamma(Y) = 1(Y)$ implies $u' \in T$.

The set $U_r(Y)$ will be often considered as a semigroup with an operation given by the concatenation of words. In fact, the semigroup $U_r(Y)$ agrees with the semigroup $F'^{\infty}(Y)$ from [6]. There is only an unessential technical difference between $U_r(Y)$ and $F'^{\infty}(Y)$. In [6], $U'(Y)$ stands for the free unary semigroup on the set $\overline{Y}$. The unary operation is denoted by $^{-1}$ in $U'(Y)$ and $F'^{\infty}(Y)$ is the smallest subsemigroup in $U'(Y)$ containing the set $\overline{Y}$ and closed under the partial operation assigning the word $(u)^{-1}$ to any word u with $r(u) = 1$ (see [6], Section 2, for the definition of $r(u)$). If we consider the unary homomorphism $\eta : U'(Y) \to U(Y)$ extending the mapping $y \mapsto y, y' \mapsto y' = (y)'$ $(y \in Y)$ then, for any $u \in U'(Y)$, the condition $r(u) = 1$ is equivalent to $\eta(u)\gamma(Y) = 1(Y)$ and the restriction of η to $F'^{\infty}(Y)$ is an isomorphism between $F'^{\infty}(Y)$ and $U_r(Y)$.

If $(S, \cdot)$ is a regular semigroup, then a mapping $\vartheta : \overline{Y} \to S$ is called matched if $\vartheta(y) \cdot \vartheta(y') \cdot \vartheta(y) = \vartheta(y)$ and $\vartheta(y') \cdot \vartheta(y) \cdot \vartheta(y') = \vartheta(y')$ for all $y \in Y$.

To any matched mapping $\vartheta : \overline{Y} \to S$, where S is a regular E-solid semigroup, we now define a homomorphism $\theta : U_r(Y) \to S$ as follows. We proceed by induction with respect to the complexity of words from $U_r(Y)$, and we put

(i) $\theta(z) = \vartheta(z)$ $(z \in \overline{Y})$

(ii) $\theta(uv) = \theta(u)\theta(v)$ $(u, v \in U_r(Y))$

(iii) $\theta(u') = (\theta(u))^{-1}$ $(u \in U_r(Y), u\gamma(Y) = 1(Y))$,

where $(\theta(u))^{-1}$ denotes the group inverse of $\theta(u)$ in the maximal subgroup of S containing $\theta(u)$. Of course, we must show that this $\theta(u)$ really lies in a subgroup of S. This will be the content of the next result. We will then see that θ is well defined and we will call the homomorphism θ the extension of the matched mapping $\vartheta : \overline{Y} \to S$ to $U_r(Y)$.

3.2 Result. ([6], Lemma 2.1) *The above definition is correct, that is, for any $u \in U_r(Y)$ with $u\gamma(Y) = 1(Y)$, the element $\theta(u)$ lies in a subgroup of S, provided that S is a regular E-solid semigroup.*

By a biidentity over Y we will mean any pair $u \doteq v$ of words $u, v \in U_r(Y)$. We will say that a biidentity $u \doteq v$ is satisfied in a regular E-solid semigroup S if, for any matched mapping $\vartheta : \overline{Y} \to S$, we have $\theta(u) = \theta(v)$ where $\theta : U_r(Y) \to S$ is the extension of ϑ to $U_r(Y)$. As usual, we will say that a biidentity is satisfied in a class $\mathcal{V}$ of regular E-solid semigroups if it is satisfied in each member of $\mathcal{V}$.

Given a class $\mathcal{V}$ of regular E-solid semigroups, put

$$\varrho(\mathcal{V}, Y) = \{(u, v) \in U_r(Y) \times U_r(Y) | \text{ the biidentity } u \doteq v \text{ is satisfied in } \mathcal{V}\}.$$

For any set $\Sigma \subseteq U_r(Y) \times U_r(Y)$ of biidentities, put

$$[\Sigma] = \{S \in \mathbf{ES} | \; S \text{ satisfies all biidentities in } \Sigma\}.$$

We will write $u(y_1, y_1', \dots, y_n, y_n')$ to indicate that only elements $y_1, \dots, y_n \in Y, y_1', \dots, y_n' \in Y'$ may occur in $u \in U_r(Y)$. If $u = u(y_1, y_1', \dots, y_n, y_n') \in U_r(Y)$ and $p_1, q_1, \dots, p_n, q_n \in U_r(Y)$ then $u(p_1, q_1, \dots, p_n, q_n)$ is obtained by substituting $p_1, q_1, \dots, p_n, q_n$ into u for $y_1, y_1', \dots, y_n, y_n'$ respectively. It is clear that $u(p_1, q_1, \dots, p_n, q_n) \in U(Y)$. It is easy to see that if $(p_i q_i)\gamma(Y) = 1(Y)$ $(1 \le i \le n)$ then $u(p_1, q_1, \dots, p_n, q_n) \in U_r(Y)$.

A congruence ρ on $U_r(Y)$ will be called biinvariant if $\varrho(\mathbf{ES}, Y) \subseteq \rho$ and it has the following property: whenever $u, v, p_1, q_1, \dots, p_n, q_n \in U_r(Y)$ such that

$$u(y_1, y_1', \dots, y_n, y_n') \; \rho \; v(y_1, y_1', \dots, y_n, y_n')$$

and

$$p_iq_ip_i\ \varrho(\mathbf{ES}, Y)\ p_i, q_ip_iq_i\ \varrho(\mathbf{ES}, Y)\ q_i \text{ for } i = 1, 2, \ldots, n$$

then also

$$u(p_1, q_1, \ldots, p_n, q_n)\ \rho\ v(p_1, q_1, \ldots, p_n, q_n).$$

Observe that the second assumption implies $(p_iq_i)\gamma(Y) = 1(Y)$ for $i = 1, \ldots, n$, as the class of all groups is contained in **ES**, so that this definition is correct.

The set of all fully invariant congruences on the unary semigroup $U(Y)$ will be denoted by $FIC\,U(Y)$ and the set of all biinvariant congruences on the semigroup $U_r(Y)$ will be denoted by $BIC\,U_r(Y)$.

Now, we can present the syntax of our multiplication.

Let $X = \{x_1, x_2, \ldots\}$ be a set of variables. Given $\rho \in FIC\,U(X)$, define a new alphabet $X_\rho = U(X)/\rho \times X$.

Define a left action $*$ of $U(X)$ on $U(X_\rho)$ by

$$\begin{aligned} u * (v\rho, x) &= (uv\rho, x) \\ u * ab &= (u * a)(u * b) \\ u * a' &= (u * a)' \end{aligned}$$

for $u, v \in U(X), x \in X, a, b \in U(X_\rho)$.

We will frequently use the following lemma without references.

3.3 Lemma. *Let $\rho \in FIC\,U(X), u, v \in U(X)$. Then*

(i) *$u\ \rho\ v$ implies $u * a = v * a$ for all $a \in U(X_\rho)$*
(ii) *$u * (v * a) = (uv) * a$ for all $a \in U(X_\rho)$.*

Proof. The assertions are clear. □

3.4 Lemma. *Let $\rho \in FIC\,U(X), u \in U(X)$. If $a \in U_r(X_\rho)$, then $u * a \in U_r(X_\rho)$.*

Proof. By induction with respect to a. Let $v \in U(X), x \in X$. Then $u * (v\rho, x) = (uv\rho, x) \in U_r(X_\rho), u * (v\rho, x)' = (uv\rho, x)' \in U_r(X_\rho)$.

Let $b, c \in U_r(X_\rho), u * b, u * c \in U_r(X_\rho)$. Then $u * bc = (u * b)(u * c) \in U_r(X_\rho)$.

Let $b \in U_r(X_\rho), b\gamma(X_\rho) = 1(X_\rho), u * b \in U_r(X_\rho)$. Then $u * b' = (u * b)' \in U_r(X_\rho)$, since $(u * b)\gamma(X_\rho) = 1(X_\rho)$ ($\gamma(X_\rho) \in FICU(X_\rho)$ and $c \mapsto u * c$ is an endomorphism on $U(X_\rho)$). □

Now, let $\rho \in FIC\,U(X), \rho \supseteq \iota(X)$. Define

$$\pi_\rho : U(X) \to U(X_\rho)$$

by

$$\begin{aligned} \pi_\rho(x) &= (xx'\rho, x) \\ \pi_\rho(uv) &= (uvv'u' * \pi_\rho(u))(u * \pi_\rho(v)) \\ \pi_\rho(u') &= u' * (\pi_\rho(u))' \end{aligned}$$

where $x \in X, u, v \in U(X)$.

