Cycles and Stability
(full version)

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March 8, 2006

Abstract

We prove a number of Turán and Ramsey type stability results for cycles, in particular, the following one:

Let $n \geq 4$, $0 < \beta \leq 1/2 - 1/2n$, and the edges of $K_{\lfloor (2-\beta)n \rfloor}$ be 2-colored so that no mononchromatic C_n exists. Then, for some $q \in ((1-\beta)n - 1, n)$, we may drop a vertex v so that in $K_{\lfloor (2-\beta)n \rfloor} - v$ one of the colors induces $K_{q, \lfloor (2-\beta)n \rfloor - q - 1}$, while the other one induces $K_q \cup K_{\lfloor (2-\beta)n \rfloor - q - 1}$.

We also derive the following Ramsey type result.

If n is sufficiently large and G is a graph of order $2n-1$, with minimum degree $\delta(G) \geq (2 - 10^{-6})n$, then for every 2-coloring of $E(G)$ one of the colors contains cycles C_t for all $t \in [3, n]$.

1 Introduction

Our graph theoretic notation is standard (e.g., see [2]). We write $c(G)$ for the length of the longest cycle in a graph G. Given a graph G and disjoint sets $A, B \subset V(G)$, we write $E_G(A, B)$ for the set of $A - B$ edges of G, and, abusing notation, $A \times B$ for the set of all possible $A - B$ edges. $A^{(k)}$ stands for the family of k-subsets of a set A.

Given a graph G, a 2-*coloring* of $E(G)$ is a partition $E(G) = E(R) \cup E(B)$, where R and B are graphs with $V(R) = V(B) = V(G)$. Given a 2-coloring $E(G) = E(R) \cup E(B)$, a statement $S(R, B)$ involving R and B is said to be true *up to color,* if either $S(R, B)$ or $S(B, R)$ is true.

In this paper we study stability results about cycles. Stability is a topic studied mostly in extremal problems of Turán type, but also appears neatly in Ramsey problems for cycles, as shown below, and for books, as shown in [8]. In addition to being interesting for their own sake, stability theorems are excellent tools for obtaining new extremal results, like Theorem 3 below. In fact, our initial motivation came from another application that will be presented in a sequel to this paper. It should be noted that our most applicable results - Theorems 12 and 13 - are too technical to be stated in the Introduction.

Our first stability result is of Turán type.

Theorem 1 *Let $0 < \gamma < 10^{-5}$. If $G = G(n)$ is a graph with $e(G) > n^2/4$, then one of the following two conditions hold:*

(a) $C_t \subset G$ for every $t \in [3, \lceil (1/2 + \gamma) n \rceil]$;

(b) there exists a vertex v such that $G - v = H_1 \cup H_2$, where H_1 and H_2 are vertex-disjoint graphs satisfying

$$\left(\frac{1}{2} - 900\gamma\right) n < |H_1| \leq |H_2| < \left(\frac{1}{2} + 900\gamma\right) n.$$

The following theorem is a stability result of Ramsey type. It states that if p is close to $2n$, and the edges of K_p are 2-colored so that no monochromatic cycle C_n exists, then we may remove a vertex from K_p so that, for some q close to n, one of the colors induces $K_{q,p-q-1}$, while the other one induces $K_q \cup K_{p-q-1}$.

Theorem 2 *Let $n \geq 4$, $0 < \beta \leq \lfloor n/2 \rfloor / n$, and $E\left(K_{\lfloor (2-\beta)n \rfloor}\right) = E(R) \cup E(B)$ be a 2-coloring such that $C_n \not\subseteq R$ and $C_n \not\subseteq B$. Then there exist a vertex $u \in V\left(K_{\lfloor (2-\beta)n \rfloor}\right)$ and a partition $V\left(K_{\lfloor (2-\beta)n \rfloor}\right) = U_1 \cup U_2 \cup \{u\}$ with*

$$(1 - \beta) n - 1 < |U_1| \leq |U_2| < n \tag{1}$$

satisfying, up to color,

$$E(R - v) = U_1^{(2)} \cup U_2^{(2)} \quad \text{and} \quad E(B - v) = U_1 \times U_2.$$

We also derive the following Ramsey type result.

