

**t -ANALOGUE OF q -CHARACTERS, BASES OF QUANTUM
CLUSTER ALGEBRAS, AND A CORRECTION
TECHNIQUE**

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ABSTRACT. We first study a new family of graded quiver varieties together with a new t -deformation of the associated Grothendieck rings. This provides the geometric foundations for a joint paper by Yoshiyuki Kimura and the author.

We further generalize the result of that paper to any acyclic quantum cluster algebra with arbitrary nondegenerate coefficients. In particular, we obtain the generic basis, the dual PBW basis, and the dual canonical basis. The method consists in a correction technique, which works for general quantum cluster algebras.

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1. INTRODUCTION

1.1. Motivation. The theory of cluster algebras, which was invented by Fomin and Zelevinsky [FZ02], has its origin in Lie theory and combinatorics. Since its very beginning, the theory of (quantum) cluster algebras has been related to many areas, such as Poisson Geometry, discrete dynamical systems, higher Teichmüller spaces, combinatorics, commutative and non-commutative algebraic geometries, and representation theory, *cf.* [Kel12].

One of the main motivations of the study of quantum cluster algebras [BZ05] is to provide an algebraic framework for the dual canonical bases of quantum groups. Therefore, bases of cluster algebra deserve to be put under scrutiny. The following two questions arise naturally.

Question 1.1.1. *1) How to construct bases of (quantum) cluster algebras?*

2) What are the corresponding structure constants and transition matrices?

Question 1.1.2. *1) Can we construct a “dual canonical basis” of the quantum cluster algebra, such that its structure constants are positive and it contains all the quantum cluster monomials.*

2) Can we identify this basis with a subset of the dual canonical basis of certain quantum group?

Remark 1.1.3. *We expect this basis for acyclic quantum cluster algebra to be identified with a proper subset of the dual canonical basis, because the acyclic cluster algebras are “smaller” than general quantum groups, cf. [GLS11].*

Furthermore, by [FZ07] [Tra09], (quantum) cluster monomials vary little when the choice of the coefficient pattern changes. If such a result holds for the bases as well, our research on the bases will be largely simplified. Surprisingly, to best of the author’s knowledge, this problem seems has not been studied in literature yet.

Question 1.1.4. *How does the bases data depend on the choice of the coefficients and quantization? Are they controllable?*

The final motivation of this paper comes from the study of quantum cluster characters. In [Qin10], the author defined quantum cluster characters over the rigid objects of certain cluster category, and showed that these characters describe the quantum cluster monomials of acyclic quantum cluster algebras. Nagao proposed a more general formula to describe the quantum cluster variables of general (quantum) cluster algebras based on the theory of non-commutative Donaldson-Thomas invariants developed by Kontsevich and Soibelman (up to some conjectures), *cf.* [KS08] [Nag10]. The following question is natural from the representation theoretic point of view.

Question 1.1.5. *Can we extend the quantum cluster character to generic objects, such that we obtain a generic basis of the quantum cluster algebra containing all the quantum cluster monomials?*

1.2. Previous context, strategies, and results.

Previous context. Despite the many successful applications of (quantum) cluster algebras to other areas, the basis construction problems which motivate their introduction remain largely open.

We have limited knowledge of the constructions of bases of the classical cluster algebras. In the approach via preprojective algebra, *cf.* [GLS11], Geiß, Leclerc, and Schroëer have shown that if G is a semi-simple complex algebraic group and $N \subset G$ a maximal nilpotent subgroup, then the coordinate algebra $\mathbb{C}[N]$ admits a canonical classical cluster structure whose coefficient type is specific. They further constructed the generic basis of $\mathbb{C}[N]$, which contain the cluster monomials, and identified it with Lusztig’s dual semicanonical basis of $\mathbb{C}[N]$ [Lus00]. As another approach, recently, Musiker, Schiffler, and Williams constructed bases for classical cluster algebras arising from unpunctured surfaces with coefficients whose exchange matrix is of full rank, [MSW11].

Our knowledge of the bases of the quantum cluster algebras is even more limited. [Lam11a] [Lam11b] [DX11] [HL11] obtained partial results of Question 1.1.2 for quivers of finite and affine type. Standard bases and partial results of triangular bases have been obtained for acyclic quantum cluster, *cf.* [BFZ05] [BZ11] [BZ12].

Deformed monoidal pseudo-categorification. In this paper, we use (deformed) monoidal pseudo-categorifications to give a new approach to bases of (quantum) cluster algebras.

Monoidal categorification was used by Hernandez and Leclerc as a new approach to the positivity conjecture of cluster algebras, *cf.* [HL10]. For a given cluster algebra \mathcal{A} , we want to find a tensor category \mathcal{C} together with a well behaved algebra isomorphism from \mathcal{A} to the Grothendieck ring R of \mathcal{C} . In particular, each cluster monomial should be sent to the class of a simple module. In the original work [HL10], the tensor category \mathcal{C} is the tensor category of certain finite dimensional modules of certain quantum affine algebra, and the isomorphism is obtained by comparing the (truncated) q -characters of these modules with cluster characters.

Therefore, we easily arrive at the following naive idea: in order to construct the bases of (quantum) cluster algebras, it suffices to study bases of the (deformed) Grothendieck ring, and then apply the algebra isomorphism.

In this paper, we use a similar category \mathcal{C} (closely related to quantum loop algebras) and construct a linear map identifying the (truncated) qt -characters with quantum cluster characters studied by the author in [Qin10]. Our construction holds for general coefficients and quantizations, where the linear map fails to be algebraic, thus the name “pseudo-categorification”. A key ingredient is the observation that the failures are mild and controllable. Therefore, in practice, we can still follow the above naive idea to study bases.

Graded quiver varieties. In order to construct this monoidal pseudo-categorification, we need to understand the deformed Grothendieck ring and the qt -characters. A second key ingredient of the paper is that we can understand them geometrically by using Nakajima's quiver varieties, *cf.* [Nak01a] [Nak04].

Recall that Nakajima's quiver variety is a natural generalization of the ADHM-construction of instantons. His graded quiver variety is the fixed point subvariety with respect to a \mathbb{C}^* -action. In [Nak11], he required the quiver to be bipartite when constructing the graded quiver varieties.

In Section 4, we introduce a new family of graded quiver varieties for acyclic quivers which are not necessarily bipartite. In order to remove the bipartite restriction, we carefully change the definition of the \mathbb{C}^* -action, such that the resulting new graded quiver varieties (fixed point sets) still have good properties. Then we establish the geometric foundations of our discussion, and proceed to study the deformed Grothendieck rings and qt -characters following the arguments of Nakajima in [Nak01a] [Nak04] [Nak11].

Furthermore, as an important application, these constructions lead to an affirmative answer of Question 1.1.2 for acyclic quantum cluster algebras of specific coefficients and quantization, which will be presented in the joint work [KQ12] by Yoshiyuki Kimura and the author. As a corollary, the result of Nakajima [Nak11] can be generalized as the following:

When the quantum cluster algebra contains an acyclic seed, the positivity conjecture is true. Namely, the coefficients of the cluster expansions of the quantum cluster variables are positive with respect to any chosen seed.

Its proof depends on the Fourier-Deligne-Sato transform, *cf.* the recent work by Efimov [Efi11] for an independent different approach. We shall use some useful results obtained in the proof.

Main results. Via the pseudo-categorification approach introduced above, we use t -analogue of q -characters to realize three bases of the acyclic quantum cluster algebras for any choice of coefficients and quantizations: the generic basis, the dual PBW basis, and the canonical basis, *cf.* Theorem 8.1.8 8.2.1. For such (quantum) cluster algebras, we obtain affirmative answers and solutions of Question 1.1.1, 1.1.2(1), 1.1.5, *cf.* section 8.2 8.3.

In the last section of the paper, we develop the correction technique for general quantum cluster algebras, *cf.* Theorem 9.1.2, which has been implicitly used in previous sections when we measure the failures of the monoidal categorification in [KQ12]. In acyclic case, we obtain an affirmative answer to Question 1.1.4, *cf.* section 8.2 8.3.

Remark 1.2.1. *Notice that, by the joint work of Kimura with the author [KQ12], 1.1.2(2) has an affirmative answer for quantum cluster algebra of a specific choice of coefficients and quantization, while for a general choice, a reasonable identification which preserves the multiplications does not exist.*

It should be mentioned that, despite that the results on the generic quantum cluster characters and the quantum generic basis seem very natural, their proof needs the full power of the graded quiver varieties and the existence of the monoidal categorification. For the moment, the author does not see any alternative approach.

1.3. Plan of the paper. In section 2, we recall notations of quantum cluster algebras arising from ice quivers.

In section 3, we show how the cluster expansions of quantum cluster variables (monomials) of quantum cluster algebras with arbitrary compatible pairs can be deduced from those with unittally compatible pairs.

In section 4, we study a family of new graded quiver varieties by choosing a new torus action. Then we follow the statements of [Nak11] to construct the deformed Grothendieck ring R_t .

In section 5, we define the t -analogue of q -characters for these graded quiver varieties.

In section 6, we collect useful results from [KQ12] on generic characters.

In section 7, we measure the failure of these characters being isomorphism. Our constructions allow to quantize [Nak11] by comparing the qt -characters with the quantum cluster characters in [Qin10].

In section 8, we construct bases of acyclic quantum cluster algebra for any choice of coefficients and quantizations. We also study their transition matrices and structure constants.

Finally, in section 9, we establish the correction technique for general quantum cluster algebras.

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2. PRELIMINARIES

We refer the readers to [Qin12] or [KQ12, Section 2] for details, whose conventions will be briefly recalled here.

2.1. Quantum cluster algebras. We use (Λ, \tilde{B}) to denote a *compatible pair*, namely we have

$$(1) \quad \Lambda(-\tilde{B}) = \begin{bmatrix} D \\ 0 \end{bmatrix},$$

where the B -matrix \tilde{B} is an $m \times n$ integer matrix and the Λ -matrix Λ is an $m \times m$ integer matrix for some integers m, n , and D is a diagonal matrix with strictly positive integers on the diagonal. The *principal part* B of \tilde{B} is defined to be its upper $n \times n$ submatrix.

We use v to denote the formal parameter $q^{\frac{1}{2}}$, while v^2 is sometimes denoted by q . The *quantum torus* $\mathcal{T} = \mathcal{T}(\Lambda)$ associated with the Λ -matrix Λ is the Laurent polynomial ring $\mathbb{Z}[v^{\pm}][x_1^{\pm}, \dots, x_m^{\pm}]$, whose usual product \cdot is often omitted. Let $\mathbb{Z}P[q^{\pm\frac{1}{2}}]$ denotes its subring $\mathbb{Z}[v^{\pm}][x_{n+1}^{\pm}, \dots, x_m^{\pm}]$, which we call the *coefficient ring*.

The matrix product $g^T \Lambda h$, $g, h \in \mathbb{Z}^m$, is denoted by $\Lambda(g, h)$, where we use g^T to denote the matrix transposition of g . Then we endow \mathcal{T} with the

twisted product $*$ such that we have

$$x^g * x^h = v^{\Lambda(g,h)} x^{g+h}$$

for any degrees g and h in \mathbb{Z}^m . The natural involution of \mathcal{T} which sends v to v^{-1} is denoted by $\overline{(\)}$.

We fix an n -regular tree \mathbb{T}_n with root t_0 . Recursively, we can associate a *quantum seed* $(\Lambda(t), \tilde{B}(t), x(t))$ with each vertex t of the n -regular tree such that we have

- (1) $(\Lambda(t_0), \tilde{B}(t_0), x(t_0)) = (\Lambda, \tilde{B}, x)$, where $x = (x_1, \dots, x_m)$, and
- (2) if there exists an edge labeled k connecting two vertices t and t' , then the two quantum seeds $(\Lambda(t'), \tilde{B}(t'), X(t'))$ and $(\Lambda(t), \tilde{B}(t), X(t))$ are related by the *mutation* at k , cf. [KQ12].

We define the x -variables to be $x_i(t)$, $1 \leq i \leq m$, $t \in \mathbb{T}_n$. Those x -variables $x_i(t)$ with $1 \leq i \leq n$ are called the *quantum cluster variables*. The *quantum cluster monomials* are the monomials of the quantum cluster variables contained in any common quantum seed.

Let \mathcal{F} denote the skew-field of fractions of the quantum torus \mathcal{T} . The *quantum cluster algebra* \mathcal{A}^q over (R, v) is the $(+, *)$ -subalgebra of the skew-field generated by the x -variables and the elements x_j^{-1} for all $j > n$. According to the *Quantum Laurent phenomenon*, cf. [BZ05, Section 5], \mathcal{A}^q is contained in the quantum torus \mathcal{T} .

When we specialize v to 1, we obtain the *classical cluster algebra* $\mathcal{A} = \mathcal{A}^q|_{v \rightarrow 1}$, which we also denote by $\mathcal{A}^{\mathbb{Z}}$.

2.2. Coefficient types (frozen patterns). We use Q to denote a *quiver* whose set of vertices is labeled by $I = \{1, \dots, n\}$. Any arrow h of Q points from its source $s(h)$ to its target $t(h)$. By reversing the direction of h , we obtain a new arrow \bar{h} . By reversing the arrow directions, we obtain the quiver Q^{op} .

Let \tilde{Q} be a quiver with vertices $\{1, \dots, m\}$ which contains Q as a full sub-quiver. Q is called the *principal part* of \tilde{Q} and \tilde{Q} is called an *ice quiver*. Then the difference $\tilde{Q} - Q$ is called the *coefficient type* (or *frozen pattern*) of the ice quiver \tilde{Q} .

We associate with \tilde{Q} an $m \times n$ integer matrix¹ $\tilde{B} = (b_{ij})$ such that we have

$$b_{ij} = \#\{h \in \tilde{Q} | s(h) = i, t(h) = j\} - \#\{h \in \tilde{Q} | s(h) = j, t(h) = i\}.$$

Similarly, we associate with Q an $n \times n$ -matrix $B = B_Q$.

Q is called *acyclic* if it has no oriented cycles. We refer the reader to [KQ12] or [Qin12] for the definition of *bipartite quivers* and *level l ice quivers with z -pattern*.

Example 2.2.1. (1) *Figure 1 is an example of acyclic quiver (which is not bipartite).*

(2) *Figure 2 is a example of an level 1 ice quiver with z -pattern whose principal is given by Figure 1. Notice that since $B_{\tilde{Q}}$ is invertible, its inverse provides a canonical choice of Λ such that (Λ, \tilde{B}) is a compatible pair.*

¹Notice that this convention is opposite to that of [Nak11].

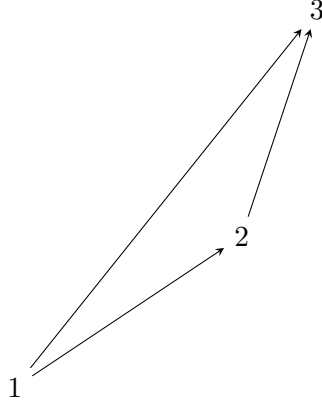


FIGURE 1. An acyclic quiver which is not bipartite

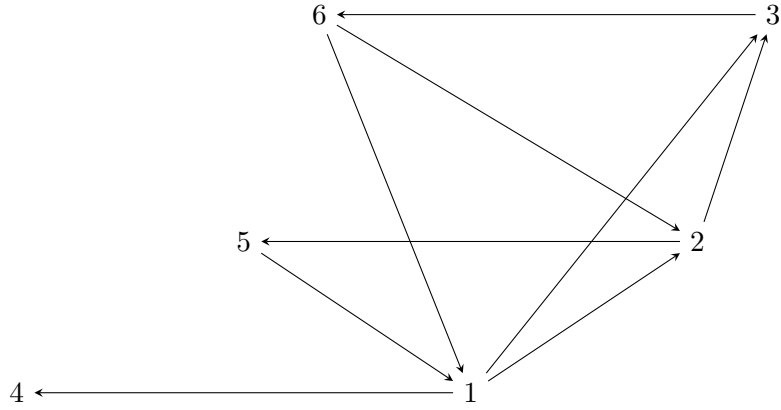


FIGURE 2. A level 1 ice quiver with z-pattern

3. QUANTUM CLUSTER VARIABLES VIA NORMALIZATION

Let \tilde{Q} be an ice quiver with principal part Q and \tilde{B} its associated $m \times n$ matrix. Let \mathcal{A}^q be the quantum cluster algebra associated with a compatible pair (Λ, \tilde{B}) such that $\Lambda(-\tilde{B}) = \begin{bmatrix} D \\ 0 \end{bmatrix}$.

Following [Pla10b], let $\mathcal{C}_{\tilde{Q}}$ be the cluster category² associated with the ice quiver \tilde{Q} . Then to each $i \in I$ and $t \in \mathcal{T}_n$, associate an object $M_i(t)$ as in the cluster category such that $M_i(t_0)[-1]$ is the projective module P_i via the canonical embedding of $\text{mod } -\mathbb{C}Q$ into $\mathcal{C}_{\tilde{Q}}$. Furthermore, we have a map $\text{ind}(\)$ sending each object in $\mathcal{C}_{\tilde{Q}}$ to its index $\text{ind}(M_i(t))$ in \mathbb{Z}^m such that $\text{ind}(M_i(t_0)) = e_i$.

²We have to choose a generic potential \tilde{W} associated with the ice quiver \tilde{Q} in order to construct this category.

First choose $\tilde{Q} - Q$ and Λ such that D is the identity matrix $\mathbf{1}_n$. The quantum cluster variables take the form

$$(2) \quad x_i(t) = \sum_{v \in \mathbb{N}^n} c_{i,v}(q^{\frac{1}{2}}) x^{\text{ind}(M_i(t)) + \tilde{B}v},$$

for some bar-invariant Laurent polynomials $c_{i,v} \in \mathbb{Z}[t^{\pm}]$. For general $\tilde{Q} - Q$ and Λ , we have the following result.

Theorem 3.0.2. *Assume D is a matrix whose diagonal entries are equal to $\delta \in \mathbb{Z}_{>0}$. Then in \mathcal{A}^q , we have*

$$(3) \quad x_i(t) = \sum_{v \in \mathbb{N}^n} c_{i,v}(q^{\frac{\delta}{2}}) x^{\text{ind}(M_i(t)) + \tilde{B}v}.$$

Proof. Consider the $2n \times 2n$ -matrix

$$\begin{pmatrix} B & -\mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix}.$$

Denote its inverse by $-\Lambda_1$. Its submatrix $\tilde{B}' = \begin{pmatrix} B \\ \mathbf{1}_n \end{pmatrix}$ is the $2n \times n$ -matrix associated with the *ice quiver with principal coefficient* \tilde{Q}' and principal part Q . We have the objects $M'_i(t)$ in the cluster category $\mathcal{C}_{\tilde{Q}'}$, and the map ind' .

Notice that (Λ_1, \tilde{B}') is a unitaly compatible pair, and we have the associated quantum cluster algebra \mathcal{A}^{q_1} .

Inspired by [Tra09, Proposition 5.1], we consider the $2n \times 2n$ -matrix

$$\Lambda_2 = \begin{pmatrix} 0 & -D \\ D & BD \end{pmatrix}.$$

The pair (Λ_2, \tilde{B}') is again compatible. In fact, one can check that $\Lambda_2 = \delta \cdot \Lambda_1$. Denote the associated quantum cluster algebra by \mathcal{A}^{q_2} .

Since Λ_2 is divisible by δ , we can view \mathcal{A}^{q_2} as a quantum cluster algebra over $(\mathbb{Z}[v^{\pm}], v)$ with $v = q^{\frac{\delta}{2}}$ and the initial compatible pair is $(\Lambda_2/\delta, \tilde{B}')$. Then by sending $q^{\frac{1}{2}}$ to $q^{\frac{\delta}{2}}$, \mathcal{A}^{q_1} is identified with \mathcal{A}^{q_2} . In \mathcal{A}^{q_2} , we have

$$\begin{aligned} x_i(t) &= \sum_{v \in \mathbb{N}^n} c_{i,v}(q^{\frac{\delta}{2}}) x^{\text{ind}'(M'_i(t)) + \tilde{B}'v} \\ &= \left(\sum_v c_{i,v}(q^{\frac{\delta}{2}}) q^{-\frac{1}{2}\Lambda_2(\tilde{B}'v, \tilde{g}_{i,1})} x^{\tilde{B}'v} \right) * x^{\text{ind}'(M'_i(t))} \end{aligned}$$

for the vectors $\tilde{g}_{i,1} = \text{ind}'(M'_i(t))$.

Finally, by [Tra09, Theorem 5.3] quantum F -polynomials exist. Therefore, the quantum cluster variables in \mathcal{A}^q can be written as

$$\begin{aligned} x_i(t) &= \left(\sum_v c_{i,v}(q^{\frac{\delta}{2}}) q^{-\frac{1}{2}\Lambda_2(\tilde{B}'v, \tilde{g}_{i,1})} x^{\tilde{B}'v} \right) * x^{\text{ind}(M_i(t))} \\ &= \sum_{v \in \mathbb{N}^n} q^{\frac{1}{2}f_{i,v}} c_{i,v}(q^{\frac{\delta}{2}}) x^{\text{ind}(M_i(t)) + \tilde{B}v}. \end{aligned}$$

for some integers $f_{i,v} \in \mathbb{Z}$. Notice that $\{X^{\tilde{g}_i + \tilde{B}v} | v \in \mathbb{N}^n\}$ is linearly independent over $\mathbb{Z}[q^{\pm \frac{1}{2}}]$ because we can show that \tilde{B} is full rank. Since $X_i(t)$ and $c_{i,v}$ are bar-invariant, all $f_{i,v}$ must vanish. \square

4. GRADED QUIVER VARIETIES

This section give proofs of properties of our new graded quiver varieties, which are needed by [KQ12] and later sections of this paper.

