

THE MULTIPLICATION THEOREM AND BASES IN FINITE AND AFFINE QUANTUM CLUSTER ALGEBRAS

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ABSTRACT. We prove a multiplication theorem for quantum cluster algebras of acyclic quivers. The theorem generalizes the multiplication formula for quantum cluster variables in [19]. Moreover some \mathbb{ZP} -bases in quantum cluster algebras of finite and affine types are constructed. Under the specialization q and coefficients to 1, these bases are the integral bases of cluster algebra of finite and affine types (see [4] and [11]).

1. INTRODUCTION

Quantum cluster algebras were introduced by A. Berenstein and A. Zelevinsky [2] as a noncommutative analogue of cluster algebras [13][14] to study canonical bases. A quantum cluster algebra is generated by a set of generators called the *quantum cluster variables* inside an ambient skew-field \mathcal{F} . Under the specialization $q = 1$, the quantum cluster algebras are exactly cluster algebras which were introduced by S. Fomin and A. Zelevinsky [13][14].

Cluster algebras have a close link to quiver representations via cluster categories invented in [1]. The link is explicitly characterized by the Caldero-Chapoton map ([3]) and the Caldero-Keller multiplication theorems ([4],[5]). The Caldero-Chapoton map associates the objects in the cluster categories to some Laurent polynomials, in particular, sends rigid objects to cluster variables. The Caldero-Keller multiplication theorems show the multiplication rules between images of objects under the Caldero-Chapoton map. The theorem is remarkable. On the one hand, it is similar to the multiplication in a dual Hall algebra and unifies homological and geometric properties of cluster categories and combinatorial properties of cluster algebras. On the other hand, since cluster algebras were introduced to study canonical bases, it is important to construct integral bases of cluster algebras. The Caldero-Keller multiplication theorems are essentially important to construct integral bases of cluster algebras. Following this link, some good bases have been constructed for finite and affine cluster algebras ([4], [7], [10] and [11]).

Naturally, one can study the quantum analogue of the link. Recently, Rupel ([22]) defined a quantum analog of the Caldero-Chapoton map (called the quantum Caldero-Chapoton map) and conjectured that cluster variables could be expressed as images of indecomposable rigid objects under the quantum Caldero-Chapoton formula. A key ingredient of the conjecture is to confirm the multiplication rules between quantum cluster variables given by [2]. Most recently, the conjecture has been proved by Qin ([19]) for

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acyclic equally valued quivers. There Qin constructed a quantum cluster multiplication formula and then confirmed the multiplication rules between quantum cluster variables.

The present paper is contributed to prove a multiplication theorem (a combination of Theorem 3.5 and 3.8) for acyclic quantum cluster algebras in Section 3. The theorem generalizes the quantum cluster multiplication formula in [19] and can be viewed as a quantum analogue of the 1-dimensional Caldero-Keller multiplication theorem in [5]. Compared to the role which the Caldero-Keller multiplication theorems play for cluster algebras, our multiplication theorem is worthy of highlighting and also reflects the information and the difficulty to prove the more general quantum analog of the Caldero-Keller multiplication theorems. The main idea in the proof of the multiplication theorem is taken from [17]. Moreover, we construct some good $\mathbb{Z}\mathbb{P}$ -bases in quantum cluster algebras of finite and affine types. By specializing q and coefficients to 1, these bases induce the good bases for cluster algebras of finite[4] and affine types[11], respectively.

2. THE QUANTUM CALDERO-CHAPOTON MAP

2.1. Quantum cluster algebras. The main reference for quantum cluster algebras is [2]. Here, we also recommend [19, Section2] as a nice reference. Let L be a lattice of rank m and $\Lambda : L \times L \rightarrow \mathbb{Z}$ a skew-symmetric bilinear form. Let q be a formal variable and consider the ring of integer Laurent polynomials $\mathbb{Z}[q^{\pm 1/2}]$. Define the *based quantum torus* associated to the pair (L, Λ) to be the $\mathbb{Z}[q^{\pm 1/2}]$ -algebra \mathcal{T} with a distinguished $\mathbb{Z}[q^{\pm 1/2}]$ -basis $\{X^e : e \in L\}$ and the multiplication given by

$$X^e X^f = q^{\Lambda(e,f)/2} X^{e+f}.$$

It is easy to see that \mathcal{T} is associative and the basis elements satisfy the following relations:

$$X^e X^f = q^{\Lambda(e,f)} X^f X^e, \quad X^0 = 1, \quad (X^e)^{-1} = X^{-e}.$$

It is known that \mathcal{T} is an Ore domain, i.e., is contained in its skew-field of fractions \mathcal{F} . The quantum cluster algebra will be defined as a $\mathbb{Z}[q^{\pm 1/2}]$ -subalgebra of \mathcal{F} .

A *toric frame* in \mathcal{F} is a map $M : \mathbb{Z}^m \rightarrow \mathcal{F} \setminus \{0\}$ of the form

$$M(\mathbf{c}) = \varphi(X^{\eta(\mathbf{c})})$$

where φ is an automorphism of \mathcal{F} and $\eta : \mathbb{Z}^m \rightarrow L$ is an isomorphism of lattices. By the definition, the elements $M(\mathbf{c})$ form a $\mathbb{Z}[q^{\pm 1/2}]$ -basis of the based quantum torus $\mathcal{T}_M := \varphi(\mathcal{T})$ and satisfy the following relations:

$$M(\mathbf{c})M(\mathbf{d}) = q^{\Lambda_M(\mathbf{c},\mathbf{d})/2} M(\mathbf{c} + \mathbf{d}), \quad M(\mathbf{c})M(\mathbf{d}) = q^{\Lambda_M(\mathbf{c},\mathbf{d})} M(\mathbf{d})M(\mathbf{c}),$$

$$M(\mathbf{0}) = 1, \quad M(\mathbf{c})^{-1} = M(-\mathbf{c}),$$

where Λ_M is the skew-symmetric bilinear form on \mathbb{Z}^m obtained from the lattice isomorphism η . Let Λ_M also denote the skew-symmetric $m \times m$ matrix defined by $\lambda_{ij} = \Lambda_M(e_i, e_j)$ where $\{e_1, \dots, e_m\}$ is the standard basis of \mathbb{Z}^m . Given a toric frame M , let $X_i = M(e_i)$. Then we have

$$\mathcal{T}_M = \mathbb{Z}[q^{\pm 1/2}]\langle X_1^{\pm 1}, \dots, X_m^{\pm 1} : X_i X_j = q^{\lambda_{ij}} X_j X_i \rangle.$$

An easy computation shows that

$$M(\mathbf{c}) = q^{\frac{1}{2} \sum_{i < j} c_i c_j \lambda_{ji}} X_1^{c_1} X_2^{c_2} \dots X_m^{c_m} =: X^{\mathbf{c}} \quad (\mathbf{c} \in \mathbb{Z}^m).$$

Let Λ be an $m \times m$ skew-symmetric matrix and let \tilde{B} be an $m \times n$ matrix for some positive integer $n \leq m$. We call the pair (Λ, \tilde{B}) *compatible* if $\tilde{B}^T \Lambda = (D|0)$ is an $n \times m$ matrix with $D = \text{diag}(d_1, \dots, d_n)$ where $d_i \in \mathbb{N}$ for $1 \leq i \leq n$. The pair (M, \tilde{B}) is called a *quantum seed* if the pair (Λ_M, \tilde{B}) is compatible. Define the $m \times m$ matrix $E = (e_{ij})$ by

$$e_{ij} = \begin{cases} \delta_{ij} & \text{if } j \neq k; \\ -1 & \text{if } i = j = k; \\ \max(0, -b_{ik}) & \text{if } i \neq j = k. \end{cases}$$

For $n, k \in \mathbb{Z}$, $k \geq 0$, denote $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n - q^{-n}) \cdots (q^{n-k+1} - q^{-n+k-1})}{(q^k - q^{-k}) \cdots (q - q^{-1})}$. Let $\mathbf{c} = (c_1, \dots, c_m) \in \mathbb{Z}^m$ with $c_k \geq 0$. Define the toric frame $M' : \mathbb{Z}^m \rightarrow \mathcal{F} \setminus \{0\}$ as follows:

$$(2.1) \quad M'(\mathbf{c}) = \sum_{p=0}^{c_k} \begin{bmatrix} c_k \\ p \end{bmatrix}_{q^{d_k/2}} M(E\mathbf{c} + p\mathbf{b}^k), \quad M'(-\mathbf{c}) = M'(\mathbf{c})^{-1}.$$

where the vector $\mathbf{b}^k \in \mathbb{Z}^m$ is the k -th column of \tilde{B} . Then the quantum seed (M', \tilde{B}') is defined to be the mutation of (M, \tilde{B}) in direction k . In general, two quantum seeds (M, \tilde{B}) and (M', \tilde{B}') are mutation-equivalent if they can be obtained from each other by a sequence of mutations, denoted by $(M, \tilde{B}) \sim (M', \tilde{B}')$. Let $\mathcal{C} = \{M'(e_i) \mid (M, \tilde{B}) \sim (M', \tilde{B}'), i = 1, \dots, n\}$. The elements of \mathcal{C} are called *quantum cluster variables*. Let $\mathcal{P} = \{M(e_i) : i = n+1, \dots, m\}$ and the elements of \mathcal{P} are called *coefficients*. Given $(M', \tilde{B}') \sim (M, \tilde{B})$ and $\mathbf{c} = (c_i) \in \mathbb{Z}^m$, a element $M'(\mathbf{c})$ is called a *quantum cluster monomial* if $c_i \geq 0$ for $i = 1, \dots, n$ and 0 for $i = n+1, \dots, m$. We denote by \mathbb{P} the multiplicative group by $q^{\frac{1}{2}}$ and \mathcal{P} . Write $\mathbb{Z}\mathbb{P}$ as the ring of Laurent polynomials in the elements of \mathcal{P} with coefficients in $\mathbb{Z}[q^{\pm 1/2}]$. The *quantum cluster algebra* $\mathcal{A}_q(\Lambda_M, \tilde{B})$ is the $\mathbb{Z}\mathbb{P}$ -subalgebra of \mathcal{F} generated by \mathcal{C} . We associate (M, \tilde{B}) a \mathbb{Z} -linear *bar-involution* on \mathcal{T}_M defined by

$$\overline{q^{r/2} M(\mathbf{c})} = q^{-r/2} M(\mathbf{c}), \quad (r \in \mathbb{Z}, \mathbf{c} \in \mathbb{Z}^n).$$

It is easy to show that $\overline{XY} = \bar{Y} \bar{X}$ for all $X, Y \in \mathcal{A}_q(\Lambda_M, \tilde{B})$ and that each element of $\mathcal{C} \cup \mathcal{P}$ is *bar-invariant*.

