DUAL FUSION FRAMES

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Abstract. The definition of dual fusion frame presents technical problems related to the domain of the synthesis operator. The notion commonly used is the analogous to the canonical dual frame. Here a new concept of dual is studied in infinite-dimensional separable Hilbert spaces. It extends the commonly used notion and overcomes these technical difficulties. We show that with this definition in many cases dual fusion frames behave similar to dual frames. We exhibit examples of non-canonical dual fusion frames.

1. Introduction

Frames are systems of vectors in a separable Hilbert space $\mathcal{H}$ which are redundant. This means, they allow representations of the elements of the Hilbert space which are not necessarily unique. This property is desirable for many situations that appear in applications e.g. in signal processing when we have presence of noise, since they allow more flexibility for choosing the adequate representation. Other areas where frames are used include coding theory, communication theory, sampling theory and the development of fast algorithms. Frames appeared for the first time in the work of Duffin and Schaeffer in [7]. For more information about frame theory we refer to [1, 3, 5, 6].

When a huge amount of data has to be processed it is often advantageous to subdivide a frame system into smaller subsystems and combine locally data vectors. This gives rise to the concept of *fusion frames* (or *frames of subspaces*) [2, 4] (see also [3, Chapter 13]), which are a generalization of frames. They allow decompositions of the elements of $\mathcal{H}$ into weighted orthogonal projections onto closed subspaces.

Many concepts of classical frame theory have been generalized to the setting of fusion frames. Having a proper notion of *dual fusion frames* permits us to have different reconstruction strategies. However in the definition appears a problem connected to the domain of the synthesis operator. Also, one wants from a proper definition to yield expected duality results.

So far the most frequently used definition is the one that corresponds to the canonical dual of the classical frames. The first author of this paper proposed a new concept of dual fusion frame. This new notion extends the "canonical"

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one and solves the technical problem mentioned before. Moreover, with this definition we obtain results which are analogous to those valid for dual frames.

In [8] the new definition is studied in finite-dimensional Hilbert spaces and it is used to obtain optimal reconstructions under erasures. In the present work we study properties and provide examples in infinite-dimensional separable Hilbert spaces.

The paper is organized as follows. In Section 2 we review results about frames and fusion frames. In Section 3 we present the new definition of dual fusion frames. We then study the relation between dual fusion frames and the left inverses of the analysis operator. We present examples of dual fusion frames obtained using this relation. We show that the "canonical" dual fusion frame is a particular case of our duals. We also consider the construction of dual fusion frames from dual frames. We finally give examples of non-canonical harmonic dual fusion frames constructed from Gabor systems.

2. Preliminaries

We will now give a brief review about the concepts of frame and fusion frame

Throughout this paper $\mathcal{H}$ will be a separable Hilbert space and I a countable index set.

2.1. **Frames and dual frames.**

Definition 2.1. Let $\{f_i\}_{i\in I} \subset \mathcal{H}$. Then $\{f_i\}_{i\in I}$ is a *frame* for $\mathcal{H}$, if there exist constants $0 < \alpha \leq \beta < \infty$ such that

$$\alpha\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq \beta\|f\|^2 \text{ for all } f \in \mathcal{H}. \tag{2.1}$$

If the right inequality in (2.1) is satisfied, $\{f_i\}_{i\in I}$ is a *Bessel sequence.*

The constants α and β are called *frame bounds.* If $\alpha = \beta$, we call $\{f_i\}_{i\in I}$ an *α-tight frame*, and if $\alpha = \beta = 1$ it is a *Parseval frame.*

We associate to a Bessel sequence $\{f_i\}_{i\in I}$ the following bounded operators. The *synthesis operator*

$$T : \ell^2(I) \to \mathcal{H},\ T\{c_i\}_{i\in I} = \textstyle\sum_{i\in I} c_i f_i,$$

and the *analysis operator*

$$T^* : \mathcal{H} \to \ell^2(I),\ T^* f = \{\langle f, f_i\rangle\}_{i\in I}.$$

For a frame $\{f_i\}_{i\in I}$ the operator

$$S = TT^*,$$

called *frame operator* is positive, selfadjoint and invertible. Furthermore if $\{f_i\}_{i\in I}$ is an α-tight frame, then $S = \alpha I_{\mathcal{H}}$.

A *Riesz basis* for $\mathcal{H}$ is a frame for $\mathcal{H}$ which is also a basis.

Definition 2.2. Let $\{f_i\}_{i\in I}$ be a frame for $\mathcal{H}$ with synthesis operator T. A frame $\{g_i\}_{i\in I}$ for $\mathcal{H}$ with synthesis operator U is a dual frame of $\{f_i\}_{i\in I}$ if the following reconstruction formula holds

$$f = UT^* f = \sum_{i\in I} \langle f, f_i\rangle g_i \text{ , for all } f \in \mathcal{H}. \tag{2.2}$$

In particular, $\{S^{-1}f_i\}_{i\in I}$ is called the *canonical dual frame* of $\{f_i\}_{i\in I}$. Observe that a Riesz basis has a unique dual, the canonical one.

2.2. **Fusion frames.** Let $\{W_i\}_{i\in I}$ be a family of closed subspaces in $\mathcal{H}$, and let $\{w_i\}_{i\in I}$ be a family of weights, i.e., $w_i > 0$ for all $i \in I$. We will denote $\{W_i\}_{i\in I}$ with $\mathbf{W}$, $\{w_i\}_{i\in I}$ with $\mathbf{w}$ and $\{(W_i, w_i)\}_{i\in I}$ with $(\mathbf{W}, \mathbf{w})$. If $T \in L(\mathcal{H}, \mathcal{K})$ we will write $(T\mathbf{W}, \mathbf{w})$ for $\{(TW_i, w_i)\}_{i\in I}$.