Theorem 3 *If n is sufficiently large and $G = G(2n-1)$ is a graph with $\delta(G) \geq (2 - 10^{-6}) n$, then, for every 2-coloring $E(G) = E(R) \cup E(B)$, either $C_t \subset R$ for all $t \in [3, n]$ or $C_t \subset B$ for all $t \in [3, n]$.*

The rest of the paper is organized as follows: in Section 2 we give sufficient conditions for cycles and paths, in Section 3 we prove Turán type stability results including Theorem 1, and in Section 4 we prove Ramsey type stability results including Theorem 2 and Theorem 3.

2 Sufficient conditions for cycles and paths

In this section we list sufficient conditions for the existence of cycles and paths. Most of the them are known, but we also give a few new ones.

Theorem 4 (Erdős and Gallai [5]) *If $e(G) \geq |G|$, then $c(G) > 2e(G) / |G|$.* $\square$

This result was significantly improved for 2-connected graphs by Woodall [9], and recently by Fan, Lv, and Weng [7].

Theorem 5 (Fan, Lv, and Weng [7]) *If the graph $G = G(n)$ is 2-connected and $c(G) = c$ then*

$$e(G) \leq \binom{c+1-\lfloor c/2 \rfloor}{2} + \left\lfloor \frac{c}{2} \right\rfloor \left(n - c - 1 + \left\lfloor \frac{c}{2} \right\rfloor \right).$$

□

Theorem 6 (Bollobás [1], p. 150) *If G is a graph with $e(G) > |G|^2/4$, then $C_t \subset G$ for every $3 \leq t \leq c(G)$.* □

Implicit in [5] (see [3], p. 26) is the following theorem.

Theorem 7 (Erdős and Gallai [5]) *If G is a graph with $\delta(G) > |G|/2$, then every two vertices of G can be joined by a path of order $|G|$.* □

The following theorem follows from results of Brandt, Faudree, and Goddard ([4], p. 143).

Theorem 8 (Brandt, Faudree, and Goddard [4]) *If $n > 30$ and $G = G(n)$ is a 2-connected, nonbipartite graph with $\delta(G) > 2n/5$, then $C_t \subset G$ for all $t \in [3, c(G)]$.* □

We derive below a simple consequence of Theorem 5.

Corollary 9 *If $G = G(n)$ is a 2-connected graph, then either G is Hamiltonian or*

$$c(G) > 2n \left(1 - \sqrt{1 - \frac{2e(G)}{n^2}} \right).$$

Proof Write m for $e(G)$ and let $c(G)$ be even, say $c(G) = 2k$. Theorem 5 implies that

$$m \leq \frac{k(k+1)}{2} + k(n-k-1) = -\frac{k(k+1)}{2} + kn.$$

Hence $k^2 - 2kn + 2m \leq -k < 0$, and the assertion follows.

Let now $c(G)$ be odd, say $c(G) = 2k+1$. Theorem 5 implies that

$$\begin{aligned} m &\leq \binom{2k+2-k}{2} + k(n-2k-2+k) = \frac{(k+2)(k+1)}{2} + k(n-k-2) \\ &= -\frac{k(k+1)}{2} + kn + 1; \end{aligned}$$

hence, $k^2 - k(2n-1) + 2m - 2 \leq 0$. In view of $(2n-1)^2 - 8(m-1) \leq 4n^2 - 8m$, it follows that

$$2k+1 \geq 2n - \sqrt{(2n-1)^2 - 8(m-1)} \geq 2n - \sqrt{4n^2 - 8m},$$

completing the proof. □

We shall make use of the following simple consequence of Theorem 7.

Lemma 10 *If $G = G(n)$ is a graph with $\delta(G) \geq n/2 + 1$, every two vertices of G can be joined by paths of order n and $n-1$.*

Proof Select $u, v \in V(G)$. Theorem 7 implies that u and v may be joined by a path of order n. From $n - 1 \geq \delta(G) \geq n/2 + 1$ we see that $n \geq 4$. Select $w \neq u, v$ and consider $G' = G - w$. We have

$$\delta(G') \geq \delta(G) - 1 \geq \frac{n}{2} > \frac{n-1}{2} = \frac{|G'|}{2},$$

thus, again by Theorem 7, u and v can be joined by a path of order $n - 1$. □

Lemma 11 *Let G be a bipartite graph with vertex classes A and B, $|A| \leq |B|$, and $\delta = \delta(G) \geq |B|/2 + 1$. Then*

(i) if $x, y \in A$ or $x, y \in B$, G contains an xy-path of length t for all even $t \in [2, 2(2\delta - |A| - 1)]$;

(ii) if $x \in A$, $y \in B$, , G contains an xy-path of length t for all odd $t \in [3, 2(2\delta - |A| - 1)]$;

(iii) $C_t \subset G$ for all even $t \in [4, 2(2\delta - |A| - 1)]$.