Now assume that there exists a finite field k satisfying $|k| = q$. In the same way, we can define based quantum torus $\mathcal{T}_{|k|}$ and *specialized quantum cluster algebras* $\mathcal{A}_{|k|}(\Lambda_M, \tilde{B})$ by substituting $\mathbb{Z}[|k|^{\pm \frac{1}{2}}]$ for $\mathbb{Z}[q^{\pm \frac{1}{2}}]$ in the above definition. By [2, Corollary 5.2], $\mathcal{A}_q(\Lambda_M, \tilde{B})$ and $\mathcal{A}_{|k|}(\Lambda_M, \tilde{B})$ are subalgebras of \mathcal{T} and $\mathcal{T}_{|k|}$, respectively. There is a specialization map $ev : \mathcal{T} \rightarrow \mathcal{T}_{|k|}$ by mapping $q^{\frac{1}{2}}$ to $|k|^{\frac{1}{2}}$, which induces a bijection between quantum monomials of $\mathcal{A}_q(\Lambda_M, \tilde{B})$ and $\mathcal{A}_{|k|}(\Lambda_M, \tilde{B})$ ([19, Section 2.2]).

2.2. The quantum Caldero-Chapoton map. Let k be a finite field with cardinality $|k| = q$ and $m \geq n$ be two positive integers and \tilde{Q} an acyclic quiver with vertex set $\{1, \dots, m\}$ [19]. Denote the subset $\{n+1, \dots, m\}$ by C . The elements in C are called the *frozen vertices*, and \tilde{Q} is called an *ice quiver*. The full subquiver Q on the vertices $1, \dots, n$ is called the *principal part* of \tilde{Q} .

Let \tilde{B} be the $m \times n$ matrix associated to the ice quiver \tilde{Q} , i.e., its entry in position (i, j) is

$$b_{ij} = |\{\text{arrows } i \longrightarrow j\}| - |\{\text{arrows } j \longrightarrow i\}|$$

for $1 \leq i \leq m, 1 \leq j \leq n$. And let \tilde{I} be the left $m \times n$ matrix of the identity matrix of size $m \times m$. Further assume that there exists some antisymmetric $m \times m$ integer matrix Λ such that

$$(2.2) \quad \Lambda(-\tilde{B}) = \tilde{I} := \begin{bmatrix} I_n \\ 0 \end{bmatrix},$$

where I_n is the identity matrix of size $n \times n$. Thus, the matrix \tilde{B} is of full rank.

Let \tilde{R} and \tilde{R}^{tr} be the $m \times n$ matrix with its entry in position (i, j) is

$$\tilde{r}_{ij} = \dim_k \text{Ext}_{k\tilde{Q}}^1(S_i, S_j)$$

and

$$\tilde{r}_{ij}^* = \dim_k \text{Ext}_{k\tilde{Q}}^1(S_j, S_i)$$

for $1 \leq i \leq m, 1 \leq j \leq n$, respectively. Note that

$$\dim_k \text{Ext}_{k\tilde{Q}}^1(S_i, S_j) = |\{\text{arrows } j \longrightarrow i\}|.$$

Denote the principal parts of the matrices \tilde{B} and \tilde{R} by B and R respectively. Note that $\tilde{B} = \tilde{R}^{tr} - \tilde{R}$ and $B = R^{tr} - R$ where R^{tr} represents the transposition of the matrix R . In general, the matrix B is not of full rank so that there exists no matrix Λ compatible with B . Hence, one need add some frozen vertices to Q and then obtain an acyclic quiver \tilde{Q} with a compatible pair (\tilde{B}, Λ) .

Let $\mathcal{C}_{\tilde{Q}}$ be the cluster category of $k\tilde{Q}$, i.e., the orbit category of the derived category $\mathcal{D}^b(\tilde{Q})$ by the functor $F = \tau \circ [-1]$ where τ is the Auslander-Reiten translation and $[1]$ is the translation functor. We note that the indecomposable objects of the cluster category $\mathcal{C}_{\tilde{Q}}$ are either the indecomposable $k\tilde{Q}$ -modules or $P_i[1]$ for indecomposable projective modules $P_i (1 \leq i \leq m)$. Each object M in $\mathcal{C}_{\tilde{Q}}$ can be uniquely decomposed in the following way:

$$M \cong M_0 \oplus P_M[1]$$

where M_0 is a $k\tilde{Q}$ -module and P_M is a projective $k\tilde{Q}$ -module. Let $P_M = \bigoplus_{1 \leq i \leq m} m_i P_i$. We extend the definition of the dimension vector $\underline{\dim}$ on modules in $\text{mod } k\tilde{Q}$ to objects in $\mathcal{C}_{\tilde{Q}}$ by setting

$$\underline{\dim} M = \underline{\dim} M_0 - (m_i)_{1 \leq i \leq m}.$$

The Euler form on $k\tilde{Q}$ -modules M and N is given by

$$\langle M, N \rangle = \dim_k \text{Hom}_{k\tilde{Q}}(M, N) - \dim_k \text{Ext}_{k\tilde{Q}}^1(M, N).$$

Note that the Euler form only depends on the dimension vectors of M and N . As in [17], we define

$$[M, N] = \dim_k \text{Hom}_{k\tilde{Q}}(M, N) \text{ and } [M, N]^1 = \dim_k \text{Ext}_{k\tilde{Q}}^1(M, N).$$

The quantum Caldero-Chapoton map of an acyclic quiver \tilde{Q} has been studied in [22] and [19]. Here, we reformulate their definitions to the following map

$$X_{\tau}^{\tilde{Q}} : \text{obj}\mathcal{C}_{\tilde{Q}} \longrightarrow \mathcal{T}$$

defined by the following rule: If M is a kQ -module and P is a projective $k\tilde{Q}$ -module, then

$$X_{M \oplus P[1]}^{\tilde{Q}} = \sum_{\underline{e}} |\text{Gr}_{\underline{e}} M| q^{-\frac{1}{2}\langle \underline{e}, \underline{m} - \underline{e} \rangle} X^{\tilde{B}\underline{e} - (\tilde{I} - \tilde{R})\underline{m} + \underline{\dim} P / \text{rad} P},$$

where $\underline{\dim} M = \underline{m}$ and $\text{Gr}_{\underline{e}} M$ denotes the set of all submodules V of M with $\underline{\dim} V = \underline{e}$. Usually, we omit the upper index \tilde{Q} in the notation $X_{\tau}^{\tilde{Q}}$ (except Section 4 and Section 5) if there is no confusion. We note that

$$X_{P[1]} = X_{\tau P} = X^{\underline{\dim} P / \text{rad} P} = X^{\underline{\dim} \text{soc} I} = X_{I[-1]} = X_{\tau^{-1} I}.$$

for any projective $k\tilde{Q}$ -module P and injective $k\tilde{Q}$ -module I with $\text{soc} I = P / \text{rad} P$. Hereinafter, we denote by the corresponding underlined small letter \underline{x} the dimension vector of a kQ -module X and view \underline{x} as a column vector in \mathbb{Z}^n .

3. MULTIPLICATION THEOREMS FOR ACYCLIC QUANTUM CLUSTER ALGEBRAS

Throughout this section, assume that \tilde{Q} is an acyclic quiver and Q is its full subquiver. In this section, we will prove a multiplication theorem for any acyclic quantum cluster algebra. First, we improve Lemma 5.2.1 and Corollary 5.2.2 in [19], i.e., here we handle the dimension vector of any kQ -module while in [19] the author only deals with dimension vectors of rigid modules.