We consider the Hilbert space

$$\mathcal{W} := \bigoplus_{i\in I} W_i = \{\{f_i\}_{i\in I} : f_i \in W_i \text{ and } \{\|f_i\|\}_{i\in I} \in \ell^2(I)\}$$

with inner product $\langle \{f_i\}_{i\in I}, \{g_i\}_{i\in I}\rangle = \sum_{i\in I}\langle f_i, g_i\rangle$.

For V a closed subspace of $\mathcal{H}$, π_V is the orthogonal projection onto V.

Definition 2.3. We say that $(\mathbf{W}, \mathbf{w})$ is a *fusion frame* for $\mathcal{H}$, if there exist constants $0 < \alpha \le \beta < \infty$ such that

$$\alpha\|f\|^2 \le \sum_{i\in I} w_i^2 \|\pi_{W_i}(f)\|^2 \le \beta\|f\|^2 \text{ for all } f \in \mathcal{H}. \tag{2.3}$$

We call α and β the *fusion frame bounds*. The family $(\mathbf{W}, \mathbf{w})$ is called an *α-tight fusion frame*, if in (2.3) the constants α and β can be chosen so that $\alpha = \beta$, and a *Parseval fusion frame* provided that $\alpha = \beta = 1$.

If $(\mathbf{W}, \mathbf{w})$ possesses an upper fusion frame bound, but not necessarily a lower bound, we call it a *Bessel fusion sequence* with Bessel fusion bound β.

If $w_i = w$ for all $i \in I$, the collection $(\mathbf{W}, \mathbf{w})$ is called *w-uniform*. In this case we write $(\mathbf{W}, w)$.

We refer to a fusion frame $(\mathbf{W}, 1)$ as an *orthonormal fusion basis* if $\mathcal{H}$ is the orthogonal sum of the subspaces W_i.

We associate to a Bessel fusion sequence $(\mathbf{W}, \mathbf{w})$ the following bounded operators:

$$T_{\mathbf{W},\mathbf{w}} : \mathcal{W} \to \mathcal{H},\ T_{\mathbf{W},\mathbf{w}}\{f_i\}_{i\in I} = \textstyle\sum_{i\in I} w_i f_i,$$

called the *synthesis operator* and

$$T^*_{\mathbf{W},\mathbf{w}} : \mathcal{H} \to \mathcal{W},\ T^*_{\mathbf{W},\mathbf{w}} f = \{w_i \pi_{W_i}(f)\}_{i\in I},$$

named the *analysis operator*.

If $(\mathbf{W}, \mathbf{w})$ is a *fusion frame* we have the *fusion frame operator*

$$S_{\mathbf{W},\mathbf{w}} = T_{\mathbf{W},\mathbf{w}} T^*_{\mathbf{W},\mathbf{w}},$$

which is positive, selfadjoint and invertible. If $(\mathbf{W}, \mathbf{w})$ is an α-tight fusion frame, then $S_{\mathbf{W},\mathbf{w}} = \alpha I_{\mathcal{H}}$.

As for frames, $(\mathbf{W}, \mathbf{w})$ is a Bessel fusion sequence for $\mathcal{H}$ if and only if $T_{\mathbf{W},\mathbf{w}}$ is a well defined bounded linear operator. A Bessel fusion sequence $(\mathbf{W}, \mathbf{w})$ is a fusion frame for $\mathcal{H}$ if and only if $T_{\mathbf{W},\mathbf{w}}$ is onto.

Remark 2.4. For $\mathbf{w} \in \ell^2(I)$ it is easy to see that $(\mathbf{W}, \mathbf{w})$ is a Bessel fusion sequence for $\mathcal{H}$.

Suppose that $T_{\mathbf{W},\mathbf{w}}$ is well defined. If $T_{\mathbf{W},\mathbf{w}}$ is bounded then $\mathbf{w} \in \ell^{\infty}(I)$. In view of this, in the sequel we suppose that each family of weights is in $\ell^{\infty}(I)$.

3. Dual fusion frames

Assume that $(\mathbf{W}, \mathbf{w})$ is a fusion frame for $\mathcal{H}$. In [2, Definition 3.19] the fusion frame $(S_{\mathbf{W},\mathbf{w}}^{-1}\mathbf{W}, \mathbf{w})$ is called the *dual fusion frame* of $(\mathbf{W}, \mathbf{w})$. This family looks as the analogous to the canonical dual frame in the classical frame theory and

$$f = S_{\mathbf{W},\mathbf{w}}^{-1} S_{\mathbf{W},\mathbf{w}} f = \sum_{j \in I} w_j^2 S_{\mathbf{W},\mathbf{w}}^{-1} \pi_{W_j}(f) \text{ , for all } f \in \mathcal{H}. \tag{3.1}$$

If we like (3.1) expressed in terms of operators as in (2.2), we find the following obstacle. We have $R(T^*_{\mathbf{W},\mathbf{w}}) \subseteq \mathcal{W}$ and $D(T_{S_{\mathbf{W},\mathbf{w}}^{-1} W,\mathbf{w}}) = \bigoplus_{i \in I} S_{\mathbf{W},\mathbf{w}}^{-1} W_i$, so $T_{S_{\mathbf{W},\mathbf{w}}^{-1} W,\mathbf{w}} T^*_{\mathbf{W},\mathbf{w}}$ is generally not defined. This difficulty with the domain disappears with the following definition of dual fusion frame.

Definition 3.1. Assume that $(\mathbf{W}, \mathbf{w})$ and $(\mathbf{V}, \mathbf{v})$ are fusion frames for $\mathcal{H}$. If there exists $Q \in L(\mathcal{W}, \mathcal{V})$ such that

$$T_{\mathbf{V},\mathbf{v}} Q T^*_{\mathbf{W},\mathbf{w}} = I_{\mathcal{H}}, \tag{3.2}$$

then $(\mathbf{V}, \mathbf{v})$ is a dual fusion frame of $(\mathbf{W}, \mathbf{w})$

Sometimes we will write *Q-dual fusion frame* in case we need the operator Q to be explicit.