We begin with constructing some graded quiver varieties associated with acyclic quivers. Our graded quiver varieties are slightly different from the original ones in [Nak01a]. Thanks to our modified definition, we can show the existence of suitable stratifications for arbitrary acyclic quivers (Proposition 4.3.5). To prove this and other geometric properties, we adapt Nakajima's original arguments in [Nak98], [Nak01a].

Remark 4.0.3. *It seems likely that our definition should still enjoy the link with quantum loop algebras established in [Nak01a]. However, since Lie theory is not the major focus in this paper, we do not recheck the link. Therefore, when we follow any useful arguments in [Nak01a], the Lie theoretic expressions should be replaced with their geometric counterparts.*

4.1. Graded quiver varieties. We first give the definitions of the graded quiver varieties.

Fix a quiver Q which is acyclic, cf. Section 2.2. We label its vertices such that there is no arrows from i to j if $i \geq j$. The associated Cartan matrix $C = (c_{ij})$ is given by

$$(4) \quad c_{ij} = \begin{cases} 2 & \text{if } i = j \\ -|b_{ij}| & \text{if } i \neq j \end{cases}.$$

Let C_q denote the linear map from $\mathbb{Z}^{I \times (\frac{1}{2} + \mathbb{Z})}$ to $\mathbb{Z}^{I \times \mathbb{Z}}$ such that for any $\xi \in \mathbb{Z}^{I \times (\frac{1}{2} + \mathbb{Z})}$, the image $C_q \xi$ is given by

$$(5) \quad (C_q \xi)_k(a) = \xi_k(a - \frac{1}{2}) + \xi_k(a + \frac{1}{2}) - \sum_{j:k < j \leq n} b_{kj} \xi_j(a - \frac{1}{2}) - \sum_{i:1 \leq i < k} b_{ik} \xi_i(a + \frac{1}{2}).$$

Definition 4.1.1 (q -Cartan matrix). *The map C_q is the q -analogue of the Cartan matrix C .*

The map C_q induces a map from $\mathbb{Z}^{I \times \mathbb{R}}$ to $\mathbb{Z}^{I \times \mathbb{R}}$, which we denote by C_q by abuse of notation. It follows that $C_q[d]$ equals $[d]C_q$ for any $d \in \mathbb{R}$, where $[d]$ is the natural degree shift.

Lemma 4.1.2. *Let $\xi^1, \xi^2 \in \mathbb{Z}^{I \times \mathbb{R}}$ be any two given vectors, such that at least one of them has finite support, then we have*

$$(6) \quad \xi^2 \cdot C_q \xi^1[-\frac{1}{2}] = C_q \xi^2 \cdot \xi^1[-\frac{1}{2}].$$

Proof. To prove the equation, it suffices to expand both sides:

$$\begin{aligned}
\text{LHS} &= \sum_{k,a} \xi_k^2(a) \cdot (\xi_k^1(a) + \xi_k^1(a-1) - \sum_{i:i<k} b_{ik}\xi_i^1(a) - \sum_{j:k<j} b_{kj}\xi_j^1(a-1)) \\
&= \sum_{k,a} (\xi_k^2(a - \frac{1}{2}) + \xi_k^2(a + \frac{1}{2}) - \sum_{j:k<j} b_{kj}\xi_j^2(a - \frac{1}{2}) \\
&\quad - \sum_{i:i<k} b_{ik}\xi_i^2(a + \frac{1}{2})) \cdot \xi_k^1(a - \frac{1}{2}) \\
&= \text{RHS}.
\end{aligned}$$

□

Notice that there is a lexicographical order on the set $I \times \mathbb{R}$, which is also called the *weight order* $<_w$ in [Qin12].

Define E to be $\{\xi \in \mathbb{Z}^{I \times \mathbb{Z}} \mid \xi_k(a) = 0, \forall k \in I, a \ll 0\}$.

Lemma 4.1.3. (1) C_q induces a bijection between $E[\frac{1}{2}]$ and E .
(2) C_q induces a bijection between E and $E[-\frac{1}{2}]$.

Proof. Since C_q commutes with $[\frac{1}{2}]$, it suffices to show 1). Consider the block matrix of the restricted map $C_q[\frac{1}{2}]$ from E to E , whose components are indexed by $(i, a) \times (j, b)$, $i, j \in I$, $a, b \in \mathbb{Z}$. It is lower unitriangular with respect to the lexicographical order on $I \times \mathbb{R}$. Notice that E is bounded below. The claim follows. □

Therefore, it makes sense to talk about the inverses of the above bijections, which are both denoted by C_q^{-1} .

Next, we generalize the graded quiver varieties of [Nak11] from the bipartite to the acyclic case, modifying the original construction via the lexicographical order. We follow the convention of [Qin12] and consider bigraded dimension vectors $w = (w_i(a))_{i \in I, a \in \mathbb{Z}}$ and $v = (v_i(b))_{i \in I, b \in \mathbb{Z} + \frac{1}{2}}$. We always assume that they have non-negative components and finite supports. Let $W = \mathbb{C}^w = \bigoplus_{i,a} W_i(a)$ and $V = \mathbb{C}^v = \bigoplus_{i,b} V_i(b)$ be the associated bigraded vector spaces. If two $I \times \mathbb{R}$ -graded vectors are given, such that at least one of them has finite support, their natural inner product \cdot is well defined.

We say a pair (v, w) is *l-dominant* if the difference $w - C_q v$ is contained in $\mathbb{N}^{I \times \mathbb{Z}}$. We say (v, w) dominates (v', w') , which we denote by $(v, w) \geq (v', w')$ (*dominance order*), if there exists some $v'' \in \mathbb{N}^{I \times (\mathbb{Z} + \frac{1}{2})}$ such that $w' - C_q v' = w - C_q(v + v'')$. We denote $w' \leq w$ if $(0, w') \leq (0, w)$.

For any $v, v' \in \mathbb{N}^{I \times (\mathbb{Z} + \frac{1}{2})}$, $w \in \mathbb{N}^{I \times \mathbb{Z}}$, which have finite supports, we define

$$\begin{aligned}
\mathbf{L}^\bullet(v, v') &= \bigoplus_{(i,b)} \text{Hom}(V_i(b), V'_i(b)), \\
\mathbf{L}(w, v) &= \bigoplus_{(i,a)} \text{Hom}(W_i(a), V_i(a - \frac{1}{2})), \\
\mathbf{L}(v, w) &= \bigoplus_{(i,b)} \text{Hom}(V_i(b), W_i(b - \frac{1}{2})), \\
\mathbf{E}(v, v') &= (\bigoplus_{h \in \Omega, b} \text{Hom}(V_{s(h)}(b), V'_{t(h)}(b))) \oplus (\bigoplus_{\bar{h} \in \bar{\Omega}, a} \text{Hom}(V_{s(\bar{h})}(b), V'_{t(\bar{h})}(b-1))),
\end{aligned}$$

where Ω and $\bar{\Omega}$ are the sets of arrows of the quivers Q^{op} and Q respectively. Let H be the union of Ω and $\bar{\Omega}$, and ϵ the function on H such that it sends

Ω to 1 and $\bar{\Omega}$ to -1 respectively. Let B_h be any map indexed by some h in H . The function ϵ acts on B_h by $\epsilon B_h = \epsilon(h)B_h$.

Define the vector space

$$(7) \quad \text{Rep}^\bullet(Q^{op}, v, w) = \mathbf{E}(v, v) \oplus \mathbf{L}(w, v) \oplus \mathbf{L}(v, w),$$

whose points are given by

$$(8) \quad \begin{aligned} (B, \alpha, \beta) &= ((B_h)_{h \in H}, \alpha, \beta) \\ &= ((b_h)_{h \in \Omega}, (\bar{b}_h)_{h \in \bar{\Omega}}, (\alpha_i)_i, (\beta_i)_i) \\ &= ((\oplus_b b_{h,b})_h, (\oplus_b \bar{b}_{\bar{h},b})_{\bar{h}}, (\oplus_a \alpha_{i,a})_i, (\oplus_b \beta_{i,b})_i). \end{aligned}$$

The restriction of the moment map for ungraded quiver varieties becomes the map $\mu : \text{Rep}^\bullet(Q^{op}, v, w) \rightarrow \mathbf{L}^\bullet(v, v[-1])$ such that we have

$$(9) \quad \mu(B, \alpha, \beta) = \oplus_{i,b} \left(\sum_{h \in \Omega} (b_{h,b} b_{\bar{h},b+1} - \bar{b}_{\bar{h},b+1} b_{h,b+1}) + \alpha_{i,b+\frac{1}{2}} \beta_{i,b+1} \right).$$

Example 4.1.4. *Figure 3 provides an example of $\text{Rep}^\bullet(Q^{op}, v, w)$, whose rows and columns are indexed by I -degrees and \mathbb{R} -degrees respectively.*

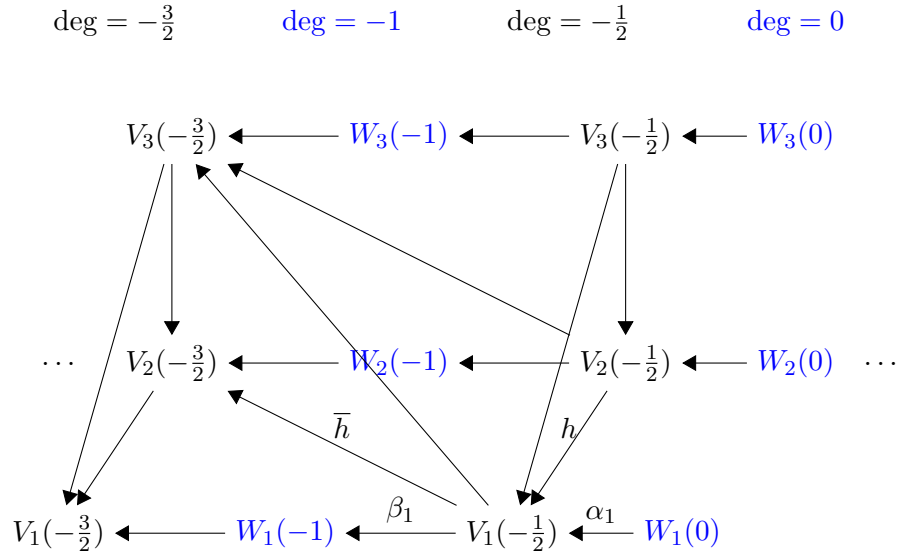


FIGURE 3. Vector space $\text{Rep}^\bullet(Q^{op}, v, w)$

The base change group $G_v = \prod_{i,a} GL(V_{i,a})$ naturally acts on $\mu^{-1}(0)$. Define χ to be the character which sends any group element g to $\prod_{i,a} (\det g_{i,a})^{-1}$.

Let $\mu^{-1}(0)^s$ denote the set of χ -stable points in $\mu^{-1}(0)$ and $\mathcal{M}^\bullet(v, w)$ the free quotient of $\mu^{-1}(0)^s/G_v$. This is a quasi-projective variety. Define $\mathcal{M}_0^\bullet(v, w)$ to be the affine variety $\text{Spec}(\mathbb{C}[\mu^{-1}(0)]^{G_v})$. Let π denote the canonical projective morphism from $\mathcal{M}^\bullet(v, w)$ to $\mathcal{M}_0^\bullet(v, w)$. For any point x in $\mathcal{M}_0^\bullet(v, w)$, denote $\pi^{-1}(x)$ by $\mathbf{m}_x^\bullet(v, w)$. We also denote $\pi^{-1}(0) = \mathcal{L}^\bullet = \mathcal{L}^\bullet(v, w)$. The varieties $\mathcal{M}^\bullet(v, w)$, $\mathcal{M}_0^\bullet(v, w)$, $\mathcal{L}^\bullet(v, w)$ are called *graded quiver varieties*.

In the rest of this section, we verify important properties of graded quiver varieties.

4.2. Ungraded quiver varieties. We first recall important properties of ungraded quiver varieties.

Let V and W be finite-dimensional I -graded complex vector spaces (without a -grading). In analogy with the previous subsection, we have vector spaces

$$\begin{aligned} L(V, V) &= \oplus_i \mathbf{Hom}(V_i, V_i) \\ L(W, V) &= \oplus_i \mathbf{Hom}(W_i, V_i) \\ L(V, W) &= \oplus_i \mathbf{Hom}(V_i, W_i) \\ E(V, V) &= \oplus_{h \in H} \mathbf{Hom}(V_{s(h)}, V_{t(h)}). \end{aligned}$$

Consider the symplectic vector space $\mathbf{Rep}(Q^{op}, V, W) = L(W, V) \oplus L(V, W) \oplus E(V, V)$. The associated moment map $\mu : \mathbf{Rep}(Q^{op}, V, W) \rightarrow L(V, V)$ takes a point (B, α, β) of $\mathbf{Rep}(Q^{op}, V, W)$ to

$$\mu(B, \alpha, \beta) = (\epsilon B)B + \alpha\beta.$$

Following the arguments of [Nak98], we consider the $GL(V)$ -variety $\mu^{-1}(0)$, and fix the character χ of $GL(V)$ such that $\chi(g) = \prod_i (\det g_i)^{-1}$. Then we can construct the geometric invariant theory quotient (GIT quotient for short) $\mathcal{M}(V, W)$ with respect to χ and the categorical quotient $\mathcal{M}_0(V, W)$ by the action of $GL(V)$ together with the projective morphism $\pi : \mathcal{M}(V, W) \rightarrow \mathcal{M}_0(V, W)$.

The points in $\mathcal{M}_0(V, W)$ are in bijection with the closed orbits in $\mu^{-1}(0)$. A point (B, α, β) in such a closed orbit is called a representative of the corresponding point in $\mathcal{M}_0(V, W)$, which is denoted by $[B, \alpha, \beta]$. Let $\mu^{-1}(0)^s$ be the open subset of $\mu^{-1}(0)$ consisting of the χ -stable points. It is well known that $GL(V)$ acts freely on $\mu^{-1}(0)^s$. Therefore, the points of $\mathcal{M}(V, W)$ are in bijection with the orbits of $\mu^{-1}(0)^s$. Again, a point (B, α, β) of such a free orbit is called a representative of the corresponding point in $\mathcal{M}(V, W)$, denoted by $[B, \alpha, \beta]$.

Proposition 4.2.1 ([Nak98, Corollary 3.12]). *The variety $\mathcal{M}(V, W)$ is smooth.*

Given any two vectors v, v' such that $v \geq v'$ with respect to the dominance order, there is a natural embedding of $\mathcal{M}_0(V, W)$ into $\mathcal{M}_0(V', W)$ given by extending the coordinates of the representatives by zero. Take all possible v and define $\mathcal{M}_0(W) = \cup_V \mathcal{M}_0(V, W)$ to be the direct limit of all the embeddings. It is possibly infinite dimensional, cf. [Nak01a, 2.5].

Let $[B, \alpha, \beta]$ be a point in $\mathcal{M}(V, W)$ and let $x = (B, \alpha, \beta)$ be its representative. Suppose that we have a B -invariant filtration of V

$$0 \subset F^0 \subset F^1 \subset \dots \subset F^t = V,$$

where $\mathbf{Im} \alpha \subset F^0$. Let $\mathrm{gr}_0 \alpha$ denote the morphism from W to F^0 such that its composition with the inclusion $F^0 \rightarrow V$ is α . Let $\mathrm{gr}_0 \beta$ denote the restriction of β to F^0 . For $1 \leq s \leq t$, let $\mathrm{gr}_s B$ denote the endomorphism which B induces on F^s/F^{s-1} and $\mathrm{gr} B = \oplus_{1 \leq s \leq t} \mathrm{gr}_s B$ the endomorphism on $\oplus_{1 \leq s \leq t} F^s/F^{s-1}$. The induced representative $\mathrm{gr} x$ is defined to be $(\mathrm{gr} B, \mathrm{gr}_0 \alpha, \mathrm{gr}_0 \beta)$, cf. [Nak98, Definition 3.19].

Proposition 4.2.2. [Nak98, Proposition 3.20] *Let $[B, \alpha, \beta]$ be a point in $\mathcal{M}(V, W)$ and let $x = (B, \alpha, \beta)$ be its representative. Then there exists a B -invariant filtration of V*

$$0 \subset F^0 \subset F^1 \subset \dots \subset F^t = V,$$

such that $\text{Im } \alpha \subset F^0$ and the induced triple $\text{gr } x = (\text{gr } B, \text{gr}_0 \alpha, \text{gr}_0 \beta)$ is a representative of $\pi([B, \alpha, \beta])$.

If \widehat{G} is a subgroup of G_v , we denote by (\widehat{G}) the conjugacy class of \widehat{G} . There is a natural stratification $\mathcal{M}_0(V, W) = \sqcup_{(\widehat{G})} \mathcal{M}_0(V, W)_{(\widehat{G})}$, such that each stratum is the set of the points $[B, \alpha, \beta]$ which have representatives (B, α, β) with the stabilizers in the conjugacy class (\widehat{G}) .

Proposition 4.2.3 (3.27, [Nak98]). *Let $[B, \alpha, \beta]$ be a point in $\mathcal{M}_0(V, W)_{(\widehat{G})}$ for some nontrivial \widehat{G} . Then there is a representative (B, α, β) and a decomposition*

$$V = V^0 \oplus (V^1)^{\oplus \widehat{v}_1} \oplus \dots \oplus (V^r)^{\oplus \widehat{v}_r},$$

such that we have

- (1) $B(V^s) \subset V^s$ for each summand V^s , $0 \leq s \leq r$;
- (2) if $s \neq s'$, there is no isomorphism from V^s to $V^{s'}$ that commutes with B ;
- (3) $\text{Im } \alpha$ is contained in V^0 , and V^s is contained in $\text{Ker } \beta$ for all $s > 0$;
- (4) the restriction of (B, α, β) to V^0 has the trivial stabilizer in $\prod_{i \in I} GL(V_i^0)$;
- (5) the subgroup $\prod_{i \in I} GL(V_i^s)$ meets \widehat{G} only in the scalar subgroup $\mathbb{C}^* \subset GL(V)$;

Remark 4.2.4. *Restricting the equation $\mu(B, \alpha, \beta) = 0$ to each summand V^s , $s > 0$, we see that V^s is a module over the preprojective algebra associated with Q^{op} with respect to the restriction of the B -action. It has the minimal possible stabilizer \mathbb{C}^* and a closed orbit under the $\prod_i GL(V_i^s)$ action. Therefore, it is a representative of a point in the categorical quotient $\mathcal{M}_0(V^s, 0)$.*

We call $\mathcal{M}_0(V, W)_{(\{1\})}$ the regular stratum and denote it by $\mathcal{M}_0^{\text{reg}}(V, W)$. It is known that the restriction of π gives an isomorphism from $\pi^{-1}(\mathcal{M}_0^{\text{reg}}(V, W))$ to $\mathcal{M}_0^{\text{reg}}(V, W)$.

Assume x is a point in $\mathcal{M}_0^{\text{reg}}(V^0, W)$, which is naturally embedded into a quotient $\mathcal{M}_0(V, W)$. Let T be the tangent space of $\mathcal{M}_0^{\text{reg}}(V, W)$ at x . Since $\mathcal{M}_0^{\text{reg}}(V^0, W)$ is non-empty, (V^0, W) is l -dominant. Define $W^\perp = \mathbb{C}^{\dim W - \dim V^0}$, $V^\perp = \mathbb{C}^{\dim V - \dim V^0}$. We have the following theorem.

Theorem 4.2.5 (Theorem 3.3.2, [Nak04]). *There exist neighborhoods \mathcal{U} , \mathcal{U}_T , \mathcal{U}^\perp of $x \in \mathcal{M}_0(V, W)$, $0 \in \mathcal{T}$, $0 \in \mathcal{M}_0(V^\perp, W^\perp)$ respectively, and biholomorphic maps $\mathcal{U} \rightarrow \mathcal{U}_T \times \mathcal{U}^\perp$, $\pi^{-1}(\mathcal{U}) \rightarrow \mathcal{U}_T \times \pi^{-1}(\mathcal{U}^\perp)$, such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{M}(V, W) \supset \pi^{-1}(\mathcal{U}) & \xrightarrow{\cong} & \mathcal{U}_T \times \pi^{-1}(\mathcal{U}^\perp) \subset T \times \mathcal{M}(V^\perp, W^\perp) \\ & & \cong \\ \pi \downarrow & & \downarrow \mathbf{1} \times \pi \\ \mathcal{M}_0(V, W) \supset \mathcal{U} & \xrightarrow{\cong} & \mathcal{U}_T \times \mathcal{U}^\perp \subset \mathcal{T} \times \mathcal{M}_0(V^\perp, W^\perp) \end{array}$$

For any two I -graded vector spaces V, V' , define $L(V, V') = \bigoplus_i \text{Hom}(V_i, V'_i)$, $E(V, V') = \bigoplus_h \text{Hom}(V_{s(h)}, V'_{t(h)})$.

For a point $(B, \alpha, \beta) \in \mu^{-1}(0)$, we consider the following complex

$$(10) \quad L(V, V) \xrightarrow{\iota} E(V, V) \oplus L(W, V) \oplus L(V, W) \xrightarrow{d\mu} L(V, V),$$

where $d\mu$ is the differential map of μ , and for $\xi \in L(V, V)$, we have

$$(11) \quad \iota(\xi) = (\bigoplus_{h \in \Omega} (B_h \xi - \xi B_h)) \oplus (-\xi \alpha) \oplus (\beta \xi).$$

Then the tangent space of $\mathcal{M}(V, W)$ at $[B, \alpha, \beta]$ is isomorphic to the middle cohomology of this complex.

4.3. Graded quiver varieties as fixed point sets. Let V and W be as in Section 4.2. Using the fixed point technique developed in [Nak01a], we can deduce the properties of our graded quiver varieties from their ungraded version in Section 4.2.