Lemma 3.1. *For any dimension vector $\underline{m}, \underline{e}, \underline{f} \in \mathbb{Z}_{\geq 0}^n$, we have*

- (1) $\Lambda((\tilde{I} - \tilde{R})\underline{m}, \tilde{B}\underline{e}) = -\langle \underline{e}, \underline{m} \rangle;$
- (2) $\Lambda(\tilde{B}\underline{e}, \tilde{B}\underline{f}) = \langle \underline{e}, \underline{f} \rangle - \langle \underline{f}, \underline{e} \rangle.$

Proof. By definition, we have

$$\begin{aligned} & \Lambda((\tilde{I} - \tilde{R})\underline{m}, \tilde{B}\underline{e}) \\ &= \underline{m}^{tr} (\tilde{I} - \tilde{R})^{tr} \Lambda \tilde{B}\underline{e} = -\underline{m}^{tr} (\tilde{I} - \tilde{R})^{tr} \begin{bmatrix} I_n \\ 0 \end{bmatrix} \underline{e} \\ &= -\underline{m}^{tr} (I_n - R)^{tr} \underline{e} = -\underline{e}^{tr} (I_n - R)\underline{m} \\ &= -\langle \underline{e}, \underline{m} \rangle. \end{aligned}$$

As for (2), the left side of the desired equation is equal to

$$\underline{e}^{tr} \tilde{B}^{tr} \Lambda \tilde{B}\underline{f} = -\underline{e}^{tr} \tilde{B}^{tr} \begin{bmatrix} I_n \\ 0 \end{bmatrix} \underline{f} = -\underline{e}^{tr} B^{tr} \underline{f}.$$

The right side is

$$\begin{aligned} & \langle \underline{e}, \underline{f} \rangle - \langle \underline{f}, \underline{e} \rangle \\ &= \underline{e}^{tr} (I_n - R)\underline{f} - \underline{f}^{tr} (I_n - R)\underline{e} \\ &= \underline{e}^{tr} (I_n - R)\underline{f} - \underline{e}^{tr} (I_n - R)^{tr} \underline{f} \\ &= \underline{e}^{tr} (R^{tr} - R)\underline{f} = -\underline{e}^{tr} (R - R^{tr})\underline{f} = -\underline{e}^{tr} B^{tr} \underline{f}. \end{aligned}$$

Thus we prove the lemma. \square

Corollary 3.2. *For any dimension vector $\underline{m}, \underline{l}, \underline{e}, \underline{f} \in \mathbb{Z}_{\geq 0}^n$, we have*

$$\begin{aligned} & \Lambda(\tilde{B}\underline{e} - (\tilde{I} - \tilde{R})\underline{m}, \tilde{B}\underline{f} - (\tilde{I} - \tilde{R})\underline{l}) \\ &= \Lambda((\tilde{I} - \tilde{R})\underline{m}, (\tilde{I} - \tilde{R})\underline{l}) + \langle \underline{e}, \underline{f} \rangle - \langle \underline{f}, \underline{e} \rangle - \langle \underline{e}, \underline{l} \rangle + \langle \underline{f}, \underline{m} \rangle. \end{aligned}$$

For any kQ -modules M, N, E , denote by ε_{MN}^E the cardinality of the set $\text{Ext}_{kQ}^1(M, N)_E$ which is the subset of $\text{Ext}_{kQ}^1(M, N)$ consisting of those equivalence classes of short exact sequences with middle term isomorphic to M ([17, Section 4]). For kQ -modules M, A and B , we denote by F_{AB}^M the number of submodules U of M such that U is isomorphic to B and M/U is isomorphic to A . Then by definition, we have

$$|\text{Gr}_{\underline{e}}(M)| = \sum_{A, B; \dim B = \underline{e}} F_{AB}^M.$$

Different from the case in cluster categories, for kQ -modules, it does not generally hold that $X_N X_M = X_{N \oplus M}$. We have the following explicit characterization, which is a generalization of [19, Proposition 5.3.2].

Theorem 3.3. *Let M and N be kQ -modules. Then*

$$q^{[M, N]^1} X_N X_M = q^{-\frac{1}{2}\Lambda((\tilde{I} - \tilde{R})\underline{m}, (\tilde{I} - \tilde{R})\underline{n})} \sum_E \varepsilon_{MN}^E X_E.$$

Proof. We apply Green's formula in [15]

$$\sum_E \varepsilon_{MN}^E F_{XY}^E = \sum_{A, B, C, D} q^{[M, N] - [A, C] - [B, D] - \langle A, D \rangle} F_{AB}^M F_{CD}^N \varepsilon_{AC}^X \varepsilon_{BD}^Y.$$

Then

$$\begin{aligned} & \sum_E \varepsilon_{MN}^E X_E \\ &= \sum_{E, X, Y} \varepsilon_{MN}^E q^{-\frac{1}{2}\langle Y, X \rangle} F_{XY}^E X^{\tilde{B}\underline{y} - (\tilde{I} - \tilde{R})\underline{e}} \\ &= \sum_{A, B, C, D, X, Y} q^{[M, N] - [A, C] - [B, D] - \langle A, D \rangle - \frac{1}{2}\langle B + D, A + C \rangle} F_{AB}^M F_{CD}^N \varepsilon_{AC}^X \varepsilon_{BD}^Y X^{\tilde{B}\underline{y} - (\tilde{I} - \tilde{R})\underline{e}}. \end{aligned}$$

Since

$$\begin{aligned} & X^{\tilde{B}\underline{y} - (\tilde{I} - \tilde{R})\underline{e}} \\ &= X^{\tilde{B}(\underline{b} + \underline{d}) - (\tilde{I} - \tilde{R})(\underline{m} + \underline{n})} \\ &= q^{-\frac{1}{2}\Lambda(\tilde{B}\underline{d} - (\tilde{I} - \tilde{R})\underline{n}, \tilde{B}\underline{b} - (\tilde{I} - \tilde{R})\underline{m})} X^{\tilde{B}\underline{d} - (\tilde{I} - \tilde{R})\underline{n}} X^{\tilde{B}\underline{b} - (\tilde{I} - \tilde{R})\underline{m}} \\ &= q^{-\frac{1}{2}\Lambda((\tilde{I} - \tilde{R})\underline{n}, (\tilde{I} - \tilde{R})\underline{m}) - \frac{1}{2}[\langle D, B \rangle - \langle B, D \rangle - \langle D, M \rangle + \langle B, N \rangle]} X^{\tilde{B}\underline{d} - (\tilde{I} - \tilde{R})\underline{n}} X^{\tilde{B}\underline{b} - (\tilde{I} - \tilde{R})\underline{m}} \\ &= q^{-\frac{1}{2}\Lambda((\tilde{I} - \tilde{R})\underline{n}, (\tilde{I} - \tilde{R})\underline{m})} q^{\frac{1}{2}\langle D, A \rangle - \frac{1}{2}\langle B, C \rangle} X^{\tilde{B}\underline{d} - (\tilde{I} - \tilde{R})\underline{n}} X^{\tilde{B}\underline{b} - (\tilde{I} - \tilde{R})\underline{m}}. \end{aligned}$$

Thus

$$\begin{aligned} & \sum_E \varepsilon_{MN}^E X_E \\ = & q^{\frac{1}{2}\Lambda((\tilde{I}-\tilde{R})\underline{m},(\tilde{I}-\tilde{R})\underline{n})} \sum_{A,B,C,D} q^{[M,N]-[A,C]-[B,D]-\langle A,D\rangle-\frac{1}{2}\langle B+D,A+C\rangle+[A,C]^1+[B,D]^1} \\ & q^{\frac{1}{2}\langle D,A\rangle-\frac{1}{2}\langle B,C\rangle} F_{AB}^M F_{CD}^N X^{\tilde{B}\underline{d}-(\tilde{I}-\tilde{R})\underline{n}} X^{\tilde{B}\underline{b}-(\tilde{I}-\tilde{R})\underline{m}}. \end{aligned}$$

Here we use the following fact

$$\sum_X \varepsilon_{AC}^X = q^{[A,C]^1}, \quad \sum_Y \varepsilon_{BD}^Y = q^{[B,D]^1}$$

Note that

$$[M, N] - [A, C] - [B, D] - \langle A, D \rangle + [A, C]^1 + [B, D]^1 = [M, N]^1 + \langle B, C \rangle.$$

Hence

$$\begin{aligned} & \sum_E \varepsilon_{MN}^E X_E \\ = & q^{\frac{1}{2}\Lambda((\tilde{I}-\tilde{R})\underline{m},(\tilde{I}-\tilde{R})\underline{n})} q^{[M,N]^1} \sum_{A,B,C,D} q^{\langle B,C\rangle-\frac{1}{2}\langle B,C\rangle-\frac{1}{2}\langle D,A\rangle+\frac{1}{2}\langle D,A\rangle-\frac{1}{2}\langle B,C\rangle} \\ & F_{CD}^N q^{-\frac{1}{2}\langle D,C\rangle} X^{\tilde{B}\underline{d}-(\tilde{I}-\tilde{R})\underline{n}} F_{AB}^M q^{-\frac{1}{2}\langle B,A\rangle} X^{\tilde{B}\underline{b}-(\tilde{I}-\tilde{R})\underline{m}} \\ = & q^{\frac{1}{2}\Lambda((\tilde{I}-\tilde{R})\underline{m},(\tilde{I}-\tilde{R})\underline{n})} q^{[M,N]^1} X_N X_M. \end{aligned}$$

This completes the proof. \square

Remark 3.4. Theorem 3.3 is similar to the multiplication formula in dual Hall algebras. It is reasonable to conjecture that it provides some PBW-type basis ([16]) in the corresponding quantum cluster algebra.

Let M, N be $k\tilde{Q}$ -modules and assume that

$$\dim_k \text{Ext}_{k\tilde{Q}}^1(M, N) = \dim_k \text{Hom}_{k\tilde{Q}}(N, \tau M) = 1.$$

Then there are two ‘‘canonical’’ exact sequences

$$\varepsilon: \quad 0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0$$

$$\varepsilon': \quad 0 \longrightarrow D_0 \longrightarrow N \longrightarrow \tau M \longrightarrow \tau A \oplus I \longrightarrow 0$$

which induces the k -bases of $\text{Ext}_{k\tilde{Q}}^1(M, N)$ and $\text{Hom}_{k\tilde{Q}}(N, \tau M)$, respectively. We fix them.