The following lemma collects some properties of dual fusion frames that are analogous to corresponding ones for dual frames with similar proofs (see, e. g., [5, Lemma 5.6.2]).

Lemma 3.2. *Let $(\boldsymbol{W}, \mathbf{w})$ and $(\boldsymbol{V}, \mathbf{v})$ be Bessel fusion sequences for $\mathcal{H}$, and let $Q \in L(\mathcal{W}, \mathcal{V})$. Then the following conditions are equivalent:*

1. $T_{\boldsymbol{V},\mathbf{v}} Q T^*_{\boldsymbol{W},\mathbf{w}} = I_{\mathcal{H}}$.
2. $T_{\boldsymbol{W},\mathbf{w}} Q^* T^*_{\boldsymbol{V},\mathbf{v}} = I_{\mathcal{H}}$.
3. $T^*_{\boldsymbol{W},\mathbf{w}}$ *is injective,* $T_{\boldsymbol{V},\mathbf{v}} Q$ *is surjective and* $$(T^*_{\boldsymbol{W},\mathbf{w}} T_{\boldsymbol{V},\mathbf{v}} Q)^2 = T^*_{\boldsymbol{W},\mathbf{w}} T_{\boldsymbol{V},\mathbf{v}} Q.$$
4. $T^*_{\boldsymbol{V},\mathbf{v}}$ *is injective,* $T_{\boldsymbol{W},\mathbf{w}} Q^*$ *is surjective and* $$(T^*_{\boldsymbol{V},\mathbf{v}} T_{\boldsymbol{W},\mathbf{w}} Q^*)^2 = T^*_{\boldsymbol{V},\mathbf{v}} T_{\boldsymbol{W},\mathbf{w}} Q^*.$$
5. $\langle f, g \rangle = \langle Q T^*_{\boldsymbol{W},\mathbf{w}} f, T^*_{\boldsymbol{V},\mathbf{v}} g \rangle = \langle Q^* T^*_{\boldsymbol{V},\mathbf{v}} f, T^*_{\boldsymbol{W},\mathbf{w}} g \rangle$ *for all* $f, g \in \mathcal{H}$.

In case any of these equivalent conditions are satisfied, $(\boldsymbol{W}, \mathbf{w})$ and $(\boldsymbol{V}, \mathbf{v})$ are fusion frames for $\mathcal{H}$, $(\boldsymbol{V}, \mathbf{v})$ is a Q-dual fusion frame of $(\boldsymbol{W}, \mathbf{w})$, and $(\boldsymbol{W}, \mathbf{w})$ is a Q^*-dual fusion frame of $(\boldsymbol{V}, \mathbf{v})$.

Note that the equivalence of conditions (1) and (2) implies that the roles of $(\mathbf{W}, \mathbf{w})$ and $(\mathbf{V}, \mathbf{v})$ can be interchanged in the definition of dual fusion frame. Conditions (3) and (4) say that the Q-mixed Gram operator is a projection if $(\mathbf{W}, \mathbf{w})$ and $(\mathbf{V}, \mathbf{v})$ are dual fusion frames. Finally, condition (5) expresses the inner product of two elements of $\mathcal{H}$ in terms of a Q-inner product of their images under the analysis operators of $(\mathbf{W}, \mathbf{w})$ and $(\mathbf{V}, \mathbf{v})$.

3.1. **Dual fusion frames obtained from left inverses of the analysis operator.** In classical frame theory, dual frames are related to the left inverses of the analysis operator. An analogous result is valid for the following special type of dual fusion frame.

Definition 3.3. Let $p_i : \mathcal{W} \to \mathcal{W}$, $p_i\{f_j\}_{j\in I} = \{\delta_{i,j} f_j\}_{j\in I}$. If Q in definition 3.1 satisfies

$$Qp_i\mathcal{W} = p_i\mathcal{V},$$

we say that Q is *component preserving* and $(\mathbf{V}, \mathbf{v})$ is a *component preserving dual fusion frame* of $(\mathbf{W}, \mathbf{w})$.

The next two lemmas links component preserving dual fusion frames of $(\mathbf{W}, \mathbf{w})$ with the left inverses of the analysis operator $T^*_{\mathbf{W},\mathbf{w}}$. They are analogous to a corresponding result for dual frames (see, e. g., [5, Lemma 5.6.3.]).

We denote the set of bounded left inverses of $T^*_{\mathbf{W},\mathbf{w}}$ with $\mathfrak{L}_{T^*_{\mathbf{W},\mathbf{w}}}$.

Lemma 3.4. *Let $(\boldsymbol{W}, \mathbf{w})$ be a fusion frame for $\mathcal{H}$. If $(\boldsymbol{V}, \mathbf{v})$ is a component preserving dual fusion frame of $(\boldsymbol{W}, \mathbf{w})$ then $V_i = Ap_i\mathcal{W}$, for each $i \in I$, where $A \in \mathfrak{L}_{T^*_{\boldsymbol{W},\mathbf{w}}}$.*

Proof. Let $Q \in L(\mathcal{W}, \mathcal{V})$ be component preserving such that $T_{\mathbf{V},\mathbf{v}}QT^*_{\mathbf{W},\mathbf{w}} = I_{\mathcal{H}}$. Let $A = T_{\mathbf{V},\mathbf{v}}Q$. Clearly, $A \in \mathfrak{L}_{T^*_{\mathbf{W},\mathbf{w}}}$. Since Q is component preserving, $Ap_i\mathcal{W} = T_{\mathbf{V},\mathbf{v}}Qp_i\mathcal{W} = T_{\mathbf{V},\mathbf{v}}p_i\mathcal{V} = V_i$. □