Proof To prove *(i)* and *(ii)* we use induction on t. If $x, y \in A$ or $x, y \in B$, then

$$|\Gamma(x) \cap \Gamma(y)| \geq d(x) + d(y) - |B| > 2\delta - |B| \geq 2,$$

and so, there exists an xy-path of length 2. If $x \in A$, $y \in B$, select $u_1 \in \Gamma(x)$. Since $|\Gamma(u_1) \cap \Gamma(y)| \geq 2$, there exists $u_2 \in (\Gamma(u_1) \cap \Gamma(y)) \setminus \{x\}$; the path x, u_1, u_2, y has length 3. To complete the induction we show that if $l < 2(2\delta - |A| - 1)$, every xy-path $P = xu_1, ...u_{l-1}, y$ of length l can be extended to a xy-path of length $l + 2$. Select $u_i, u_{i+1} \in P$ so that $u_i \in A$, $u_{i+1} \in B$. Since

$$|P \cap B| \leq \frac{l+1}{2} < 2\delta - |A| < \delta,$$

we can select a vertex $v \in \Gamma(u_i) \setminus P$. Since

$$|\Gamma(u_{i+1}) \cap \Gamma(v)| \geq 2\delta - |A| > \frac{l+1}{2} \geq |P \cap A|,$$

we can select $w \in \Gamma(u_{i+1}) \cap \Gamma(v) \setminus P$. The xy-path

$$x, u_1, ...u_i, v, w, u_{i+1}, ..., y$$

has length $l + 2$, completing the induction and the proof of *(i)* and *(ii)*.

To prove *(iii)*, select two adjacent vertices $x \in A$, $y \in B$. According to *(ii)* there exists an xy-path of odd length $t \in [3, 2(2\delta - |A| - 1) - 1]$, and consequently, a cycle of length $t + 1$, completing the proof. □

3 Turán type stability

Most results in this paper are derived from the following theorem.

Theorem 12 *Let* $0 < \alpha < 10^{-5}$, $0 \leq \beta < 10^{-5}$, *and* $n \geq \alpha^{-1}/2$. *If* $G = G(n)$ *is a graph with* $e(G) > (1/4 - \beta) n^2$, *then one of the following conditions holds:*

(i) $c(G) \geq (1/2 + \alpha) n$;

(ii) there exists a set $M \subset V(G)$ *with* $|M| < 840(\alpha + 2\beta) n$ *such that* $G - M$ *consists of two components* G_1, G_2 *satisfying*

$$\left(\frac{1}{2} - 840(\alpha + 2\beta)\beta\right) n < |G_1| \leq |G_2| < \left(\frac{1}{2} + 20(\alpha + 2\beta)\right) n, \tag{2}$$

$$\delta(G_1) \geq \frac{3n}{7}, \qquad \delta(G_2) \geq \frac{3n}{7}. \tag{3}$$

Proof Assume that *(i)* fails, i.e.,

$$c(G) < (1/2 + \alpha) n. \tag{4}$$

The rest of our proof has two phases - in the first one we find $M_1 \subset V(G)$ such that $|M_1| < 40(\alpha + \beta) n$ and $G - M_1$ consists of two components H_1, H_2 satisfying

$$\left(\frac{1}{2} - 20\alpha + 40\beta\right) n < |H_1| \leq |H_2| < \left(\frac{1}{2} + 20\alpha + 40\beta\right) n. \tag{5}$$

Then, in the second phase, we obtain G_1 and G_2 by dropping the low degree vertices from H_1 and H_2.