Choose a torus action³ of \mathbb{C}^* on $\text{Rep}(Q^{op}, V, W)$, such that for any $\varepsilon \in \mathbb{C}^*$, we have

$$\varepsilon(\alpha, \beta, b_h, b_{\bar{h}}) = (\varepsilon^n \alpha, \varepsilon^n \beta, \varepsilon^{2(s(h)-t(h))} b_h, \varepsilon^{2(n-s(h)+t(h))} b_{\bar{h}}).$$

Taking into account of the natural action of $GL(W)$ on $\text{Rep}(Q^{op}, V, W)$, we get a group action of $GL(W) \times \mathbb{C}^*$ on this space.

Since the actions of $GL(W) \times \mathbb{C}^*$ and $GL(V)$ commute, $GL(W) \times \mathbb{C}^*$ acts on the quiver varieties $\mathcal{M}(V, W)$ and $\mathcal{M}_0(V, W)$ and it commutes with the projective morphism π .

Take a pair (s, ε) , such that $s \in GL(W)$ is semisimple, and ε is not a root of unity. It generates a cyclic subgroup, whose closure with respect to the Zariski topology is denoted by A . As in [Nak01a], let $[B, \alpha, \beta]$ be a point $\mathcal{M}(V, W)^A$ and (B, α, β) be any representative of it. There exists a group homomorphism $\rho_{(B, \alpha, \beta)}$ from A to $GL(V)$ such that for any element $a \in A$, we have $a(B, \alpha, \beta) = \rho_{(B, \alpha, \beta)}(a)^{-1}(B, \alpha, \beta)$. The conjugacy class of $\rho_{(B, \alpha, \beta)}$ is independent of the choice of the representative (B, α, β) because the actions of $GL(V)$ and A commute. So we can denote it by $[\rho_{[B, \alpha, \beta]}]$.

Lemma 4.3.1 ([Nak01a, Section 4.1]). *The map from $\mathcal{M}(V, W)^A$ to the conjugacy classes of the group homomorphisms from A to $GL(V)$, sending $[B, \alpha, \beta]$ to $[\rho_{[B, \alpha, \beta]}]$, is locally constant.*

Proof. Since A is generated by the element $a = (s, \varepsilon)$, it suffices to study the conjugacy class $[\rho_{[B, \alpha, \beta]}(a)]$.

First, we show that $[\rho_{[B, \alpha, \beta]}(a)]$ is continuous in $[B, \alpha, \beta]$. Recall that $\mu^{-1}(0)^s$ is a principal $GL(V)$ -bundle over $\mathcal{M}(V, W)$. Denote the fibre map by p . Take a trivialization. Let U be any chart. For any continuous curve $[x(t)]$, $0 \leq t \leq 1$, the curve $[x^U(t)] = U \cap [x(t)]$ in $U \cap \mathcal{M}(V, W)^A$ can be lifted to a continuous curve $x^U(t) = [x^U(t)] \times \{e\}$ in $(U \times GL(V)) \cap p^{-1}(\mathcal{M}(V, W)^A)$, where e denotes the identity of $GL(V)$. Then the fibre coordinates of the continuous curve $a^{-1}x^U(t)$ are described by $\rho_{x(t)}(a)$. Recall that the transition between different charts is given by conjugating. Therefore, the conjugacy class $[\rho_{[x(t)]}(a)]$ is continuous on the curve $[x(t)]$.

³The choice is not unique. In fact, different choices might give isomorphic graded quiver varieties.

Since s is semi-simple and $\rho : A \rightarrow GL(V)$ is a group homomorphism, the Jordan form of $\rho_{x(t)}(a)$ is a discrete subset in the set of the conjugacy classes of $GL(V)$. Therefore, $[\rho_{x(t)}(a)]$ is locally constant. \square

We denote the collection of the points $[B, \alpha, \beta]$ with the common conjugacy class $[\rho]$ by $\mathcal{M}([\rho])$. Then $\mathcal{M}([\rho])$ is a union of connected components of $\mathcal{M}(V, W)^A$. It follows that we have $\pi\mathcal{M}(V, W)^A = \cup_{[\rho]}\pi\mathcal{M}([\rho]) = \sqcup_{[\rho]}\pi\mathcal{M}([\rho])$. Denote each stratum $\pi\mathcal{M}([\rho])$ by $\mathcal{M}_0([\rho])$.

Fix the conjugacy class $[\rho]$. Using the eigenvalues and the eigenspaces of s and $\rho(s, \varepsilon)$, we can endow W and V with gradings. Assume W and V have eigenspace decompositions $W = \oplus_i W_i = \oplus_{i,a \in \mathbb{Z}} W_i(a)$, $V = \oplus_i V_i = \oplus_{i,a \in \mathbb{Z}} V_i(a + \frac{1}{2})$, where $W_i(a)$ and $V_i(a + \frac{1}{2})$ have eigenvalues $\varepsilon^{2(an+i-1)}$, $\varepsilon^{2((a+\frac{1}{2})n+i-1)}$ respectively. Associate to $[\rho(s, \varepsilon)]$ the bigraded vectors $w = (\dim W_i(a))$, $v = (\dim V_i(a))$. We can identify $\mathcal{M}([\rho])$ with the graded quiver variety $\mathcal{M}^\bullet(v, w)$. Similarly, the graded categorical quotient $\mathcal{M}_0^\bullet(v, w)$ is identified with the subvariety $\mathcal{M}_0([\rho])$ of $\mathcal{M}_0(V, W)$.

Remark 4.3.2. *If we take the \mathbb{C}^* -action in [Nak11], the representatives of the fixed points in the sub-varieties $\mathcal{M}([\rho])$ do not take the form of representations of the quiver in Figure 3. For example, let the quiver Q be given by Figure 1. For simplicity, let us assume $w_2 = 0$. Then these representatives are representations of the quiver in Figure 4, where the black arrows arise from those of Q^{op} , the green arrows arise from those of Q , and the orange arrows correspond to the linear maps $\alpha_i(a)$, $\beta_i(a)$, $i \in I$, $a \in \mathbb{Z}$.*

Such representations do not suit our purpose.

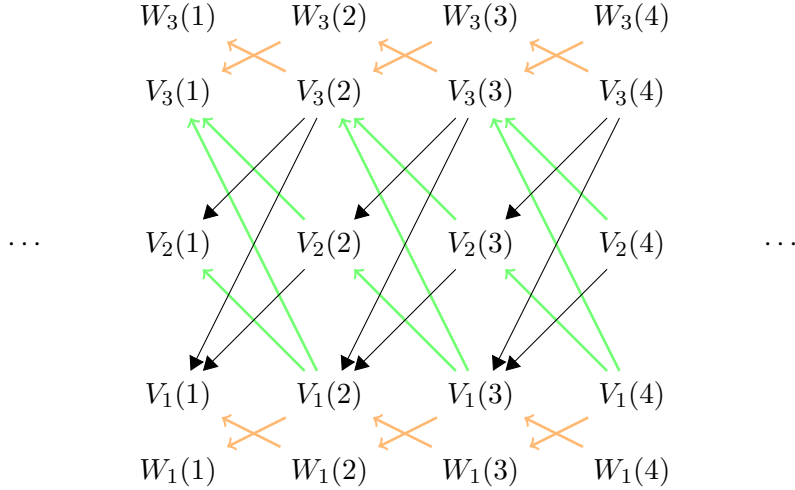


FIGURE 4. The vector space $\text{Rep}^\bullet(Q^{op}, v, w)$ for Q acyclic and the \mathbb{C}^* -action of [Nak11]

The graded version of Proposition 4.2.1 implies that the graded quiver variety $\mathcal{M}^\bullet(V, W)$ is smooth.

Proposition 4.3.3 ([Nak94, Corollary 5.5] [Nak01a, Proposition 4.1.2]). $\mathcal{M}([\rho])$ is homotopic to $\mathcal{L}(V, W) \cap \mathcal{M}([\rho])$.

Proof. By using Slodowy's technique [Slo80, Section 4.3], Nakajima has shown that $\mathcal{M}(V, W)$ is homotopic to $\mathcal{L}(V, W)$ in [Nak94], and he has also proved this proposition with a different $GL(W) \times \mathbb{C}^*$ -action in [Nak01a]. The technique remains valid here. We shall briefly recall it.

Equip the space $\text{rep}(Q^{op}, V, W)$ with the \mathbb{C}^* -action such that we have $\varepsilon(B_h, B_{\bar{h}}, \alpha, \beta) = (B_h, \varepsilon B_{\bar{h}}, \alpha, \varepsilon \beta)$ for any $\varepsilon \in \mathbb{C}^*$. It commutes with the $GL(V)$ -action. Furthermore, the set of the stable points $\mu^{-1}(0)^s$ is invariant under this action. Therefore, we obtain a \mathbb{C}^* -action on $\mathcal{M}(V, W)$. We have

$$(12) \quad \mathcal{L}(V, W) = \{[x] \in \mathcal{M}(V, W) \mid \lim_{\varepsilon \rightarrow \infty} \varepsilon[x] \text{ exists}\}.$$

Now the technique of Slodowy [Slo80, 4.3] implies that, via this \mathbb{C}^* -action, $\mathcal{M}(V, W)$ retracts to a neighborhood of $\mathcal{L}(V, W)$, such that $\mathcal{L}(V, W)$ is a strong deformation retract of this neighborhood.

Because our A -action commutes with this \mathbb{C}^* -action, we can apply the above constructions to the A -fixed subsets $\text{Rep}^\bullet(Q^{op}, v, w)$, $\mathcal{M}^\bullet(v, w)$, $\mathcal{L}^\bullet(v, w)$. Then the proposition is verified. \square

Let us define $\mathcal{M}_0^{\bullet \text{reg}}(v, w) = \mathcal{M}_0^{\bullet \text{reg}}([\rho]) = \pi(\pi^{-1}(\mathcal{M}_0^{\text{reg}}(V, W)) \cap \mathcal{M}([\rho]))$. Then the morphism π is an isomorphism from $\pi^{-1}(\mathcal{M}_0^{\bullet \text{reg}}(v, w))$ to $\mathcal{M}_0^{\bullet \text{reg}}(v, w)$.

The maps in Theorem 4.2.5 commute with the $GL(W) \times \mathbb{C}^*$ action. Restrict the maps to the subvarieties $\mathcal{M}^\bullet(v, w)$, $\mathcal{M}^\bullet(v^\perp, w^\perp)$ of $\mathcal{M}(V, W)$, $\mathcal{M}(V^\perp, W^\perp)$ respectively. We obtain a transversal slice theorem for graded quiver varieties.

For any $x \in \mathcal{M}_0^{\bullet \text{reg}}(v^0, w) \subset \mathcal{M}_0^\bullet(v, w)$, let T denote the its tangent space in $\mathcal{M}_0^{\bullet \text{reg}}(v, w)$. Define w^\perp and v^\perp to be $w - C_q v^0$ and $v^\perp = v - v^0$ respectively.

Theorem 4.3.4 (Transversal slice). *There exist neighborhoods \mathcal{U} , \mathcal{U}_T , \mathcal{U}^\perp of x in $\mathcal{M}_0^\bullet(V, W)$ and the origins in \mathcal{T} , $\mathcal{M}_0^\bullet(v^\perp, w^\perp)$ respectively, and biholomorphic maps $\mathcal{U} \rightarrow \mathcal{U}_T \times \mathcal{U}^\perp$, $\pi^{-1}(\mathcal{U}) \rightarrow \mathcal{U}_T \times \pi^{-1}(\mathcal{U}^\perp)$, such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{M}^\bullet(v, w) \supset \pi^{-1}(\mathcal{U}) & \xrightarrow{\cong} & \mathcal{U}_T \times \pi^{-1}(\mathcal{U}^\perp) \subset \mathcal{T} \times \mathcal{M}(v^\perp, w^\perp) \\ & \pi \downarrow & \downarrow \mathbf{1} \times \pi \\ \mathcal{M}_0^\bullet(v, w) \supset \mathcal{U} & \xrightarrow{\cong} & \mathcal{U}_T \times \mathcal{U}^\perp \subset \mathcal{T} \times \mathcal{M}_0^\bullet(v^\perp, w^\perp) \end{array}$$

Notice that the fibre $\pi^{-1}(x)$ is biholomorphic to the fibre $\mathcal{L}^\bullet(v^\perp, w^\perp)$ over the origin.

Proposition 4.3.5. *The affine graded quiver variety $\mathcal{M}_0^\bullet(v, w)$ admits a stratification*

$$(13) \quad \sqcup_{(v', w) \geq (v, w)} \mathcal{M}_0^{\bullet \text{reg}}(v', w).$$

Proof. It suffices to show

$$(14) \quad \pi(\mathcal{M}(V, W)^A) = \sqcup_{V'} \pi(\pi^{-1}(\mathcal{M}_0^{\text{reg}}(V', W)) \cap \mathcal{M}(V, W)^A)$$

for each pair of vector spaces (V, W) . Then we obtain (13) by restricting (14) to the subvariety $\mathcal{M}_0^\bullet(v, w) = \mathcal{M}_0([\rho])$ for the conjugacy class $[\rho]$ associated with (v, w) .

We claim that every point $[B, \alpha, \beta]$ of $\pi(\mathcal{M}(V, W)^A)$ is contained in the right hand side of (14). Using Proposition 4.2.3 and Remark 4.2.4, we see that $[B, \alpha, \beta]$ belongs to some $\mathcal{M}_0(V, W)_{(\widehat{G})}$. If \widehat{G} equals $\{e\}$, the claim is true. Otherwise, choose the representative (B, α, β) in Proposition 4.2.3 and consider the B actions on all the summand V^s , $s > 0$. If the actions are trivial, $[B, \alpha, \beta]$ lies in the regular stratum $\mathcal{M}_0^{\text{reg}}(V^0, W)$, and the claim follows easily. If the action is nontrivial for some V^s , $s > 0$, we obtain a point other than 0 in the categorical quotient $\mathcal{M}_0(V^s, 0)$. Because the $GL(W) \times \mathbb{C}^*$ -action is compatible with the decomposition of V in Proposition 4.2.3, $\rho_{(B, \alpha, \beta)}$ stabilizes the decomposition. Let v^s be the bigraded vector associated with the $\rho_{(B, \alpha, \beta)}$ -action on V^s . As in Remark 4.2.4, V^s is a representative of a nonzero point in $\mathcal{M}_0^\bullet(v^s, 0)$. However in our setting $\mathcal{M}_0^\bullet(v^s, 0)$ is the categorical quotient of the v^s -dimensional representations of some acyclic quiver, which is always equal to $\{0\}$. This contradiction implies that the B -action on V^s must be trivial. \square

Remark 4.3.6. *When Q is of Dynkin type, the ungraded version of the proposition holds, cf. [Nak01a, Proposition 2.6.3]. However, it does not necessarily hold when Q is not of Dynkin type, cf. Example 10.10 in [Nak98]. In general, whether the proposition is true or not depends on the choice of the $GL(W) \times \mathbb{C}^*$ -action.*

Let m be any integer. Following [Nak01a, Section 2.8], we define a \mathbb{C}^* -module structure $L(m)$ on \mathbb{C} by

$$(15) \quad \varepsilon \cdot v = \varepsilon^m v,$$

where $\varepsilon \in \mathbb{C}^*$, $v \in \mathbb{C}$.

For a \mathbb{C}^* -module M , we denote the \mathbb{C}^* -module $L(m) \otimes_{\mathbb{C}} M$ by $q^m M$.

As in [2.9, [Nak01a]], $\mu^{-1}(0)^s$ is a principal $GL(V)$ -bundle over $\mathcal{M}(V, W)$. Therefore, for any $i \in I$, we can view the vector space V_i as an associated vector bundle by using the natural $GL(V_i)$ action. Also, it is naturally a $GL(W) \times \mathbb{C}^*$ -equivariant vector bundle such that $GL(W)$ acts trivially. Similarly, we view W_i as a $GL(W) \times \mathbb{C}^*$ -equivariant vector bundle over $\mathcal{M}(V, W)$ by using the $GL(W_i) \times \mathbb{C}^*$ action such that \mathbb{C}^* acts trivially.

For each $k \in I$, we have the following complex $C_k^* = (\sigma_k, \tau_k)$ of $GL(W) \times \mathbb{C}^*$ -equivariant vector bundles:

$$(16) \quad C_k^* : q^{-2n} V_k \xrightarrow{\sigma_k} ((\oplus_{j:i < k} q^{2(k-i-n)} V_i^{\oplus b_{ik}}) \oplus (\oplus_{j:j > k} q^{2(j-k)} V_j^{\oplus b_{kj}}) \oplus q^{-n} W_k) \xrightarrow{\tau_k} V_k,$$

where $\sigma_k = (\oplus_{h \in H, s(h)=k} B_h) \oplus \beta_k$, $\tau_k = \sum_{h \in \Omega, t(h)=k} B_h - \sum_{\bar{h} \in \bar{\Omega}, t(\bar{h})=k} B_{\bar{h}} + \alpha_k$ are $GL(W) \times \mathbb{C}^*$ -equivariant morphisms. This complex is just the complex in [Nak98][4.2] with a modification of $GL(W) \times \mathbb{C}^*$ -action. Let the middle term be the degree 0 component.

Proposition 4.3.7 ([Nak01a, Lemma 2.9.2, Lemma 2.9.4]). *Fix a point $[B, \alpha, \beta] \in \mathcal{M}(V, W)$ and consider C_k^* as a complex of vector spaces, $k \in I$.*

- 1) *The cohomology $H^{-1}(C_k^*)$ vanishes.*
- 2) *If the image $\pi([B, \alpha, \beta]) \in \mathcal{M}_0(V, W)$ is contained in some regular stratum $\mathcal{M}_0^{\text{reg}}(V', W) \subset \mathcal{M}_0(V, W)$, then V' equals V if and only if $H^1(C_k^*)$ vanishes for all $k \in I$.*

Fix a graded quiver variety $\mathcal{M}^\bullet(v, w) = \mathcal{M}([\rho])$. Let $(C_k^*)^\bullet$ be the restriction of C_k^* to $\mathcal{M}^\bullet(v, w)$. Then the complex $(C_k^*)^\bullet$ decomposes as

$$(17) \quad (C_k^*)^\bullet = \bigoplus_{b \in \mathbb{Z} + \frac{1}{2}} C_k^*(b),$$

where the complexes $C_k^*(b) = (\sigma_k(b), \tau_k(b))$ are given by

$$(18) \quad q^{-2n} V_k(b) \xrightarrow{\sigma_k(b)} ((\bigoplus_{j:i < k} q^{2(k-i-n)} V_i(b)^{\oplus b_{ik}}) \oplus (\bigoplus_{j:j > k} q^{2(j-k)} V_j(b-1)^{\oplus b_{kj}}) \oplus q^{-n} W_k(b - \frac{1}{2})) \xrightarrow{\tau_k(b)} V_k(b-1).$$

The decomposition commutes with the A -action.

We have the following analogue of Proposition 4.3.7 in graded cases.

Proposition 4.3.8. 1) *The cohomology $H^{-1}((C_k^*)^\bullet)$ vanishes.*

2) *The image $\pi([B, \alpha, \beta]) \in \mathcal{M}_0^\bullet(v, w)$ is contained in the regular stratum $\mathcal{M}_0^{\bullet \text{reg}}(v, w)$ if and only if $H^1((C_k^*)^\bullet)$ vanishes, $\forall k \in I$.*

Proof. Part 1) follows from Proposition 4.3.7 by restriction. For part 2), we additionally use Proposition 4.3.5. \square

Theorem 4.3.9. $\mathcal{M}^\bullet(v, w)$ is connected.

Proof. The statement follows from the arguments in the proof of [Nak01a, Theorem 5.5.6]. Notice that, since our $GL(W) \times \mathbb{C}^*$ -action is different from that of [Nak01a], we don't need the condition $|b_{ij}| \leq 1$, $1 \leq i, j \leq n$, in [Nak01a, Theorem 5.5.6]. \square

As a consequence, the smooth variety $\mathcal{M}^\bullet(v, w)$ is irreducible.

Proposition 4.3.10 ([Nak01a, Corollary 5.5.5]). *On a nonempty open subset of $\mathcal{M}^\bullet(v, w)$, we have*

$$(19) \quad \text{codim } \text{Im } \tau_k(b) = \max(0, -\text{rank}(C_k^*(b))).$$

Lemma 4.3.11. $(\text{rank}(C_k^*)^\bullet)_{k \in I}$ equals $w - C_q v$.

Proposition 4.3.12. $\mathcal{M}_0^{\bullet \text{reg}}(v, w)$ is non-empty if and only if (v, w) is l -dominant.

Proof. This follows from Proposition 4.3.8 4.3.10 and Lemma 4.3.11. \square

Because the restriction of π over $\mathcal{M}_0^{\bullet \text{reg}}(v, w)$ is a local homeomorphism, $\dim \mathcal{M}_0^{\bullet \text{reg}}(v, w)$ can be calculated by Lemma 4.3.19.

Remark 4.3.13. In their ongoing work [KS13], Bernhard Keller and Sarah Scherotzke use representaiton theory to give explicit constructions of the points in $\mathcal{M}_0^{\bullet \text{reg}}(v, w)$, where (v, w) is l -dominant.

Remark 4.3.14. For any given vector w , by induction on its width with respect to the lexicographical order, one can prove that there are only finitely many v such that (v, w) is l -dominant. Then Propositions 4.3.12 and 4.3.5 imply that $\mathcal{M}_0^\bullet(w) = \cup_v \mathcal{M}_0^\bullet(v, w)$ is finite dimensional.

Corollary 4.3.15. For any l -dominant pairs (v, w) , (v^0, w) , $\mathcal{M}_0^{\bullet \text{reg}}(v^0, w)$ is contained in the closure of $\mathcal{M}_0^{\bullet \text{reg}}(v, w)$ if and only if (v^0, w) dominates (v, w) .

Proof. Proposition 4.3.12 implies that both regular strata are non-empty. By Theorem 4.3.9, $\mathcal{M}_0^\bullet(v, w)$ is irreducible. Since $\mathcal{M}_0^{\bullet\text{reg}}(v, w)$ is a non-empty open subset in $\mathcal{M}_0^\bullet(v, w)$, its closure equals $\mathcal{M}_0^\bullet(v, w)$.