Fusion frames behave differently under operators than classical frames do (see e.g. [4, 9]). A well known result in classical frame theory is that if $A \in L(\mathcal{K}, \mathcal{H})$ is surjective and $\{e_i\}_{i\in I}$ is an orthonormal basis for $\mathcal{K}$, then $\{Ae_i\}_{i\in I}$ is a frame for $\mathcal{H}$. In the context of fusion frames the situation is different. Given an orthonormal fusion basis $(\mathbf{W}, 1)$ of $\mathcal{K}$, a surjective $A \in L(\mathcal{K}, \mathcal{H})$ and a family of weights $\mathbf{v}$, the collection $(A\mathbf{W}, \mathbf{v})$ could even not be a Bessel sequence for $\mathcal{H}$. This fact does not allow to have the complete converse of the previous result in an infinite-dimensional separable Hilbert space. However, a reciprocal is valid in the following sense:

Lemma 3.5. *Let $(\boldsymbol{W}, \mathbf{w})$ be a fusion frame for $\mathcal{H}$, $A \in \mathfrak{L}_{T^*_{\boldsymbol{W},\mathbf{w}}}$ and $V_i = Ap_i\mathcal{W}$ for each $i \in I$. If $(\boldsymbol{V}, \mathbf{v})$ is a Bessel fusion sequence and*

$$Q_{A,\mathbf{v}} : \mathcal{W} \to \mathcal{V},\ Q_{A,\mathbf{v}}\{f_j\}_{j\in I} = \{\tfrac{1}{v_i}Ap_i\{f_j\}_{j\in I}\}_{i\in I}$$

is a well defined bounded operator, then $(\boldsymbol{V}, \mathbf{v})$ is a $Q_{A,\mathbf{v}}$-component preserving dual fusion frame of $(\boldsymbol{W}, \mathbf{w})$.

Proof. Let $\{f_j\}_{j\in I} \in \mathcal{W}$. We have

$$Ap_ip_{i_0}\{f_j\}_{j\in I} = \delta_{i,i_0}Ap_{i_0}\{f_j\}_{j\in I} \in Ap_{i_0}\mathcal{W} = V_{i_0}.$$

Therefore,

$$Q_{A,\mathbf{v}}p_{i_0}\{f_j\}_{j\in I} = \{\delta_{i,i_0}\tfrac{1}{v_{i_0}}Ap_{i_0}\{f_j\}_{j\in I}\}_{i\in I} \in p_{i_0}\textstyle\sum_{j\in I}\oplus V_j.$$

Consequently, $Q_{A,\mathbf{v}}p_{i_0}\mathcal{W} \subseteq p_{i_0}\sum_{j\in I}\oplus V_j$.

For the other inclusion, let $\{g_i\}_{i\in I} \in \mathcal{V}$. Then $g_i = Ap_i\{f_j^i\}_{j\in I}$ with $\{f_j^i\}_{j\in I} \in \mathcal{W}$ for each $i \in I$, and so

$$p_{i_0}\{g_i\}_{i\in I} = \{\delta_{i,i_0} Ap_{i_0}\{f_j^{i_0}\}_{j\in I}\}_{i\in I} = \{\delta_{i,i_0} A\{\delta_{j,i_0} f_{i_0}^{i_0}\}_{j\in I}\}_{i\in I}.$$

We have $v_{i_0}\{\delta_{j,i_0} f_{i_0}^{i_0}\}_{j\in I} \in p_{i_0}\mathcal{W}$ and

$$Ap_i v_{i_0}\{\delta_{j,i_0} f_{i_0}^{i_0}\}_{j\in I} = \delta_{i,i_0} v_{i_0} A\{\delta_{j,i_0} f_{i_0}^{i_0}\}_{j\in I}.$$

Thus $p_{i_0}\{g_i\}_{i\in I} = Q_{A,\mathbf{v}} v_{i_0}\{\delta_{j,i_0} f_{i_0}^{i_0}\}_{j\in I}$. So, $p_{i_0}\mathcal{V} \subseteq Q_{A,\mathbf{v}} p_{i_0}\mathcal{W}$.

This shows that $Q_{A,\mathbf{v}}$ is component preserving.

Since $(\mathbf{V}, \mathbf{v})$ is a Bessel fusion sequence, $T_{\mathbf{V},\mathbf{v}}$ is a well defined bounded linear operator. If $\{f_j\}_{j\in I} \in \mathcal{W}$, then

$$T_{\mathbf{V},\mathbf{v}} Q_{A,\mathbf{v}}\{f_j\}_{j\in I} = \sum_{i\in I} v_i \tfrac{1}{v_i} Ap_i\{f_j\}_{j\in I} = A\sum_{i\in I} p_i\{f_j\}_{j\in I} = A\{f_j\}_{j\in I}.$$

Hence $T_{\mathbf{V},\mathbf{v}} Q_{A,\mathbf{v}} = A \in \mathfrak{L}_{T^*_{\mathbf{W},\mathbf{w}}}$. So $(\mathbf{V}, \mathbf{v})$ is a $Q_{A,\mathbf{v}}$-component preserving dual fusion frame of $(\mathbf{W}, \mathbf{w})$. □

Remark 3.6. Let A, $(\mathbf{V}, \mathbf{v})$ and $Q_{A,\mathbf{v}}$ as in Lemma 3.5. We can give the following sufficient conditions for $(\mathbf{V}, \mathbf{v})$ being a Bessel fusion sequence and for $Q_{A,\mathbf{v}}$ being a well defined bounded operator:

(1) Let $\gamma(A)$ be the reduced minimum modulus of A, i.e., $\gamma(A) = \inf\{\|Ax\| : \|x\| = 1, x \in N(A)^\perp\}$. Assume $\gamma(Ap_i) > 0$ and there exists $\delta > 0$ such that $\delta \le v_i^{-2}\gamma(Ap_i)^2$ for all $i \in I$. Since $\{(p_i\mathcal{W}, 1)\}_{i\in I}$ is an orthonormal fusion basis for $\mathcal{W}$, by [9, Theorem 3.6] $(\mathbf{V}, \mathbf{v})$ is a Bessel fusion sequence for $\mathcal{H}$ with upper bound $\frac{\|A\|^2}{\delta}$.