Setting

$$M_0 = \{v : v \in V(G),\ d(v) \leq 9n/40\},$$

our first goal is to prove that

$$|M_0| < (20\alpha + 40\beta) n. \tag{6}$$

Indeed, Lemma 4 implies that

$$2e(G - M_0) < c(G - M_0)(n - |M_0|) \leq c(G)(n - |M_0|) < \left(\frac{1}{2} + \alpha\right)(n - |M_0|) n,$$

and so,

$$\left(\frac{1}{4} + \frac{\alpha}{2}\right) n^2 - \frac{1}{4}|M_0| n \geq \frac{1}{2}\left(\frac{1}{2} + \alpha\right)(n - |M_0|) n > e(G - M_0) \geq e(G) - \sum_{u \in M_0} d(u)$$
$$> \left(\frac{1}{4} - \beta\right) n^2 - \frac{9n}{40}|M_0|,$$

implying (6).

From

$$\left(\frac{11}{10}-8\beta\right)n \geq (20\alpha+40\beta)\,n \geq (1-4\beta)\,20\,(\alpha+2\beta)\,n \geq (1-4\beta)\,|M_0|$$

we deduce that

$$\left(\frac{1}{4}-\beta\right)n^2-\frac{9}{40}\,|M_0|\,n \geq \left(\frac{1}{4}-\beta\right)(n-|M_0|)^2 .$$

If $\kappa\,(G-M_0)\geq 2$, Corollary 9 implies that

$$\begin{aligned}
c\,(G) &\geq 2\,(n-|M_0|)\left(1-\sqrt{1-\frac{2e\,(G-M_0)}{(n-|M_0|)^2}}\right)\\
&\geq 2\,(1-20\alpha-40\beta)\left(1-\sqrt{\frac{1}{2}+2\beta}\right)n\\
&\geq 2\,(1-20\alpha-40\beta)\left(1-\frac{1+2\beta}{\sqrt{2}}\right)n\\
&\geq \left(2-\sqrt{2}\right)(1-20\alpha-40\beta)\left(1-2\left(\sqrt{2}+1\right)\beta\right)n\\
&\geq \left(2-\sqrt{2}\right)(1-20\alpha-45\beta)\,n \geq \left(\frac{1}{2}+\alpha\right)n,
\end{aligned}$$

contradicting (4).

Hence, there exists $K\subset V\,(G)$ with $|K|\leq 1$ such that the graph $G'=G-M_0-K$ is disconnected. Observe that $\alpha+2\beta<3\times 10^{-5}$ and $n>10^5$ imply

$$\begin{aligned}
\delta\,(G') = \delta\,(G-M_0-K) &> \frac{9n}{40}-|M_0|-1 > \frac{9n}{40}-20\,(\alpha+2\beta)\,n-1\\
&\geq \left(\frac{9}{40}-\frac{20\times 3}{10^5}\right)n-1 \geq \frac{n}{5}.
\end{aligned}$$

Case 1: G' has a component G'' with $|G''|\leq n/3$

Then, by Lemma 4,

$$2e\,(G'-G'')\leq c\,(G'-G'')\,(|G'|-|G''|)\leq c\,(G)\,(n-|M_0|-|G''|)\,.$$

In view of $\Delta\,(G'')<n/3$, we see that

$$\begin{aligned}
(n-|M_0|-|G''|)\left(\frac{1}{4}+\frac{\alpha}{2}\right)n &\geq e\,(G'-G'')\geq e\,(G')-e\,(G'')\\
&\geq \left(\frac{1}{4}-\beta\right)n^2-\frac{9n}{40}\,|M_0|-n\,|K|-\frac{|G''|\,n}{6},
\end{aligned}$$

and therefore,

$$\left(\frac{1}{4}+\frac{\alpha}{2}\right)n-\left(\frac{1}{4}+\frac{\alpha}{2}\right)|M_0|-\left(\frac{1}{4}+\frac{\alpha}{2}\right)|G''|\geq\left(\frac{1}{4}-\beta\right)n-\frac{9}{40}|M_0|-|K|-\frac{|G''|}{6},$$

implying that

$$\begin{aligned}\left(\frac{\alpha+2\beta}{2}\right)n-\left(\frac{1}{40}+\frac{\alpha}{2}\right)|M_0|+|K| &\geq \left(\frac{1}{12}+\frac{\alpha}{2}\right)|G''| \\ &> \left(\frac{1}{12}+\frac{\alpha}{2}\right)\delta\left(G'\right) > \left(\frac{1}{12}+\frac{\alpha}{2}\right)\frac{1}{5}n.\end{aligned}$$

This gives

$$6\left(\alpha+2\beta\right)n+12\geq\frac{1+6\alpha}{5}n,$$

and, in view of $\alpha<10^{-5},\beta\leq10^{-5}$, it follows that

$$n\leq24\alpha n+60\beta n+60<\frac{84}{10^5}n+60.$$

This inequality is a contradiction, as $n\geq\alpha^{-1}/2>10^5/2$.