If $(v^0, w) \geq (v, w)$, $\mathcal{M}_0^\bullet(v^0, w)$ is naturally embedded into $\mathcal{M}_0^\bullet(v, w)$. Thus the only if part holds.

Conversely, if $\mathcal{M}_0^{\bullet\text{reg}}(v^0, w) \subset \overline{\mathcal{M}_0^{\bullet\text{reg}}(v, w)} = \mathcal{M}_0^\bullet(v, w)$, using proposition 4.3.5 we obtain the if part. \square

Following [Nak04, (4.9)], for any given pairs of vectors (v^1, w^1) and (v^2, w^2) , we define the following complex of vector bundles over $\mathcal{M}^\bullet(v^1, w^1) \times \mathcal{M}^\bullet(v^2, w^2)$:

$$(20) \quad \mathbf{L}^\bullet(v^1, v^2) \xrightarrow{\sigma^{21}} \mathbf{E}(v^1, v^2) \oplus \mathbf{L}(w^1, v^2) \oplus \mathbf{L}(v^1, w^2) \xrightarrow{\tau^{21}} \mathbf{L}^\bullet(v^1, v^2[-1]),$$

where the middle term has degree 0, and we denote

$$(21) \quad \sigma^{21}(\xi) = (B^2\xi - \xi B^1) \oplus (-\xi\alpha^1) \oplus \beta^2\xi,$$

$$(22) \quad \tau^{21}(C \oplus I \oplus J) = \epsilon B^2 C + \epsilon C B^1 + \alpha^2 J + I \beta^1.$$

The complex is exact on the left and on the right, cf. the argument in [Nak98, 3.10].

Lemma 4.3.16. *The quotient $\ker \tau^{21} / \text{Im } \sigma^{21}$ is a vector bundle over $\mathcal{M}^\bullet(V^1, W^1) \times \mathcal{M}^\bullet(V^2, W^2)$.*

Proof. By [CBJ05], any subvariety of a vector bundle over the smooth variety $\mathcal{M}^\bullet(v^1, w^1) \times \mathcal{M}^\bullet(v^2, w^2)$, whose intersection with every fibre is a vector subspace of constant dimension, is a sub-bundle. Therefore, $\ker \tau^{21}$ is a vector bundle. Let $(\sigma^{21})^*$ be the transpose (or dual) of σ^{21} restricted to $\ker \tau^{21}$. Its kernel $(\ker \tau^{21} / \text{Im } \sigma^{21})^*$ is a sub-bundle of $(\ker \tau^{21})^*$. Therefore, $\ker \tau^{21} / \text{Im } \sigma^{21}$ is again a vector bundle. \square

Denote the rank of the vector bundle $\ker \tau^{21} / \text{Im } \sigma^{21}$ by $d((v^1, w^1), (v^2, w^2))$.

Lemma 4.3.17. *The rank $d((v^1, w^1), (v^2, w^2))$ is given by*

$$(23) \quad (w^1 - C_q v^1) \cdot v^2[-\frac{1}{2}] + v^1 \cdot w^2[-\frac{1}{2}].$$

Proof. It suffices to calculate the rank of the complex. We have

$$\begin{aligned} d((v^1, w^1), (v^2, w^2)) &= \sum_{k \in I, a \in \mathbb{Z}} v_k^2(a - \frac{1}{2}) \left(\sum_{i: i < k} b_{ik} v_i^1(a + \frac{1}{2}) + \sum_{j: j > k} b_{kj} v_j^1(a - \frac{1}{2}) \right) \\ &\quad + \sum_{k \in I, a \in \mathbb{Z}} (w_k^1(a) v_k^2(a - \frac{1}{2}) + v_k^1(a - \frac{1}{2}) w_k^2(a - 1)) \\ &\quad - \sum_{k \in I, a \in \mathbb{Z}} v_k^2(a - \frac{1}{2}) (v_k^1(a - \frac{1}{2}) + v_k^1(a + \frac{1}{2})) \\ &= \sum_{k, a} (v_k^2(a - \frac{1}{2}) (-C_q v^1)_k(a) + w_k^1(a) v_k^2(a - \frac{1}{2}) + w_k^2(a) v_k^1(a + \frac{1}{2})) \\ &= -v^2[-\frac{1}{2}] \cdot C_q v^1 + w^1 \cdot v^2[-\frac{1}{2}] + w^2 \cdot v^1[\frac{1}{2}] \\ &= \text{RHS}. \end{aligned}$$

\square

Remark 4.3.18. Using equation (6), we can easily rewrite the quadratic form (23) as

$$(24) \quad d((v^1, w^1), (v^2, w^2)) = v^1 \left[\frac{1}{2} \right] \cdot (w^2 - C_q v^2) + w^1 \left[\frac{1}{2} \right] \cdot v^2.$$

The reader is invited to compare this expression with [Nak04, (2.1)].

Let $[B, \alpha, \beta]$ be any point in $\mathcal{M}^\bullet(v, w) = \mathcal{M}([\rho]) \subset \mathcal{M}(V, W)$. The tangent space of $\mathcal{M}^\bullet(v, w)$ at $[B, \alpha, \beta]$ is the A -fixed part of the tangent space of $\mathcal{M}(V, W)$ at the same point. The A -fixed part of complex (10) over this point is just

$$(25) \quad \mathbf{L}(v, v[-1]) \xrightarrow{\iota} \mathbf{E}(v, v) \oplus \mathbf{L}(v, w) \oplus \mathbf{L}(w, v) \xrightarrow{d\mu} \mathbf{L}(v, v[1]).$$

Comparing it with the complex (20), we have the following result.

Lemma 4.3.19. The dimension of $\mathcal{M}^\bullet(v, w)$ equals $d((v, w), (v, w))$.

Theorem 4.3.20 ([Nak01a, Theorem 7.4.1]). The odd homology of $\mathcal{L}^\bullet(v, w)$ vanishes. And we have a nondegenerate pairing between $H_*(\mathcal{L}^\bullet(v, w))$ and $H_*(\mathcal{M}^\bullet(v, w))$.

Proof. We refer the reader to [Nak01a, Section 7] for the proof. This theorem is just a consequence of [Nak01a, Theorem 7.4.1]. \square

5. qt -CHARACTERS

In the study of finite dimensional representations of quantum affine algebras, Nakajima invented the t -analogues of the q -characters (qt -characters for short) [Nak04], which are defined as the generating series of the Betti numbers of the graded quiver varieties. In this section, we generalize his constructions for our graded quiver varieties and introduce a new t -deformation.

5.1. Quadratic forms. For any pairs $(v, w), (v', w')$, recall that we have the quadratic form $d((v, w), (v', w'))$ given by (23), whose geometric meaning is explained by Lemma 4.3.19. Let us further define the quadratic forms $\tilde{d}'((v, w), (v', w')), \tilde{d}'_W(w, w')$ to be $\mathcal{E}'(w, w')$ such that

$$\begin{aligned} \tilde{d}'_W(w, w') &= -w \left[\frac{1}{2} \right] \cdot C_q^{-1} w', \\ \mathcal{E}'(w, w') &= -w \left[\frac{1}{2} \right] \cdot C_q^{-1} w' + w' \left[\frac{1}{2} \right] \cdot C_q^{-1} w, \\ \tilde{d}'((v, w), (v', w')) &= d((v, w), (v', w')) + \tilde{d}'_W(w, w'). \end{aligned}$$

Our definitions are slightly different from those of [Nak04]. It follows from our definitions that $\tilde{d}'((0, w), (0, w')) = \tilde{d}'_W(w, w')$.

Lemma 5.1.1. The form $\tilde{d}'((v, w), (v', w'))$ equals $\tilde{d}'((0, w - C_q v), (0, w' - C_q v'))$.

Proof. We have

$$\begin{aligned}
 \tilde{d}'((v, w), (v', w')) &= (w - C_q v) \cdot v'[-\frac{1}{2}] + v \cdot w'[-\frac{1}{2}] - w[\frac{1}{2}] \cdot C_q^{-1} w' \\
 &= (w - C_q v) \cdot v'[-\frac{1}{2}] + v \cdot C_q C_q^{-1} w'[-\frac{1}{2}] - w[\frac{1}{2}] \cdot C_q^{-1} w' \\
 &\stackrel{(6)}{=} (w - C_q v) \cdot v'[-\frac{1}{2}] + C_q v \cdot C_q^{-1} w'[-\frac{1}{2}] - w[\frac{1}{2}] \cdot C_q^{-1} w' \\
 &= -(w - C_q v)[\frac{1}{2}] \cdot C_q^{-1} (w' - C_q v') \\
 &= \tilde{d}'((0, w - C_q v), (0, w' - C_q v')).
 \end{aligned}$$

□

Define $e_{k,a}$ to be the unit $I \times \mathbb{Z}$ -graded vector concentrated at the degree (k, a) . Define $p_{i,j} = \dim \text{Hom}_{\text{mod } \mathbb{C}Q}(P_i, P_j)$.

Lemma 5.1.2. 1) For any $a \in \mathbb{Z}$ and $i, j \in I$, we have

$$(26) \quad \mathcal{E}'(e_{i,a}, e_{j,a}) = 0,$$

$$(27) \quad \mathcal{E}'(e_{i,a}, e_{j,a-1}) = -p_{ji},$$

$$(28) \quad \mathcal{E}'(e_{i,a-1} + e_{i,a}, e_{j,a-1} + e_{j,a}) = p_{ij} - p_{ji}.$$

2) Let $B_{\tilde{Q}}$ be the $2n \times 2n$ -matrix associated with \tilde{Q}_1^z the level 1 ice quiver with z -pattern whose principal part is Q . Define Λ^z to be the $2n \times 2n$ matrix such that for $1 \leq i, j \leq n$, we have

$$(29) \quad \Lambda^z_{i,j} = -\mathcal{E}'(e_{i,0}, e_{j,0}) = 0,$$

$$(30) \quad \Lambda^z_{i+n,j} = -\mathcal{E}'(e_{i,-1} + e_{i,0}, e_{j,0}) = -p_{ij},$$

$$(31) \quad \Lambda^z_{i,j+n} = -\mathcal{E}'(e_{i,0}, e_{j,-1} + e_{j,0}) = p_{ji},$$

$$(32) \quad \Lambda^z_{i+n,j+n} = -\mathcal{E}'(e_{i,-1} + e_{i,0}, e_{j,-1} + e_{j,0}) = -p_{ij} + p_{ji}.$$

Then Λ^z is the inverse of $-B_{\tilde{Q}}$.

Proof. 1) The claim follows from straightforward computation.

2) Apply the results of 1) for $a = 0$. The entries of $B_{\tilde{Q}}$ are given by

$$\begin{aligned}
 (B_{\tilde{Q}})_{i,j} &= b_{ij}, \\
 (B_{\tilde{Q}})_{i+n,j} &= -\delta_{ij} + [b_{ji}]_+, \\
 (B_{\tilde{Q}})_{i,j+n} &= -(B_{\tilde{Q}})_{j+n,i}, \\
 (B_{\tilde{Q}})_{i+n,j+n} &= 0,
 \end{aligned}$$

for any $1 \leq i, j \leq n$. Here $[]_+$ is defined to be $\max\{ \cdot, 0 \}$.

We have, for any $1 \leq i, k \leq n$,

$$\begin{aligned}
(-\Lambda^z B_{\tilde{Q}})_{i,k} &= \sum_{1 \leq j \leq n} \Lambda^z_{i,j+n} \cdot (-B_{\tilde{Q}})_{j+n,k} \\
&= \sum_{1 \leq j \leq n} p_{ji} \delta_{jk} - \sum_{1 \leq j \leq n} p_{ji} [b_{kj}]_+ \\
&= p_{ki} - \sum_j [b_{kj}]_+ p_{ji} \\
&= \delta_{ik},
\end{aligned}$$

$$\begin{aligned}
(-\Lambda^z B_{\tilde{Q}})_{i+n,k} &= \sum_{1 \leq j \leq n} (-\Lambda^z_{i+n,j}) \cdot (B_{\tilde{Q}})_{j,k} + \sum_{1 \leq j \leq n} (-\Lambda^z_{i+n,j+n}) \cdot (B_{\tilde{Q}})_{j+n,k} \\
&= \sum_{1 \leq j \leq n} p_{ij} b_{jk} + \sum_{1 \leq j \leq n} (p_{ij} - p_{ji})(-\delta_{jk} + [b_{kj}]_+) \\
&= \sum_{j:j>k} p_{ij}(-b_{kj}) + \sum_{j:j<k} p_{ij} b_{jk} + (-p_{ik} + p_{ki}) \\
&\quad + \sum_j (p_{ij} - p_{ji}) [b_{kj}]_+ \\
&= \left(\sum_{j:j>k} p_{ij}(-b_{kj}) + \sum_j p_{ij} [b_{kj}]_+ \right) + \left(\sum_{j:j<k} p_{ij} b_{jk} - p_{ik} \right) \\
&\quad + \left(p_{ki} - \sum_j [b_{kj}]_+ p_{ji} \right) \\
&= 0 - \delta_{ik} + \delta_{ik} \\
&= 0,
\end{aligned}$$

$$(-\Lambda^z B_{\tilde{Q}})_{i,k+n} = (\Lambda^z B_{\tilde{Q}})_{k+n,i} = 0,$$

$$\begin{aligned}
(-\Lambda^z B_{\tilde{Q}})_{i+n,k+n} &= \sum_{1 \leq j \leq n} p_{ij} (\delta_{kj} - [b_{jk}]_+) \\
&= p_{ik} - \sum_j p_{ij} [b_{jk}]_+ \\
&= \delta_{ik}.
\end{aligned}$$

□

5.2. Deformed Grothendieck ring. Recall that, by Proposition 4.3.5, we have the stratification

$$\mathcal{M}_0^\bullet(w) = \cup_v \mathcal{M}_0^\bullet(v, w) = \sqcup_{v:(v,w) \text{ is } l\text{-dominant}} \mathcal{M}_0^{\bullet \text{reg}}(v, w).$$

Let $IC(v, w)$ denote the intersection cohomology sheaf associated with $\mathcal{M}_0^{\bullet \text{reg}}(v, w)$ for l -dominant pair (v, w) and $1_{\mathcal{M}^\bullet(v,w)}$ the perverse sheaf $\mathbb{C}[\dim \mathcal{M}^\bullet(v, w)]$. Applying the decomposition theorem [BBD82] to the sheaf $\pi(v, w) = \pi_!(1_{\mathcal{M}^\bullet(v,w)})$, we obtain the following result.

Theorem 5.2.1. *We have a decomposition*

$$(33) \quad \pi(v, w) = \bigoplus_{v':(v',w) \text{ is } l\text{-dominant}, d \in \mathbb{Z}} IC(v', w)[d]^{\oplus a_{v',w}^d}.$$

Moreover, we have $a_{v,v';w}^d \in \mathbb{Z}_{\geq 0}$, $a_{v,v';w}^d = a_{v,v';w}^{-d}$, $a_{v,v;w}^d = \delta_{d0}$ for l -dominant (v, w) , and $a_{v,v';w}^d \neq 0$ only if $(v, w) \leq (v', w)$.

Proof. By Proposition 4.3.5, Proposition 4.3.12, Corollary 4.3.15, the summands appearing in the decomposition of $\pi_!(1_{\mathcal{M}^\bullet(v,w)})$ are shifts of simple perverse sheaves generated by local systems over $\mathcal{M}_0^{\bullet\text{reg}}(v', w)$ for l -dominant pairs (v', w) .

The argument in the proof of [Nak01a, Theorem 14.3.2] implies the following results:

- (1) $a_{v,v;w}^d = \delta_{d0}$ if (v, w) is l -dominant, since π is an isomorphism between $\pi^{-1}\mathcal{M}_0^{\bullet\text{reg}}(v, w) \subset \mathcal{M}^\bullet(v, w)$ and $\mathcal{M}_0^{\bullet\text{reg}}(v, w)$;
- (2) Because we have Theorem 4.3.4, the transversal slice technique in the proof of [Nak01a, Theorem 14.3.2] is effective. It follows that the decomposition summands are shifts of simple perverse sheaves $IC(v', w)$ generated by trivial local systems.

Finally, since $\mathbf{1}_{\mathcal{M}^\bullet(v,w)}$ is invariant under the Verdier duality D , the sheaf $\pi(v, w)$ is invariant as well, and therefore we have $a_{v,v';w}^d = a_{v,v';w}^{-d}$. \square

We construct a positive Laurent polynomial from the coefficients $a_{v,v';w}$:

$$a_{v,v';w}(t) = \sum_{d \in \mathbb{Z}} a_{v,v';w}^d t^d \in \mathbb{N}[t^\pm].$$

Theorem 4.3.4 implies that we have $a_{v,v';w}(t) = a_{v-v',0;w-C_q v}(t)$.

For any given w , we have a finite set by Remark 4.3.14:

$$\mathcal{P}_w = \{IC(v, w) \mid \text{the pair } (v, w) \text{ is } l\text{-dominant,}\}.$$

As usual, we consider the bounded derived category $\mathcal{D}_c(\mathcal{M}_0^\bullet(w))$ of constructible sheaves on $\mathcal{M}_0^\bullet(w)$ and its full subcategory \mathcal{Q}_w whose objects are the direct sums of $IC(v, w)[d]$, $IC(v, w) \in \mathcal{P}_w$, $d \in \mathbb{Z}$. Let \mathcal{K}_w be the Grothendieck group of \mathcal{Q}_w . It is naturally a free $\mathbb{Z}[t^\pm]$ -module whose module structure is given by $t(L) = (L[1])$, $L \in \mathcal{K}_w$. The Verdier duality D induces an involution $(\bar{\quad})$ on \mathcal{K}_w .

By Theorem 5.2.1, the Grothendieck group \mathcal{K}_w has a $\mathbb{Z}[t^\pm]$ -basis $\{IC(v, w)\}$ and a $\mathbb{Z}[t^\pm]$ -basis $\{\pi(v, w) \mid (v, w) \text{ is } l\text{-dominant}\}$. It follows that the group $\mathcal{K}_w^* = \text{Hom}_{\mathbb{Z}[t^\pm]}(\mathcal{K}_w, \mathbb{Z}[t^\pm])$ has two dual bases $\{L(v, w)\}$ and $\{\chi(v, w)\}$ respectively. Define the third basis

$$\{M(v, w) \mid (v, w) \text{ is } l\text{-dominant}\}$$

whose elements are given by

$$\langle M(v, w), L \rangle = \sum_k t^{\dim \mathcal{M}_0^{\bullet\text{reg}}(v,w) - k} \dim H^k(i_{x_{v,w}}^! L),$$

where $x_{v,w}$ is any regular point in $\mathcal{M}_0^{\bullet\text{reg}}(v, w)$, $i_{x_{v,w}}$ the inclusion, $L \in \mathcal{Q}_w$, and $\langle \cdot, \cdot \rangle$ the natural pairing. Notice that, by Theorem 4.3.4, $M(v, w)$ does not depend on the choice of $x_{v,w}$. Furthermore, we have

$$\langle M(v', w), IC(v, w) \rangle = \langle M(v'^\perp, w^\perp), IC(v^\perp, w^\perp) \rangle.$$

The properties of perverse sheaves imply

$$(34) \quad L(v, w) \in M(v, w) + \sum_{(v', w) < (v, w)} t^{-1} \mathbb{Z}[t^{-1}] M(v', w).$$

In particular, $\{M(v, w)\}$ is a $\mathbb{Z}[t^\pm]$ -basis.

Following [Nak04, Section 8], let us denote the inverse basis transform by

$$(35) \quad M(v, w) = L(v, w) + \sum_{(v', w) < (v, w)} Z_{v, v'; w}(t) L(v', w),$$

where $Z_{v, v'; w}(t) \in t^{-1} \mathbb{Z}[t^{-1}]$.

As in [Nak11, 3.3], we define

$$(36) \quad \mathcal{R}_t = \{f = (f_w) \in \prod_w \mathcal{K}_w^* \mid \langle f_w, IC(v, w) \rangle = \langle f_{w^\perp}, IC(v^\perp, w^\perp) \rangle, \forall IC(v, w) \in \mathcal{P}_w\}.$$

It is a free $\mathbb{Z}[t^\pm]$ -module. Define $M(w) = (f_{w'})_{w'}$ to be the element in \mathcal{R}_t such that $f_w = M(0, w)$ and $L(w) = (f_{w'})_{w'}$ the element such that $f_w = L(0, w)$. Then $\{M(w)\}$ and $\{L(w)\}$ are $\mathbb{Z}[t^\pm]$ -bases of \mathcal{R}_t .

Next, we endow \mathcal{R}_t with an associative multiplication \otimes . By [VV03] (cf. also [Nak11, Section 3.5]), there is a restriction functor for any given decomposition $w = w^1 + w^2$:

$$\widetilde{\text{Res}}_{w^1, w^2}^w : \mathcal{D}_c(\mathcal{M}_0^\bullet(w)) \rightarrow \mathcal{D}_c(\mathcal{M}_0^\bullet(w^1)) \times \mathcal{D}_c(\mathcal{M}_0^\bullet(w^2)).$$

In particular, $\widetilde{\text{Res}}_{w^1, w^2}^w(\pi(v, w))$ equals

$$\oplus_{v^1 + v^2 = v} \pi(v^1, w^1) \boxtimes \pi(v^2, w^2) [d((v^2, w^2), (v^1, w^1)) - d((v^1, w^1), (v^2, w^2))].$$

Define the functors Res^w to the the shifted sums $\sum_{w^1 + w^2 = w} \widetilde{\text{Res}}_{w^1, w^2}^w[-\mathcal{E}'(w^1, w^2)]$. Then they are compatible with Theorem 4.3.4 and induce a multiplication \otimes of \mathcal{R}_t .