3.5 Remark. The mapping π_ρ is defined unambiguously:
Let $u, v, w \in U(X)$. Suppose that the values $\pi_\rho(u), \pi_\rho(v), \pi_\rho(w), \pi_\rho(uv)$ and $\pi_\rho(vw)$ are determined unambiguously. We show that $\pi_\rho((uv)w) = \pi_\rho(u(vw))$:

$$\begin{aligned}
\pi_\rho((uv)w) &= ((uv)ww'(uv)' * \pi_\rho(uv))\,(uv * \pi_\rho(w)) \\
&= (uvww'v'u' * ((uvv'u' * \pi_\rho(u))(u * \pi_\rho(v))))\,(uv * \pi_\rho(w)) \\
&= (uvww'\underline{v'u'uvv'u'} * \pi_\rho(u))\,(u\underline{vww'v'u'u} * \pi_\rho(v))\,(uv * \pi_\rho(w)) \\
&= (uvww'v'u' * \pi_\rho(u))\,(\underline{uu'u}vww'v' * \pi_\rho(v))\,(uv * \pi_\rho(w)) \\
&= (uvww'v'u' * \pi_\rho(u))\,(uvww'v' * \pi_\rho(v))\,(uv * \pi_\rho(w)), \\
\pi_\rho(u(vw)) &= (u(vw)(vw)'u' * \pi_\rho(u))\,(u * \pi_\rho(vw)) \\
&= (uvww'v'u' * \pi_\rho(u))\,(u * ((vww'v' * \pi_\rho(v))(v * \pi_\rho(w)))) \\
&= (uvww'v'u' * \pi_\rho(u))\,(uvww'v' * \pi_\rho(v))\,(uv * \pi_\rho(w)).
\end{aligned}$$

The following lemma will be also often used without references.

3.6 Lemma. *Let $\rho \in FIC\,U(X), \rho \supseteq \iota(X), u \in U(X)$. Then $uu' * \pi_\rho(u) = \pi_\rho(u)$.*

Proof. By induction with respect to u. Let $x \in X$. Then

$$\begin{aligned}
xx' * \pi_\rho(x) &= xx' * (xx'\rho, x) \\
&= (xx'xx'\rho, x) \\
&= (xx'\rho, x) \\
&= \pi_\rho(x).
\end{aligned}$$

Let $u, v \in U(X), vv' * \pi_\rho(v) = \pi_\rho(v)$. Then

$$\begin{aligned}
uv(uv)' * \pi_\rho(uv) &= uvv'u' * ((uvv'u' * \pi_\rho(u))(u * \pi_\rho(v))) \\
&= (uv\underline{v'u'uvv'u'} * \pi_\rho(u))(u\underline{vv'u'u} * \pi_\rho(v)) \\
&= (uvv'u' * \pi_\rho(u))(\underline{uu'u}vv' * \pi_\rho(v)) \\
&= (uvv'u' * \pi_\rho(u))(u * (vv' * \pi_\rho(v))) \\
&= (uvv'u' * \pi_\rho(u))(u * \pi_\rho(v)) \\
&= \pi_\rho(uv).
\end{aligned}$$

Let $u \in U(X)$. Then

$$\begin{aligned}
u'(u')' * \pi_\rho(u') &= u'u * (u' * (\pi_\rho(u))') \\
&= u'uu' * (\pi_\rho(u))' \\
&= u' * (\pi_\rho(u))' \\
&= \pi_\rho(u').
\end{aligned}$$

$\square$

3.7 Lemma. *Let $\rho \in FIC\,U(X), \rho \supseteq \iota(X), u, v \in U(X)$. If $u\ \iota(X)\ v$, then $\pi_\rho(u)\ \iota(X_\rho)\ \pi_\rho(v)$.*

Proof. Having in mind that the variety **I** is the class of all unary semigroups satisfying the identities (id 1) — (id 4) we prove the lemma in the following seven steps.

1. $\pi_\rho((u')')\ \iota(X_\rho)\ \pi_\rho(u)\ \ (u \in U(X))$

$$\begin{aligned}
\pi_\rho((u')') &= (u')' * (\pi_\rho(u'))' \\
&= u * (u' * (\pi_\rho(u))')' \\
&= ((uu' * \pi_\rho(u))')' \\
&= ((\pi_\rho(u))')' \\
&\iota(X_\rho)\ \pi_\rho(u).
\end{aligned}$$

2. $\pi_\rho((uv)')\ \iota(X_\rho)\ \pi_\rho(v'u')\ \ (u,v\in U(X))$
$$\begin{aligned}\pi_\rho((uv)') &= (uv)' * (\pi_\rho(uv))'\\ &= v'u' * ((uvv'u' * \pi_\rho(u))(u * \pi_\rho(v)))'\\ &= ((v'u' * \pi_\rho(u))(v'u'u * \pi_\rho(v)))'\\ &\iota(X_\rho)\ (v'u'u * \pi_\rho(v))'(v'u' * \pi_\rho(u))'\\ &= (v'u'uvv' * \pi_\rho(v))'(v'u' * \pi_\rho(u))'\\ &= (v'u'uv * (v' * (\pi_\rho(v))'))(v' * (u' * (\pi_\rho(u))'))\\ &= (v'u'(u')'(v')' * \pi_\rho(v'))(v' * \pi_\rho(u'))\\ &= \pi_\rho(v'u').\end{aligned}$$

3. $\pi_\rho(uu'u)\ \iota(X_\rho)\ \pi_\rho(u)\ \ (u\in U(X))$
$$\begin{aligned}\pi_\rho(uu'u) &= (u(u'u)(u'u)'u' * \pi_\rho(u))\ (u * \pi_\rho(u'u))\\ &= (uu'uu'uu' * \pi_\rho(u))\ (uu'uu'(u')' * \pi_\rho(u'))\ (uu' * \pi_\rho(u))\\ &= \pi_\rho(u)(uu'uu'uu' * \pi_\rho(u))'\pi_\rho(u)\\ &= \pi_\rho(u)(\pi_\rho(u))'\pi_\rho(u)\\ &\iota(X_\rho)\ \pi_\rho(u).\end{aligned}$$

4. $\pi_\rho(uu'vv')\ \iota(X_\rho)\ \pi_\rho(vv'uu')\ \ (u,v\in U(X))$
Let $w\in U(X)$. Then
$$\begin{aligned}\pi_\rho(ww') &= (ww'(w')'w' * \pi_\rho(w))(w * \pi_\rho(w'))\\ &= (ww'ww' * \pi_\rho(w))(ww' * \pi_\rho(w))'\\ &= \pi_\rho(w)(\pi_\rho(w))'.\end{aligned}$$
Further,
$$\begin{aligned}\pi_\rho(uu'vv') &= (uu'vv'(vv')'(uu')' * \pi_\rho(uu'))(uu' * \pi_\rho(vv'))\\ &= (uu'vv'vv'uu' * \pi_\rho(uu'))(uu'vv'(vv')' * \pi_\rho(vv'))\\ &= (uu'vv' * \pi_\rho(uu'))(uu'vv' * \pi_\rho(vv'))\\ &= uu'vv' * \pi_\rho(u)(\pi_\rho(u))'\pi_\rho(v)(\pi_\rho(v))',\\ \pi_\rho(vv'uu') &= vv'uu' * \pi_\rho(v)(\pi_\rho(v))'\pi_\rho(u)(\pi_\rho(u))'\\ &= uu'vv' * \pi_\rho(v)(\pi_\rho(v))'\pi_\rho(u)(\pi_\rho(u))'\\ &\iota(X_\rho)\ uu'vv' * \pi_\rho(u)(\pi_\rho(u))'\pi_\rho(v)(\pi_\rho(v))'\\ &= \pi_\rho(uu'vv').\end{aligned}$$

5. $u\,\iota(X)\,v$ and $\pi_\rho(u)\ \iota(X_\rho)\ \pi_\rho(v)$ implies $\pi_\rho(wu)\ \iota(X_\rho)\ \pi_\rho(wv)\ \ (u,v,w\in U(X))$
$\pi_\rho(wu) = (wuu'w' * \pi_\rho(w))(w * \pi_\rho(u))$
$\pi_\rho(wv) = (wvv'w' * \pi_\rho(w))(w * \pi_\rho(v))$
Since $u\,\iota(X)\,v$, we have $wuu'w' * \pi_\rho(w) = wvv'w' * \pi_\rho(w)$.
Since $\pi_\rho(u)\ \iota(X_\rho)\ \pi_\rho(v)$, we have $w * \pi_\rho(u)\ \iota(X_\rho)\ w * \pi_\rho(v)$.
So, $\pi_\rho(wu)\ \iota(X_\rho)\ \pi_\rho(wv)$.

6. $u\,\iota(X)\,v$ and $\pi_\rho(u)\ \iota(X_\rho)\ \pi_\rho(v)$ implies $\pi_\rho(uw)\ \iota(X_\rho)\ \pi_\rho(vw)\ \ (u,v,w\in U(X))$
$\pi_\rho(uw) = (uww'u' * \pi_\rho(u))(u * \pi_\rho(w))$
$\pi_\rho(vw) = (vww'v' * \pi_\rho(v))(v * \pi_\rho(w))$
Since $\pi_\rho(u)\ \iota(X_\rho)\ \pi_\rho(v)$, we have $uww'u' * \pi_\rho(u)\ \iota(X_\rho)\ uww'u' * \pi_\rho(v)$. Further, from $u\,\iota(X)\,v$ we get $uww'u' * \pi_\rho(v) = vww'v' * \pi_\rho(v)$. So, $uww'u' * \pi_\rho(u)\,\iota(X_\rho)\,vww'v' * \pi_\rho(v)$. Finally, $u * \pi_\rho(w) = v * \pi_\rho(w)$, since $u\,\iota(X)\,v$. Now, we see that $\pi_\rho(uw)\ \iota(X_\rho)\ \pi_\rho(vw)$.