Case 2: The order of each component of G' is greater than $n/3$

Therefore, G' has exactly two components - H_1 and H_2; let, say $|H_1|\leq|H_2|$. Setting $M_1=M_0\cup K$, we see that

$$|M_1|\leq20\left(\alpha+2\beta\right)n+1\leq20\left(\alpha+2\beta\right)n+2\alpha n<40\left(\alpha+\beta\right)n.$$

We shall prove that inequalities (5) hold. From Lemma 4 we have

$$e\left(H_2\right)\leq v\left(H_2\right)c\left(H_2\right)\leq\left(n-|M_1|-|H_1|\right)\left(\frac{1}{4}+\frac{\alpha}{2}\right)n.$$

Thus, in view of

$$e\left(H_2\right)=e\left(G-M_1-H_1\right)>\left(\frac{1}{4}-\beta\right)n^2-\frac{9n}{40}|M_0|-n-\frac{|H_1|^2}{2},$$

and the previous inequality we have

$$\left(n-|M_1|-|H_1|\right)\left(\frac{1}{2}+\alpha\right)n>\left(\frac{1}{2}-2\beta\right)n^2-\frac{9n}{20}|M_0|-2n-|H_1|^2,$$

and so,

$$|H_1|^2-\frac{1}{2}n|H_1|+\left(\alpha+2\beta\right)n^2+2n\geq\alpha|H_1|+\left(\frac{1}{20}+\alpha\right)|M_0|n+\left(\frac{1}{2}+\alpha\right)|K|>0.$$

Solving the quadratic inequality with respect to $|H_1|$ we see that

$$|H_1| \geq \frac{1+\sqrt{1-16(\alpha+2\beta)-32/n}}{4}n \tag{7}$$

or

$$|H_1| \leq \frac{1-\sqrt{1-16(\alpha+2\beta)-32/n}}{4}n.$$

Since

$$\begin{aligned}\frac{1-\sqrt{1-16(\alpha+2\beta)-32/n}}{4} &\leq \frac{1-1+16(\alpha+2\beta)+32/n}{4} \\ &= 4(\alpha+2\beta)+\frac{32}{n} < 1/3,\end{aligned}$$

we see that precisely (7) holds. From

$$1 > 1-16(\alpha+2\beta)-\frac{32}{n} > 0,$$

we deduce that

$$\sqrt{1-16(\alpha+2\beta)-32/n} > 1-16(\alpha+2\beta)-32/n,$$

and so,

$$\begin{aligned}|H_1| &\geq \frac{1+\sqrt{1-16(\alpha+2\beta)-32/n}}{4}n \geq \frac{1+1-16(\alpha+2\beta)-32/n}{4}n \\ &= \left(\frac{1}{2}-4(\alpha+2\beta)\right)n-8 \geq \left(\frac{1}{2}-20(\alpha+2\beta)\right)n.\end{aligned}$$

This, together with

$$|H_2| \leq n-|H_1| \leq \left(\frac{1}{2}+20(\alpha+2\beta)\right)n,$$

completes the proof of (5).

To complete the proof of the theorem, we shall remove all low degree vertices from $H_1 \cup H_2$. Letting

$$M_2 = \left\{v : v \in V(H_1 \cup H_2),\ d_{H_1 \cup H_2}(v) \leq \frac{9}{20}n\right\},$$