Set $M = M' \oplus P_0, A_0 = A \oplus P_0$ where P_0 is a projective $k\tilde{Q}$ -module, A and M' have no projective summands. The exact sequences also provide the two non-split triangles in $\mathcal{C}_{\tilde{Q}}$:

$$N \longrightarrow E \longrightarrow M \longrightarrow N[1] = \tau N$$

and

$$M \longrightarrow D_0 \oplus A_0 \oplus I[-1] \longrightarrow N \longrightarrow \tau M.$$

Now we state the first part of our multiplication theorem for acyclic quantum cluster algebras, which can be viewed as a quantum analogue of the one-dimensional Caldero-Keller multiplication theorem in [5]. The main idea in the proof comes from [17].

Theorem 3.5. *With the above notation, assume that $\text{Hom}_{k\tilde{Q}}(D_0, \tau A_0 \oplus I) = \text{Hom}_{k\tilde{Q}}(A_0, I) = 0$. Then the following formula holds*

$$X_N X_M = q^{\frac{1}{2}\Lambda((\tilde{I}-\tilde{R})\underline{n}, (\tilde{I}-\tilde{R})\underline{m})} X_E + q^{\frac{1}{2}\Lambda((\tilde{I}-\tilde{R})\underline{n}, (\tilde{I}-\tilde{R})\underline{m}) + \frac{1}{2}\langle M, N \rangle - \frac{1}{2}\langle A_0, D_0 \rangle} X_{D_0 \oplus A_0 \oplus I[-1]}.$$

Here, we note that

$$\frac{q^{[M, N]^1} - 1}{q - 1} X_N X_M = X_N X_M.$$

Proof. By definition, we have

$$\begin{aligned} & X_N X_M \\ &= \sum_{C, D} q^{-\frac{1}{2}\langle D, C \rangle} F_{CD}^N X^{\tilde{B}\underline{d} - (\tilde{I}-\tilde{R})\underline{n}} \sum_{A, B} q^{-\frac{1}{2}\langle B, A \rangle} F_{AB}^M X^{\tilde{B}\underline{b} - (\tilde{I}-\tilde{R})\underline{m}} \\ &= \sum_{A, B, C, D} F_{AB}^M F_{CD}^N q^{-\frac{1}{2}\langle D, C \rangle - \frac{1}{2}\langle B, A \rangle + \frac{1}{2}\Lambda(\tilde{B}\underline{d} - (\tilde{I}-\tilde{R})\underline{n}, \tilde{B}\underline{b} - (\tilde{I}-\tilde{R})\underline{m})} X^{\tilde{B}(\underline{b} + \underline{d}) - (\tilde{I}-\tilde{R})(\underline{m} + \underline{n})} \\ &= \sum_{A, B, C, D} F_{AB}^M F_{CD}^N q^{-\frac{1}{2}\langle B + D, A + C \rangle} q^{\langle B, C \rangle} q^{\frac{1}{2}\Lambda((\tilde{I}-\tilde{R})\underline{n}, (\tilde{I}-\tilde{R})\underline{m})} X^{\tilde{B}(\underline{b} + \underline{d}) - (\tilde{I}-\tilde{R})(\underline{m} + \underline{n})}. \end{aligned}$$

We set

$$s_1 := \sum_{E \neq M \oplus N} \frac{\varepsilon_{MN}^E}{q - 1} X_E = \sum_{X, Y, E \neq M \oplus N} \frac{\varepsilon_{MN}^E}{q - 1} F_{XY}^E q^{-\frac{1}{2}\langle Y, X \rangle} X^{\tilde{B}\underline{y} - (\tilde{I}-\tilde{R})\underline{e}}$$

As in the proof of Theorem 3.3, we have

$$\begin{aligned} & \sum_{X, Y, E} \varepsilon_{MN}^E X_E \\ &= \sum_{A, B, C, D, X, Y} q^{[M, N] - [A, C] - [B, D] - \langle A, D \rangle - \frac{1}{2}\langle B + D, A + C \rangle} F_{AB}^M F_{CD}^N \varepsilon_{AC}^X \varepsilon_{BD}^Y X^{\tilde{B}\underline{y} - (\tilde{I}-\tilde{R})\underline{e}} \\ &= \sum_{A, B, C, D} q^{[M, N]^1 + \langle B, C \rangle - \frac{1}{2}\langle B + D, A + C \rangle} F_{AB}^M F_{CD}^N X^{\tilde{B}\underline{y} - (\tilde{I}-\tilde{R})\underline{e}}. \end{aligned}$$

On the other hand

$$X_{M \oplus N} = \sum_{A, B, C, D} q^{[B, C] - \frac{1}{2}\langle B + D, A + C \rangle} F_{AB}^M F_{CD}^N X^{\tilde{B}(\underline{b} + \underline{d}) - (\tilde{I}-\tilde{R})(\underline{m} + \underline{n})}.$$

Thus

$$s_1 = \sum_{A, B, C, D} \frac{q^{[M, N]^1} - q^{[B, C]^1}}{q - 1} q^{\langle B, C \rangle - \frac{1}{2}\langle B + D, A + C \rangle} F_{AB}^M F_{CD}^N X^{\tilde{B}(\underline{b} + \underline{d}) - (\tilde{I}-\tilde{R})(\underline{m} + \underline{n})}.$$

Thirdly we compute the term

$$s_2 := \sum_{A, D_0, I, D_0 \neq N} \frac{|\text{Hom}_{k\tilde{Q}}(N, \tau M)_{D_0 AI}|}{q - 1} X_{A_0 \oplus D_0 \oplus I[-1]}.$$

Here, we use the following notation as in [17]

$$\text{Hom}_{k\tilde{Q}}(N, \tau M)_{D_0 AI} := \{f \neq 0 : N \longrightarrow \tau M \mid \ker f \cong D_0, \text{coker } f \cong \tau A \oplus I\}.$$

Note that $\dim_k \text{Hom}_{k\tilde{Q}}(N, \tau M) = 1$, we have the following exact sequences

$$0 \longrightarrow B_0 \longrightarrow M \longrightarrow A_0 \longrightarrow 0$$

$$0 \longrightarrow C \longrightarrow \tau B_0 \longrightarrow I \longrightarrow 0$$

where $C = \text{im} f$, $\ker f = D_0$.

$$\begin{aligned} s_2 &= \frac{|\text{Hom}_{k\tilde{Q}}(N, \tau M)| - 1}{q - 1} X_{A_0 \oplus D_0 \oplus I[-1]} \\ &= \sum_{X, Y, K, L} \frac{|\text{Hom}_{k\tilde{Q}}(N, \tau M)| - 1}{q - 1} F_{XY}^{D_0} F_{KL}^{A_0} q^{[L, X] - \frac{1}{2}\langle Y+L, K+X \rangle} X^{\tilde{B}(\underline{y}+\underline{l}+\underline{b}_0) - (\tilde{I}-\tilde{R})(\underline{m}+\underline{n})} \\ &= \sum_{A, B, C, D} \frac{q^{[C, \tau B]} - 1}{q - 1} F_{AB}^M F_{CD}^N q^{[L, X] - \frac{1}{2}\langle Y+L, K+X \rangle} X^{\tilde{B}(\underline{y}+\underline{l}+\underline{b}_0) - (\tilde{I}-\tilde{R})(\underline{m}+\underline{n})}. \end{aligned}$$

Here, $Y = D, K = A, B = B_0 + L$ in the above expression and the equality can be illustrated by the following diagram:

$$\begin{array}{ccccccccc} & & Y & \xlongequal{\quad} & Y & & \tau A & \xlongequal{\quad} & \tau A & & \\ & & \downarrow & & \downarrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & D_0 & \longrightarrow & N & \longrightarrow & \tau M & \longrightarrow & \tau A_0 \oplus I & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & X & \longrightarrow & C & \longrightarrow & \tau B & \longrightarrow & \tau L \oplus I & \longrightarrow & 0 \end{array}$$

We must to check the relation between

$$-\frac{1}{2}\langle Y + L, K + X \rangle + [L, X]$$

and

$$-\frac{1}{2}\langle B + D, A + C \rangle + \langle B, C \rangle.$$

In this case, note that $D = Y, L = A_0 - A, K = A, [L, X]^1 = [X, \tau L] = 0$. We have

$$\begin{aligned} -\frac{1}{2}\langle Y + L, K + X \rangle + [L, X] &= -\frac{1}{2}\langle Y + L, K + X \rangle + \langle L, X \rangle \\ &= -\frac{1}{2}\langle D + A_0 - A, A + D_0 - D \rangle + \langle A - A_0, D_0 - D \rangle \\ &= -\frac{1}{2}\langle D, A \rangle - \frac{1}{2}\langle D, D_0 \rangle + \frac{1}{2}\langle D, D \rangle - \frac{1}{2}\langle A_0, A \rangle + \frac{1}{2}\langle A_0, D_0 \rangle \\ &\quad - \frac{1}{2}\langle A_0, D \rangle + \frac{1}{2}\langle A, A \rangle - \frac{1}{2}\langle A, D_0 \rangle + \frac{1}{2}\langle A, D \rangle. \end{aligned}$$