Theorem 5.2.2. *The structure constants of the basis $\{L(w)\}$ are positive:*

$$(37) \quad L(w^1) \otimes L(w^2) = \sum_{w^3} b_{w^1, w^2}^{w^3}(t) L(w^3), \quad b_{w^1, w^2}^{w^3}(t) \in \mathbb{N}[t^\pm].$$

Proof. The statement follows from the argument of [VV03, Section 5.1 Lemma 1(b)]. \square

5.3. Rings of characters. Define the ring

$$(38) \quad \widehat{\mathcal{Y}} = \mathbb{Z}[t^\pm][[W_i(a), V_i(b)]_{i \in \mathbb{1}, a \in \mathbb{Z}, b \in (\mathbb{Z} + \frac{1}{2})}],$$

where $t, W_i(a), V_i(b)$ are indeterminates. We denote its product by \cdot , and often omit this notation.

For any (v, w) , we let $m(v, w)$ denote the monomial

$$m = m(v, w) = \prod_{i, a} W_i(a)^{w_i(a)} \prod_{i, b} V_i(b)^{v_i(b)}.$$

Endow $\widehat{\mathcal{Y}}$ with the twisted product $*$ and the bar involution $\overline{(\)}$ such that we have

$$(39) \quad \bar{t} = t^{-1}, \quad \bar{m} = m,$$

$$(40) \quad m^1 * m^2 = t^{-\tilde{d}'(m^1, m^2) + \tilde{d}'(m^2, m^1)} m^1 m^2,$$

where m, m^1, m^2 are any monomials, and we define

$$\tilde{d}'(m^1(v^1, w^1), m^2(v^2, w^2)) = \tilde{d}'((v^1, w^1), (v^2, w^2)).$$

We follow the notation of Section 5.2. Moreover, as in Theorem 4.3.4, we write $\mathbf{m}^\bullet_x(v, w)$ for the fibre $\pi^{-1}(x)$ of a point x under $\pi : \mathcal{M}^\bullet(v, w) \rightarrow \mathcal{M}_0^\bullet(v, w)$. Notice that the homology of the fibre $H_*(\mathbf{m}^\bullet_{x_{v,w}}(v', w), \mathbb{C})$ is isomorphic to $H^{\dim \mathcal{M}^\bullet(v', w) - *}(i_{x_{v,w}}^! \pi(v', w), \mathbb{C})$, cf. [CG97, 8.5.4]. Therefore, Theorem 4.3.4 implies that

$$\langle M(v, w), \pi(v', w) \rangle = \langle M(v^\perp, w^\perp), \pi(v' - v^0, w^\perp) \rangle.$$

Definition 5.3.1 (*t-analogue of q-characters*). For any given w , define $\widehat{\chi}_{q,t}(\)$ to be the $\mathbb{Z}[t^\pm]$ -linear map from \mathcal{K}_w^* to $\widehat{\mathcal{Y}}$ such that

$$(41) \quad \widehat{\chi}_{q,t}(\) = \sum_v \langle \ , \pi(v, w) \rangle W^w V^v.$$

We can compute

$$\begin{aligned} \widehat{\chi}_{q,t}(L(v, w)) &= \sum_{v'} a_{v', v; w}(t) W^w V^{v'} \\ &= \sum_{v'-v} a_{v'-v, 0; w}(t) W^w V^{v'}. \end{aligned}$$

Since $W^w V^{v'}$ and $a_{v', v; w}(t)$ are bar-invariant, $\widehat{\chi}_{q,t}(\)$ is bar-invariant as well. Similarly, we have

$$\begin{aligned} M^{\widehat{\mathcal{Y}}}(v, w) &= \widehat{\chi}_{q,t}(M(v, w)) \\ &= \sum_{v', k} t^{\dim \mathcal{M}^\bullet(v, w) - k} H^k(i_{x_{v,w}}^! \pi(v', w), \mathbb{C}) W^w V^{v'} \\ &= \sum_{v', k} t^{\dim \mathcal{M}^\bullet(v, w) - \dim \mathcal{M}^\bullet(v', w) - k} H^k(i_{x_{v,w}}^! \pi(v', w), \mathbb{C}) W^w V^{v'} \\ &= \sum_{v', l} t^{\dim \mathcal{M}^\bullet(v, w) - \dim \mathcal{M}^\bullet(v', w) - l} H_l(\mathbf{m}^\bullet_{x_{v,w}}(v', w), \mathbb{C}) W^w V^{v'} \\ &\stackrel{4.3.20}{=} \sum_{v'} t^{\dim \mathcal{M}^\bullet(v, w) - \dim \mathcal{M}^\bullet(v', w)} P_t(\mathbf{m}^\bullet_{x_{v,w}}(v', w)) W^w V^{v'} \\ &= \sum_{v'-v} t^{-\dim \mathcal{M}^\bullet(v'-v, w - C_q v)} P_t(\mathcal{L}^\bullet(v'-v, w - C_q v)) W^w V^{v'}. \end{aligned}$$

Here $P_t(\)$ is the Poincaré polynomial $\sum_k (-t)^k \dim H_k(\)$.

Definition 5.3.2 (dual PBW elements). For any given w , define $M^{\widehat{\mathcal{Y}}}(w)$ to be the generating series

$$(42) \quad M^{\widehat{\mathcal{Y}}}(w) = \widehat{\chi}_{q,t}(M(0, w)) = \sum_v P_t(\mathcal{L}^\bullet(v, w)) t^{-d((v, w), (v, w))} W^w V^v.$$

Notice that the summation runs over all v and (v, w) is not necessarily l -dominant.

Remark 5.3.3. *By Theorem 4.3.20, the parameter $-t$ in $P_t(\cdot)$ in Definition 5.3.2 can be replaced by t .*

Let $E(\cdot; x, y) \in \mathbb{Z}[x, y]$ be the virtual Hodge Polynomial, cf. e.g. [HV08]. Define $p_t(\cdot)$ to be the virtual Poincaré polynomial (called the E -polynomial in [Qin10]), i.e. $p_t(\mathcal{L}^\bullet(v, w)) = E(\mathcal{L}^\bullet(v, w); t, t)$. It is additive with respect to α -partitions in the sense of [Nak01a, Section 7.1], and multiplicative with respect to fibre products.

Proposition 5.3.4 ([Nak04, Lemma 5.2]). *We have $p_t(\mathcal{L}^\bullet(v, w)) = P_t(\mathcal{L}^\bullet(v, w))$ and $p_t(\mathcal{M}^\bullet(v, w)) = P_t(\mathcal{M}^\bullet(v, w))$.*

Recall that, in \mathcal{K}_w^* , we have $M(v, w) = \sum_{v'} Z_{(v,w)(v',w)}(t) L(v', w)$, where the matrix of coefficients $(Z_{(v,w)(v',w)})$ is an upper uni-triangular matrix with respect to the dominance order. Therefore, we can solve the following equation recursively

$$\overline{M(v, w)} = \sum_{v':(v',w) \leq (v,w)} u_{(v,w),(v',w)} M(v', w),$$

where the coefficients $u_{(v,w),(v',w)} \in \mathbb{Z}[t^\pm]$.

Finally, we apply $\widehat{\chi}_{q,t}$ to the above equation and obtain

$$(43) \quad \overline{M^{\widehat{\mathcal{Y}}}(v, w)} = \sum_{v':(v',w) \leq (v,w)} u_{(v,w),(v',w)} M^{\widehat{\mathcal{Y}}}(v', w),$$

Remark 5.3.5. *In fact, we can compute the coefficient matrix $(u_{(v,w)(v',w)})$ and the basis transition matrix $(Z_{(v,w)(v',w)})$ from $P_t(\mathcal{L}^\bullet(v, w))$, cf. [Nak04, Section 8].*

Proposition 5.3.6 ([Nak04, Proposition 6.2]). *If for all $i, j \in I$, $a, b \in \mathbb{Z}$, $a > b$, either $(w^1)_i(a)$ or $(w^2)_j(b)$ vanishes, then we have*

$$(44) \quad M^{\widehat{\mathcal{Y}}}(w^2) * M^{\widehat{\mathcal{Y}}}(w^1) = t^{\mathcal{E}'(w^1, w^2)} M^{\widehat{\mathcal{Y}}}(w^1 + w^2).$$

Proof. As in the proof of [Nak04, Proposition 6.2], for any monomials $m^1 = W^{w^1} V^{v^1}$, $m^2 = W^{w^2} V^{v^2}$, we have a complex vector bundle $\widetilde{\mathcal{Z}}^\bullet(v^1, w^1; v^2, w^2)$ of rank $\text{Ker } \tau^{21} / \text{Im } \sigma^{21} = d((v^1, w^1), (v^2, w^2))$ over each $\mathcal{L}^\bullet(v^1, w^1) \times \mathcal{L}^\bullet(v^2, w^2)$. The set of $\widetilde{\mathcal{Z}}^\bullet(v^1, w^1; v^2, w^2)$ for various v^1, v^2 , gives a stratification of

$\mathcal{L}^\bullet(v^1 + v^2, w^1 + w^2)$, cf. [Nak01b, 6.12]. Using the additivity and multiplicativity of $p_t(\cdot)$, we deduce

$$\begin{aligned}
 & M^{\widehat{\mathcal{Y}}}(w^2) * M^{\widehat{\mathcal{Y}}}(w^1) \\
 &= \sum_{v^1, v^2} p_t(\mathcal{L}^\bullet(v^1, w^1)) \times p_t(\mathcal{L}^\bullet(v^2, w^2)) t^{-d(m^1, m^1) - d(m^2, m^2)} m^2 * m^1 \\
 &= \sum_{v^1, v^2} p_t(\mathcal{L}^\bullet(v^1, w^1) \times \mathcal{L}^\bullet(v^2, w^2)) t^{-d(m^1, m^1) - d(m^2, m^2)} \\
 &\quad t^{\widetilde{d}'(m^1, m^2) - \widetilde{d}'(m^2, m^1)} m^1 m^2 \\
 &= \sum_{v^1, v^2} p_t(\widetilde{\mathcal{Z}}^\bullet(v^1, w^1; v^2, w^2)) t^{-2d(m^1, m^2)} t^{-d(m^1, m^1) - d(m^2, m^2)} \\
 &\quad t^{\mathcal{E}'(w^1, w^2) + d(m^1, m^2) - d(m^2, m^1)} m^1 m^2 \\
 &= \sum_{v^1, v^2} p_t(\widetilde{\mathcal{Z}}^\bullet(v^1, w^1; v^2, w^2)) t^{\mathcal{E}'(w^1, w^2)} t^{-d(m^1 m^2, m^1 m^2)} m^1 m^2 \\
 &= \text{RHS.}
 \end{aligned}$$

□

In order to study quantum cluster algebras, we also need the following ring of formal power series

$$(45) \quad \mathcal{Y} = \mathbb{Z}[t^\pm][[Y_i(a)^\pm]]_{i \in I, a \in \mathbb{Z}},$$

where t , $Y_i(a)$ are indeterminates. We often omit its usual product \cdot . We associate the monomial $m(v, w) = Y^{w - C_q v}$ to any given pair (v, w) . Then similar to the case of $\widehat{\mathcal{Y}}$, we endow \mathcal{Y} with a bar involution $(\bar{\cdot})$ and a twisted product $*$ arising from the bilinear form \widetilde{d}' .

Define the $\mathbb{Z}[t^\pm]$ -linear map $\widehat{\Pi}$ from $\widehat{\mathcal{Y}}$ to \mathcal{Y} such that for any (v, w) , it sends any monomial $m(v, w) \in \widehat{\mathcal{Y}}$ to the monomial $m(v, w) = Y^{w - C_q v} \in \mathcal{Y}$. Then it is a ring homomorphism. Define the t -analogue of the q -character map to be the $\mathbb{Z}[t^\pm]$ -linear map $\chi_{q,t}(\cdot)$ from \mathcal{R}_t to \mathcal{Y} given by

$$(46) \quad \chi_{q,t}(\cdot) = \sum_v \langle \cdot, \pi(v, w) \rangle Y^{w - C_q v}.$$

Theorem 5.3.7. 1) $\widehat{\chi}_{q,t}(\cdot)$ is injective.

2) $\chi_{q,t}(\cdot)$ is an injective algebra homomorphism from \mathcal{R}_t to \mathcal{Y} .

Proof. The statements follow from the same arguments in [VV03] and [Nak04, Theorem 3.5]. □

6. RESULTS ON GENERIC CHARACTERS

From this section, let us restrict to level 1 case. Namely, we always assume $w \in \mathbb{N}^{I \times \{-1, 0\}}$. Furthermore, we consider only those v contained in $\mathbb{N}^{I \times \{-\frac{1}{2}\}} \simeq \mathbb{N}^I$. By putting these restrictions on w, v appearing in $\widehat{\chi}_{q,t}$ and $\chi_{q,t}$, we arrive at the *truncated character* maps $\widehat{\chi}_{q,t}^{\leq 0}$ and $\chi_{q,t}^{\leq 0}$. Proposition 5.3.6 and Theorem 5.3.7 are still valid after truncations, cf. [HL10, Proposition 6.1].

By default, we consider Q^{op} -representations, and the ice quiver \tilde{Q} is of level 1 with z -pattern.

For any $m = (m_i)_{i \in I} \in \mathbb{N}^I$, we define $I^m = \bigoplus_{i \in I} I_i^{\oplus m_i}$ (resp. $P^m = \bigoplus_{i \in I} P_i^{\oplus m_i}$) to be the corresponding injective (resp. projective) Q^{op} -representation.

For any $w, w' \in \mathbb{N}^{I \times \{-1, 0\}}$, we define the integer $r_{w, w'}$ to be given as in [KQ12, 3.2.1], which are determined by Fourier-Deligne-Sato transformations. Then we have

- $r_{w, w} = 1$,
- for any given w , only finitely many of the $r_{w, w'}$ are nonzero,
- the matrix $(r_{w, w'})$ is upper triangular with respect to the dominance order.

We define the *almost simple pseudo-module* $\mathbb{L}(w)$ to be the element in the Grothendieck group \mathcal{R}_t such that we have

$$(47) \quad \mathbb{L}(w) = \sum_{w'} r_{w, w'} L(w').$$

Denote the truncated qt -characters $\chi_{q, t}^{\leq 0}(\mathbb{L}(w))$ by $\mathbb{L}^{\mathcal{Y}}(w)$.

The following is the main theorem of the first part of [KQ12].

Theorem 6.0.8 ([KQ12]). *The truncated qt -characters of the almost simple pseudo-modules in \mathcal{Y} are given by*

$$(48) \quad \mathbb{L}^{\mathcal{Y}}(w) = \sum_v t^{-\dim(\text{Gr}_v^\sigma W)} P_t(\text{Gr}_v^\sigma W) Y^{w - C_q v},$$

where $\text{Gr}_v^\sigma W$ is the quiver Grassmannian determined by Q , v , w .

Definition 6.0.9. *For any given $w \in \mathbb{N}^{I \times \{-1, 0\}}$, its pure coefficient part ${}^f w$ is the maximal element in $\text{span}_{\mathbb{N}}\{e_{i, -1} + e_{i, 0} | i \in I\}$ such that $w - {}^f w \in \mathbb{N}^{I \times \{-1, 0\}}$. We denote the difference $w - {}^f w$ by ${}^\phi w$ and call it the coefficient-free part of w . Let \mathcal{J} denote the set $\{w | {}^f w = 0\}$.*

The truncated characters of almost simple pseudo-modules, which are also called the *generic characters*, have the following properties.

Proposition 6.0.10 ([KQ12]). *1) If $\mathbb{L}^{\mathcal{Y}}(w)$ is a quantum cluster monomial, then we have $\mathbb{L}(w) = L(w)$.*

2) In the notation of Definition 6.0.9, we have a factorization

$$(49) \quad \mathbb{L}^{\mathcal{Y}}(w) = \mathbb{L}^{\mathcal{Y}}({}^\phi w) \cdot \mathbb{L}^{\mathcal{Y}}({}^f w).$$

7. FROM CHARACTERS TO THE QUANTUM TORUS

In this section, we compare the ring of formal power series $\widehat{\mathcal{Y}}$ in which qt -characters live with the quantum tori $D(\mathcal{T})$ and \mathcal{T} in which quantum cluster algebras live. Although our construction fails to preserve some properties of the products of qt -characters, we find that the failures are bearable and controllable. This observation will play a crucial role in later sections.

7.1. Dual PBW basis elements. As in Section 2.2, let \tilde{Q} be an ice quiver whose principal part Q is acyclic.

Denote $I_i[-1]$ by T_i . Following the convention of section 3, for any M in the cluster category which appears in a triangle

$$I^m[-1] \rightarrow I^{m'}[-1] \rightarrow M \rightarrow I^m$$

for some $m, m' \in \mathbb{N}^m$, its index $\text{ind } M$ is $m - m' \in \mathbb{Z}^n \oplus \mathbb{N}^{m-n}$, cf. [Pla10a].

By abuse of notation, we use $\text{ind}(\)$ to denote the map from $\mathbb{N}^{I \times \{-1,0\}}$ to $\mathbb{Z}^n \oplus \mathbb{N}^{m-n}$ which sends $w = (w_{i,a}) \in \mathbb{N}^{I \times \{-1,0\}}$ to $\text{ind}(\oplus_{i,a} T_i[a]^{w_{i,a}})$.

When \tilde{Q} is of z -pattern in the sense of [KQ12], the index map $\text{ind}(\)$ is denoted by $\text{ind}^z(\)$ and the compatible pair by (\tilde{B}^z, Λ^z) . In this section, we shall see how the variation of the coefficients attached to the principal part Q affects our computation.

Lemma 7.1.1. *We have, for $1 \leq k \leq n$,*

$$(50) \quad \text{ind}^z(e_{k,0}) = e_k,$$

$$(51) \quad \text{ind}^z(e_{k,-1}) = e_{k+n} - e_k.$$

Proof. The first equation is clear.

Consider the triangle

$$T_k \xrightarrow{f} T_{k+n} \rightarrow M \rightarrow \Sigma T_k$$

in $\mathcal{C}_{\tilde{Q}}$, where f is non-zero. Then f is a left approximation of T_k in $\text{add}(T_{n+i}, 1 \leq i \leq n)$ in the sense of [IY08]. It follows that M is the coefficient-free object lifting $T_i[1]$ and so we have $\text{ind}^z(e_{k,-1}) = \text{ind } M = e_{k+n} - e_k$. \square

Let pr_n denote the projection of \mathbb{Z}^m (respectively \mathbb{Z}^{2n}) to the first n coordinates. Then $\text{pr}_n \text{ind}$ is independent of the choice of the coefficient $\tilde{Q} - Q$.

Lemma 7.1.2. *We have*

$$(52) \quad \text{ind}^z(w - C_q v) = \text{ind}^z(w) + \tilde{B}^z v.$$

Proof. It suffices to check the case of $w = 0$ and $v = e_{k,-\frac{1}{2}}$, $1 \leq k \leq n$. We have

$$\begin{aligned} \text{ind}^z(-C_q v) &= -\text{ind}^z(e_{k,-1} + e_{k,0}) + \sum_{i:1 \leq i < k} b_{ik} \text{ind}^z(e_{i,0}) \\ &\quad + \sum_{j:k < j \leq n} b_{kj} \text{ind}^z(e_{j,-1}) \\ &= -e_{k+n} + \sum_{1 \leq i < k} b_{ik} e_i + \sum_{k < j \leq n} b_{kj} (e_{j+n} - e_j) \\ &= b_{k+n,k} e_{k+n} + \sum_{1 \leq i < k} b_{ik} e_i + \sum_{k < j \leq n} b_{j+n,k} e_{j+n} + \sum_{k < j \leq n} b_{jk} e_j \\ &= \tilde{B}^z e_k. \end{aligned}$$

\square

Let δ be a strictly positive integer. Denote the product of δ with the rank m identity matrix I_m by D . Let (\tilde{B}, Λ) be a compatible pair such that $\Lambda(-\tilde{B}) = \begin{bmatrix} D \\ 0 \end{bmatrix}$.

Define $N = \max\{2n, m\}$. Extend Λ into an $N \times N$ matrix by putting zero entries in the extra rows and columns indexed by either $\{m+1, \dots, N\}$ or $\{n+1, \dots, N\}$. We extend the $m \times n$ matrix \tilde{B} and the $2n \times n$ matrix \tilde{B}^z to $N \times N$ matrices similarly. Notice that $\Lambda(e_i, \tilde{B}v) = 0$, for any $n+1 \leq i \leq N$ and $v \in \mathbb{N}^n$.

Definition 7.1.3. *The double quantum torus is the Laurent polynomial ring*

$$(53) \quad D(\mathcal{T}) = \mathbb{Z}[q^{\pm \frac{1}{2}}][x_1^\pm, \dots, x_N^\pm, y_1^\pm, \dots, y_n^\pm],$$

together with the twisted product $*$ such that for any $g^1, g^2 \in \mathbb{Z}^N$, $v^1, v^2 \in \mathbb{Z}^n$, we have

$$x^{g^1} y^{v^1} * x^{g^2} y^{v^2} = q^{\frac{1}{2}\Lambda(g^1 + \tilde{B}v^1, g^2 + \tilde{B}v^2)} x^{g^1 + g^2} y^{v^1 + v^2}.$$

It has the bar involution $\overline{(\)}$ given by $\overline{(q^{\frac{1}{2}} x^g y^v)} = q^{-\frac{1}{2}} x^g y^v$.

Define the quantum torus \mathcal{T} to be the $\mathbb{Z}[q^{\pm \frac{1}{2}}]$ -subalgebra of $D(\mathcal{T})(+, *)$ generated by x_1^\pm, \dots, x_N^\pm . Recall that the coefficient ring is $\mathbb{Z}P[q^{\pm \frac{1}{2}}] = \mathbb{Z}[q^{\pm \frac{1}{2}}][x_{n+1}^\pm, \dots, x_m^\pm]$. Let $\mathbb{Z}P$ denote its semi-classical limit under the specialization $q^{\frac{1}{2}} \mapsto 1$.

Next, we construct a map from $\hat{\mathcal{Y}}$ to $D(\mathcal{T})$.