7. $u\,\iota(X)\,v$ and $\pi_\rho(u)\,\iota(X_\rho)\,\pi_\rho(v)$ implies $\pi_\rho(u')\,\iota(X_\rho)\,\pi_\rho(v')$ $(u, v \in U(X))$
$\pi_\rho(u') = u' * (\pi_\rho(u))'$
$\pi_\rho(v') = v' * (\pi_\rho(v))'$
Since $u\,\iota(X)\,v$, we get $u'\,\iota(X)\,v'$ and then $u' * (\pi_\rho(u))' = v' * (\pi_\rho(u))'$. Since $\pi_\rho(u)\,\iota(X_\rho)\,\pi_\rho(v)$, we get $(\pi_\rho(u))'\,\iota(X_\rho)\,(\pi_\rho(v))'$ and $v' * (\pi_\rho(u))'\,\iota(X_\rho)\,v' * (\pi_\rho(v))'$. So, $\pi_\rho(u')\,\iota(X_\rho)\,\pi_\rho(v')$. $\square$

3.8 Corollary. *Let $\rho \in FIC\,U(X), \rho \supseteq \iota(X), u \in U(X)$. If $u\gamma(X) = 1(X)$, then $\pi_\rho(u)\gamma(X_\rho) = 1(X_\rho)$.*

Proof. Let $u \in U(X), u\gamma(X) = 1(X)$. We know that $u^2\,\iota(X)\,u$ (see 3.1). From 3.7 we get $\pi_\rho(u^2)\,\iota(X_\rho)\,\pi_\rho(u)$. Further,

$$\begin{aligned}\pi_\rho(u^2) &= (uuu'u' * \pi_\rho(u))(u * \pi_\rho(u))\\ &= (uu' * \pi_\rho(u))(uuu' * \pi_\rho(u))\\ &= (uu' * \pi_\rho(u))(uu' * \pi_\rho(u))\\ &= (\pi_\rho(u))^2.\end{aligned}$$

Thus, $(\pi_\rho(u))^2\,\iota(X_\rho)\,\pi_\rho(u), \pi_\rho(u)\gamma(X_\rho) = 1(X_\rho)$. $\square$

3.9 Corollary. *Let $\rho \in FIC\,U(X), \rho \supseteq \iota(X), u \in U(X)$. If $u \in U_r(X)$, then $\pi_\rho(u) \in U_r(X_\rho)$.*

Proof. By induction with respect to u. Let $x \in X$. Then $\pi_\rho(x) = (xx'\rho, x) \in U_r(X_\rho)$,
$\pi_\rho(x') = x' * (xx'\rho, x)' = (x'xx'\rho, x)' = (x'\rho, x)' \in U_r(X_\rho)$.

Let $u, v \in U_r(X), \pi_\rho(u), \pi_\rho(v) \in U_r(X_\rho)$.
$\pi_\rho(uv) = (uvv'u' * \pi_\rho(u))(u * \pi_\rho(v))$

We know that $uvv'u' * \pi_\rho(u) \in U_r(X_\rho), u * \pi_\rho(v) \in U_r(X_\rho)$ (see 3.4). Thus, $\pi_\rho(uv) \in U_r(X_\rho)$.

Let $u \in U_r(X), u\gamma(X) = 1(X), \pi_\rho(u) \in U_r(X_\rho)$.
$\pi_\rho(u') = u' * (\pi_\rho(u))'$
We know that $\pi_\rho(u)\gamma(X_\rho) = 1(X_\rho)$ (see 3.8). Then $(\pi_\rho(u))' \in U_r(X_\rho)$. Using 3.4 we obtain $\pi_\rho(u') \in U_r(X_\rho)$. $\square$

Now, let $\rho \in FIC\,U(X), \rho \supseteq \iota(X), \sigma \in BIC\,U_r(X_\rho)$. Define

$$\sigma \,\square\, \rho \subseteq U_r(X) \times U_r(X)$$

by

$$u\,(\sigma \,\square\, \rho)\,v \iff u\,\rho\,v \text{ and } \pi_\rho(u)\,\sigma\,\pi_\rho(v)$$

$(u, v \in U_r(X))$.

The correctness of the definition is based on 3.9.

3.10 Remark. If $\rho \in FIC\,U(X), \rho \supseteq \iota(X), \sigma \in BIC\,U_r(X_\rho)$, then $\sigma \,\square\, \rho \in BIC\,U_r(X)$. We will prove it in 4.10.

4. Relationships between syntax and semantics

4.1 Result. ([6], Corollary 2.11) *For any infinite set Y, the rules*

$$\mathcal{V} \mapsto \varrho(\mathcal{V}, Y) \text{ and } \rho \mapsto [\rho]$$

define mutually inverse order–reversing bijections between all e-varieties of regular E-solid semigroups and all biinvariant congruences on $U_r(Y)$.

We will denote the one–to–one correspondence from 4.1 simply by the symbol $\leftrightarrow$. Since it causes no confusion, we will use the symbol $\leftrightarrow$ also for the well–known one–to–one correspondence between all varieties of unary semigroups and all fully invariant congruences on the free unary semigroup $U(Y)$.

Now, we recall the notion of a bifree object. Let $\mathcal{V}$ be a class of regular semigroups. By a bifree object in $\mathcal{V}$ on a non–empty set Y, we mean a pair (S, ι) where $S \in \mathcal{V}$ and $\iota : \overline{Y} \to S$ is a matched mapping such that the following universal property is satisfied: for any semigroup $T \in \mathcal{V}$ and any matched mapping $\vartheta : \overline{Y} \to T$, there exists a unique homomorphism $\psi : S \to T$ such that $\psi \circ \iota = \vartheta$. In cases when the mapping ι is obvious, we omit it and we term simply S to be a bifree object in $\mathcal{V}$ on Y. Note that in any class of regular semigroups, there exists, up to isomorphism, at most one bifree object on any non–empty set.

4.2 Result. ([6], Theorem 2.5) *If Y is an infinite set and $\mathcal{V}$ is a class of regular E-solid semigroups closed under taking regular subsemigroups and direct products then $U_r(Y)/\varrho(\mathcal{V}, Y)$ is a bifree object in $\mathcal{V}$ on Y.*

4.3 Lemma. *Let $\rho \in FIC\,U(X), \sigma \in BIC\,U_r(X_\rho)$. Then the mapping*

$$\varphi : U(X)/\rho \to (\mathrm{End}(U_r(X_\rho)/\sigma), \circ)$$

given by

$$\varphi(u\rho)(a\sigma) = (u * a)\sigma \ \ (u \in U(X), a \in U_r(X_\rho))$$

is a correctly defined homomorphism.

Proof.

1. correctness of the definition:
   It follows from 3.4 that $u \in U(X)$ and $a \in U_r(X_\rho)$ implies $u * a \in U_r(X_\rho)$. Now, let $u, v \in U(X), a, b \in U_r(X_\rho), u\,\rho\,v, a\,\sigma\,b$. We will show that $u * a\,\sigma\,v * b$. We have $u * a = v * a$. Since $\sigma \in BIC\,U_r(X_\rho)$, we get $v * a\,\sigma\,v * b$. So, $u * a\,\sigma\,v * b$.

2. $\varphi(u\rho) : U_r(X_\rho)/\sigma \to U_r(X_\rho)/\sigma$ is an endomorphism (for any $u \in U(X)$):
   Let $a, b \in U_r(X_\rho)$. Then
   $$\begin{aligned}\varphi(u\rho)((a\sigma)(b\sigma)) &= \varphi(u\rho)(ab\sigma) = (u * (ab))\sigma = (u * a)(u * b)\sigma \\ &= ((u * a)\sigma)((u * b)\sigma) = \varphi(u\rho)(a\sigma)\varphi(u\rho)(b\sigma).\end{aligned}$$

3. φ is a homomorphism:
   Let $u, v \in U(X), a \in U_r(X_\rho)$. Then
   $$\begin{aligned}\varphi(u\rho)(\varphi(v\rho)(a\sigma)) &= \varphi(u\rho)((v * a)\sigma) = (u * (v * a))\sigma = ((uv) * a)\sigma \\ &= \varphi(uv\rho)(a\sigma) = \varphi((u\rho)(v\rho))(a\sigma).\end{aligned}$$

$\square$

4.4 Lemma. *Let $\rho \in FIC\,U(X), \rho \supseteq \iota(X), \sigma \in BIC\,U_r(X_\rho)$. Further, let $\varphi : U(X)/\rho \to (\mathrm{End}(U_r(X_\rho)/\sigma), \circ)$ be the homomorphism from 4.3. Finally, let $\vartheta : \overline{X} \to U_r(X_\rho)/\sigma \times_\varphi U(X)/\rho$ be given by*

$$y \mapsto (\pi_\rho(y)\sigma, y\rho) \;\; (y \in \overline{X}) .$$

Then

(i) *ϑ is a matched mapping*
(ii) *$\theta(u) = (\pi_\rho(u)\sigma, u\rho)$ for all $u \in U_r(X)$*
(where θ is the extension of the matched mapping ϑ).

Proof. Note that $U(X)/\rho \in \mathbf{I}, U_r(X_\rho)/\sigma \in \mathbf{ES}, U_r(X_\rho)/\sigma \times_\varphi U(X)/\rho \in \mathbf{ES}$ (see 4.1,4.2 and 2.4).