And

$$\begin{aligned}
& -\frac{1}{2}\langle B+D, A+C \rangle + \langle B, C \rangle \\
&= -\frac{1}{2}\langle M-A+D, A+N-D \rangle + \langle M-A, N-D \rangle \\
&= -\frac{1}{2}\langle M, A \rangle - \frac{1}{2}\langle M, D \rangle + \frac{1}{2}\langle A, A \rangle - \frac{1}{2}\langle A, N \rangle + \frac{1}{2}\langle A, D \rangle \\
&\quad -\frac{1}{2}\langle D, A \rangle - \frac{1}{2}\langle D, N \rangle + \frac{1}{2}\langle D, D \rangle + \frac{1}{2}\langle M, N \rangle.
\end{aligned}$$

Hence it is equivalent to compare

$$-\frac{1}{2}\langle D, D_0 \rangle - \frac{1}{2}\langle A_0, A \rangle + \frac{1}{2}\langle A_0, D_0 \rangle - \frac{1}{2}\langle A_0, D \rangle - \frac{1}{2}\langle A, D_0 \rangle$$

and

$$-\frac{1}{2}\langle D, N \rangle - \frac{1}{2}\langle M, A \rangle + \frac{1}{2}\langle M, N \rangle - \frac{1}{2}\langle M, D \rangle - \frac{1}{2}\langle A, N \rangle.$$

We claim that

$$\langle D, N \rangle + \langle M, D \rangle = \langle D, D_0 \rangle + \langle A_0, D \rangle$$

and

$$\langle A_0, A \rangle + \langle A, D_0 \rangle = \langle M, A \rangle + \langle A, N \rangle.$$

Indeed, we have

$$\begin{aligned}
\langle D, N - D_0 \rangle &= \langle D, \tau M - \tau A_0 - I \rangle \\
&= \langle D, \tau M - \tau A_0 \rangle = \langle A_0 - M, D \rangle.
\end{aligned}$$

In the same way, we also have

$$\langle A, N - D_0 \rangle = \langle A_0 - M, A \rangle.$$

Thus

$$s_2 = q^{\frac{1}{2}\langle A_0, D_0 \rangle - \frac{1}{2}\langle M, N \rangle} \sum_{A, B, C, D} \frac{q^{[B, C]^1} - 1}{q - 1} q^{\langle B, C \rangle - \frac{1}{2}\langle B+D, A+C \rangle} F_{AB}^M F_{CD}^N X^{\tilde{B}(\underline{b}+\underline{d}) - (\tilde{I}-\tilde{R})(\underline{m}+\underline{n})}.$$

Therefore we have the following multiplication formula

$$X_N X_M = q^{\frac{1}{2}\Lambda((\tilde{I}-\tilde{R})\underline{n}, (\tilde{I}-\tilde{R})\underline{m})} X_E + q^{\frac{1}{2}\Lambda((\tilde{I}-\tilde{R})\underline{n}, (\tilde{I}-\tilde{R})\underline{m}) + \frac{1}{2}\langle M, N \rangle - \frac{1}{2}\langle A_0, D_0 \rangle} X_{D_0 \oplus A_0 \oplus I[-1]}.$$

□

There are three canonical special cases satisfying the assumption $\text{Hom}_{k\tilde{Q}}(D_0, \tau A \oplus I) = \text{Hom}_{k\tilde{Q}}(A_0, I) = 0$ in Theorem 3.5.

Special case I. Assume that $A_0 = 0 = I$. Then $L = K = 0 = A$. If $B \neq M$, i.e., $B \subsetneq M$, then there exists $f_1 : N \rightarrow \tau M$ induced by the above diagram which is not surjective. It is a contradiction to the assumption $\dim_k \text{Hom}_{k\tilde{Q}}(N, \tau M) = 1$. In this case, the multiplication formula is

$$X_N X_M = q^{\frac{1}{2}\Lambda((\tilde{I}-\tilde{R})\underline{n}, (\tilde{I}-\tilde{R})\underline{m})} X_E + q^{\frac{1}{2}\Lambda((\tilde{I}-\tilde{R})\underline{n}, (\tilde{I}-\tilde{R})\underline{m}) + \frac{1}{2}\langle M, N \rangle} X_{D_0}.$$

Special case II. Assume that $D_0 = 0$ and $\text{Hom}_{k\tilde{Q}}(A_0, I) = 0$. Then $Y = X = 0, C = N$. In this case, the multiplication formula is

$$X_N X_M = q^{\frac{1}{2}\Lambda((\tilde{I}-\tilde{R})\underline{\mathbb{L}}, (\tilde{I}-\tilde{R})\underline{\mathbb{M}})} X_E + q^{\frac{1}{2}\Lambda((\tilde{I}-\tilde{R})\underline{\mathbb{L}}, (\tilde{I}-\tilde{R})\underline{\mathbb{M}}) + \frac{1}{2}\langle M, N \rangle} X_{A_0 \oplus I[-1]}.$$

Special case III. Assume that M, N are indecomposable rigid $k\tilde{Q}$ -mod and

$$\dim_k \text{Ext}_{\tilde{C}}^1(M, N) = 1.$$

Since $D_0 \oplus A_0 \oplus I[-1]$ is rigid, then the assumption $\text{Hom}_{k\tilde{Q}}(D_0, \tau A \oplus I) = \text{Hom}_{k\tilde{Q}}(A_0, I) = 0$ in Theorem 3.5 holds.

Lemma 3.6. *With the assumption in Special case III, we have $\frac{1}{2}\langle A_0, D_0 \rangle - \frac{1}{2}\langle M, N \rangle = \frac{1}{2}$.*

Proof. Note that we have

$$\frac{1}{2}\langle A_0, D_0 \rangle - \frac{1}{2}\langle M, N \rangle = \frac{1}{2}\langle A_0, N - N/D_0 \rangle - \frac{1}{2}\langle M, N \rangle.$$

We need to confirm the two equations

- (1) $\langle M, N \rangle = \langle A_0, N \rangle - 1$ and
- (2) $\langle A_0, N/D_0 \rangle = 0$.

Note that $A_0 \oplus N$ is rigid, thus $[A_0, N]^1 = 0$. We have the following exact sequences

$$\begin{aligned} 0 \longrightarrow N/D_0 \longrightarrow \tau M \longrightarrow \tau A_0 \oplus I \longrightarrow 0 \\ 0 \longrightarrow D_0 \longrightarrow N \longrightarrow N/D_0 \longrightarrow 0 \end{aligned}$$

Applying the functor $\text{Hom}_{k\tilde{Q}}(N, -)$, we have the following exact sequences

$$\begin{aligned} [N, N/D_0]^1 \longrightarrow [N, \tau M]^1 \longrightarrow [N, \tau A_0 \oplus I]^1 \longrightarrow 0 \\ [N, N]^1 \longrightarrow [N, N/D_0]^1 \longrightarrow 0. \end{aligned}$$

Hence

$$\langle M, N \rangle = [M, N] - 1 = [A_0, N] - 1 = \langle A_0, N \rangle - 1.$$

As for the second equation, apply the functor $\text{Hom}_{k\tilde{Q}}(A_0, -)$ to the exact sequence

$$0 \longrightarrow D_0 \longrightarrow N \longrightarrow N/D_0 \longrightarrow 0$$

We have the following exact sequence

$$[A_0, N]^1 \longrightarrow [A_0, N/D_0]^1 \longrightarrow 0$$

Thus $[A_0, N/D_0]^1 = 0$. Applying the functor $\text{Hom}_{k\tilde{Q}}(\tau M, -)$ to the exact sequence

$$0 \longrightarrow N/D_0 \longrightarrow \tau M \longrightarrow \tau A_0 \oplus I \longrightarrow 0$$

We have the following exact sequence

$$[\tau M, \tau M]^1 \longrightarrow [\tau M, \tau A_0 \oplus I]^1 \longrightarrow 0$$

Thus we have

$$[M, A_0]^1 = [A_0, \tau M] = 0.$$

Again applying the functor $\text{Hom}_{k\tilde{Q}}(A_0, -)$, we have the exact sequence

$$0 \longrightarrow [A_0, N/D_0] \longrightarrow [A_0, \tau M] = 0$$

Hence $[A_0, N/D_0] = 0$. □

By Lemma 3.6, we obtain the following multiplication theorem between quantum cluster variables in [19].

Corollary 3.7. *Let M and N be indecomposable rigid kQ -modules and $\dim_k \text{Ext}_{\mathcal{C}_{\tilde{Q}}}^1(M, N) = 1$. Let*

$$N \longrightarrow E \longrightarrow M \longrightarrow N[1] = \tau N$$

and

$$M \longrightarrow D_0 \oplus A_0 \oplus I[-1] \longrightarrow N \longrightarrow \tau M$$

be two non-split triangles in $\mathcal{C}_{\tilde{Q}}$ as above. Then we have

$$X_N X_M = q^{\frac{1}{2}\Lambda((\tilde{I}-\tilde{R})\underline{n}, (\tilde{I}-\tilde{R})\underline{m})} X_E + q^{\frac{1}{2}\Lambda((\tilde{I}-\tilde{R})\underline{n}, (\tilde{I}-\tilde{R})\underline{m}) - \frac{1}{2}} X_{D_0 \oplus A_0 \oplus I[-1]}.$$

Now let M be a kQ -module and P be a projective $k\tilde{Q}$ -module with $[P, M] = [M, I] = 1$, where $I = \nu(P)$, here $\nu = \text{DHom}_{k\tilde{Q}}(-, k\tilde{Q})$ is the Nakayama functor. It is well-known that I is an injective module with $\text{soc} I = P/\text{rad} P$. Fix two nonzero morphisms $f \in \text{Hom}_{k\tilde{Q}}(P, M)$ and $g \in \text{Hom}_{k\tilde{Q}}(M, I)$. The two morphisms induce the following exact sequences

$$0 \longrightarrow P' \longrightarrow P \xrightarrow{f} M \longrightarrow A \longrightarrow 0$$

and

$$0 \longrightarrow B \longrightarrow M \xrightarrow{g} I \longrightarrow I' \longrightarrow 0.$$

These correspond to two non-split triangles in $\mathcal{C}_{\tilde{Q}}$

$$M \longrightarrow E' \longrightarrow P[1] \longrightarrow M[1]$$

and

$$I[-1] \longrightarrow E \longrightarrow M \longrightarrow I,$$

respectively, where $E \simeq B \oplus I'[-1]$ and $E' \simeq A \oplus P'[1]$.