we find that

$$\left(\frac{1}{2}-2\beta\right)n^2 < 2e(G) = \sum_{u\in V(G)} d(u) = \sum_{u\in V(H_1\cup H_2)\setminus M_2} d(u) + \sum_{u\in M_2} d(u) + \sum_{u\in M_1} d(u)$$
$$< (n-|M_1|-|M_2|)|H_2| + \frac{9}{20}|M_2|n + |M_1|n$$
$$\leq (n-|M_1|-|M_2|)\left(\frac{1}{2}+20(\alpha+2\beta)\right)n + \frac{9}{20}|M_2|n + |M_1|n$$
$$\leq \left(\frac{1}{2}+20(\alpha+2\beta)\right)n^2 + \frac{1}{2}|M_1|n - \frac{1}{20}|M_2|n$$
$$\leq \left(\frac{1}{2}+20(\alpha+2\beta)\right)n^2 + 20(\alpha+\beta)n^2 - \frac{1}{20}|M_2|n$$
$$= \left(\frac{1}{2}+40\alpha+60\beta\right)n^2 - \frac{1}{20}|M_2|n,$$

and hence, $|M_2| \leq (800\alpha+1240\beta)n$. Setting

$$M = M_1 \cup M_2, \quad G_1 = H_1 - M_2, \quad G_2 = H_2 - M_2,$$

we see that

$$|M| = |M_1| + |M_2| \leq (840\alpha+1280\beta)n < 840(\alpha+2\beta)n,$$
$$|G_1| \geq |H_1| - |M_2| \geq \left(\frac{1}{2} - 840(\alpha+2\beta)\right)n,$$
$$|G_2| \leq |H_2| \leq \left(\frac{1}{2}+20(\alpha+2\beta)\right)n,$$
$$\delta(G_1\cup G_2) \geq \frac{9}{20}n - |M_2| \geq \left(\frac{9}{20} - 800\alpha - 1240\beta\right)n$$
$$\geq \left(\frac{9}{20} - \frac{800}{10^5} - \frac{1240}{10^5}\right)n > \frac{3}{7}n.$$

Since

$$\frac{3}{7}n > \frac{1}{2}\left(\frac{1}{2}+20(\alpha+2\beta)\right)n \geq \frac{1}{2}\max\{|G_2|,|G_1|\},$$

it follows that G_1 and G_2 are connected, completing the proof. □

3.1 Proof of Theorem 1

Assume that $C_l \nsubseteq G$ for some $l \in [3, \lceil(1/2+\gamma)n\rceil]$; then Theorem 6 implies

$$c(G) \leq \lfloor(1/2+\gamma)n\rfloor. \tag{8}$$

First we shall show that the assertion of the theorem holds for $n < \gamma^{-1}/2$. Indeed, by Theorem 4, for n even, say $n = 2k$, we have $\lfloor k + \gamma n \rfloor \geq c(G) > k$, contradicting (8) for $n < \gamma^{-1}/2 < \gamma^{-1}$. Similarly, for n odd, say $n = 2k+1$, we have

$$\left\lfloor k + \frac{1}{2} + \gamma n \right\rfloor \geq c(G) \geq k+1,$$

contradicting (8) for $n < \gamma^{-1}/2$.

In view of $n \geq \gamma^{-1}/2$ and (8), Theorem 12, with $\alpha = \gamma$ and $\beta = 0$, implies that there exists $M \subset V(G)$ with $|M| < 840\gamma n$ such that $G - M$ consists of two components G_1 and G_2 satisfying

$$\left(\frac{1}{2} - 840\gamma\right) n < |G_1| \leq |G_2| < \left(\frac{1}{2} + 20\gamma\right) n$$
$$\delta(G_1) \geq 3n/7, \qquad \delta(G_1) \geq 3n/7.$$

From

$$\frac{3n}{7} \geq \frac{1}{2}\left(\left(\frac{1}{2} + 20\gamma\right) n + 1\right) \geq \frac{1}{2}\left(|G_2| + 1\right),$$

and Lemma 10, we see that G_1 and G_2 are Hamiltonian connected.

Suppose there are two vertex disjoint paths $P(u_1, v_1)$ and $P(u_2, v_2)$ joining vertices from G_1 to vertices from G_2, say

$$P(v_1, u_1) \cap G_1 = \{u_1\}, \quad P(u_1, v_1) \cap G_2 = \{v_1\},$$
$$P(u_2, v_2) \cap G_1 = \{u_2\}, \quad P(u_2, v_2) \cap G_2 = \{v_2\}.$$

Let $Q_1(u_1, u_2)$ and $Q_2(v_2, v_1)$ be Hamiltonian paths within G_1 and G_2. Then the length of the cycle

$$Q_1(u_1, u_2)\, P(u_2, v_2)\, Q_2(v_2, v_1)\, P(v_1, u_1)$$

is at least

$$|G_1| + |G_2| = n - |M| > (1 - 840\gamma) n > \left(\frac{1}{2} + \gamma\right) n,$$

contradicting (8).