Definition 7.1.4 (Correspondence map). *The \mathbb{Z} -linear map cor from $\hat{\mathcal{Y}}$ to $D(\mathcal{T})$ is given by*

$$(54) \quad \text{cor}(t^\lambda W^w V^v) = q^{\frac{\delta}{2}\lambda} x^{\text{ind}(w)} y^v,$$

for any w, v , and integer λ .

Notice that cor commutes with the bar involutions. Although cor does not commute with twisted products $*$, we can measure the failure as below.

Lemma 7.1.5 (Failure 1). *We have, for any monomials $m^i = W^{w^i} V^{v^i}$, $i = 1, 2$,*

$$(55) \quad \begin{aligned} \text{cor}(W^{w^1} V^{v^1} * W^{w^2} V^{v^2}) &= q^{\frac{\delta}{2}\Lambda^z(\text{ind}^z(w^1), \text{ind}^z(w^2)) - \frac{1}{2}\Lambda(\text{ind}(w^1), \text{ind}(w^2))} \\ &\quad \text{cor}(W^{w^1} V^{v^1}) * \text{cor}(W^{w^2} V^{v^2}). \end{aligned}$$

In particular, the q -power does not depend on v^1, v^2 .

Proof. Notice that we always have $\Lambda(\text{ind } w, \tilde{B}v) = -\delta \text{pr}_n \text{ind } w \cdot v$, and $\Lambda(\tilde{B}v^1, \tilde{B}v^2) = \delta \langle v^1, v^2 \rangle - \delta \langle v^2, v^1 \rangle$, cf. [Qin10, 5.2.1]. Using Lemma 5.1.2

and 7.1.2, we obtain

$$\begin{aligned}
 \text{LHS} &= \text{cor}(t^{-\tilde{d}'(m^1, m^2) + \tilde{d}'(m^2, m^1)} m^1 m^2) \\
 &= \text{cor}(t^{-\mathcal{E}'(w^1 - C_q v^1, w^2 - C_q v^2)} m^1 m^2) \\
 &= q^{\frac{\delta}{2} \Lambda^z(\text{ind}^z(w^1 - C_q v^1), \text{ind}^z(w^2 - C_q v^2))} X^{\text{ind } w^1} Y^{v^1} X^{\text{ind } w^2} Y^{v^2} \\
 &= q^{\frac{\delta}{2} \Lambda^z(\text{ind}^z w^1 + \tilde{B}^z v_1, \text{ind}^z w^2 + \tilde{B}^z v_2)} q^{-\frac{1}{2} \Lambda(\text{ind } w^1 + \tilde{B} v^1, \text{ind } w^2 + \tilde{B} v^2)} \\
 &\quad X^{\text{ind } w^1} Y^{v^1} * X^{\text{ind } w^2} Y^{v^2} \\
 &= q^{\frac{\delta}{2} \Lambda^z(\text{ind}^z w^1, \text{ind}^z w^2)} q^{-\frac{1}{2} \Lambda(\text{ind } w^1, \text{ind } w^2)} X^{\text{ind } w^1} Y^{v^1} * X^{\text{ind } w^2} Y^{v^2} \\
 &= \text{RHS}.
 \end{aligned}$$

□

Definition 7.1.6. We define the $\mathbb{Z}[q^{\pm \frac{1}{2}}]$ -linear map $\widehat{\Pi}$ from $D(\mathcal{T})$ to \mathcal{T} such that it sends $x^g y^v$ to $x^{g + \tilde{B}v}$.

Notice that $\widehat{\Pi}$ is an algebra homomorphism with respect to both the usual products and the twisted products.

Let us define

$$(56) \quad M^{D(\mathcal{T})}(v, w) = \text{cor} \widehat{\chi}_{q,t}^{\leq 0}(M(v, w)),$$

$$(57) \quad M^{\mathcal{T}}(v, w) = \widehat{\Pi} M^{D(\mathcal{T})}(v, w).$$

When $v = 0$, we also denote them by $M^{D(\mathcal{T})}(w)$ and $M^{\mathcal{T}}(w)$. Explicitly, we have

$$\begin{aligned}
 M^{D(\mathcal{T})}(w) &= \sum_v P_{q^{\frac{\delta}{2}}}(\mathcal{L}(v, w)) q^{-\frac{\delta}{2} \dim \mathcal{M}^\bullet(v, w)} x^{\text{ind}(w)} y^v \\
 &= \sum_v \text{cor}(\langle M(0, w), \pi(v, w) \rangle) x^{\text{ind}(w)} y^v.
 \end{aligned}$$

Recall that we have a similar map from $\widehat{\mathcal{Y}}$ to \mathcal{Y} , which is also denoted by $\widehat{\Pi}$. In general, we do *not* have $\widehat{\Pi} \widehat{\chi}_{q,t} = \chi_{q,t} \widehat{\Pi}$. This failure is measured by the following result.

Lemma 7.1.7 (Failure 2). We have

$$(58) \quad M^{\mathcal{T}}(v, w) = M^{\mathcal{T}}(w - C_q v) x^{\tilde{B}v + \text{ind } C_q v}.$$

Proof. Straightforward calculation shows

$$\begin{aligned}
 M^{\mathcal{T}}(v, w) &= \sum_{v': v' - v \in \mathbb{N}^I} \widehat{\Pi} \text{cor}(\langle M(v, w), \pi(v', w) \rangle) x^{\text{ind}(w)} x^{\tilde{B}v'}. \\
 &= \sum_{v' - v} \widehat{\Pi} \text{cor}(\langle M(0, w - C_q v), \pi(v' - v, w) \rangle) x^{\text{ind}(w - C_q v)} x^{\tilde{B}(v' - v)} \\
 &\quad x^{\tilde{B}v + \text{ind}(w) - \text{ind}(w - C_q v)}. \\
 &= M^{\mathcal{T}}(w - C_q v) x^{\tilde{B}v + \text{ind } C_q v}.
 \end{aligned}$$

□

Notice that the correction factor $x^{\tilde{B}v + \text{ind } C_q v}$ is contained in $\mathbb{Z}P[q^{\pm \frac{1}{2}}]$.

Proposition 7.1.8. *We have*

$$(59) \quad \overline{M^{\mathcal{T}}(w)} = \sum_{(v,w) \leq (0,w)} u_{(0,w),(v,w)} x^{\tilde{B}v + \text{ind } C_q v} M^{\mathcal{T}}(w - C_q v).$$

Proof. If we apply $\widehat{\Pi}\text{cor}$ to (43), the statement follows from (58). \square

Proposition 7.1.9. *Fix w^1 and w^2 . If for all $i, j \in I$ and $a > b \in \mathbb{Z}$, either $(w^1)_i(a)$ or $(w^2)_j(b)$ vanishes, then the multiplicative property holds:*

$$M^{\mathcal{T}}(w^2) * M^{\mathcal{T}}(w^1) = q^{\frac{1}{2}\Lambda(\text{ind}(w^2), \text{ind}(w^1)) - \frac{\delta}{2}\Lambda^z(\text{ind}^z(w^2), \text{ind}^z(w^1))} q^{\frac{\delta}{2}\mathcal{E}'(w^1, w^2)} M^{\mathcal{T}}(w^1 + w^2).$$

Proof. Using (55), (58), and Proposition 5.3.6, we obtain

$$\begin{aligned} M^{\mathcal{T}}(w^2) * M^{\mathcal{T}}(w^1) &= \widehat{\Pi}(\text{cor}\widehat{\chi}_{q,t}^{\leq 0}(M(w^2)) * \text{cor}\widehat{\chi}_{q,t}^{\leq 0}(M(w^1))) \\ &= q^{\frac{1}{2}\Lambda(\text{ind}(w^2), \text{ind}(w^1)) - \frac{\delta}{2}\Lambda^z(\text{ind}^z(w^2), \text{ind}^z(w^1))} \\ &\quad \widehat{\Pi}\text{cor}(\widehat{\chi}_{q,t}^{\leq 0}(M(w^2)) * \widehat{\chi}_{q,t}^{\leq 0}(M(w^1))) \\ &= \text{RHS}. \end{aligned}$$

\square

Let \mathcal{A}_{sub}^q denote the vector space spanned by the standard basis elements $M^{\mathcal{T}}(w)$ over $\mathbb{Z}P[q^{\pm\frac{1}{2}}]$. The following is the main result of this section.

Proposition 7.1.10. *The vector space \mathcal{A}_{sub}^q is closed under the involution $\overline{(\)}$ and the twisted multiplication $*$.*

Proof. The first assertion follows from Proposition 7.1.8. It remains to verify the second one.

Recall that $w_i(a)$ is the (i, a) -th component of w , and $w(a) = \oplus_i w_i(a)$, where $i \in I$, $a \in \{-1, 0\}$. We have $M(w_i(a)) = L(w_i(a))$. Consequently, $M^{\mathcal{T}}(w_i(a)) = \widehat{\Pi}\text{cor}\widehat{\chi}_{q,t}^{\leq 0}(L(w_i(a)))$ is bar-invariant. Applying Proposition 7.1.9, we obtain that $M^{\mathcal{T}}(w(a))$ is bar-invariant.

Use Propositions 7.1.8 and 7.1.9. For any two elements $M^{\mathcal{T}}(w^1)$, $M^{\mathcal{T}}(w^2)$, up to specified invertible elements in $\mathbb{Z}[q^{\pm\frac{1}{2}}]$, $M^{\mathcal{T}}(w^1) * M^{\mathcal{T}}(w^2)$ becomes

$$\begin{aligned} &M^{\mathcal{T}}(w^1(0)) * M^{\mathcal{T}}(w^1(-1)) * M^{\mathcal{T}}(w^2(0)) * M^{\mathcal{T}}(w^2(-1)) \\ &= M^{\mathcal{T}}(w^1(0)) * \overline{\overline{M^{\mathcal{T}}(w^2(0)) * \overline{M^{\mathcal{T}}(w^1(-1))}} * M^{\mathcal{T}}(w^2(-1)) \\ &= M^{\mathcal{T}}(w^1(0)) * \overline{\overline{M^{\mathcal{T}}(w^2(0)) * \overline{M^{\mathcal{T}}(w^1(-1))}}} * M^{\mathcal{T}}(w^2(-1)) \\ &= M^{\mathcal{T}}(w^1(0)) * \overline{\overline{M^{\mathcal{T}}(w^2(0) + w^1(-1))}} * M^{\mathcal{T}}(w^2(-1)) \\ &= M^{\mathcal{T}}(w^1(0)) * \left(\sum_{w'} u_{w,w'} x^{\tilde{B}v + \text{ind } C_q v} M^{\mathcal{T}}(w') \right) * M^{\mathcal{T}}(w^2(-1)), \end{aligned}$$

where we write $w = w^2(0) + w^1(-1)$, $w' = w - C_q v$, $u_{w,w'} = u_{(0,w),(v,w)}$. The monomial $x^{\tilde{B}v + \text{ind } C_q v}$ is contained in $\mathbb{Z}P[q^{\pm\frac{1}{2}}]$ and quasi-commutes with the other factors. Therefore, up to specified invertible elements in $\mathbb{Z}[q^{\pm\frac{1}{2}}]$, the

above result becomes

$$\begin{aligned} & \sum_{w'} u_{w,w'} x^{\tilde{B}v + \text{ind } C_q v} M^{\mathcal{T}}(w^1(0)) * M^{\mathcal{T}}(w'(0)) * M^{\mathcal{T}}(w'(-1)) \\ & \quad * M^{\mathcal{T}}(w^2(-1)) \\ & = \sum_{w'} u_{w,w'} x^{\tilde{B}v + \text{ind } C_q v} M^{\mathcal{T}}(w^1(0) + w'(0) + w'(-1) + w^2(-1)). \end{aligned}$$

□

Remark 7.1.11. (1) *The above proof shows that the twisted product * of the standard basis elements is determined by Proposition 5.3.6 and 7.1.8.*

(2) *Notice that the map $\widehat{\Pi\text{cor}}_{\chi_{q,t}^{\leq 0}}$ is algebraic when \tilde{Q} is of z -pattern. So Theorem 9.1.2 below provides another proof to Proposition 7.1.10.*

7.2. Dual canonical basis elements. For any l -dominant pair (v, w) , define $L^{\mathcal{T}}(v, w) = \widehat{\Pi\text{cor}}_{\chi_{q,t}^{\leq 0}}(L(v, w))$ and denote it by $L^{\mathcal{T}}(w)$ when $v = 0$. Then $L^{\mathcal{T}}(w)$ is given by

$$L^{\mathcal{T}}(w) = \sum_v a_{v,0;w}(q^{\frac{\delta}{2}}) x^{\text{ind}(w) + \tilde{B}v},$$

where the Laurent polynomials $a_{v,0;w}(t) = \sum_{d \in \mathbb{Z}} a_{v,0;w}^d t^d$ are given by 33. We shall see all the quantum cluster monomials essentially take this form.

Lemma 7.2.1. *We have*

$$L^{\mathcal{T}}(v, w) = L^{\mathcal{T}}(w - C_q v) x^{\tilde{B}v + \text{ind } C_q v}.$$

Proof. 1) Similar to (58), we compute

$$\begin{aligned} L^{\mathcal{T}}(v, w) & = \sum_{v'} \widehat{\Pi\text{cor}}(\langle L(v, w), \pi(v', w) \rangle) x^{\text{ind}(w)} x^{\tilde{B}v'} \\ & = \sum_{v'-v} \widehat{\Pi\text{cor}}(\langle L(0, w - C_q v), \pi(v' - v, w - C_q v) \rangle) x^{\text{ind}(w - C_q v)} x^{\tilde{B}(v' - v)} x^{\tilde{B}v + \text{ind}(w) - \text{ind}(w - C_q v)} \\ & = L^{\mathcal{T}}(w - C_q v) x^{\tilde{B}v + \text{ind } C_q v}. \end{aligned}$$

□

We already know that $M(0, w) = \sum_v Z_{(0,w)(v,w)} L(v, w)$. If we apply $\widehat{\Pi\text{cor}}_{\chi_{q,t}^{\leq 0}}$ to it, we obtain

$$M^{\mathcal{T}}(w) = \sum_v Z_{(0,w)(v,w)} L^{\mathcal{T}}(w - C_q v) x^{\tilde{B}v + \text{ind } C_q v}.$$

7.3. Generic basis elements. The generic basis elements $\mathbb{L}^{\mathcal{T}}(w)$ are defined to be

$$(60) \quad \mathbb{L}^{\mathcal{T}}(w) = \sum_{v:(v,w) \text{ is } l\text{-dominant}} r_{w,w'} x^{\tilde{B}v + \text{ind } C_q v} L^{\mathcal{T}}(w - C_q v),$$

where $w' = w - C_q v$, and the integer $r_{w,w'}$ is given in [KQ12, 3.2.1]. We have

$$\begin{aligned}
\mathbb{L}^{\mathcal{T}}(w) &= \sum_v r_{w,w-C_q v} L^{\mathcal{T}}(w - C_q v) x^{\tilde{B}v + \text{ind } C_q v} \\
&= \sum_{v,v'} r_{w,w-C_q v} a_{v',0;w-C_q v} (q^{\frac{\delta}{2}}) x^{\text{ind}(w-C_q v) + \tilde{B}v'} x^{\tilde{B}v + \text{ind } C_q v} \\
&= \sum_{v,v'} r_{w,w-C_q v} a_{v'+v,v;w} (q^{\frac{\delta}{2}}) x^{\text{ind}(w)} x^{\tilde{B}(v+v')} \\
&= \sum_{v+v'=v''} b_{v''} (q^{\frac{\delta}{2}}) x^{\text{ind}(w)} x^{\tilde{B}v''},
\end{aligned}$$

where the coefficient $b_{v''} = \sum_{(v,w) \text{ is } l\text{-dominant}} r_{w,w-C_q v} \cdot a_{v'',v;w}$.

In particular, $b_{v''}$ is independent of the frozen pattern.

8. BASES OF ACYCLIC QUANTUM CLUSTER ALGEBRAS

We keep the assumptions on the ice quiver \tilde{Q} and the vectors v, w as in Section 7.

8.1. Generic basis. The results in Section 6 and 8.1 imply the following theorem.

Theorem 8.1.1. 1) *The generic basis elements have the following expansion*

$$(61) \quad \mathbb{L}^{\mathcal{T}}(w) = \sum_v P_{q^{\frac{\delta}{2}}}(\text{Gr}_v(\sigma W)) q^{-\frac{\delta}{2} \dim \text{Gr}_v(\sigma W)} x^{\text{ind } w + \tilde{B}v}.$$

2) *We have a factorization*

$$(62) \quad \mathbb{L}^{\mathcal{T}}(w) = \mathbb{L}^{\mathcal{T}}(\phi w) \cdot \mathbb{L}^{\mathcal{T}}(f w).$$

3) *If $\mathbb{L}^{\mathcal{T}}(w)$ becomes a quantum cluster monomial when we choose \tilde{Q} to be of z -pattern, then we have $\mathbb{L}^{\mathcal{T}}(w) = L^{\mathcal{T}}(w)$.*

For any given $w \in \mathcal{J}$, define $O(w)$ to be the unique object in the cluster category such that we have the following triangle

$$O(w) \rightarrow I^{w(-1)} \xrightarrow{f} I^{w(0)} \xrightarrow{O} (w)$$

where the map f is generic. We define the associated *generic quantum cluster characters* to be

$$(63) \quad \begin{aligned} \mathbb{L}^{\mathcal{A}}(w) &= \mathbb{L}^{\mathcal{T}}(w) \cdot x^{\text{ind } O(w) - \text{ind } w} \\ &= \sum_v P_{q^{\frac{1}{2}\delta}}(\text{Gr}_v(\sigma W)) q^{-\frac{1}{2} \dim \text{Gr}_v(\sigma W)} x^{\text{ind } O(w) + \tilde{B}v}. \end{aligned}$$

Remark 8.1.2. *We can denote $\mathbb{L}^{\mathcal{A}}(w)$ by $X_{O(w)}$. This definition naturally generalizes the quantum cluster character formula in [Qin10, Definition 1.2.1] to generic objects.*

It is not clear if one can extend this quantum character to arbitrary objects of the presentable cluster category in a reasonable way, for example, such that its image is still contained in \mathcal{A}^q . Some results for Dynkin quivers are discussed in [Din10].

Proposition 8.1.3. *For any $w \in \mathcal{J}$, we have $\text{pr}_n \text{ind } O(w) = \text{pr}_n \text{ind } w$. When the ice quiver \tilde{Q} is the level 1 ice quiver with z -pattern, we have $\text{ind } O(w) = \text{ind } w$.*

Proof. Because $\text{pr}_n \text{ind } O(w)$ and $\text{pr}_n \text{ind } w$ do not depend on the coefficients $\tilde{Q} - Q$, it suffices to prove the second statement.

First, the index $\text{ind } w$ is linear in w .

Second, the index $\text{ind } O(w)$ is linear with respects to the components in the canonical decomposition of the generic objects $O(w)$, which are either generic modules or the object $P_i[1]$, $i \in I$.

Therefore, it suffices to study the indices of modules. Let $O(w) = M$ be any Q^{op} -module with the minimal injective resolution

$$(64) \quad 0 \rightarrow M \xrightarrow{f} I^{w(-1)} \xrightarrow{g} I^{w(0)} \rightarrow 0.$$

Let B denote the Jacobi algebra of (\tilde{Q}, \tilde{W}) . View the resolution as a short exact sequence in the category of B^{op} -modules. Denote the simple B^{op} -modules by S_j , $1 \leq j \leq 2n$. If we apply $\text{Hom}_{B^{op}\text{-mod}}(S_j, \)$ to the above short exact sequence, we obtain a long exact sequence

$$\begin{aligned} \dots \rightarrow \text{Hom}_{B^{op}\text{-mod}}(S_j, M) \xrightarrow{f_0} \text{Hom}_{B^{op}\text{-mod}}(S_j, I^{w(-1)}) \rightarrow \text{Hom}_{B^{op}\text{-mod}}(S_j, I^{w(0)}) \\ \xrightarrow{w_0} \text{Ext}_{B^{op}\text{-mod}}^1(S_j, M) \xrightarrow{g_1} \text{Ext}_{B^{op}\text{-mod}}^1(S_j, I^{w(-1)}) \rightarrow \text{Ext}_{B^{op}\text{-mod}}^1(S_j, I^{w(0)}) \rightarrow \dots \end{aligned}$$

Then $(\text{ind } M)_j = -\dim \text{Hom}_{B^{op}\text{-mod}}(S_j, M) + \dim \text{Ext}_{B^{op}\text{-mod}}^1(S_j, M)$, cf. [Pal08].

Because M is supported on the principal part Q , $\dim \text{Hom}_{B^{op}\text{-mod}}(S_j, M)$ vanishes unless $1 \leq j \leq n$. The fact that (64) is the minimal injective resolution of Q^{op} -modules implies f_0 is an isomorphism. It follows that w_0 is injective.

When the quiver is of z -pattern, we have $\text{ind } I_i = -e_i + e_{i+n}$, cf. Lemma 7.1.1. Therefore $\text{Ext}_{B^{op}\text{-mod}}^1(S_j, I^{w(-1)})$ equals $w_{j-n}(-1)$ if $j > n$ and vanishes elsewhere. Furthermore, the fact that (64) is the minimal injective resolution of the Q^{op} -module M implies that for any $i \in I$, either $\dim \text{Ext}_{B^{op}\text{-mod}}^1(S_{i+n}, I^{w(-1)}) = w_i(-1)$ or $\dim \text{Ext}_{B^{op}\text{-mod}}^1(S_{i+n}, I^{w(0)}) = w_i(0)$ is zero. Therefore, g_1 is surjective.