(i) Choose $x \in X$. Then $\vartheta(x)\vartheta(x')\vartheta(x) =$
$= ((xx'\rho, x)\sigma, x\rho)\,((x'\rho, x)'\sigma, x'\rho)\,((xx'\rho, x)\sigma, x\rho)$
$= (\varphi(xx'xx'\rho)((xx'\rho, x)\sigma)\varphi(x\rho)((x'\rho, x)'\sigma), xx'\rho)\,((xx'\rho, x)\sigma, x\rho)$
$= ((xx' * (xx'\rho, x))(x * (x'\rho, x)')\sigma, xx'\rho)\,((xx'\rho, x)\sigma, x\rho)$
$= ((xx'\rho, x)(xx'\rho, x)'\sigma, xx'\rho)\,((xx'\rho, x)\sigma, x\rho)$
$= (\varphi(xx'xx'xx'\rho)((xx'\rho, x)(xx'\rho, x)'\sigma)\varphi(xx'\rho)((xx'\rho, x)\sigma), xx'x\rho)$
$= ((xx' * (xx'\rho, x)(xx'\rho, x)')(xx' * (xx'\rho, x))\sigma, x\rho)$
$= ((xx'\rho, x)(xx'\rho, x)'(xx'\rho, x)\sigma, x\rho)$
$= ((xx'\rho, x)\sigma, x\rho)$
$= \vartheta(x).$
Similarly, $\vartheta(x')\vartheta(x)\vartheta(x') = \vartheta(x')$.

(ii) We proceed by induction with respect to u. Let $u, v \in U_r(X), \theta(u) = (\pi_\rho(u)\sigma, u\rho), \theta(v) = (\pi_\rho(v)\sigma, v\rho)$. Then $\theta(uv) = \theta(u)\theta(v)$
$= (\pi_\rho(u)\sigma, u\rho)(\pi_\rho(v)\sigma, v\rho)$
$= (\varphi(uvv'u'\rho)(\pi_\rho(u)\sigma)\varphi(u\rho)(\pi_\rho(v)\sigma), uv\rho)$
$= ((uvv'u' * \pi_\rho(u))(u * \pi_\rho(v))\sigma, uv\rho)$
$= (\pi_\rho(uv)\sigma, uv\rho)$.
Let $u \in U_r(X), u\gamma(X) = 1(X), \theta(u) = (\pi_\rho(u)\sigma, u\rho)$. Note that $\pi_\rho(u) \in U_r(X_\rho)$ by 3.9 and $\pi_\rho(u)\gamma(X_\rho) = 1(X_\rho)$ by 3.8. We want to prove $\theta(u') = (\pi_\rho(u')\sigma, u'\rho)$. In view of $\theta(u') = (\theta(u))^{-1}$ we have to show that
$(\pi_\rho(u)\sigma, u\rho)(\pi_\rho(u')\sigma, u'\rho)(\pi_\rho(u)\sigma, u\rho) = (\pi_\rho(u)\sigma, u\rho)$,
$(\pi_\rho(u')\sigma, u'\rho)(\pi_\rho(u)\sigma, u\rho)(\pi_\rho(u')\sigma, u'\rho) = (\pi_\rho(u')\sigma, u'\rho)$,
$(\pi_\rho(u)\sigma, u\rho)(\pi_\rho(u')\sigma, u'\rho) = (\pi_\rho(u')\sigma, u'\rho)(\pi_\rho(u)\sigma, u\rho)$.
We see that $(\pi_\rho(u)\sigma, u\rho)(\pi_\rho(u')\sigma, u'\rho)(\pi_\rho(u)\sigma, u\rho) =$
$= (\varphi(uu'uu'\rho)(\pi_\rho(u)\sigma)\varphi(u\rho)(\pi_\rho(u')\sigma), uu'\rho)\,(\pi_\rho(u)\sigma, u\rho)$
$= ((uu' * \pi_\rho(u))(u * (u' * (\pi_\rho(u))'))\sigma, uu'\rho)\,(\pi_\rho(u)\sigma, u\rho)$
$= (\pi_\rho(u)(\pi_\rho(u))'\sigma, uu'\rho)\,(\pi_\rho(u)\sigma, u\rho)$
$= (\varphi(uu'uu'uu'\rho)(\pi_\rho(u)(\pi_\rho(u))'\sigma)\varphi(uu'\rho)(\pi_\rho(u)\sigma), uu'u\rho)$
$= ((uu' * \pi_\rho(u))(uu' * \pi_\rho(u))'(uu' * \pi_\rho(u))\sigma, u\rho)$
$= (\pi_\rho(u)(\pi_\rho(u))'\pi_\rho(u)\sigma, u\rho)$
$= (\pi_\rho(u)\sigma, u\rho)$, since $\sigma \supseteq \varrho(\mathbf{ES}, X_\rho)$ and $\pi_\rho(u)(\pi_\rho(u))'\pi_\rho(u)\ \varrho(\mathbf{ES}, X_\rho)\ \pi_\rho(u)$.

Similarly, $(\pi_\rho(u')\sigma, u'\rho)(\pi_\rho(u)\sigma, u\rho)(\pi_\rho(u')\sigma, u'\rho) = (\pi_\rho(u')\sigma, u'\rho)$. Further,
$(\pi_\rho(u)\sigma, u\rho)(\pi_\rho(u')\sigma, u'\rho) =$
$= (\varphi(uu'uu'\rho)(\pi_\rho(u)\sigma)\varphi(u\rho)(\pi_\rho(u')\sigma), uu'\rho)$
$= ((uu' * \pi_\rho(u))(u * (u' * (\pi_\rho(u))'))\sigma, uu'\rho)$
$= (\pi_\rho(u)(\pi_\rho(u))'\sigma, uu'\rho)$,
$(\pi_\rho(u')\sigma, u'\rho)(\pi_\rho(u)\sigma, u\rho) =$
$= (\varphi(u'uu'u\rho)(\pi_\rho(u')\sigma)\varphi(u'\rho)(\pi_\rho(u)\sigma), u'u\rho)$.
We know that $u^2\ \iota(X)\ u$ (see 3.1). So, $u\ \iota(X)\ u'$, $u\ \rho\ u'$.
Then $(\pi_\rho(u')\sigma, u'\rho)(\pi_\rho(u)\sigma, u\rho) =$
$= ((uu' * (u' * (\pi_\rho(u))'))(u * (uu' * \pi_\rho(u)))\sigma, uu'\rho)$
$= ((u(u^2)' * \pi_\rho(u))'(u^2u' * \pi_\rho(u))\sigma, uu'\rho)$
$= ((\pi_\rho(u))'\pi_\rho(u)\sigma, uu'\rho)$
$= (\pi_\rho(u)(\pi_\rho(u))'\sigma, uu'\rho)$,
since $\sigma \supseteq \varrho(\mathbf{ES}, X_\rho)$ and $\pi_\rho(u)(\pi_\rho(u))'\ \varrho(\mathbf{ES}, X_\rho)\ (\pi_\rho(u))'\pi_\rho(u)$. $\square$

4.5 Corollary. *Let $\rho \in FIC\,U(X), \rho \supseteq \iota(X), \sigma \in BIC\,U_r(X_\rho)$. Let $\mathcal{U} \subseteq \mathbf{ES}$ be an e-variety and $\mathcal{V} \subseteq \mathbf{I}$ be a variety such that $\mathcal{U} \leftrightarrow \sigma, \mathcal{V} \leftrightarrow \rho$. Then*

$$\sigma \,\square\, \rho \supseteq \varrho(\mathcal{U} \,\square\, \mathcal{V}, X)\,.$$

Proof. Let $u, v \in U_r(X)$, $u\,\varrho(\mathcal{U} \,\square\, \mathcal{V}, X)\,v$. We will show that $u\,(\sigma \,\square\, \rho)\,v$, i.e. $u\,\rho\,v$ and $\pi_\rho(u)\,\sigma\,\pi_\rho(v)$. Note that $U_r(X_\rho)/\sigma \in \mathcal{U}$, $U(X)/\rho \in \mathcal{V}$ (see 4.1 and 4.2). We use the homomorphism $\varphi : U(X)/\rho \to (\mathrm{End}\,(U_r(X_\rho)/\sigma), \circ)$ from 4.3 and the matched mapping $\vartheta : \overline{X} \to U_r(X_\rho)/\sigma \times_\varphi U(X)/\rho$ from 4.4. Now, $U_r(X_\rho)/\sigma \times_\varphi U(X)/\rho \in \mathcal{U} \square \mathcal{V}$. Thus the biidentity $u \doteq v$ is satisfied in $U_r(X_\rho)/\sigma \times_\varphi U(X)/\rho$, and therefore $\theta(u) = \theta(v)$ (where θ is the extension of the matched mapping ϑ). Hence, by 4.4(ii), $(\pi_\rho(u)\sigma, u\rho) = (\pi_\rho(v)\sigma, v\rho)$. $\square$

4.6 Lemma. *Let $S \in \mathbf{ES}, T \in \mathbf{I}, \varphi : (T, \cdot) \to (\mathrm{End}\,(S), \circ)$ be a homomorphism, $\vartheta : \overline{X} \to S \times_\varphi T$ be a matched mapping such that*

$$\begin{aligned} \vartheta(x_i) &= (s_i, t_i) \\ \vartheta(x_i') &= (p_i, q_i) \qquad (\text{for } i = 1, 2, \ldots). \end{aligned}$$

Let $\rho \in FIC\,U(X), \rho \supseteq \iota(X)$. Suppose that all identities from ρ are satisfied in T. Let $\vartheta_2 : X \to T$ be given by

$$x_i \mapsto t_i\,.$$

Let $\vartheta_1 : \overline{X_\rho} \to S$ be given by

$$\begin{aligned} (u\rho, x_i) &\mapsto \varphi(\theta_2(u))(s_i) \\ (u\rho, x_i)' &\mapsto \varphi(\theta_2(ux_i))(p_i) \qquad (\text{for } (u\rho, x_i) \in X_\rho), \end{aligned}$$

where $\theta_2 : U(X) \to T$ is the unary homomorphism extending ϑ_2.
Finally, let $\theta : U_r(X) \to S \times_\varphi T$ be the extension of the matched mapping ϑ. Then

(i) the mapping ϑ_1 is matched

(ii) $\theta_1(u*a)=\varphi(\theta_2(u))(\theta_1(a))$ *for all* $u\in U(X), a\in U_r(X_\rho)$ (θ_1 *denotes the extension of the matched mapping* ϑ_1)

(iii) $\theta(u)=(\theta_1(\pi_\rho(u)),\theta_2(u))$ *for all* $u\in U_r(X)$.