Now we state the second part of our multiplication theorem for acyclic quantum cluster algebras.

Theorem 3.8. *With the above notations, assume that $[B, I'] = [P', A] = 0$. Then we have*

$$X_{\tau P} X_M = q^{\frac{1}{2}\Lambda(\underline{\dim} P/\text{rad} P, -(\tilde{I}-\tilde{R})\underline{m})} X_E + q^{\frac{1}{2}\Lambda(\underline{\dim} P/\text{rad} P, -(\tilde{I}-\tilde{R})\underline{m}) - \frac{1}{2}} X_{E'}.$$

Proof.

$$\begin{aligned} & X_{\tau P} X_M \\ &= X^{\underline{\dim} P/\text{rad} P} \sum_{G, H} q^{-\frac{1}{2}\langle H, G \rangle} F_{GH}^M X^{\tilde{B}\underline{h} - (\tilde{I}-\tilde{R})\underline{m}} \\ &= \sum_{G, H} q^{-\frac{1}{2}\langle H, G \rangle} F_{GH}^M q^{\frac{1}{2}\Lambda(\underline{\dim} P/\text{rad} P, \tilde{B}\underline{h} - (\tilde{I}-\tilde{R})\underline{m})} X^{\tilde{B}\underline{h} - (\tilde{I}-\tilde{R})\underline{m} + \underline{\dim} P/\text{rad} P} \\ &= q^{\frac{1}{2}\Lambda(\underline{\dim} P/\text{rad} P, -(\tilde{I}-\tilde{R})\underline{m})} \sum_{G, H} q^{-\frac{1}{2}\langle H, G \rangle} q^{-\frac{1}{2}[P, H]} F_{GH}^M X^{\tilde{B}\underline{h} - (\tilde{I}-\tilde{R})\underline{m} + \underline{\dim} P/\text{rad} P}. \end{aligned}$$

Here we use the following fact

$$\Lambda(\underline{\dim} P/\text{rad} P, \tilde{B}\underline{h}) = (\underline{\dim} P/\text{rad} P)^{tr} \Lambda \tilde{B}\underline{h} = -(\underline{\dim} P/\text{rad} P)^{tr} \begin{bmatrix} I_n \\ 0 \end{bmatrix} \underline{h} = -[P, H].$$

Note that by assumption $[P, M] = 1$, we have that $[P, H] = 0$ or 1 .

We firstly compute the term

$$X_E = X_{B \oplus I'[-1]} = \sum_{X, Y} q^{-\frac{1}{2}\langle Y, X \rangle} F_{XY}^B X^{\tilde{B}\underline{y} - (\tilde{I} - \tilde{R})\underline{b} + \underline{\dim} \text{soc} I'}.$$

We have the following diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & Y & \xlongequal{\quad} & Y & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & B & \longrightarrow & M & \xrightarrow{\theta} & I \longrightarrow I' \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & X & \longrightarrow & G & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

and a short exact sequence

$$0 \longrightarrow \text{im} \theta \longrightarrow I \longrightarrow I' \longrightarrow 0.$$

As we assume that $[B, I'] = 0$, thus $[H, I'] = 0$. Then

$$\langle Y, X \rangle - \langle H, G \rangle = \langle H, X \rangle - \langle H, G \rangle = \langle H, X - G \rangle = \langle H, B - M \rangle = -\langle H, \text{im} \theta \rangle.$$

Applying the functor $[H, -]$ to the above short exact sequence, we have

$$0 \longrightarrow [H, \text{im} \theta] \longrightarrow [H, I] \longrightarrow [H, I'] \longrightarrow [H, \text{im} \theta]^1 \longrightarrow 0$$

Note that $[H, I] = [H, I'] = 0$, thus $\langle H, \text{im} \theta \rangle = 0$. Hence

$$X_E = \sum_{G, H, [P, H]=0} q^{-\frac{1}{2}\langle H, G \rangle} F_{GH}^M X^{\tilde{B}\underline{h} - (\tilde{I} - \tilde{R})\underline{m} + \underline{\dim} P / \text{rad} P}.$$

Now compute the term

$$X_{E'} = X_{A \oplus P'[1]} = \sum_{X, Y} q^{-\frac{1}{2}\langle Y, X \rangle} F_{XY}^A X^{\tilde{B}\underline{y} - (\tilde{I} - \tilde{R})\underline{a} + \underline{\dim} P' / \text{rad} P'}.$$

We have the following diagram

$$\begin{array}{ccccccc}
& & & 0 & & 0 & \\
& & & \downarrow & & \downarrow & \\
& & P & \longrightarrow & H & \longrightarrow & Y \longrightarrow 0 \\
& & \parallel & & \downarrow & & \downarrow \\
0 & \longrightarrow & P' & \longrightarrow & P & \xrightarrow{\theta'} & M \longrightarrow A \longrightarrow 0 \\
& & & & \downarrow & & \downarrow \\
& & & & G & \xlongequal{\quad} & X \\
& & & & \downarrow & & \downarrow \\
& & & & 0 & & 0
\end{array}$$

Applying the functor $[P', -]$ to the following short exact sequence

$$0 \longrightarrow Y \longrightarrow A \longrightarrow G \longrightarrow 0.$$

we have

$$0 \longrightarrow [P', Y] \longrightarrow [P', A] \longrightarrow [P', G] \longrightarrow 0$$

As we assume that $[P', A] = 0$, thus $[P', G] = 0$. Then $\langle P', G \rangle = 0$. Hence we have

$$\begin{aligned}
& \langle Y, X \rangle - \langle H, G \rangle = \langle Y, G \rangle - \langle H, G \rangle = \langle Y - H, G \rangle = \langle A - M, G \rangle \\
& = \langle P' - P, G \rangle = \langle P', G \rangle = 0.
\end{aligned}$$

Therefore

$$X'_E = \sum_{G, H, [P, H]=1} q^{-\frac{1}{2}\langle H, G \rangle} F_{GH}^M X^{\tilde{B}h - (\tilde{I} - \tilde{R})_{\underline{m} + \underline{\dim} P / \text{rad} P}}.$$

This completes the proof. \square

Note that if M is indecomposable rigid object in $\mathcal{C}_{\tilde{Q}}$ and $[P, M] = [M, I] = 1$, then both $E = B \oplus I'[-1]$ and $E' = A \oplus P'[1]$ are rigid. Thus the assumptions $\text{Hom}_{k\tilde{Q}}(B, I') = \text{Hom}_{k\tilde{Q}}(P', A) = 0$ in Theorem 3.8 naturally hold. The quantum cluster multiplication theorem in [19] deals with this special case.

4. \mathbb{ZP} -BASES IN SPECIALIZED QUANTUM CLUSTER ALGEBRAS OF FINITE TYPE

Let k be a finite field with cardinality $|k| = q$ and $m \geq n$ be two positive integers and \tilde{Q} an acyclic quiver with vertex set $\{1, \dots, m\}$. The full subquiver Q on the vertices $\{1, \dots, n\}$ is the principal part of \tilde{Q} . Let $\mathcal{A}_{|k|}(\tilde{Q})$ be the corresponding specialized quantum cluster algebra of Q with coefficients. Then the main theorem in [19] shows that $\mathcal{A}_{|k|}(\tilde{Q})$ is the \mathbb{ZP} -subalgebra of \mathcal{F} generated by

$$\begin{aligned}
& \{X_M \mid M \text{ is indecomposable rigid } kQ\text{-mod}\} \cup \\
& \{X_{\tau P_i}, 1 \leq i \leq n \mid P_i \text{ is indecomposable projective } k\tilde{Q}\text{-mod}\}.
\end{aligned}$$

Let i be a sink or a source in \tilde{Q} . We define the reflected quiver $\sigma_i(\tilde{Q})$ by reversing all the arrows ending at i . An *admissible sequence of sinks (resp. sources)* is a sequence (i_1, \dots, i_l) such that i_1 is a sink (resp. source) in \tilde{Q} and i_k is a sink (resp. source) in $\sigma_{i_{k-1}} \cdots \sigma_{i_1}(\tilde{Q})$

for any $k = 2, \dots, l$. A quiver \tilde{Q}' is called *reflection-equivalent* to \tilde{Q} if there exists an admissible sequence of sinks or sources (i_1, \dots, i_l) such that $\tilde{Q}' = \sigma_{i_l} \cdots \sigma_{i_1}(\tilde{Q})$. Note that mutations can be viewed as generalizations of reflections, i.e, if i is a sink or a source in a quiver \tilde{Q} , then $\mu_i(\tilde{Q}) = \sigma_i(\tilde{Q})$ where μ_i denotes the mutation in the direction i . Thus if \tilde{Q}' is a quiver mutation-equivalent to \tilde{Q} , there is a natural canonical isomorphism between $\mathcal{A}_{|k|}(\tilde{Q})$ and $\mathcal{A}_{|k|}(\tilde{Q}')$, denoted by

$$\Phi_i : \mathcal{A}_{|k|}(\tilde{Q}) \rightarrow \mathcal{A}_{|k|}(\tilde{Q}').$$