Therefore, we have

$$\begin{aligned} (\text{ind } M)_j &= -\dim \text{Hom}_{B^{op}\text{-mod}}(S_j, I^{w(-1)}) + \dim \text{Hom}_{B^{op}\text{-mod}}(S_j, I^{w(0)}) \\ &\quad + \dim \text{Ext}_{B^{op}\text{-mod}}^1(S_j, I^{w(0)}) \\ &= \begin{cases} -w_j(-1) + w_j(0) & \text{if } j \leq n \\ w_j(0) + w_{j-n}(-1) & \text{if } j > n, \end{cases} \\ &= (\text{ind } w)_j. \end{aligned}$$

□

Remark 8.1.4 (Failure 3). *For general \tilde{Q} , we have*

$$(65) \quad \text{ind } w \neq \text{ind } O(w).$$

Lemma 8.1.5. *The elements in the coefficient ring $\mathbb{Z}P[q^{\pm \frac{1}{2}}]$ quasi-commute with $M^T(w)$, $L^T(w)$, $\mathbb{L}^T(w)$, $w \in \mathbb{N}^{I \times \{-1, 0\}}$.*

Proof. It follows from the fact that x_i , $n < i \leq N$, commutes with $x^{\tilde{B}v}$ for any $v \in \mathbb{N}^n$. \square

Proposition 8.1.6. *The set $\{\mathbb{L}^{\mathcal{T}}(w), w \in \mathcal{J}\}$ is a $\mathbb{Z}P[q^{\pm\frac{1}{2}}]$ -basis of the vector space \mathcal{A}_{sub}^q .*

Proof. Notice that the matrix $(r_{w,w'})$ is upper unitriangular. Furthermore, each element $\mathbb{L}^{\mathcal{T}}(w)$ divides into the composition of the coefficient free part $\mathbb{L}^{\mathcal{T}}(\phi w)$ and the pure coefficient part $\mathbb{L}^{\mathcal{T}}(f w) \in \mathbb{Z}P[q^{\pm\frac{1}{2}}]$. Therefore, \mathcal{A}_{sub}^q is spanned by $\{\mathbb{L}^{\mathcal{T}}(w), w \in \mathcal{J}\}$ over $\mathbb{Z}P[q^{\pm\frac{1}{2}}]$.

Take any vectors w, w' in \mathcal{J} , such that $w \neq w'$. By proposition 8.1.3, the vector $\text{pr}_n \text{ind}^z w$ is the index of a generic object in the cluster category of Q . Furthermore, we have $\text{pr}_n \text{ind} w \neq \text{pr}_n \text{ind} w'$, cf. the proof of Proposition 8.1.3 or [Pla10b]. Because $\text{ind} w$ is the leading term of $\mathbb{L}^{\mathcal{T}}(w)$ and \tilde{B} is of full rank, the set $\{\mathbb{L}^{\mathcal{T}}(w), w \in \mathcal{J}\}$ is $\mathbb{Z}P[q^{\pm\frac{1}{2}}]$ -linearly independent. \square

Proposition 8.1.7. *\mathcal{A}_{sub}^q equals \mathcal{A}^q .*

Proof. \mathcal{A}_{sub}^q is a subspace of \mathcal{A}^q because the latter contains $\{M^{\mathcal{T}}(w)\}$. If we take any two elements $\mathbb{L}^{\mathcal{T}}(\phi w^1) * f_1, \mathbb{L}^{\mathcal{T}}(\phi w^2) * f_2$, where $f_1, f_2 \in \mathbb{Z}P[q^{\pm\frac{1}{2}}]$, it follows from Proposition 7.1.10 that their twisted product still belongs to \mathcal{A}_{sub}^q . Therefore, \mathcal{A}_{sub}^q is a subalgebra of \mathcal{A}^q with respect to the twisted product. But it contains all the quantum cluster variables. Therefore, the two algebras must agree. \square

Theorem 8.1.8. *$\{\mathbb{L}^{\mathcal{A}}(w), w \in \mathcal{J}\}$ is a $\mathbb{Z}P[q^{\pm\frac{1}{2}}]$ -basis of \mathcal{A}^q . It is called the generic basis. Furthermore, it contains all the quantum cluster monomials.*

Proof. The statement follows from Proposition 8.1.6 and 8.1.7. \square

8.2. Dual canonical basis and dual PBW basis. Similar to the treatment of generic quantum cluster characters, for each $w \in \mathcal{J}$, we normalize the quantum torus elements $M^{\mathcal{T}}(w)$ and $L^{\mathcal{T}}(w)$ by defining

$$(66) \quad M^{\mathcal{A}}(w) = M^{\mathcal{T}}(w) \cdot x^{\text{ind } O(w) - \text{ind } w},$$

$$(67) \quad L^{\mathcal{A}}(w) = L^{\mathcal{T}}(w) \cdot x^{\text{ind } O(w) - \text{ind } w}.$$

$$(68)$$

Theorem 8.2.1. *The sets $\{M^{\mathcal{A}}(w), w \in \mathcal{J}\}, \{L^{\mathcal{A}}(w), w \in \mathcal{J}\}$ are $\mathbb{Z}P[q^{\pm\frac{1}{2}}]$ -bases of \mathcal{A}^q . They are called the dual PBW basis and the dual canonical basis of the quantum cluster algebra \mathcal{A}^q respectively. Furthermore, all the quantum cluster monomials are contained in $\{L^{\mathcal{A}}(w), w \in \mathcal{J}\}$.*

Proof. It suffices to show that $\{M^{\mathcal{T}}(w), w \in \mathcal{J}\}, \{L^{\mathcal{T}}(w), w \in \mathcal{J}\}$ are $\mathbb{Z}P[q^{\pm\frac{1}{2}}]$ -bases of \mathcal{A}^q . Applying the truncated character map to (35), we obtain

$$(69) \quad M^{\mathcal{T}}(w) = L^{\mathcal{T}}(w) + \sum_{(v,w) \text{ is } l\text{-dominant}} Z_{w,w'}(q^{\frac{\delta}{2}}) x^{\tilde{B}v + \text{ind } C_q v} L^{\mathcal{T}}(w'),$$

where $w' = w - C_q v$ and $Z_{w,w'}(t) = Z_{0,v;w}(t) \in t^{-1}\mathbb{Z}[t^{-1}]$. Denote the matrix of the coefficients $Z_{w,w'}(q^{\frac{\delta}{2}}) x^{\tilde{B}v + \text{ind } C_q v}$ by $\tilde{Z}(q^{\frac{\delta}{2}})$.

Similarly, denote the matrix of the coefficients $r_{w,w'}x^{\tilde{B}v+\text{ind } C_q v}$ in (60) by R and its inverse by R^{-1} .

Define the $\mathcal{J} \times \mathcal{J}$ matrix ϕR^{-1} such that its entry in position (w, w') is

$$(\phi R^{-1})_{w,w'} = \sum_{w'' \in \mathbb{N}^I \times \{-1,0\}: \phi w'' = w'} R_{w,w''}^{-1} x^{\text{ind}(f w'')}.$$

Denote the product $\tilde{Z}(q^{\frac{\delta}{2}})R^{-1}$ by $S(q^{\frac{\delta}{2}})$. Similarly, the $\mathcal{J} \times \mathcal{J}$ matrix $\phi S(q^{\frac{\delta}{2}})$ is defined such that its entry in position (w, w') is

$$(\phi S(q^{\frac{\delta}{2}}))_{w,w'} = \sum_{w'' \in \mathbb{N}^I \times \{-1,0\}: \phi w'' = w'} S(q^{\frac{\delta}{2}})_{w,w''} x^{\text{ind}(f w'')}.$$

The matrix transition between $\{\mathbb{L}^{\mathcal{T}}(w), w \in \mathcal{J}\}$ and $\{L^{\mathcal{T}}(w), w \in \mathcal{J}\}$ is given by the matrix ϕR^{-1} . The matrix transition between $\{\mathbb{L}^{\mathcal{T}}(w), w \in \mathcal{J}\}$ and $\{M^{\mathcal{T}}(w), w \in \mathcal{J}\}$ is given by the matrix $\phi S(q^{\frac{\delta}{2}})$.

With respect to the dominance order, the matrices R^{-1} and S are upper unitriangular. It follows that ϕR^{-1} and ϕS are upper triangular. Notice that, if for w, w' in $\mathbb{N}^I \times \{-1,0\}$, we have $w' \leq w$ with respect to the dominance order, then w' must be equal to w . It follows that ϕR^{-1} and ϕS are upper unitriangular. Therefore, we obtain the statements from Theorem 8.1.8. \square

Remark 8.2.2. *The transition matrices vary little when the choice of the coefficients and quantization change:*

1) *The matrix Z can be solved recursively by a combinatorial algorithm, cf. [Lus90, 7.10] or [Nak04, Section 8]. Z only depends on Q . Also, the geometrically defined integer matrix $(r_{w,w'})$ only depends on Q . The set $\{w' : \phi w' = w''\}$ for given w'' can be calculated easily.*

2) *We have $\tilde{Z}(q^{\frac{\delta}{2}}) = Z(q^{\frac{\delta}{2}})$ and $R = (r_{w,w'})$ if the quiver is of z -pattern.*

8.3. Structure constants. In [Kim12], Kimura studied the factorization of the dual canonical basis \mathbf{B}^{up} of a certain quantum unipotent subgroup. Because in [KQ12], the authors identified $L(w)$ with elements in \mathbf{B}^{up} , we can translate his result into our setting.

Theorem 8.3.1 ([Kim12, Theorem 6.21]). *Up to t -powers, we have the factorization of simples*

$$(70) \quad L(w) = L(\phi w) \cdot L(f w).$$

For any w^1, w^2, w^3 in \mathcal{J} , define an element in $\mathbb{Z}P[q^{\pm \frac{1}{2}}]$

$$(71) \quad \phi b_{w^1, w^2; \tilde{B}}^{w^3} = x^{\text{ind } O(w^1) + \text{ind } O(w^2) - \text{ind } O(w^3) - \text{ind}(w^1 + w^2 - w^3)} \sum_{w: \phi w = w^3} b_{w^1, w^2}^w x^{\text{ind } f w} x^{\tilde{B}v + \text{ind } C_q v}$$

where v is determined by

$$(72) \quad w = w^1 + w^2 - C_q v$$

and the integers b_{w^1, w^2}^w by (37).

Theorem 8.3.2 (Positive basis). *The structure constants of the dual canonical basis $\{L^{\mathcal{T}}(w), w \in \mathcal{J}\}$ are given by, for any w^1, w^2 in \mathcal{J} ,*

$$L^{\mathcal{A}}(w^1) * L^{\mathcal{A}}(w^2) = q^{\frac{1}{2}\Lambda(\text{ind}(w^1), \text{ind}(w^2)) - \frac{\delta}{2}\Lambda^z(\text{ind}^z(w^1), \text{ind}^z(w^2))} \cdot \sum_{w^3 \in \mathcal{J}} \phi b_{w^1, w^2; \tilde{B}}^{w^3}(q^{\frac{\delta}{2}}) L^{\mathcal{A}}(w^3).$$

In particular, they are contained in $\mathbb{N}[q^{\pm\frac{1}{2}}][x_{n+1}^{\pm}, \dots, x_m^{\pm}]$.

Proof. First consider the quantum cluster algebra \mathcal{A}^q associated with (\tilde{B}^z, Λ^z) . Then we have $\widehat{\Pi} \widehat{\chi}_{q,t} = \chi_{q,t} \widehat{\Pi}$. The truncated character $\chi_{q,t}^{\leq 0}$ is algebraic from R_t to $D(\mathcal{T})$. Therefore, the $\mathbb{Z}[t^{\pm}]$ -linear independent set $\{L^{\mathcal{T}}(w)\}$ in \mathcal{T} is closed under the twisted product $*$ and the corresponding positive structure constants are b_{w^1, w^2}^w as given by (37). Using Theorem 8.3.1, we obtain the structure constants of the dual canonical basis $\{L^{\mathcal{T}}(w), w \in \mathcal{J}\}$.

Finally, the correction technique in Theorem 9.1.2 implies the above result for any acyclic quantum cluster algebra. \square

We obtain that, whatever the coefficient pattern $\tilde{Q} - Q$ and the quantization Λ we choose, the quantum cluster algebras containing acyclic seeds are *strongly positive*: they admit a basis in which the structure constants are positive.

The following important consequence of the deformed monoidal categorification is an easy generalization of the first main result of the author's joint work with Kimura [KQ12] for arbitrary choice of coefficients and quantization.

Corollary 8.3.3. (Quantum positivity [KQ12]) *Any quantum cluster monomial m can be written as a Laurent polynomial of the quantum cluster variables x_i , $1 \leq i \leq n$, in any given seed with coefficients in $\mathbb{N}[q^{\pm\frac{\delta}{2}}, x_{n+1}^{\pm}, \dots, x_m^{\pm}]$.*

Proof. By the quantum Laurent phenomenon, we have

$$m = \frac{\sum_{m_*=(m_i)} c_{m_*} \prod_{1 \leq i \leq n} x_i^{m_i}}{\prod_i x_i^{d_i}},$$

where $m_* = (m_i)_{i \in I}$, $d_* = (d_i)_{i \in I}$ are sequences of nonnegative integers and the coefficients c_{m_*} are contained in $\mathbb{Z}P[q^{\pm\frac{1}{2}}]$. Notice that we use the usual product \cdot in this expression.

The quantum cluster monomial m equals $L^{\mathcal{A}}(w)$ for some w . Also, the quantum X -variable x_i , $1 \leq i \leq m$, equals $L^{\mathcal{A}}(w_i)$ for some w_i . We can rewrite the above equation as

$$\begin{aligned} \sum_{m_*=(m_i)} c_{m_*} L^{\mathcal{A}}\left(\sum_i m_i w_i\right) &= \prod_i L^{\mathcal{A}}(w_i)^{d_i} \cdot L^{\mathcal{A}}(w) \\ &= q^{-\frac{1}{2}\Lambda(\text{ind}(\sum_i d_i w_i), \text{ind}(w))} L^{\mathcal{A}}\left(\sum_i d_i w_i\right) * L^{\mathcal{A}}(w). \end{aligned}$$

The statement follows from Theorem (8.3.2). \square

The almost simple pseudo-modules $\mathbb{L}(w)$ introduced in Section 6 form a $\mathbb{Z}[q^{\pm\frac{1}{2}}]$ -basis of \mathcal{R}_t . Notice that, when the ice quiver is of z -pattern, we have

$\widehat{\Pi} \text{cor} \chi_{q,t} \leq 0 \mathbb{L}(w) = \mathbb{L}^{\mathcal{T}}(w) = \mathbb{L}^{\mathcal{A}}(w)$. Denote the structure constants of this basis by $c_{w^1, w^2}^w(t)$: for any w^1, w^2 in $\mathbb{N}^I \times \{-1, 0\}$, we have

$$(73) \quad \mathbb{L}(w^1) \otimes \mathbb{L}(w^2) = \sum_w c_{w^1, w^2}^w \mathbb{L}(w).$$

Because the matrix $(r_{w, w'})$ and the matrix of the structure constants of $\{\mathbb{L}^{\mathcal{T}}(w)\}$ are upper unitriangular, the matrix (c_{w^1, w^2}^w) is upper unitriangular as well. Therefore, for every nonzero term in the sum, we always have $w = w^1 + w^2 - C_q v$ for some v .

For any w^1, w^2, w^3 in \mathcal{J} , define an element in $\mathbb{Z}P[q^{\pm \frac{1}{2}}]$

$$(74) \quad \phi c_{w^1, w^2; \tilde{B}}^{w^3} = x^{\text{ind } O(w^1) + \text{ind } O(w^2) - \text{ind } O(w^3) - \text{ind}(w^1 + w^2 - w^3)} \sum_{w: \phi w = w^3} c_{w^1, w^2}^w x^{\text{ind } f w} x^{\tilde{B}v + \text{ind } C_q v}$$

where v is determined by $w = w^1 + w^2 - C_q v$.

Theorem 8.3.4. *The structure constants of the generic basis $\{\mathbb{L}^{\mathcal{T}}(w), w \in \mathcal{J}\}$ are given by, for any w^1, w^2 in \mathcal{J} ,*

$$\begin{aligned} \mathbb{L}^{\mathcal{A}}(w^1) * \mathbb{L}^{\mathcal{A}}(w^2) &= q^{\frac{1}{2} \Lambda(\text{ind}(w^1), \text{ind}(w^2)) - \frac{\delta}{2} \Lambda^z(\text{ind}^z(w^1), \text{ind}^z(w^2))} \\ &\quad \cdot \sum_{w^3 \in \mathcal{J}} \phi c_{w^1, w^2; \tilde{B}}^{w^3} (q^{\frac{\delta}{2}}) \mathbb{L}^{\mathcal{A}}(w^3). \end{aligned}$$

Proof. This theorem is a consequence of (62) and Theorem 9.1.2. The proof is similar to that of Theorem 8.3.2. \square

9. CORRECTION TECHNIQUE

9.1. Corrections of algebraic relations. Given any (quantum) cluster algebra, if some basis of it is known, we want to ask what happens if we change the coefficients and the quantization. As we have seen in Theorem 8.1.8 Theorem 8.2.1 and Remark 8.2.2, the bases and their transition matrices vary little the $\tilde{Q} - Q$ and Λ change. We shall show that this phenomenon is true in general, by generalizing the correction factors in the previous failures (55) (58) (65).

Let $n, m^{(1)}, m^{(2)}$ be three integers, such that $0 < n \leq m^{(1)}, m^{(2)}$. For $i = 1, 2$, let $\tilde{B}^{(i)}$ be a rank $m^{(i)} \times n$ matrix. Let $\Lambda^{(i)}$ be a rank $m^{(i)} \times m^{(i)}$ skew-symmetric integer matrix.

Definition 9.1.1 (weakly compatible pair). *The pair $(\tilde{B}^{(i)}, \Lambda^{(i)})$ is called weakly compatible if there exists an $n \times n$ integer matrix $D^{(i)}$ such that we have*

$$(75) \quad \Lambda^{(i)}(-\tilde{B}^{(i)}) = \begin{bmatrix} D^{(i)} \\ 0 \end{bmatrix}.$$

Assume the matrices $\tilde{B}^{(i)}$ have the common principal part B , the pairs $(\tilde{B}^{(i)}, \Lambda^{(i)})$ are weakly compatible, and $D^{(2)} = \delta D^{(1)}$ for some integer δ . A priori, $\text{rank } D^{(1)}$ is no less than $\text{rank } D^{(2)}$.

Define the associated quantum tori

$$\mathcal{T}^{(i)} = \mathcal{T}^{(i)}(\Lambda^{(i)}) = \mathbb{Z}[q^{\pm\frac{1}{2}}][x_1^\pm, \dots, x_{m^{(i)}}^\pm]$$

such that twisted product is determined by $\Lambda^{(i)}$.

Let $\max(m^{(1)}, m^{(2)})$ be denoted by m . As in (53), enlarge $\tilde{B}^{(1)}, \tilde{B}^{(2)}$ into $m \times n$ matrices and $\Lambda^{(2)}$ into $m \times m$ matrix by adding zero entries, and define the associated enlarged quantum torus

$$\mathcal{T} = \mathcal{T}(\Lambda^{(2)}) = \mathbb{Z}[q^{\pm\frac{1}{2}}][x_1^\pm, \dots, x_m^\pm],$$

such that its twisted product is determined by the enlarged matrix $\Lambda^{(2)}$.

For each i , and for any integer $s \geq 3$, integer $1 \leq j \leq s$, vector $g^{(i)}_j \in \mathbb{Z}^{n_i}$, and polynomial

$$F_j(t; y_1, \dots, y_n) = \sum_{v \in \mathbb{N}^n} b_j(t; v) y^v \in \mathbb{Z}[t^\pm][y_1, \dots, y_n],$$

where t, y_1, \dots, y_n are indeterminates, $b_j(t; v) \in \mathbb{Z}[t^\pm]$, we define the following element in the quantum torus $\mathcal{T}^{(1)}$:

$$\begin{aligned} M_j^{(1)} &= x^{g_j^{(1)}} F_j(q^{\frac{1}{2}}; x^{\tilde{B}^{(1)}e_1}, \dots, x^{\tilde{B}^{(1)}e_n}) \\ &= x^{g_j^{(1)}} \sum_{v \in \mathbb{N}^n} b_j(q^{\frac{1}{2}}; v) x^{\tilde{B}^{(1)}v}, \end{aligned}$$

where e_k is the k -th unite vector in \mathbb{Z}^n , and a similar element in $\mathcal{T}^{(2)}$:

$$\begin{aligned} M_j^{(2)} &= x^{g_j^{(2)}} F_j(q^{\frac{\delta}{2}}; x^{\tilde{B}^{(2)}e_1}, \dots, x^{\tilde{B}^{(2)}e_n}) \\ &= x^{g_j^{(2)}} \sum_{v \in \mathbb{N}^n} b_j(q^{\frac{\delta}{2}}; v) x^{\tilde{B}^{(2)}v}. \end{aligned}$$

Assume the first n coordinates of $g_j^{(1)}$ and $g_j^{(2)}$ are equal, *i.e.* we have $\text{pr}_n g_j^{(1)} = \text{pr}_n g_j^{(2)} = g_j$ for some vector $g_j \in \mathbb{Z}^n$.