Proof.

(i) Let $u,v\in U(X), u\rho v$. We have to show that $\theta_2(u)=\theta_2(v)$. But all identities from ρ are satisfied in T, which implies $\theta_2(u)=\theta_2(v)$.
Now, let $(u\rho,x_i)\in X_\rho$. We have to show that
$\vartheta_1((u\rho,x_i))\vartheta_1((u\rho,x_i)')\vartheta_1((u\rho,x_i))=\vartheta_1((u\rho,x_i))$ and
$\vartheta_1((u\rho,x_i)')\vartheta_1((u\rho,x_i))\vartheta_1((u\rho,x_i)')=\vartheta_1((u\rho,x_i)')$ i.e.
$\varphi(\theta_2(u))(s_i)\varphi(\theta_2(ux_i))(p_i)\varphi(\theta_2(u))(s_i)=\varphi(\theta_2(u))(s_i)$ and
$\varphi(\theta_2(ux_i))(p_i)\varphi(\theta_2(u))(s_i)\varphi(\theta_2(ux_i))(p_i)=\varphi(\theta_2(ux_i))(p_i)$ i.e.
$\varphi(\theta_2(u))(s_i\varphi(t_i)(p_i)s_i)=\varphi(\theta_2(u))(s_i)$ and
$\varphi(\theta_2(u))(\varphi(t_i)(p_i)s_i\varphi(t_i)(p_i))=\varphi(\theta_2(u))(\varphi(t_i)(p_i))$.
We know that $(s_i,t_i)(p_i,q_i)(s_i,t_i)=(s_i,t_i)$ and $(p_i,q_i)(s_i,t_i)(p_i,q_i)=(p_i,q_i)$.
We get $(\varphi(t_iq_it_it_i'q_i't_i't_iq_iq_i't_i')(s_i)\varphi(t_iq_it_it_i'q_i't_i't_i)(p_i)\varphi(t_iq_i)(s_i),t_iq_it_i)=(s_i,t_i)$ and $(\varphi(q_it_iq_iq_i't_i'q_i'q_it_it_i'q_i')(p_i)\varphi(q_it_iq_iq_i't_i'q_i'q_i)(s_i)\varphi(q_it_i)(p_i),q_it_iq_i)=(p_i,q_i)$.
Thus $q_i=t_i', s_i\varphi(t_i)(p_i)s_i=s_i, p_i\varphi(t_i')(s_i)p_i=p_i$.
The last equality implies $\varphi(t_i)(p_i)s_i\varphi(t_i)(p_i)=\varphi(t_i)(p_i)$.

(ii) We proceed by induction with respect to a. Let $(v\rho,x_i)\in X_\rho$. Then

$$\begin{aligned}\theta_1(u*(v\rho,x_i))&=\theta_1((uv\rho,x_i))=\varphi(\theta_2(uv))(s_i)\\&=\varphi(\theta_2(u))(\varphi(\theta_2(v))(s_i))=\varphi(\theta_2(u))(\theta_1((v\rho,x_i)))\end{aligned}$$

and

$$\begin{aligned}\theta_1(u*(v\rho,x_i)')&=\theta_1((uv\rho,x_i)')=\varphi(\theta_2(uvx_i))(p_i)\\&=\varphi(\theta_2(u)(\varphi(\theta_2(vx_i))(p_i))=\varphi(\theta_2(u))(\theta_1((v\rho,x_i)')).\end{aligned}$$

Let $a,b\in U_r(X_\rho),\theta_1(u*a)=\varphi(\theta_2(u))(\theta_1(a)),\theta_1(u*b)=\varphi(\theta_2(u))(\theta_1(b))$. Then

$$\begin{aligned}\theta_1(u*ab)&=\theta_1((u*a)(u*b))=\theta_1(u*a)\theta_1(u*b)\\&=\varphi(\theta_2(u))(\theta_1(a))\varphi(\theta_2(u))(\theta_1(b))=\varphi(\theta_2(u))(\theta_1(a)\theta_1(b))\\&=\varphi(\theta_2(u))(\theta_1(ab)).\end{aligned}$$

Let $a\in U_r(X_\rho), a\gamma(X_\rho)=1(X_\rho),\theta_1(u*a)=\varphi(\theta_2(u))(\theta_1(a))$. Then

$$\begin{aligned}\theta_1(u*a')&=\theta_1((u*a)')=(\varphi(\theta_2(u))(\theta_1(a)))^{-1}\\&=\varphi(\theta_2(u))((\theta_1(a))^{-1})=\varphi(\theta_2(u))(\theta_1(a')).\end{aligned}$$

(iii) By induction:

$$\begin{aligned}(\theta_1(\pi_\rho(x_i)),\theta_2(x_i))&=(\theta_1((x_ix_i'\rho,x_i)),\theta_2(x_i))=(\varphi(\theta_2(x_ix_i'))(s_i),t_i)\\&=(\varphi(t_it_i')(s_i),t_i)=(s_i,t_i)=\theta(x_i),\\(\theta_1(\pi_\rho(x_i')),\theta_2(x_i'))&=(\theta_1((x_i'\rho,x_i)'),\theta_2(x_i'))\\&=(\varphi(\theta_2(x_i'x_i))(p_i),q_i)=(\varphi(q_iq_i')(p_i),q_i)\\&=(p_i,q_i)=\theta(x_i').\end{aligned}$$

Let $u,v\in U_r(X),\theta(u)=(\theta_1(\pi_\rho(u)),\theta_2(u)),\theta(v)=(\theta_1(\pi_\rho(v)),\theta_2(v))$. Then

$$\theta(uv)=\theta(u)\theta(v)$$

$= (\theta_1(\pi_\rho(u)), \theta_2(u))(\theta_1(\pi_\rho(v)), \theta_2(v))$
$= (\varphi(\theta_2(uvv'u'))(\theta_1(\pi_\rho(u)))\varphi(\theta_2(u))(\theta_1(\pi_\rho(v))), \theta_2(u)\theta_2(v))$.

It was proved in section (ii) that $\varphi(\theta_2(uvv'u'))(\theta_1(\pi_\rho(u))) = \theta_1(uvv'u' * \pi_\rho(u))$, $\varphi(\theta_2(u))(\theta_1(\pi_\rho(v))) = \theta_1(u * \pi_\rho(v))$.

Now,

$$\theta(uv) = (\theta_1((uvv'u' * \pi_\rho(u))(u * \pi_\rho(v))), \theta_2(uv)) = (\theta_1(\pi_\rho(uv)), \theta_2(uv)).$$

Finally, let $u \in U_r(X), u\gamma(X) = 1(X), \theta(u) = (\theta_1(\pi_\rho(u)), \theta_2(u))$. We have to show that

$(\theta_1(\pi_\rho(u)), \theta_2(u))(\theta_1(\pi_\rho(u')), \theta_2(u'))(\theta_1(\pi_\rho(u)), \theta_2(u)) = (\theta_1(\pi_\rho(u)), \theta_2(u))$,
$(\theta_1(\pi_\rho(u')), \theta_2(u'))(\theta_1(\pi_\rho(u)), \theta_2(u))(\theta_1(\pi_\rho(u')), \theta_2(u')) = (\theta_1(\pi_\rho(u')), \theta_2(u'))$,
$(\theta_1(\pi_\rho(u)), \theta_2(u))(\theta_1(\pi_\rho(u')), \theta_2(u')) = (\theta_1(\pi_\rho(u')), \theta_2(u'))(\theta_1(\pi_\rho(u)), \theta_2(u))$.