(2) If $v_i > \delta > 0$ for each $i \in I$, then $Q_{A,\mathbf{v}}$ is a well defined bounded operator with $\|Q_{A,\mathbf{v}}\| \le \frac{\|A\|}{\delta}$. To see this, note that if $\{f_j\}_{j\in I} \in \mathcal{W}$, then

$$\sum_{i\in I} \|\tfrac{1}{v_i} Ap_i\{f_j\}_{j\in I}\| \le \tfrac{\|A\|^2}{\delta^2}\sum_{i\in I}\|p_i\{f_j\}_{j\in I}\|^2 = \tfrac{\|A\|^2}{\delta^2}\|\{f_i\}_{i\in I}\|^2.$$

Example 3.7. Any reconstruction formula of the form $f = AT^*_{\mathbf{W},\mathbf{w}} f$ involves a left inverse A of $T^*_{\mathbf{W},\mathbf{w}}$, and then, in view of Lemma 3.5, it could be expressed in terms of a $Q_{A,\mathbf{v}}$-component preserving dual of $(\mathbf{W}, \mathbf{w})$. The present example is based on this observation.

Since $(S^{-1}_{\mathbf{W},\mathbf{w}} T_{\mathbf{W},\mathbf{w}})T^*_{\mathbf{W},\mathbf{w}} = I_{\mathcal{H}}$, then $A = S^{-1}_{\mathbf{W},\mathbf{w}} T_{\mathbf{W},\mathbf{w}} \in \mathfrak{L}_{T^*_{\mathbf{W},\mathbf{w}}}$. We have

$$Ap_i\mathcal{W} = S^{-1}_{\mathbf{W},\mathbf{w}} W_i,\ \forall i \in I$$

and that

$$Q_{A,\mathbf{w}} : \mathcal{W} \to \bigoplus_{i\in I} S^{-1}_{\mathbf{W},\mathbf{w}} W_i,\ Q_{A,\mathbf{w}}\{f_j\}_{j\in I} = \{S^{-1}_{\mathbf{W},\mathbf{w}} f_i\}_{i\in I}$$

is a well defined bounded operator with $\|Q_{A,\mathbf{w}}\| \le \|S^{-1}_{\mathbf{W},\mathbf{w}}\|$. So, by Lemma 3.5, the dual fusion frame introduced in [2], $(S^{-1}_{\mathbf{W},\mathbf{w}}\mathbf{W}, \mathbf{w})$, is a component preserving dual fusion frame of $(\mathbf{W}, \mathbf{w})$.

Note that $Q_{A,\mathbf{w}}$ results bounded without any restriction on the weights w_i.

In the sequel we refer to $(S^{-1}_{\mathbf{W},\mathbf{w}}\mathbf{W}, \mathbf{w})$ as the canonical dual and to

$$Q^*_{S^{-1}_{\mathbf{W},\mathbf{w}} T_{\mathbf{W},\mathbf{w}},\mathbf{w}} T^*_{S^{-1}_{\mathbf{W},\mathbf{w}}\mathbf{W},\mathbf{w}} f = T^*_{\mathbf{W},\mathbf{w}} S^{-1}_{\mathbf{W},\mathbf{w}} f \in \mathcal{W}$$

as the *fusion frame coefficients* of $f \in \mathcal{H}$.

The next lemma is about the minimality of the fusion frame coefficients and has its analogous in classical frame theory with a similar proof (see e.g. [5, Lemma 5.4.2]).

Lemma 3.8. *Let* $(\boldsymbol{W},\mathbf{w})$ *be a fusion frame for* $\mathcal{H}$ *and* $f \in \mathcal{H}$. *For all* $\{f_j\}_{j\in I} \in \mathcal{W}$ *satisfying* $T_{\boldsymbol{W},\mathbf{w}}\{f_j\}_{j\in I} = f$ *we have* $\|T^*_{\boldsymbol{W},\mathbf{w}}S^{-1}_{\boldsymbol{W},\mathbf{w}}f\| \leq \|\{f_j\}_{j\in I}\|$.

Example 3.9. Assume $\{e_j\}_{j=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}$. Fix $N \in \mathbb{N}$ and define $W_j = \overline{\text{span}}\{e_1,\dots,e_j\}$ for $1 \leq j \leq N$ and $W_j = \overline{\text{span}}\{e_{j-N+1},\dots,e_j\}$ for $j > N$.

Then $(\mathbf{W},1)$ is a $1-$uniform N-tight fusion frame for $\mathcal{H}$. To simplify the exposition we consider in the sequel $N = 3$ and $e_{-1} = e_0 = 0$.

In this case $\mathcal{W} = \{\{\sum_{k=0}^{2} c_{j,j-k}e_{j-k}\}_{j=1}^{\infty} : \sum_{j=1}^{\infty}\sum_{k=0}^{2}|c_{j,j-k}|^2 < \infty\}$ and $T^*_{W,1} : \mathcal{H} \to \mathcal{W}$, $T^*_{W,1}\sum_{j=1}^{\infty} c_je_j = \{\sum_{k=0}^{2} c_{j-k}e_{j-k}\}_{j=1}^{\infty}$.

Consider $A : \mathcal{W} \to \mathcal{H}$, $A\{\sum_{k=0}^{2} c_{j,j-k}e_{j-k}\}_{j=1}^{\infty} = \sum_{j=1}^{\infty} c_{j,j}e_j$. Then $A \in \mathfrak{L}_{T^*_{\mathbf{W},1}}$, $V_i = Ap_i\mathcal{W} = \overline{\text{span}}\{e_i\}$, $(\mathbf{V},1)$ is a $1-$uniform orthonormal fusion basis for $\mathcal{H}$ and $Q_{A,1} : \mathcal{W} \to \mathcal{V}$, $Q_{A,1}\{\sum_{k=0}^{2} c_{j,j-k}e_{j-k}\}_{j=1}^{\infty} = \{c_{i,i}e_i\}_{i=1}^{\infty}$ is bounded.