Theorem 9.1.2. *Assume that $\tilde{B}^{(1)}$ is of full rank and $b_j(t; 0)$ does not vanish for any $3 \leq j \leq s$. If the following equation holds in $\mathcal{T}^{(1)}$:*

$$(76) \quad q^{-\frac{1}{2}\Lambda^{(1)}(g_1^{(1)}, g_2^{(1)})} M_1^{(1)} * M_2^{(1)} = \sum_{3 \leq j \leq s} c_j^{(1)}(q^{\frac{1}{2}}) M_j^{(1)}$$

for some coefficients $c_j^{(1)}(t) \in \mathbb{Z}[t^\pm][x_{n+1}^\pm, \dots, x_{m^{(1)}}^\pm]$, then there exist unique vectors $u_j \in \mathbb{N}^n$ such that $g_j^{(1)} = g_1^{(1)} + g_2^{(1)} + \tilde{B}^{(1)}u_j$, and we have

$$(77) \quad q^{-\frac{1}{2}\Lambda^{(2)}(g_1^{(2)}, g_2^{(2)})} M_1^{(2)} * M_2^{(2)} = \sum_{3 \leq j \leq s} c_j^{(2)}(q^{\frac{\delta}{2}}) M_j^{(2)}$$

such that the coefficients in $\mathbb{Z}[t^\pm][x_{n+1}^\pm, \dots, x_{m^{(2)}}^\pm]$ are given by

$$(78) \quad c_j^{(2)}(q^{\frac{\delta}{2}}) = c_j^{(1)}(q^{\frac{\delta}{2}}) x^{g_1^{(2)} + g_2^{(2)} + \tilde{B}^{(2)}u_j - g_j^{(2)}}.$$

Proof. Expanding LHS of (76), we obtain

$$\begin{aligned}
 \text{LHS} &= \sum_{v_1, v_2: v_1+v_2=v} q^{-\frac{1}{2}\Lambda^{(1)}(g_1^{(1)}, g_2^{(1)}) + \frac{1}{2}\Lambda^{(1)}(g_1^{(1)} + \tilde{B}^{(1)}v_1, g_2^{(1)} + \tilde{B}^{(1)}v_2)} \\
 &\quad \cdot b_1(q^{\frac{1}{2}}; v_1)b_2(q^{\frac{1}{2}}; v_2)x^{g_1^{(1)}+g_2^{(1)}+\tilde{B}^{(1)}v} \\
 &= \sum_{v_1, v_2: v_1+v_2=v} q^{\frac{1}{2}(-g_1^T D^{(1)}v_2 + g_2^T D^{(1)}v_1 + v_1^T B D^{(1)}v_2)} \\
 &\quad \cdot b_1(q^{\frac{1}{2}}; v_1)b_2(q^{\frac{1}{2}}; v_2)x^{g_1^{(1)}+g_2^{(1)}+\tilde{B}^{(1)}v}. \\
 &= \sum_{v_1, v_2: v_1+v_2=v} q^{\frac{1}{2\delta}(-g_1^T D^{(2)}v_2 + g_2^T D^{(2)}v_1 + v_1^T B D^{(2)}v_2)} \\
 &\quad \cdot b_1(q^{\frac{1}{2}}; v_1)b_2(q^{\frac{1}{2}}; v_2)x^{g_1^{(1)}+g_2^{(1)}+\tilde{B}^{(1)}v}.
 \end{aligned}$$

In (76), since $b_j(t; 0)$ is nonzero, the monomial $x^{g_j^{(1)}}$ in RHS must be killed by either another monomial $x^{g_{j'}^{(1)}+\tilde{B}^{(1)}v_{j'}}$ in RHS or a monomial in LHS. In the former case, repeat this argument for $x^{g_{j'}^{(1)}}$. Then, after finite steps, we obtain that for some $u_j \in \mathbb{N}^n$, $x^{g_j^{(1)}}$ must equal the monomial $x^{g_1^{(1)}+g_2^{(1)}+\tilde{B}^{(1)}u_j}$ in LHS. Therefore, we can rewrite RHS as

$$\text{RHS} = \sum_j c_j^{(1)}(q^{\frac{1}{2}}) \sum_{v_j} b_j(q^{\frac{1}{2}}; v_j) x^{g_1^{(1)}+g_2^{(1)}+\tilde{B}^{(1)}(u_j+v_j)}.$$

Viewed the both sides as usual Laurent polynomial and embed them into the enlarged quantum torus \mathcal{T} . We can rewrite them as

$$\begin{aligned}
 \text{LHS} &= \sum_{v_1, v_2: v_1+v_2=v} (q^{\frac{1}{2\delta}})^{-\Lambda^{(2)}(g_1^{(2)}, g_2^{(2)}) + \Lambda^{(2)}(g_1^{(2)} + \tilde{B}^{(2)}v_1, g_2^{(2)} + \tilde{B}^{(2)}v_2)} \\
 &\quad \cdot b_1(q^{\frac{1}{2}}; v_1)b_2(q^{\frac{1}{2}}; v_2)x^{g_1^{(2)}+g_2^{(2)}+\tilde{B}^{(2)}v} \\
 &\quad \cdot x^{g_1^{(1)}+g_2^{(1)}-g_1^{(2)}-g_2^{(2)}} x^{(\tilde{B}^{(1)}-\tilde{B}^{(2)})v}, \\
 \text{RHS} &= \sum_{j, v_j: u_j+v_j=v} c_j^{(1)}(q^{\frac{1}{2}}) b_j(q^{\frac{1}{2}}; v_j) x^{g_1^{(2)}+g_2^{(2)}+\tilde{B}^{(2)}(u_j+v_j)} \\
 &\quad \cdot x^{g_1^{(1)}+g_2^{(1)}-g_1^{(2)}-g_2^{(2)}} x^{(\tilde{B}^{(1)}-\tilde{B}^{(2)})(u_j+v_j)}.
 \end{aligned}$$

In the both sides, we replace the indeterminate $q^{\frac{1}{2}}$ by $q^{\frac{\delta}{2}}$ and delete the factor

$$x^{g_1^{(1)}+g_2^{(1)}-g_1^{(2)}-g_2^{(2)}} x^{(\tilde{B}^{(1)}-\tilde{B}^{(2)})v}$$

in each monomial. Then we still have LHS = RHS, which now become

$$\begin{aligned}
 \text{LHS} &= q^{-\frac{1}{2}\Lambda^{(2)}(g_1^{(2)}, g_2^{(2)})} M_1^{(2)} * M_2^{(2)} \\
 \text{RHS} &= \sum_j \sum_{v_j} c_j^{(1)}(q^{\frac{\delta}{2}}) b_j(q^{\frac{\delta}{2}}; v_j) x^{g_1^{(2)}+g_2^{(2)}+\tilde{B}^{(2)}u_j - g_j^{(2)}} x^{g_j^{(2)}+\tilde{B}^{(2)}v_j} \\
 &= \sum_j c_j^{(1)}(q^{\frac{\delta}{2}}) x^{g_1^{(2)}+g_2^{(2)}+\tilde{B}^{(2)}u_j - g_j^{(2)}} M_j.
 \end{aligned}$$

Thus, (77) is proved. \square

Remark 9.1.3. *The quotient between the q -powers on the left of (76) and (77) should be viewed as the generalization of the q -power correction factor in Failure 1 (55). The correction factor $x^{g_1^{(2)}+g_2^{(2)}+\tilde{B}^{(2)}u_j-g_j^{(2)}}$ should be viewed as the generalization of the factor in Failure 2 (58). The difference $g_j^{(2)}-g_j^{(1)}$ should be viewed as the generalization of the difference of the two sides of (65).*

9.2. Structure constants.

Theorem 9.2.1. *For $i = 1, 2$, assume that the pair $(\tilde{B}^{(i)}, \Lambda^{(i)})$, $i = 1, 2$, in Theorem 9.1.2 is either compatible or the matrix $\Lambda^{(i)}$ is zero. Denote the corresponding quantum or classical cluster algebras by $\mathcal{A}^{q(i)}$.*

For $i = 1, 2$, let $\mathbf{B}^{(i)}$ be a $\mathbb{Z}[q^{\pm\frac{1}{2}}][x_{n+1}^{\pm}, \dots, x_{m^{(i)}}^{\pm}]$ -basis of $\mathcal{A}^{q(i)}$ such that its elements take the form $M^{(i)}_j$. If the structure constants of $\mathbf{B}^{(1)}$ are described by (76), then the structure constants of $\mathbf{B}^{(2)}$ are described by the corresponding equation (77).

Proof. The statement is obtained by taking (76) to be the multiplication of basis elements. \square

Remark 9.2.2. *By this theorem, in order to study quantum cluster variables and bases, it suffices to look at a special coefficient pattern with special quantizations, e.g. the principal coefficients with the canonical quantization.*

9.3. Acyclic case. Now we can apply the correction techniques to the results of Section 8. We simplify the proofs of previous results on bases of acyclic quantum cluster algebras, and we also present results about the bases of acyclic classical cluster algebras.

Let the compatible pairs $(\tilde{B}^{(i)}, \Lambda^{(i)})$, $i = 1, 2$, be given as in Theorem 9.2.1. Further assume that $B = B_Q$ for some acyclic quiver, and choose $(\tilde{B}^{(1)}, \Lambda^{(1)})$ to be (\tilde{B}^z, Λ^z) . Then we can choose the $\mathbb{Z}[q^{\pm\frac{1}{2}}]$ -linearly independent subset⁴ $\mathbf{B}^{(1)}$ of the elements of $\mathcal{A}^{q(1)}$ to be the set $\{\mathbb{L}^A(w)\}$, the set $\{M^A(w)\}$, or the set $\{L^A(w)\}$. Notice that the elements of $\mathbf{B}^{(1)}$ take the form $M_j^{(1)}$ of Theorem 9.1.2. Define $\mathbf{B}^{(2)}$ to be the subset $\{\mathbb{L}^A(w), w \in \mathcal{J}\}$, $\{M^A(w), w \in \mathcal{J}\}$, $\{L^A(w), w \in \mathcal{J}\}$ in $\mathcal{T}^{(2)}$ respectively if $\Lambda^{(2)}$ is nonzero, or the corresponding classical limit if $\Lambda^{(2)} = 0$. Then the elements of $\mathbf{B}^{(2)}$ take the form $M_j^{(2)}$.

Theorem 9.3.1. *If $\tilde{B}^{(2)}$ is of full rank, then $\mathbf{B}^{(2)}$ is a $\mathbb{Z}P[q^{\pm\frac{1}{2}}]$ -basis of $\mathcal{A}^{q(2)}$. Furthermore, if $\mathbf{B}^{(2)}$ is $\{\mathbb{L}^A(w), w \in \mathcal{J}\}$ or $\{L^A(w), w \in \mathcal{J}\}$, then the structure constants of $\mathbf{B}^{(2)}$ can be deduced from (77) which corresponds to the structure constants equation (76).*

Proof. By the proof of 8.1.6, the leading terms (terms with $v = 0$) of the elements in $\mathbf{B}^{(2)}$ are different. Because $\tilde{B}^{(2)}$ is of full rank, in each $M_j^{(2)}$,

⁴The subset $\mathbf{B}^{(1)}$ is a basis of the subalgebra of $\mathcal{A}^{q(1)}$ generated by the quantum cluster variables and the frozen variables x_{n+1}, \dots, x_{2n} .

the leading term can not be killed by the other terms. Therefore, $\mathbf{B}^{(2)}$ is linearly independent. Because $\mathbf{B}^{(1)}$ is contained in $\mathcal{A}^{q(1)}$, by Theorem 9.1.2, $\mathbf{B}^{(2)}$ is contained in $\mathcal{A}^{q(2)}$.

Assume $\mathbf{B}^{(2)}$ is either $\{\mathbb{L}^{\mathcal{A}}(w), w \in \mathcal{J}\}$ or $\{L^{\mathcal{A}}(w), w \in \mathcal{J}\}$. Then it contains all the quantum cluster monomials by section 6. Moreover, it generates a $\mathbb{Z}P[q^{\pm\frac{1}{2}}]$ -subalgebra of $\mathcal{A}^{q(2)}$ by Theorem 9.1.2 and (49) or (70). The statement concerning its structure constants also follows from Theorem 9.1.2.

Using the upper unitriangular basis transition matrices studied before, we deduce that $\{M^{\mathcal{A}}(w), w \in \mathcal{J}\}$ is also a basis. \square

Remark 9.3.2. *If we take $\Lambda^{(2)} = 0$, then $\mathcal{A}^{q(2)}$ is a classical cluster algebra, which we denote by $\mathcal{A}^{(2)}$.*

1) *If we take $\mathbf{B}^{(1)}$ to be the dual canonical basis, then we obtain that $\mathbf{B}^{(2)}$ is the dual canonical basis of $\mathcal{A}^{(2)}$ with positive structure constants. All cluster monomials are contained in $\mathbf{B}^{(2)}$.*

2) *If we take $\mathbf{B}^{(1)}$ to be the generic basis and the dual PBW basis, then this theorem generalizes the results of the dual semicanonical basis and the dual PBW basis in [GLS11, 16.1] to the integral form and general coefficient type.*

Recently, another proof of the generic basis for $\mathcal{A}^{(2)}$ is obtained by [GLS12] [Pla11], which is based on cluster categories and a combinatorial result of [BFZ05].

REFERENCES

- [Ami09] Claire Amiot, *Cluster categories for algebras of global dimension 2 and quivers with potential*, Annales de l’institut Fourier **59** (2009), no. 6, 2525–2590, [arXiv:0805.1035](#).
- [BBD82] Alexander A. Beilinson, Joseph Bernstein, and Pierre Deligne, *Analyse et topologie sur les espaces singuliers*, Astérisque, vol. 100, Soc. Math. France, 1982 (French).
- [BFZ05] Arkady Berenstein, Sergey Fomin, and Andrei Zelevinsky, *Cluster algebras. III. Upper bounds and double Bruhat cells*, Duke Math. J. **126** (2005), no. 1, 1–52.
- [BZ05] Arkady Berenstein and Andrei Zelevinsky, *Quantum cluster algebras*, Adv. Math. **195** (2005), no. 2, 405–455, [arXiv:math/0404446v2](#).
- [BZ11] ———, *Quantum cluster algebras*, lectures in Osaka, 2011.
- [BZ12] ———, *Triangular bases in quantum cluster algebras*, [arXiv:1206.3586](#).
- [CC06] Philippe Caldero and Frédéric Chapoton, *Cluster algebras as Hall algebras of quiver representations*, Comment. Math. Helv. **81** (2006), no. 3, 595–616, [arXiv:math.RT/0410187v2](#).
- [CK08] Philippe Caldero and Bernhard Keller, *From triangulated categories to cluster algebras*, Inv. Math. **172** (2008), 169–211.
- [CG97] N. Chriss and V. Ginzburg, *Representation theory and complex geometry*, Progress in Math., Birkhäuser, 1997.
- [CBJ05] William Crawley-Boevey and Bernt Tore Jensen, *A note on sub-bundles of vector bundles*, 2005, [arXiv:math/0505149v2](#).
- [DWZ08] Harm Derksen, Jerzy Weyman, and Andrei Zelevinsky, *Quivers with potentials and their representations I: Mutations*, Selecta Mathematica **14** (2008), 59–119, [arXiv:0704.0649v4](#).
- [DWZ10] ———, *Quivers with potentials and their representations II: Applications to cluster algebras*, J. Amer. Math. Soc. **23** (2010), no. 3, 749–790, [arXiv:0904.0676v2](#).
- [Din10] Ming Ding, *On quantum cluster algebras of finite type*, [arXiv:1011.1606](#).

- [DX11] Ming Ding and Fan Xu, *A quantum analogue of generic bases for affine cluster algebras*, [arXiv:1105.2421v1](#).
- [Ef11] Alexander Ivanovich Efimov, *Quantum cluster variables via vanishing cycles*, 2011, [arXiv:1112.3601v2](#).
- [FZ02] Sergey Fomin and Andrei Zelevinsky, *Cluster algebras. I. Foundations*, J. Amer. Math. Soc. **15** (2002), no. 2, 497–529 (electronic), [arXiv:math/0104151v1](#).
- [FZ07] ———, *Cluster algebras IV: Coefficients*, Compositio Mathematica **143** (2007), 112–164, [arXiv:math/0602259v3](#).
- [GLS11] Christof Geiß, Bernard Leclerc, and Jan Schröer, *Kac-Moody groups and cluster algebras*, Advances in Mathematics **228** (2011), no. 1, 329–433, [arXiv:1001.3545v2](#).
- [GLS12] ———, *Generic bases for cluster algebras and the Chamber Ansatz*, J. Amer. Math. Soc. **25** (2012), no. 1, 21–76, [arXiv:1004.2781v3](#).
- [HV08] Tamás Hausel and Fernando Rodriguez Villegas, *Mixed Hodge polynomials of character varieties*, Invent. Math. **174** (2008), no. 3, 552–624, with an appendix by Nicholas M. Katz, [arXiv:math/0612668v3](#).
- [HL10] David Hernandez and Bernard Leclerc, *Cluster algebras and quantum affine algebras*, Duke Math. J. **154** (2010), no. 2, 265–341, [arXiv:0903.1452](#).
- [HL11] ———, *Quantum Grothendieck rings and derived Hall algebras*, 2011, [arXiv:1109.0862v1](#).
- [IY08] Osamu Iyama and Yuji Yoshino, *Mutations in triangulated categories and rigid Cohen-Macaulay modules*, Inv. Math. **172** (2008), 117–168.
- [Kel12] Bernhard Keller, *Cluster algebras and derived categories*, 2012, [arXiv:1202.4161](#).
- [Kim12] Yoshiyuki Kimura, *Quantum unipotent subgroup and dual canonical basis*, Kyoto J. Math. **52** (2012), no. 2, 277–331, [arXiv:1010.4242](#), [doi:10.1215/21562261-1550976](#).
- [KQ12] Yoshiyuki Kimura and Fan Qin, *Graded quiver varieties, quantum cluster algebras and dual canonical basis*, 2012, [arXiv:1205.2066](#).
- [KS08] Maxim Kontsevich and Yan Soibelman, *Stability structures, Donaldson-Thomas invariants and cluster transformations*, 2008, [arXiv:0811.2435v1](#).
- [KS13] Bernhard Keller and Sarah Scherotzke, *Quiver varieties and derived categories*, in preparation.
- [KY11] Bernhard Keller and Dong Yang, *Derived equivalences from mutations of quivers with potential*, Adv. Math. **226** (2011), no. 3, 2118–2168, [arXiv:0906.0761v3](#).
- [Lam11a] Philipp Lampe, *A quantum cluster algebra of Kronecker type and the dual canonical basis*, Int. Math. Res. Not. IMRN (2011), no. 13, 2970–3005, [arXiv:1002.2762v2](#), [doi:10.1093/imrn/rnq162](#).
- [Lam11b] ———, *Quantum cluster algebras of type A and the dual canonical basis*, 2011, [arXiv:1101.0580v1](#).
- [Lus90] G. Lusztig, *Canonical bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. **3** (1990), no. 2, 447–498.
- [Lus00] ———, *Semicanonical bases arising from enveloping algebras*, Adv. Math. **151** (2000), no. 2, 129–139.
- [MSW11] Gregg Musiker, Ralf Schiffler, and Lauren Williams, *Bases of cluster algebras from surfaces*, 2011, [arXiv:1110.4364v2](#).
- [Nag10] Kentaro Nagao, *Donaldson-Thomas theory and cluster algebras*, 2010, [arXiv:1002.4884](#).
- [Nak94] Hiraku Nakajima, *Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras*, Duke Math. J. **76** (1994), no. 2.
- [Nak98] ———, *Quiver varieties and Kac-Moody algebras*, Duke Math. J. **91** (1998), no. 3.
- [Nak01a] ———, *Quiver varieties and finite-dimensional representations of quantum affine algebras*, J. Amer. Math. Soc. **14** (2001), no. 1, 145–238 (electronic), [arXiv:math/9912158](#).
- [Nak01b] ———, *Quiver varieties and tensor products*, Invent. Math. **146** (2001), no. 2, 399–449, [arXiv:math.QA/0103008](#).

- [Nak04] ———, *Quiver varieties and t -analogs of q -characters of quantum affine algebras*, Ann. of Math. (2) **160** (2004), no. 3, 1057–1097, [arXiv:math/0105173v2](#).
- [Nak11] ———, *Quiver varieties and cluster algebras*, Kyoto J. Math. **51** (2011), no. 1, 71–126, [arXiv:0905.0002v5](#).
- [Pal08] Yann Palu, *Cluster characters for 2-Calabi-Yau triangulated categories*, Ann. Inst. Fourier (Grenoble) **58** (2008), no. 6, 2221–2248, [arXiv:math/0703540v2](#).
- [Pla10a] Pierre-Guy Plamondon, *Cluster algebras via cluster categories with infinite-dimensional morphism spaces*, 2010, [arXiv:1004.0830v1](#).
- [Pla10b] ———, *Cluster characters for cluster categories with infinite-dimensional morphism spaces*, 2010, [arXiv:1002.4956v2](#).
- [Pla11] ———, *Catégories amassées aux espaces de morphismes de dimension infinie, applications*, Ph.D. thesis, Université Paris Diderot - Paris 7, 2011, http://people.math.jussieu.fr/~plamondon/plamondon_these.pdf.
- [Qin10] Fan Qin, *Quantum cluster variables via Serre polynomials*, with an appendix by Bernhard Keller, to appear in Journal für die reine und angewandte Mathematik (Crelle's Journal), [arXiv:1004.4171](#), [doi:10.1515/CRELLE.2011.129](#).
- [Qin12] ———, Fan Qin, *Algèbres amassées quantiques acycliques*, Ph.D. thesis, Université Paris Diderot - Paris 7, 2012, http://www.math.jussieu.fr/~qinfan/doc/QIN_Thesis_manuscript.pdf.
- [Rei08] Markus Reineke, *Framed quiver moduli, cohomology, and quantum groups*, J. Algebra **320** (2008), no. 1, 94–115, [doi:10.1016/j.jalgebra.2008.01.025](#). MR 2417980 (2009d:16021)
- [Slo80] P. Slodowy, *Four lectures on simple groups and singularities*, Communications of the Mathematical Institute, Mathematical Institute, Rijksuniversitet Utrecht, 1980.
- [Tra09] Thao Tran, *F-polynomials in quantum cluster algebras*, 2009, [arXiv:0904.3291v1](#).
- [VV03] M. Varagnolo and E. Vasserot, *Perverse sheaves and quantum Grothendieck rings*, Studies in memory of Issai Schur (Chevaleret/Rehovot, 2000), Progr. Math., vol. 210, Birkhäuser Boston, Boston, MA, 2003, pp. 345–365, [arXiv:math/0103182v3](#). MR MR1985732 (2004d:17023)

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