We see that $(\theta_1(\pi_\rho(u)), \theta_2(u))(\theta_1(\pi_\rho(u')), \theta_2(u'))(\theta_1(\pi_\rho(u)), \theta_2(u)) =$
$= (\varphi(\theta_2(uu'uu'))(\theta_1(\pi_\rho(u)))\varphi(\theta_2(u))(\theta_1(\pi_\rho(u'))), \theta_2(uu'))$
$(\theta_1(\pi_\rho(u)), \theta_2(u))$
$= (\theta_1(\pi_\rho(u))\theta_1((\pi_\rho(u))'), \theta_2(uu'))\, (\theta_1(\pi_\rho(u)), \theta_2(u))$
$= (\varphi(\theta_2(uu'uu'uu'))(\theta_1(\pi_\rho(u))\theta_1((\pi_\rho(u))'))\varphi(\theta_2(uu'))(\theta_1(\pi_\rho(u)))$,
$\theta_2(uu'u))$
$= (\theta_1(\pi_\rho(u))\theta_1((\pi_\rho(u))')\theta_1(\pi_\rho(u)), \theta_2(uu'u))$
$= (\theta_1(\pi_\rho(u)), \theta_2(u))$.

Similarly, $(\theta_1(\pi_\rho(u')), \theta_2(u'))(\theta_1(\pi_\rho(u)), \theta_2(u))(\theta_1(\pi_\rho(u')), \theta_2(u')) =$
$= (\theta_1(\pi_\rho(u')), \theta_2(u'))$.

Further, $(\theta_1(\pi_\rho(u)), \theta_2(u))(\theta_1(\pi_\rho(u')), \theta_2(u')) =$
$= (\theta_1(\pi_\rho(u))\theta_1((\pi_\rho(u))'), \theta_2(uu'))$,
$(\theta_1(\pi_\rho(u')), \theta_2(u'))(\theta_1(\pi_\rho(u)), \theta_2(u)) =$
$= (\varphi(\theta_2(u'uu'u))(\theta_1(\pi_\rho(u')))\varphi(\theta_2(u'))(\theta_1(\pi_\rho(u))), \theta_2(u'u))$
$= (\theta_1(u' * (\pi_\rho(u))')\theta_1(u' * \pi_\rho(u)), \theta_2(u'u))$
$= (\theta_1(uu' * (\pi_\rho(u))')\theta_1(uu' * \pi_\rho(u)), \theta_2(u'u))$
$= (\theta_1((\pi_\rho(u))')\theta_1(\pi_\rho(u)), \theta_2(u'u))$
$= (\theta_1(\pi_\rho(u))\theta_1((\pi_\rho(u))'), \theta_2(uu'))$.

We used the following facts:
$u^2\ \iota(X)\ u$ (see 3.1), $u^2\ \rho\ u, u\ \rho\ u', \theta_2(u) \in E(T), \theta_2(u') \in E(T)$. □

4.7 Corollary. *Let $\rho \in FIC\,U(X), \rho \supseteq \iota(X), \sigma \in BIC\,U_r(X_\rho)$. Let $S \in$ **ES**, $T \in$ **I**. Suppose that all identities from ρ are satisfied in T and all biidentities from σ are satisfied in S. Finally, let $\varphi : (T, \cdot) \to (\mathrm{End}\,(S), \circ)$ be a homomorphism. Then $\sigma \,\Box\, \rho \subseteq \varrho(\{S \times_\varphi T\}, X)$.*

Proof. Let $u, v \in U_r(X), u\ (\sigma \,\Box\, \rho)\ v, \vartheta : \overline{X} \to S \times_\varphi T$ be a matched mapping. We have to show that $\theta(u) = \theta(v)$, where $\theta : U_r(X) \to S \times_\varphi T$ is the extension of ϑ. We know that $u\ \rho\ v, \pi_\rho(u)\ \sigma\ \pi_\rho(v)$. Consider the mappings ϑ_1 and ϑ_2 from 4.6. The mapping ϑ_1 is matched by 4.6(i). Let $\theta_1 : U_r(X_\rho) \to S$ be the extension of ϑ_1 and $\theta_2 : U(X) \to T$ be the unary homomorphism extending ϑ_2. Then $\theta_1(\pi_\rho(u)) = \theta_1(\pi_\rho(v))$ and $\theta_2(u) = \theta_2(v)$. Thus $\theta(u) = \theta(v)$ (by 4.6(iii)). □

4.8 Result. ([2], Lemma 1) *Let $\psi : Q \to W$ be a surjective homomorphism of regular semigroups and let $a, b \in W, aba = a, bab = b$. Then there exist $c, d \in Q$ such that $cdc = c, dcd = d$ and $\psi(c) = a, \psi(d) = b$.*

4.9 Corollary. *Let $\rho \in FIC\, U(X), \rho \supseteq \iota(X), \sigma \in BIC\, U_r(X_\rho)$. Let $\mathcal{U} \subseteq \mathbf{ES}$ be an e-variety, $\mathcal{V} \subseteq \mathbf{I}$ be a variety such that $\mathcal{U} \leftrightarrow \sigma, \mathcal{V} \leftrightarrow \rho$. Then*

$$\sigma \,\square\, \rho \subseteq \varrho(\mathcal{U} \,\square\, \mathcal{V}, X)\,.$$

Proof. Let $u, v \in U_r(X), u\ (\sigma \,\square\, \rho)\ v, W \in \mathcal{U} \,\square\, \mathcal{V}$ and let $\vartheta : \overline{X} \to W$ be a matched mapping. We will show that $\theta(u) = \theta(v)$, where $\theta : U_r(X) \to W$ is the extension of ϑ. It follows from 2.7 that there exist $S \in \mathcal{U}, T \in \mathcal{V}$, a homomorphism $\varphi : (T, \cdot) \to (\mathrm{End}\,(S), \circ)$, a regular subsemigroup Q in $S \times_\varphi T$ and a surjective homomorphism $\psi : Q \to W$. By 4.8, there is a matched mapping $\overline{\vartheta} : \overline{X} \to Q$ such that $\psi(\overline{\vartheta}(y)) = \vartheta(y)$ for all $y \in \overline{X}$. Then $\psi(\overline{\theta}(w)) = \theta(w)$ for all $w \in U_r(X)$ ($\overline{\theta} : U_r(X) \to Q$ is the extension of $\overline{\vartheta}$). Now, we use 4.7. We have $\overline{\theta}(u) = \overline{\theta}(v)$. Thus $\psi(\overline{\theta}(u)) = \psi(\overline{\theta}(v)), \theta(u) = \theta(v)$. □

4.10 Theorem. *Let $\rho \in FIC\, U(X), \rho \supseteq \iota(X), \sigma \in BIC\, U_r(X_\rho)$. Let $\mathcal{U} \subseteq \mathbf{ES}$ be an e-variety, $\mathcal{V} \subseteq \mathbf{I}$ be a variety such that $\mathcal{U} \leftrightarrow \sigma, \mathcal{V} \leftrightarrow \rho$. Then*

(i) $\sigma \,\square\, \rho \in BIC\, U_r(X)$
(ii) $\mathcal{U} \,\square\, \mathcal{V} \leftrightarrow \sigma \,\square\, \rho$
(iii) *The mapping $\psi : U_r(X)/\sigma \,\square\, \rho \to U_r(X_\rho)/\sigma \times_\varphi U(X)/\rho$ defined by*

$$\psi(u(\sigma \,\square\, \rho)) = (\pi_\rho(u)\sigma, u\rho)\,,$$

where φ is the homomorphism from 4.3, is an embedding.

Proof.

(i) and (ii) Note that $\mathcal{U} \,\square\, \mathcal{V} \subseteq \mathbf{ES}$ (see 2.4). By 4.5 and 4.9 we have $\sigma \,\square\, \rho = \varrho(\mathcal{U} \,\square\, \mathcal{V}, X)$. Thus $\sigma \,\square\, \rho \in BIC\, U_r(X)$ and $\mathcal{U} \,\square\, \mathcal{V} \leftrightarrow \sigma \,\square\, \rho$ by 4.1.

(iii) It follows immediately from the definition of $\sigma \,\square\, \rho$ that ψ is a correctly defined injective mapping.
ψ is a homomorphism:
Let $u, v \in U_r(X)$. Then

$$\begin{aligned}\psi((u(\sigma \,\square\, \rho))(v(\sigma \,\square\, \rho))) &= \psi(uv(\sigma \,\square\, \rho))\\ &= (\pi_\rho(uv)\sigma, uv\rho) = ((uvv'u' * \pi_\rho(u))(u * \pi_\rho(v))\sigma, uv\rho)\\ &= (\varphi(uvv'u'\rho)(\pi_\rho(u)\sigma)\varphi(u\rho)(\pi_\rho(v)\sigma), uv\rho)\\ &= (\pi_\rho(u)\sigma, u\rho)(\pi_\rho(v)\sigma, v\rho)\\ &= \psi(u(\sigma \,\square\, \rho))\psi(v(\sigma \,\square\, \rho)).\end{aligned}$$ □

4.11 Remark. Theorem 4.10 together with Result 4.2 show that bifree objects in $\mathcal{U} \,\square\, \mathcal{V}$ are isomorphic to some subsemigroups in suitable semidirect products of bifree objects in $\mathcal{U}$ by free objects in $\mathcal{V}$, for any e-variety $\mathcal{U} \subseteq \mathbf{ES}$ and any variety $\mathcal{V} \subseteq \mathbf{I}$.

This section is concluded with a corollary of Theorem 4.10. First, the following result ensures that if $\mathcal{U}$ and $\mathcal{V}$ are varieties of inverse semigroups then $\mathcal{U} \square \mathcal{V}$ is also a variety of inverse semigroups.