Note that since $(\mathbf{V},1)$ is an orthonormal fusion basis for $\mathcal{H}$, it coincides with its unique component preserving dual fusion frame. On the other hand, by Lemma 3.2, $(\mathbf{W},1)$ gives an example of dual of $(\mathbf{V},1)$, which is not component preserving.

Remark 3.10. The following shows that in Lemma 3.5, the hypotheses $(\mathbf{V},\mathbf{v})$ to be a Bessel fusion sequence and $Q_{A,\mathbf{v}}$ to be bounded, cannot be avoided:

(1) Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of a Hilbert space $\mathcal{H}$ and consider $W_1 = \overline{\text{span}}\{e_k : k \geq 2\} = \{e_1\}^{\perp}$ and $W_k = \overline{\text{span}}\{e_1,e_k\}$ for $k \geq 2$.

It is proved in [9, Example 7.5] that if $(\mathbf{W},\mathbf{w})$ is a Bessel fusion sequence, then $\mathbf{w} \in \ell^2(\mathbb{N})$. Moreover, the frame operator $S_{\mathbf{W},\mathbf{w}}$ is diagonal with respect to $\{e_n\}_{n\in\mathbb{N}}$ and so it is also $S^{-1}_{\mathbf{W},\mathbf{w}}$.

Now, in particular, this implies that $S^{-1}_{\mathbf{W},\mathbf{w}}W_k = W_k$ for all $k \in \mathbb{N}$. So if $A = S^{-1}_{\mathbf{W},\mathbf{w}}T_{\mathbf{W},\mathbf{w}}$ then $V_k := Ap_k\mathcal{W} = S^{-1}_{\mathbf{W},\mathbf{w}}W_k$. Thus, if $\mathbf{v} \in \ell^{\infty}(\mathbb{N}) \setminus \ell^2(\mathbb{N})$, then $(\mathbf{V},\mathbf{v})$ is not a Bessel fusion sequence.

(2) There exist weights v_i so that $Q_{S^{-1}_{\mathbf{W},\mathbf{w}}T_{\mathbf{W},\mathbf{w}},\mathbf{v}}$ is unbounded. Specifically, let $v_i \leq \frac{w_i}{i\|S_{\mathbf{W},\mathbf{w}}\|}$. If $\{f_j^{(i)}\}_{j\in I} \in p_i\mathcal{W}$, then

$$\|Q_{S^{-1}_{\mathbf{W},\mathbf{w}}T_{\mathbf{W},\mathbf{w}},\mathbf{v}}\{f_j^{(i)}\}_{j\in I}\| = \frac{w_i}{v_i}\|S^{-1}_{\mathbf{W},\mathbf{w}}f_i^{(i)}\| \geq \frac{w_i}{v_i\|S_{\mathbf{W},\mathbf{w}}\|}\|f_i^{(i)}\|$$
$$= \frac{w_i}{v_i\|S_{\mathbf{W},\mathbf{w}}\|}\|\{f_j^{(i)}\}_{j\in I}\| \geq i\|\{f_j^{(i)}\}_{j\in I}\|.$$

Let $(\mathbf{W},\mathbf{w})$ be a fusion frame for $\mathcal{H}$. It can be seen as in [5, Lemma 5.6.4], that the bounded left inverses of $T^*_{\mathbf{W},\mathbf{w}}$ are the operators A of the form

$$A = S^{-1}_{\mathbf{W},\mathbf{w}}T_{\mathbf{W},\mathbf{w}} + R\left(I_{\mathcal{W}} - T^*_{\mathbf{W},\mathbf{w}}S^{-1}_{\mathbf{W},\mathbf{w}}T_{\mathbf{W},\mathbf{w}}\right),$$

where $R \in L(\mathcal{W}, \mathcal{H})$. This fact, Lemma 3.4 and Remark 3.6 yield the following description for component preserving dual fusion frames:

Theorem 3.11. *Let $(\boldsymbol{W}, \mathbf{w})$ be a fusion frame for $\mathcal{H}$. Suppose that $v_i > \delta > 0$ for each $i \in I$. Then the component preserving dual fusion frames of $(\boldsymbol{W}, \mathbf{w})$ are the Bessel fusion sequences $(\boldsymbol{V}, \mathbf{v})$ where*

$$V_i = \left[S_{\mathbf{W},\mathbf{w}}^{-1} T_{\mathbf{W},\mathbf{w}} + R\left(I_{\mathcal{W}} - T_{\mathbf{W},\mathbf{w}}^{*} S_{\mathbf{W},\mathbf{w}}^{-1} T_{\mathbf{W},\mathbf{w}}\right)\right](p_i \mathcal{W})$$

and $R \in L(\mathcal{W}, \mathcal{H})$.

3.2. **Dual fusion frames obtained from dual frames.** The following theorem provides a method to obtain dual fusion frames.