Let $\Sigma_i^+ : \text{mod}(\tilde{Q}) \rightarrow \text{mod}(\tilde{Q}')$ be the standard BGP-reflection functor and $R_i^+ : \mathcal{C}_{\tilde{Q}} \rightarrow \mathcal{C}_{\tilde{Q}'}$ be the extended BGP-reflection functor defined by [24]:

$$R_i^+ : \begin{cases} X \mapsto \Sigma_i^+(X), & \text{if } X \not\cong S_i \text{ is a } kQ\text{-module,} \\ S_i \mapsto P_i[1], \\ P_j[1] \mapsto P_j[1], & \text{if } j \neq i, \\ P_i[1] \mapsto S_i. \end{cases}$$

By Rupel [22], the following holds:

Theorem 4.1. [22, Theorem 2.4, Lemma 5.6] *For any $X_M^{\tilde{Q}} \in \mathcal{A}_{|k|}(\tilde{Q})$, we have $\Phi_i(X_M^{\tilde{Q}}) = X_{R_i^+ M}^{\tilde{Q}'}$.*

Definition 4.2 ([4]). Let Q be an acyclic quiver with associated matrix B . Q will be called *graded* if there exists a linear form ϵ on \mathbb{Z}^n such that $\epsilon(B\alpha_i) < 0$ for any $1 \leq i \leq n$ where α_i still denotes the i -th vector of the canonical basis of \mathbb{Z}^n .

If Q is a graded quiver, then it is proved in [4] that we can endow the cluster algebra of Q with a grading. Namely, the results are the following:

For any Laurent polynomial P in the variables X_i , the $\text{supp}(P)$ of P is defined as the set of points $\lambda = (\lambda_i, 1 \leq i \leq n)$ of \mathbb{Z}^n such that the λ -component, that is, the coefficient of $\prod_{1 \leq i \leq n} X_i^{\lambda_i}$ in P is nonzero. For any λ in \mathbb{Z}^n , let C_λ be the convex cone with vertex λ and edge vectors generated by the $B\alpha_i$ for any $1 \leq i \leq n$. Then we have the following two propositions as the quantum versions of Proposition 5 and Proposition 7 in [4] respectively.

Proposition 4.3. *Let Q be a graded acyclic quiver with no multiple arrows and $M = M_0 \oplus P_M[1]$ with M_0 is kQ -module and P_M projective $k\tilde{Q}$ -module. Then, $\text{supp}(X_{M_0 \oplus P_M[1]})$ is in C_{λ_M} with $\lambda_M := (-\langle \alpha_i, \underline{\dim} M_0 \rangle + \langle \underline{\dim} P_M, \alpha_i \rangle)_{1 \leq i \leq n}$. Moreover, the λ_M -component of $X_{M_0 \oplus P_M[1]}$ is some nonzero monomials in $\{|k|^{\pm \frac{1}{2}}, X_{n+1}^{\pm 1}, \dots, X_m^{\pm 1}\}$.*

Proposition 4.4. *Let Q be a graded acyclic quiver with no multiple arrows. For any $m \in \mathbb{Z}$, set*

$$F_m = \left(\bigoplus_{\epsilon(\nu) \leq m} \mathbb{Z}\mathbb{P} \prod_{1 \leq i \leq n} u_i^{\nu_i} \right) \cap \mathcal{A}_{|k|}(\tilde{Q}),$$

then the set $(F_m)_{m \in \mathbb{Z}}$ defines a \mathbb{Z} -filtration of $\mathcal{A}_{|k|}(\tilde{Q})$.

For any $\underline{d} \in \mathbb{Z}^n$, define $\underline{d}^+ = (d_i^+)_{1 \leq i \leq n}$ such that $d_i^+ = d_i$ if $d_i > 0$ and $d_i^+ = 0$ if $d_i \leq 0$ for any $1 \leq i \leq n$. Dually, we set $\underline{d}^- = \underline{d}^+ - \underline{d}$. The following proposition 4.5 can be viewed as the categorification of [2, Theorem 7.3].

Proposition 4.5. *Let \tilde{Q} be an acyclic quiver. Then the set $\{\prod_{i=1}^n X_{S_i}^{d_i^+} X_{P_i[1]}^{d_i^-} \mid (d_1, \dots, d_n) \in \mathbb{Z}^n\}$ is a $\mathbb{Z}\mathbb{P}$ -basis of $\mathcal{A}_{|k|}(\tilde{Q})$.*

Proof. For any $1 \leq i \leq n$, it is easy to check that the following set is a cluster

$$\{X_{\tau P_1}, \dots, X_{\tau P_{i-1}}, X_{S_i}, X_{\tau P_{i+1}}, \dots, X_{\tau P_n}\}$$

obtained by the mutation in direction i of the cluster

$$\{X_{\tau P_1}, \dots, X_{\tau P_{i-1}}, X_{\tau P_i}, X_{\tau P_{i+1}}, \dots, X_{\tau P_n}\}.$$

Then the proposition immediately follows from [2, Theorem 7.3] and [19, Theorem 5.4.3]. \square

The main result is the following theorem showing the $\mathbb{Z}\mathbb{P}$ -basis in a quantum cluster algebra of finite type. It is the good bases in a cluster algebra of finite type in [4] by specializing q and coefficients to 1 and the existence of Hall polynomials for representation direct algebras [21].

Theorem 4.6. *Let Q is a simple-laced Dynkin quiver with $Q_0 = \{1, 2, \dots, n\}$. Then the set $\mathcal{B}(Q) := \{X_M \mid M = M_0 \oplus P_M[1] \text{ with } M_0 \text{ is } kQ\text{-module, } P_M \text{ projective } k\tilde{Q}\text{-module, } M \text{ rigid object in } \mathcal{C}_{\tilde{Q}}\}$ is a $\mathbb{Z}\mathbb{P}$ -basis of $\mathcal{A}_{|k|}(\tilde{Q})$.*

Proof. It is obvious to see that there exists an orientation such that Q' is a graded quiver where Q' is reflection-equivalent to Q . Assume that $\sigma_{i_l} \cdots \sigma_{i_1}(Q') = Q$. For any $X_M \in \mathcal{B}(Q)$ with dimension vector $\underline{\dim} M = \underline{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n$, we know that $X_M \in \mathcal{A}_{|k|}(\tilde{Q}')$. Then by Proposition 4.5 we have

$$X_M^{\tilde{Q}'} = b_{\underline{m}} \prod_{i=1}^n (X_{S_i}^{\tilde{Q}'})^{m_i^+} (X_{P_i[1]}^{\tilde{Q}'})^{m_i^-} + \sum_{\epsilon(\underline{l}) < \epsilon(\underline{m})} b_{\underline{l}} \prod_{i=1}^n (X_{S_i}^{\tilde{Q}'})^{l_i^+} (X_{P_i[1]}^{\tilde{Q}'})^{l_i^-}$$

where $\underline{l} = (l_i^+ - l_i^-)_{i \in Q_0}$, $b_{\underline{m}}$ and $b_{\underline{l}} \in \mathbb{Z}\mathbb{P}$. As Q' is a graded quiver, then by Proposition 4.3, Proposition 4.4, it follows that $b_{\underline{m}}$ must be some nonzero monomial in $\{q^{\pm \frac{1}{2}}, X_{n+1}^{\pm 1}, \dots, X_m^{\pm 1}\}$. Therefore, $\mathcal{B}(Q)$ is a $\mathbb{Z}\mathbb{P}$ -basis of $\mathcal{A}_{|k|}(\tilde{Q}')$. There is a natural isomorphism: $\Phi_{i_l} \cdots \Phi_{i_1} : \mathcal{A}_{|k|}(\tilde{Q}') \rightarrow \mathcal{A}_{|k|}(\tilde{Q})$. By Theorem 4.1, we obtain that

$$\Phi_{i_l} \cdots \Phi_{i_1}(X_M^{\tilde{Q}'}) = X_{R_{i_l}^+ \cdots R_{i_1}^+(M)}^{\tilde{Q}}.$$

Hence, $\mathcal{B}(Q)$ is a $\mathbb{Z}\mathbb{P}$ -basis of $\mathcal{A}_{|k|}(\tilde{Q})$. \square

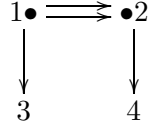
5. $\mathbb{Z}\mathbb{P}$ -BASES IN QUANTUM CLUSTER ALGEBRAS OF AFFINE TYPE

5.1. The case in the Kronecker quiver. Let Q be the tame quiver of type $\tilde{A}_1^{(1)}$ as follows

$$1 \bullet \rightleftarrows \bullet 2$$

It is well known that the regular indecomposable modules decomposes into a direct sum of homogeneous tubes indexed by the projective line \mathbb{P}^1 . We denote the regular indecomposable modules in the homogeneous tube for $p \in \mathbb{P}^1$ of degree 1 by $R_p(n)$ where $n \in \mathbb{N}$ and $\underline{\dim} R_p(n) = (n, n)$.