4.12 Result. ([1], Proposition 1) *Let S, T be inverse semigroups, $\varphi : (T, \cdot) \to (\mathrm{End}\,(S), \circ)$ be a homomorphism. Then $S \times_\varphi T$ is also an inverse semigroup.*

4.13 Corollary. *Let $\rho \in FIC\,U(X), \rho \supseteq \iota(X), \sigma \in FIC\,U(X_\rho), \sigma \supseteq \iota(X_\rho)$. Let $\mathcal{U}, \mathcal{V} \subseteq \mathbf{I}$ be varieties such that $\mathcal{U} \leftrightarrow \sigma, \mathcal{V} \leftrightarrow \rho$. Denote by $\sigma \square \rho$ the fully invariant congruence on $U(X)$ corresponding to the variety $\mathcal{U} \square \mathcal{V}$. Then*

$$u\,(\sigma \square \rho)\,v \Longleftrightarrow u\,\rho\,v \text{ and } \pi_\rho(u)\,\sigma\,\pi_\rho(v) \quad (\text{for all } u, v \in U(X))\,.$$

Proof. Let σ_0 be the biinvariant congruence on $U_r(X_\rho)$ corresponding to the e-variety $\mathcal{U}$. It follows from 4.10(ii) that $\sigma_0 \square \rho$ is the biinvariant congruence on $U_r(X)$ corresponding to the e-variety $\mathcal{U} \square \mathcal{V}$. Clearly, $\sigma_0 = \sigma \cap (U_r(X_\rho) \times U_r(X_\rho)), \sigma_0 \square \rho = (\sigma \square \rho) \cap (U_r(X) \times U_r(X))$. Let $u, v \in U(X)$. There are $u_0, v_0 \in U_r(X)$ such that $u_0\,\iota(X)\,u, v_0\,\iota(X)\,v$. Then $u_0\,\rho\,u, v_0\,\rho\,v, u_0\,(\sigma \square \rho)\,u, v_0\,(\sigma \square \rho)\,v$. Further, $\pi_\rho(u_0)\,\iota(X_\rho)\,\pi_\rho(u), \pi_\rho(v_0)\,\iota(X_\rho)\,\pi_\rho(v)$ (by 3.7). Thus $\pi_\rho(u_0)\,\sigma\,\pi_\rho(u), \pi_\rho(v_0)\,\sigma\,\pi_\rho(v)$. Now,

$$\begin{aligned} u\,(\sigma \square \rho)\,v &\Leftrightarrow u_0\,(\sigma \square \rho)\,v_0 \\ &\Leftrightarrow u_0\,(\sigma_0 \square \rho)\,v_0 \\ &\Leftrightarrow u_0\,\rho\,v_0 \text{ and } \pi_\rho(u_0)\,\sigma_0\,\pi_\rho(v_0) \\ &\Leftrightarrow u\,\rho\,v \text{ and } \pi_\rho(u_0)\,\sigma\,\pi_\rho(v_0) \\ &\Leftrightarrow u\,\rho\,v \text{ and } \pi_\rho(u)\,\sigma\,\pi_\rho(v). \end{aligned}$$

We used also the facts that $\pi_\rho(u_0), \pi_\rho(v_0) \in U_r(X_\rho)$ (by 3.9). □

5. Associativity

We specify our notation in this section. Let Y be a countable set, $\rho \in FICU(Y)$, $\rho \supseteq \iota(Y)$. Put

$$Y_\rho = U(Y)/\rho \times Y$$

and define

$$\pi(Y, \rho) : U(Y) \to U(Y_\rho)$$

in the same way as the mapping π_ρ in the section 3 (of course, we replace the set $X = \{x_1, x_2, \ldots\}$ by an arbitrary countable set Y).

Throughout this section, let $\mathcal{U} \subseteq \mathbf{ES}$ be an e-variety and $\mathcal{V}, W \subseteq \mathbf{I}$ be varieties. We will prove syntactically that

$$(U \square \mathcal{V}) \square \mathcal{W} = \mathcal{U} \square (\mathcal{V} \square \mathcal{W})\,.$$

Note that $\mathcal{V} \square \mathcal{W}$ is a variety of inverse semigroups by 4.12 and so the right side of the equation mentioned above is meaningful.

Let
$\rho \in FIC\,U(X), \rho \supseteq \iota(X), \rho \leftrightarrow \mathcal{W}$,
$\sigma \in FIC\,U(X_\rho), \sigma \supseteq \iota(X_\rho), \sigma \leftrightarrow \mathcal{V}$,

$\tau \in BIC\, U_r(X_{\sigma\Box\rho}), \tau \leftrightarrow \mathcal{U}$,
$\tau' \in BIC\, U_r((X_\rho)_\sigma), \tau' \leftrightarrow \mathcal{U}$.

In view of 4.10 we have to prove that

$$(\tau' \Box \sigma) \Box \rho = \tau \Box (\sigma \Box \rho).$$

Choose $u, v \in U_r(X)$. Then (by the definition of $\Box$ and by 4.13)

$$\begin{aligned} u\,(\tau \Box (\sigma \Box \rho))\, v \iff & u\,(\sigma \Box \rho)\, v \\ & \pi(X, \sigma \Box \rho)(u)\ \tau\ \pi(X, \sigma \Box \rho)(v) \\ \iff & u\, \rho\, v \\ & \pi(X, \rho)(u)\ \sigma\ \pi(X, \rho)(v) \\ & \pi(X, \sigma \Box \rho)(u)\ \tau\ \pi(X, \sigma \Box \rho)(v) \end{aligned}$$

and

$$\begin{aligned} u\,((\tau' \Box \sigma) \Box \rho)\, v \iff & u\, \rho\, v \\ & \pi(X, \rho)(u)\ (\tau' \Box \sigma)\ \pi(X, \rho)(v) \\ \iff & u\, \rho\, v \\ & \pi(X, \rho)(u)\ \sigma\ \pi(X, \rho)(v) \\ & \pi(X_\rho, \sigma)(\pi(X, \rho)(u))\ \tau'\ \pi(X_\rho, \sigma)(\pi(X, \rho)(v)). \end{aligned}$$

Clearly, it suffices to prove that

$$\pi(X, \sigma \Box \rho)(u)\ \tau\ \pi(X, \sigma \Box \rho)(v)$$

is equivalent to

$$\pi(X_\rho, \sigma)(\pi(X, \rho)(u))\ \tau'\ \pi(X_\rho, \sigma)(\pi(X, \rho)(v))\,.$$

Define $\alpha : X_{\sigma\Box\rho} \to (X_\rho)_\sigma$ by

$$(w(\sigma \Box \rho), x) \mapsto (\pi(X, \rho)(w)\sigma, (w\rho, x))$$

$(w \in U(X), x \in X)$.

5.1 Lemma. *α is a correctly defined injective mapping.*

Proof.

1. Let $u, v \in U(X), u\,(\sigma \Box \rho)\, v$. We want to show: $u\, \rho\, v, \pi(X, \rho)(u)\ \sigma\ \pi(X, \rho)(v)$. It follows immediately from 4.13.
2. Let $u, v \in U(X), x, y \in X, (\pi(X, \rho)(u)\sigma, (u\rho, x)) = (\pi(X, \rho)(v)\sigma, (v\rho, y))$. We want to show that $(u(\sigma \Box \rho), x) = (v(\sigma \Box \rho), y)$.
   $(u\rho, x) = (v\rho, y)$ implies $u\, \rho\, v, x = y$.
   $\pi(X, \rho)(u)\sigma = \pi(X, \rho)(v)\sigma$ together with $u\, \rho\, v$ implies $u\,(\sigma \Box \rho)\, v$ (see 4.13)$\Box$

Now, we extend the mapping

$$\alpha : X_{\sigma\Box\rho} \to (X_\rho)_\sigma$$

to the unary homomorphism

$$\alpha : U(X_{\sigma\Box\rho}) \to U((X_\rho)_\sigma)\,.$$

5.2 Lemma. *$\alpha(u*\pi(X,\sigma\Box\rho)(v)) = (uvv'u'*\pi(X,\rho)(u))*\pi(X_\rho,\sigma)(u*\pi(X,\rho)(v))$ for any $u, v \in U(X)$.*

Proof. By induction with respect to v:

1. Let $x \in X$. Then $\alpha(u*\pi(X,\sigma\Box\rho)(x)) = \alpha(u*(xx'(\sigma\Box\rho),x))$
$= \alpha((uxx'(\sigma\Box\rho),x)) = (\pi(X,\rho)(uxx')\sigma,(uxx'\rho,x))$
$= ((uxx'xx'u'*\pi(X,\rho)(u))(u*\pi(X,\rho)(xx'))\sigma,(uxx'\rho,x))$
$= (uxx'u'*\pi(X,\rho)(u))*((u*\pi(X,\rho)(xx'))\sigma,(uxx'\rho,x))$.
Now, $u*\pi(X,\rho)(xx') =$
$= u*(xx'xx'*(xx'\rho,x))(x*(x'\rho,x)')$
$= (uxx'\rho,x)(uxx'\rho,x)'$.
So, $\alpha(u*\pi(X,\sigma\Box\rho)(x)) =$
$= (uxx'u'*\pi(X,\rho)(u))*((uxx'\rho,x)(uxx'\rho,x)'\sigma,(uxx'\rho,x))$
$= (uxx'u'*\pi(X,\rho)(u))*\pi(X_\rho,\sigma)(u*\pi(X,\rho)(x))$.