First recall that if $\{f_i\}_{i\in I} \subset \mathcal{H}$ is a Bessel sequence with bound β, then

$$\|\sum_{i\in I} c_i f_i\|^2 \leq \beta \|c\|^2 \text{ for all } c \in \ell^2(I). \tag{3.3}$$

Theorem 3.12. *For each $i \in I$, let $w_i > 0$, $v_i > 0$, and let W_i and V_i be closed subspaces of $\mathcal{H}$. Let $\{f_i^\ell\}_{l\in L_i}$ be a frame for W_i and $\{\tilde{f}_i^l\}_{l\in L_i}$ be a frame for V_i, with frame bounds α_i, β_i, $\tilde{\alpha}_i$ and $\tilde{\beta}_i$, respectively. Suppose that $0 < \alpha = \inf_{i\in I} \alpha_i \leq \beta = \sup_{i\in I} \beta_i < \infty$ and $0 < \tilde{\alpha} = \inf_{i\in I} \tilde{\alpha}_i \leq \tilde{\beta} = \sup_{i\in I} \tilde{\beta}_i < \infty$. Let $Q : \mathcal{W} \longrightarrow \mathcal{V}$, $Q\{h_i\}_{i\in I} := \{\sum_{l\in L_i} \langle h_i, f_i^l\rangle \tilde{f}_i^l\}_{i\in I}$. The following conditions are equivalent.*

1. *$\{w_i f_i^\ell\}_{i\in I, l\in L_i}$ and $\{v_i \tilde{f}_i^\ell\}_{i\in I, l\in L_i}$ are dual frames in $\mathcal{H}$.*
2. *$(\boldsymbol{V}, \mathbf{v})$ is a Q-dual fusion frame of $(\boldsymbol{W}, \mathbf{w})$.*

Proof. By [2, Theorem 3.2] it only remains to see the duality. If $\{h_i\}_{i\in I} \in \mathcal{W}$, by (3.3) and (2.1),

$$\sum_{i\in I} \| \sum_{l\in L_i} \langle h_i, f_i^l\rangle \tilde{f}_i^l\|^2 \leq \sum_{i\in I} \tilde{\beta}_i^2 \beta_i^2 \|h_i\|^2 \leq \tilde{\beta}^2 \beta^2 \sum_{i\in I} \|h_i\|^2 < \infty.$$

So Q is a well defined bounded operator.

Using that $\langle \pi_{W_i}(f), f_i^l\rangle = \langle f, f_i^l\rangle$, we obtain

$$T_{\mathbf{V},\mathbf{v}} Q T_{\mathbf{W},\mathbf{w}}^{*}(f) = \sum_{i\in I} \sum_{l\in L_i} \langle f, w_i f_i^l\rangle v_i \tilde{f}_i^l.$$

Finally, the last term is equal to f for all $f \in \mathcal{H}$ if and only if $\{w_i f_i^l\}_{i\in I, l\in L_i}$ and $\{v_i \tilde{f}_i^l\}_{i\in I, l\in L_i}$ are dual frames in $\mathcal{H}$. □

In the following example we exhibit harmonic fusion frames with non-canonical dual fusion frames, which are also harmonic.

Example 3.13. Let $a \in \mathbb{R}$, $g \in L^2(\mathbb{R})$, $E_a g(x) := e^{2\pi i a x} g(x)$ and $T_a g(x) := g(x-a)$. Suppose that the so-called Gabor system $\{E_{am} T_n g\}_{m,n\in\mathbb{Z}}$ is a Parseval frame. Fix $N \in \mathbb{N}$ and define

$$W_i = \overline{\text{span}}\{E_{a(Nm+i)} T_n g\}_{m,n\in\mathbb{Z}}, \quad 0 \leq i \leq N-1.$$

We have $W_0 = E_a W_{N-1}$ and $W_{i+1} = E_a W_i$ for $0 \leq i \leq N-2$. The family $(\mathbf{W}, 1)$ is the finite harmonic fusion frame considered in [2, Example 6.4].

Let now $d \in \mathbb{C}$ and $N \in \mathbb{N}$ be such that $\frac{1}{\sqrt{2}} < |d| < 1$, $|d|^2 N \in \mathbb{N}$ and $|d|^2 N > 1$. Let $c_i \in \mathbb{C}$ for $i = 1, \ldots, N - |d|^2 N$, with some $c_i \neq 0$, such that $\sum_{i=1}^{N-|d|^2 N} c_i = 0$. Set

$$g = d\chi_{[0,1)} \quad \text{and} \quad h = d\chi_{[0,1)} + \sum_{i=1}^{N-|d|^2N} c_i\chi_{[1+\frac{i-1}{|d|^2N},1+\frac{i}{|d|^2N})}.$$

Let $W_i = \overline{\text{span}}\{E_{|d|^2(Nm+i)}T_n g\}_{m,n\in\mathbb{Z}}$ and $V_i = \overline{\text{span}}\{E_{|d|^2(Nm+i)}T_n h\}_{m,n\in\mathbb{Z}}$, for $0 \le i \le N-1$. We are going to show that the finite harmonic fusion frame $(\mathbf{V}, 1)$ is a dual fusion frame of $(\mathbf{W}, 1)$, and it is not the canonical dual.

(1) We have $\sum_{n\in\mathbb{Z}} |g(x-n)| = |d|^2$ and $\sum_{n\in\mathbb{Z}} g(x-n)\overline{g(x-n-\frac{k}{|d|^2})} = 0$ a.e.. By [5, Theorems 9.5.2 (ii) and 8.3.1 (ii)], $\{E_{|d|^2m}T_n g\}_{m,n\in\mathbb{Z}}$ is a Parseval Gabor frame but not a Riesz basis for $L^2(\mathbb{R})$.

(2) The family $\{E_{|d|^2m}T_n g\}_{m,n\in\mathbb{Z}}$ is a Bessel sequence for $L^2(\mathbb{R})$ and

$$\{E_{|d|^2mN}T_n g\}_{m,n\in\mathbb{Z}} \subseteq \{E_{|d|^2m}T_n g\}_{m,n\in\mathbb{Z}},$$

so $\{E_{|d|^2mN}T_n g\}_{m,n\in\mathbb{Z}}$ is a Bessel sequence for W_0. Moreover, it is an orthogonal system with elements of equal norm $|d|$. Consequently, the associated well defined frame operator is $|d|^2 I_{L^2(\mathbb{R})}$ and $\{E_{|d|^2mN}T_n g\}_{m,n\in\mathbb{Z}}$ is an orthogonal basis for W_0. Thus, since $E_{|d|^2Ni}$ is a unitary operator, $\{E_{|d|^2N(mN+i)}T_n g\}_{m,n\in\mathbb{Z}}$ is a $|d|^2$-tight frame for W_i, $0 \le i \le N-1$.