We consider the following ice quiver \tilde{Q} with frozen vertices 3 and 4:



Thus we have

$$\tilde{R} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

An easy calculation shows that the following antisymmetric 4×4 integer matrix

$$\Lambda = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -2 \\ 0 & -1 & 2 & 0 \end{pmatrix}$$

satisfying

$$(5.1) \quad \Lambda(-\tilde{B}) = \tilde{I} := \begin{bmatrix} I_2 \\ 0 \end{bmatrix},$$

where I_2 is the identity matrix of size 2×2 . Then we have the following result.

Lemma 5.1. *Let $R_p(1)$ be the indecomposable regular module of degree 1 as above. Then*

$$X_{R_p(1)} = X_{S_1} X_{S_2} - q^{-\frac{3}{2}} X_1 X_2 X_3.$$

Proof. By definition, we have

$$\begin{aligned} X_{S_1} &= X^{(-1,2,1,0)} + X^{(-1,0,0,0)}; \\ X_{S_2} &= X^{(0,-1,0,1)} + X^{(2,-1,0,0)}; \\ X_{R_p(1)} &= X^{(-1,1,1,1)} + X^{(1,-1,0,0)} + X^{(-1,-1,0,1)}. \end{aligned}$$

Hence the lemma follows from a direct calculation. □

By Lemma 3.1, the expression of $X_{R_p(1)}$ is independent of the choice of $p \in \mathbb{P}_k^1$ of degree 1. Hence, we set

$$X_\delta := X_{R_p(1)}.$$

Remark 5.2. (1) By Lemma 5.1, we know that X_δ belongs to $\mathcal{A}_{|k|}(\tilde{Q})$.

(2) By the following Theorem 5.3, $\mathcal{B}(Q)$ is a $\mathbb{Z}\mathbb{P}$ -basis in the quantum cluster algebra $\mathcal{A}_{|k|}(\tilde{Q})$. Moreover, if specializing q and coefficients to 1, $\mathcal{B}(Q)$ is exactly the generic basis in the sense of [10].

Note that there is an alternative choice of (Λ, \tilde{B}) , i.e., $\tilde{Q} = Q$ and set $\Lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\tilde{B} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$. Then we have $\Lambda(-\tilde{B}) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Hence, one should consider the category of KQ -representations for a field K with $|K| = q^2$. In this way, we obtain a quantum cluster algebra of Kronecker type without coefficients. The multiplication and the bar-invariant bases in this algebra have been thoroughly studied in [9]. Moreover, under the specialization $q = 1$, the bases in [9] induce the canonical basis, semicanonical basis and generic basis of the cluster algebra of the Kronecker quiver in the sense of [23],[7] and [10], respectively.

5.2. The case in affine types. An *affine quiver* is an acyclic quiver whose underlying diagram in an extended Dynkin diagram. One can refer to [12][8][20] for the theory of representations of affine quivers. We recall some useful background concerning representation theory of affine quivers. In this section we always assume that Q is an affine quiver with trivial valuation. The category $\text{rep}(kQ)$ of finite-dimensional representations can be identified with the category of $\text{mod-}kQ$ of finite-dimensional modules over the path algebra kQ . It is well-known that indecomposable kQ -module contains (up to isomorphism) three families: the component of indecomposable regular modules $\mathcal{R}(Q)$, the component of the preprojective modules $\mathcal{P}(Q)$ and the component of the preinjective modules $\mathcal{I}(Q)$. If $P \in \mathcal{P}(Q)$, $I \in \mathcal{I}(Q)$ and $R \in \mathcal{R}(Q)$, then

$$\text{Hom}_{kQ}(R, P) \simeq \text{Hom}_{kQ}(I, R) \simeq \text{Hom}_{kQ}(I, P) = 0,$$

and

$$\text{Ext}_{kQ}^1(P, R) \simeq \text{Ext}_{kQ}^1(R, I) \simeq \text{Ext}_{kQ}^1(P, I) = 0.$$

If M and N are two regular indecomposable modules in different tubes, then

$$\text{Hom}_{kQ}(M, N) = 0 \text{ and } \text{Ext}_{kQ}^1(M, N) = 0.$$

There are at most 3 non-homogeneous tubes for Q . We denote these tubes by $\mathcal{T}_1, \dots, \mathcal{T}_t$. Let r_i be the rank of \mathcal{T}_i and the regular simple modules in \mathcal{T}_i be $E_1^{(i)}, \dots, E_{r_i}^{(i)}$ such that $\tau E_2^{(i)} = E_1^{(i)}, \dots, \tau E_1^{(i)} = E_{r_i}^{(i)}$ for $i = 1, \dots, t$. If we restrict the discussion to one tube, we will omit the index i for convenience. Given a regular simple module E in a tube, $E[i]$ is the indecomposable regular module with quasi-socle E and quasi-length i for any $i \in \mathbb{N}$.

Define the set

$$\mathbf{D}(Q) = \{\underline{d} \in \mathbb{N}^{Q_0} \mid \exists \text{ a regular rigid module } R \text{ and regular simple modules } E_1, \dots, E_r$$

in a non-homogeneous tube with rank r such that $\underline{\dim}((E_1 \oplus \dots \oplus E_r)^n \oplus R) = \underline{d}$.

Set $\mathbf{E}(Q) = \{\underline{d} \in \mathbb{Z}^{Q_0} \mid \exists M = M_0 \oplus P_M[1] \text{ with } M_0 \text{ is } kQ\text{-module, } P_M \text{ projective } k\tilde{Q}\text{-module, } M \text{ rigid object in } \mathcal{C}_{\tilde{Q}} \text{ with } \underline{\dim} M = \underline{d}\}$. By the main theorem in [11], we have that \mathbb{Z}^{Q_0} is the disjoint union of $\mathbf{D}(Q)$ and $\mathbf{E}(Q)$. We make an assignment, i.e., a map

$$\phi : \mathbb{Z}^{Q_0} \rightarrow \text{obj}(\mathcal{C}_{\tilde{Q}})$$

and set

$$X_{\phi(\underline{d})} := (X_{E_1} \cdots X_{E_r})^n X_R$$

if $\underline{d} \in \mathbf{D}(Q)$ and $|Q_0| > 2$;

$$X_{\phi(\underline{d})} := X_{\delta}^n$$

for some δ in a homogeneous tube of degree 1 if $\underline{d} \in \mathbf{D}(Q)$ and Q is the Kronecker quiver;

$$X_{\phi(\underline{d})} := X_M$$

if $\underline{d} \in \mathbf{E}(Q)$. It is clear that the above assignment is not unique. For simplicity and without confusion, we omit ϕ in the notation $X_{\phi(\underline{d})}$.

Theorem 5.3. *Let Q be an affine quiver with $Q_0 = \{1, 2, \dots, n\}$ and fix an assignment as above. Then the set*

$$\mathcal{B}(Q) := \{X_{\underline{d}} \mid \underline{d} \in \mathbb{Z}^{Q_0}\}$$

is a $\mathbb{Z}\mathbb{P}$ -basis of $\mathcal{A}_{|k|}(\tilde{Q})$.

Proof. By [10], there exists an orientation such that Q' is a graded quiver where Q' is reflection-equivalent to Q .

When Q' is a Kronecker quiver, by Remark 5.2, we know that $X_{\delta}^n (n \in \mathbb{N})$ is in $\mathcal{A}_{|k|}(\tilde{Q}')$. If Q' is not a Kronecker quiver, we consider the non-homogeneous tubes. By Theorem 3.3, X_R is in $\mathcal{A}_{|k|}(\tilde{Q}')$. Thus $(X_{E_1} \cdots X_{E_r})^n X_R$ is in $\mathcal{A}_{|k|}(\tilde{Q}')$. Note that for any $\underline{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n$, $X_{\underline{m}} \in \mathcal{B}(Q')$. Then by Proposition 4.5 we have

$$X_{\underline{m}}^{\tilde{Q}'} = b_{\underline{m}} \prod_{i=1}^n (X_{S_i}^{\tilde{Q}'})^{m_i^+} (X_{P_i[1]}^{\tilde{Q}'})^{m_i^-} + \sum_{\epsilon(\underline{l}) < \epsilon(\underline{m})} b_{\underline{l}} \prod_{i=1}^n (X_{S_i}^{\tilde{Q}'})^{l_i^+} (X_{P_i[1]}^{\tilde{Q}'})^{l_i^-}$$

where $b_{\underline{m}}, b_{\underline{l}} \in \mathbb{Z}\mathbb{P}$. As Q' is a graded quiver, then by Proposition 4.3, Proposition 4.4, it follows that $b_{\underline{d}}$ must be some nonzero monomial in $\{q^{\pm \frac{1}{2}}, X_{n+1}^{\pm 1}, \dots, X_m^{\pm 1}\}$. Therefore, $\mathcal{B}(Q')$ is a $\mathbb{Z}\mathbb{P}$ -basis of $\mathcal{A}_{|k|}(\tilde{Q}')$. By Theorem 4.1, we obtain that $\mathcal{B}(Q)$ is a $\mathbb{Z}\mathbb{P}$ -basis of $\mathcal{A}_{|k|}(\tilde{Q})$. \square

By [6, Proposition 5], the quiver Grassmannians $\text{Gr}_{\underline{e}}(M)$ of a kQ -module M are some polynomials in $\mathbb{Z}[q]$. Then by specializing q and coefficients to 1, the bases in Theorem 5.3 induces the integral bases in affine cluster algebras ([11]).

Remark 5.4. Theorem 5.3 does not provide the quantum version for generic bases of affine type in [10]. In order to achieve it, one need to prove a quantum analogue of the difference property [10, Definition 3.24].

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