2. Let $v, w \in U(X)$. We suppose that
$\alpha(u*\pi(X,\sigma\Box\rho)(v)) = (uvv'u'*\pi(X,\rho)(u))*\pi(X_\rho,\sigma)(u*\pi(X,\rho)(v))$,
$\alpha(u*\pi(X,\sigma\Box\rho)(w)) = (uww'u'*\pi(X,\rho)(u))*\pi(X_\rho,\sigma)(u*\pi(X,\rho)(w))$
for all $u \in U(X)$.
Now, choose an arbitrary $u \in U(X)$. Then $\alpha(u*\pi(X,\sigma\Box\rho)(vw)) =$
$= \alpha((uvww'v'*\pi(X,\sigma\Box\rho)(v))(uv*\pi(X,\sigma\Box\rho)(w)))$
$= ((uvww'v'vv'vww'v'u'*\pi(X,\rho)(uvww'v'))*$
$*\pi(X_\rho,\sigma)(uvww'v'*\pi(X,\rho)(v)))$
$((uvww'v'u'*\pi(X,\rho)(uv))*\pi(X_\rho,\sigma)(uv*\pi(X,\rho)(w)))$.
Now, $uvww'v'\underline{vv'v}ww'v'u'*\pi(X,\rho)(uvww'v') =$
$= uvww'v'(uvww'v')'*\pi(X,\rho)(uvww'v')$
$= \pi(X,\rho)(uvww'v')$
$= (uv\underline{ww'v'v}ww'v'u'*\pi(X,\rho)(u))(u*\pi(X,\rho)(vww'v'))$
$= (u\underline{vv'v}w\underline{w'ww'}v'u'*\pi(X,\rho)(u))(u*\pi(X,\rho)(vww'v'))$
$= (uvww'v'u'*\pi(X,\rho)(u))(u*\pi(X,\rho)(vww'v'))$
and $uvww'v'u'*\pi(X,\rho)(uv) =$
$= (uvww'v'\underline{u'uvv'}u'*\pi(X,\rho)(u))(u\underline{vww'v'u'u}*\pi(X,\rho)(v))$
$= (uvww'\underline{v'vv'u'uu'}*\pi(X,\rho)(u))(\underline{uu'u}vww'v'*\pi(X,\rho)(v))$
$= (uvww'v'u'*\pi(X,\rho)(u))(uvww'v'*\pi(X,\rho)(v))$.
So, $\alpha(u*\pi(X,\sigma\Box\rho)(vw)) = (uvww'v'u'*\pi(X,\rho)(u))*$
$*((u*\pi(X,\rho)(vww'v'))*\pi(X_\rho,\sigma)(uvww'v'*\pi(X,\rho)(v)))$
$((uvww'v'*\pi(X,\rho)(v))*\pi(X_\rho,\sigma)(uv*\pi(X,\rho)(w)))$.
We will show that $(uvww'v'*\pi(X,\rho)(v))(uv*\pi(X,\rho)(w))$
$(uv*\pi(X,\rho)(w))'(uvww'v'*\pi(X,\rho)(v))'\ \iota(X_\rho)$
$\iota(X_\rho)\ u*\pi(X,\rho)(vww'v')$: $(uvww'v'*\pi(X,\rho)(v))(uv*\pi(X,\rho)(w))$
$(uv*\pi(X,\rho)(w))'(uvww'v'*\pi(X,\rho)(v))'\ \iota(X_\rho)$
$\iota(X_\rho)\ (uvww'v'*\pi(X,\rho)(v))(uv*\pi(X,\rho)(w))$
$((uvww'v'*\pi(X,\rho)(v))(uv*\pi(X,\rho)(w)))'$
$= u*\pi(X,\rho)(vw)(\pi(X,\rho)(vw))'$
$= u*(vw(vw)'*\pi(X,\rho)(vw))(vw*((vw)'*(\pi(X,\rho)(vw))'))$
$= u*(vw(vw)'vw(vw)'*\pi(X,\rho)(vw))(vw*\pi(X,\rho)((vw)'))$

$= u * \pi(X,\rho)(vw(vw)')$
$\iota(X_\rho)\ u * \pi(X,\rho)(vww'v')$
(note that $\pi(X,\rho)(vw(vw)')\ \iota(X_\rho)\ \pi(X,\rho)(vww'v')$ by 3.7 and the mapping $a \mapsto u * a$ is an endomorphism on $U(X_\rho)$).
Since $\sigma \supseteq \iota(X_\rho)$, we get $\alpha(u * \pi(X, \sigma \,\square\, \rho)(vw)) =$
$= (uvww'v'u' * \pi(X,\rho)(u)) * \pi(X_\rho,\sigma)((uvww'v' * \pi(X,\rho)(v))(uv * \pi(X,\rho)(w)))$
$= (uvww'v'u' * \pi(X,\rho)(u)) * \pi(X_\rho,\sigma)(u * \pi(X,\rho)(vw))$.

3. Let $v \in U(X)$. We suppose that $\alpha(u * \pi(X,\sigma \,\square\, \rho)(v)) = (uvv'u' * \pi(X,\rho)(u))*$ $*\pi(X_\rho,\sigma)(u * \pi(X,\rho)(v))$ for all $u \in U(X)$.
Now, choose an arbitrary $u \in U(X)$. Then
$\alpha(u * \pi(X,\sigma \,\square\, \rho)(v')) = \alpha((uv' * \pi(X,\sigma \,\square\, \rho)(v))')$
$= ((uv'\underline{vv'v}u' * \pi(X,\rho)(uv')) * \pi(X_\rho,\sigma)(uv' * \pi(X,\rho)(v)))'$
$= (uv'vu' * \pi(X,\rho)(uv')) * (\pi(X_\rho,\sigma)(uv' * \pi(X,\rho)(v)))'$
$= (u\underline{v'vu'u}v'vu' * \pi(X,\rho)(u))(u\underline{v'vu'u}v' * (\pi(X,\rho)(v))')*$
$\quad *(\pi(X_\rho,\sigma)(uv' * \pi(X,\rho)(v)))'$
$= (\underline{uu'u}v'\underline{vv'v}u' * \pi(X,\rho)(u))(\underline{uu'uv'vv'} * \pi(X,\rho)(v))'*$
$\quad *(\pi(X_\rho,\sigma)(uv' * \pi(X,\rho)(v)))'$
$= (uv'vu' * \pi(X,\rho)(u))(uv' * \pi(X,\rho)(v))'*$
$\quad *(\pi(X_\rho,\sigma)(uv' * \pi(X,\rho)(v)))'$
$= (uv'vu' * \pi(X,\rho)(u)) * ((uv' * \pi(X,\rho)(v))' * (\pi(X_\rho,\sigma)(uv' * \pi(X,\rho)(v)))')$
$= (uv'vu' * \pi(X,\rho)(u)) * \pi(X_\rho,\sigma)((uv' * \pi(X,\rho)(v))')$
$= (uv'vu' * \pi(X,\rho)(u)) * \pi(X_\rho,\sigma)(u * \pi(X,\rho)(v'))$. □

5.3 Corollary. *$\alpha(\pi(X,\sigma \,\square\, \rho)(w)) = \pi(X_\rho,\sigma)(\pi(X,\rho)(w))$ for any $w \in U(X)$.*

Proof. Using 5.2 we obtain
$\alpha(\pi(X,\sigma \,\square\, \rho)(w)) = \alpha(ww' * \pi(X,\sigma \,\square\, \rho)(w))$
$\quad = (ww'ww'ww' * \pi(X,\rho)(ww')) * \pi(X_\rho,\sigma)(ww' * \pi(X,\rho)(w))$
$\quad = (ww'ww' * \pi(X,\rho)(w))(ww' * (\pi(X,\rho)(w))') * \pi(X_\rho,\sigma)(\pi(X,\rho)(w))$
$\quad = (ww' * \pi(X,\rho)(w))(ww' * \pi(X,\rho)(w))' * \pi(X_\rho,\sigma)(\pi(X,\rho)(w))$
$\quad = \pi(X,\rho)(w)(\pi(X,\rho)(w))' * \pi(X_\rho,\sigma)(\pi(X,\rho)(w))$
$\quad = \pi(X_\rho,\sigma)(\pi(X,\rho)(w))$. □

5.4 Theorem. *Let $\mathcal{U} \subseteq \mathbf{ES}$ be an e-variety and $\mathcal{V},\mathcal{W} \subseteq \mathbf{I}$ be varieties. Then*

$$\mathcal{U} \,\square\, (\mathcal{V} \,\square\, \mathcal{W}) = (\mathcal{U} \,\square\, \mathcal{V}) \,\square\, \mathcal{W}\,.$$

Proof. It follows from 5.1 that $\pi(X,\sigma \,\square\, \rho)(u)\ \tau\ \pi(X,\sigma \,\square\, \rho)(v)$ is equivalent to $\alpha(\pi(X,\sigma \,\square\, \rho)(u))\ \tau'\ \alpha(\pi(X,\sigma \,\square\, \rho)(v))$ $(u,v \in U_r(X))$. Now, we use 5.3 and the proof is complete. □

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