(3) By [5, Theorem 8.3.1 (i)], $W_0 \neq L^2(\mathbb{R})$. Hence, since $E_{|d|^2Ni}$ is a unitary operator, $W_i \neq L^2(\mathbb{R})$ for $0 \le i \le N-1$.

(4) By [5, Proposition 5.3.5], using that $\{E_{|d|^2(Nm+i)}T_n g\}_{m,n\in\mathbb{Z}}$ is a $|d|^2$-tight frame for W_i, $0 \le i \le N-1$, and $\{E_{|d|^2m}T_n g\}_{m,n\in\mathbb{Z}}$ is a Parseval Gabor frame for $L^2(\mathbb{R})$, we obtain $S_{W,1} = |d|^{-2} I_{L^2(\mathbb{R})}$. Thus $(\mathbf{W}, 1)$ is a finite harmonic $|d|^{-2}$-tight fusion frame, so it coincides with its canonical dual.

(5) Let $\chi^{|d|^{-2}}_{[1,|d|^{-2})}$ be the $|d|^{-2}$-periodic extension to the real line of the restriction of $\chi_{[1,|d|^{-2})}$ to $[0, |d|^{-2})$. Then,

$$1 - \sum_{m\in\mathbb{Z}} |d|^2\langle\chi_{[0,1)}, E_{|d|^2m}\chi_{[0,1)}\rangle E_{|d|^2m} = \chi^{|d|^{-2}}_{[1,|d|^{-2})}.$$

The function $f = \sum_{i=1}^{N-|d|^2N} c_i\chi_{[1+\frac{i-1}{|d|^2N},1+\frac{i}{|d|^2N})}$ belongs to the Wiener space, i.e., $\sum_{k\in Z} \|f\chi_{[k,(k+1))}\|_\infty < \infty$. As a consequence of [5, Proposition 8.5.2], $\{E_{|d|^2mN}T_n f\}_{m,n\in\mathbb{Z}}$ is a Bessel sequence.

We have $h = g + f(1 - \sum_{m\in\mathbb{Z}} |d|^2\langle\chi_{[0,1)}, E_{|d|^2m}\chi_{[0,1)}\rangle E_{|d|^2m})$ and, by [5, Proposition 9.3.8], $\{E_{|d|^2m}T_n h\}_{m,n\in\mathbb{Z}}$ is a dual frame of $\{E_{|d|^2m}T_n g\}_{m,n\in\mathbb{Z}}$.

(6) The restriction to the intervals of the form $[n, n+1)$ of any function in W_0 is $\frac{1}{|d|^2N}$-periodic with $|d|^2N$ periods. So $h \notin W_0$ and consequently, $V_0 \neq W_0$. Since $V_i = E_{|d|^2i}V_0$ and $W_i = E_{|d|^2i}W_0$, it follows that $V_i \neq W_i$ for $0 \le i \le N-1$.

(7) The collection $\{E_{|d|^2m}T_n h\}_{m,n\in\mathbb{Z}}$ is a Bessel sequence for $L^2(\mathbb{R})$ and $\{E_{|d|^2mN}T_n h\}_{m,n\in\mathbb{Z}} \subseteq \{E_{|d|^2m}T_n h\}_{m,n\in\mathbb{Z}}$, thus $\{E_{|d|^2mN}T_n h\}_{m,n\in\mathbb{Z}}$ is a Bessel sequence for V_0. Hence it has a well defined bounded frame operator. We have

$$\sum_{m,n\in\mathbb{Z}}\langle E_{|d|^2m'N}T_{n'}h, E_{|d|^2mN}T_n h\rangle E_{|d|^2mN}T_n h = \|h\|^2 E_{|d|^2m'N}T_{n'}h,$$

thus $\{E_{|d|^2mN}T_n h\}_{m,n\in\mathbb{Z}}$ is a $\|h\|^2$-tight frame for V_0. Since $E_{|d|^2i}$ is a unitary operator, $\{E_{|d|^2(mN+i)}T_n h\}_{m,n\in\mathbb{Z}}$ is a $\|h\|^2$-tight frame for V_i, $0 \le i \le N-1$.

From (1)-(7), by Theorem 3.12, $(\mathbf{V}, 1)$ is a dual fusion frame of $(\mathbf{W}, 1)$, and it is not the canonical dual.

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References

[1] P. G. Casazza, *The art of frame theory*, Taiwanese Jour. of Math. 4 (2) (2000), 129-202.
[2] P. G. Casazza, G. Kutyniok, *Frames of subspaces*, Contemp. Math. 345 (2004), 87-113.
[3] P. G. Casazza and G. Kutyniok (Eds.), *Finite Frames. Theory and Applications*, Birkhäuser, Boston, 2012.
[4] P. G. Casazza, G. Kutyniok, S. Li, *Fusion frames and distributed processing*, Appl. Comput. Harmon. Anal. 25 (2008), 114-132.
[5] O. Christensen, *An introduction to frames and Riesz bases*, Appl. Numer. Harmon. Anal., Birkhäuser, Boston, 2003.
[6] I. Daubechies, *Ten Lectures on Wavelets*, SIAM, Philadelphia, PA, 1992.
[7] R. J. Duffin and A. C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc. 72 (1952), 341-366.
[8] S. B. Heineken, P. M. Morillas, *Properties of finite dual fusion frames*, Linear Algebra Appl. 453 (2014), 1-27.
[9] M. A. Ruiz, D. Stojanoff, *Some properties of frames of subspaces obtained by operator theory methods*, J. Math. Anal. Appl. 343 (2008), 366-378.

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