Therefore, no two vertex-disjoint paths join vertices from G_1 to vertices from G_2. By Menger's theorem, there exists a vertex $u \in V(G)$ separating G_1 and G_2. Clearly, $V(G_1) \setminus \{u\}$ induces a connected subgraph in $G - u$; let H_1 be the component containing $V(G_1) \setminus \{u\}$, and H_2 be the union of the remaining components of $G - u$. Observing that

$$\left(\frac{1}{2} + 840\gamma\right) n > n - |G_2| \geq |H_1| \geq |G_1| - 1 > \left(\frac{1}{2} - 900\gamma\right) n,$$
$$\left(\frac{1}{2} + 840\gamma\right) n \geq n - |G_1| \geq |H_2| \geq |G_2| - 1 > \left(\frac{1}{2} - 900\gamma\right) n,$$

we complete the proof. $\square$

4 Ramsey type results

Theorem 13, presented in the beginning of this section, is essentially a stability result of Turán type. However, it is placed in this section, since it is the main tool to derive Theorem 3 - a distinctive Ramsey type result.

Theorem 13 *Let $0 < \alpha < 5 \times 10^{-6}$, $0 \leq \beta \leq \alpha/25$, and $n \geq \alpha^{-1}$. If $G = G(n)$ is a graph with $e(G) > (1/4 - \beta)n^2$, then one of the following conditions hold:*

(i) $C_t \subset G$ for every $t \in [3, \lceil (1/2+\alpha)n \rceil]$;

(ii) there exists a partition $V(G) = V_0 \cup V_1 \cup V_2$ such that

$$|V_0| < 2000\alpha n, \tag{9}$$

$$\left(\frac{1}{2} - 10\sqrt{\alpha+\beta}\right)n < |V_1| \leq |V_2| < \left(\frac{1}{2} + 10\sqrt{\alpha+\beta}\right)n, \tag{10}$$

$$\delta(G - V_0) \geq 2n/5, \tag{11}$$

and either

$$E(G - V_0) \subset V_1^{(2)} \cup V_2^{(2)} \quad \text{or} \quad E(G - V_0) \subset V_1 \times V_2.$$

Proof Setting

$$M = \left\{ v : v \in V(G),\ d(v) \leq \frac{9n}{20} \right\},$$

our first goal is to prove that

$$|M| < 20(\alpha + 2\beta)n. \tag{12}$$

Indeed, assume for contradiction that $|M| \geq 20(\alpha+2\beta)n > 24\beta n$ and select $M_0 \subset |M|$ with $|M_0| = \lceil 24\beta n \rceil$. We shall show that

$$e(G - M_0) > \frac{1}{4}(n - |M_0|)^2. \tag{13}$$

Indeed, otherwise we have

$$\left(\frac{1}{4} - \beta\right)n^2 < e(G) \leq e(G - M_0) + \sum_{u \in M_0} d(u) \leq e(G - M_0) + \frac{9n}{20}|M_0|$$
$$\leq \frac{1}{4}(n - |M_0|)^2 + \frac{9n}{20}|M_0| = \frac{1}{4}n^2 - \frac{1}{20}n|M_0| + \frac{|M_0|^2}{4},$$

and so,

$$20\beta n^2 - n|M_0| + 5|M_0|^2 \geq 0.$$

Solving this quadratic inequality with respect to $|M_0|$, we see that either

$$|M_0| \leq \frac{1 - \sqrt{1 - 400\beta}}{10}n < 24\beta n \leq \lceil 24\beta n \rceil$$

or

$$|M_0| \geq \frac{1+\sqrt{1-400\beta}}{10}n > \frac{1}{10}n > 24\beta n + 1 > \lceil 24\beta n \rceil .$$

Since both inequalities contradict our choice of M_0, inequality (13) holds.

Note that, in view of

$$(\alpha - 12\beta - 48\alpha\beta)\, n \geq \left(\alpha - \frac{12}{25}\alpha - \frac{2\cdot 48\alpha}{25\cdot 100,000}\right) n \geq \frac{51}{100}\alpha n > \frac{1}{2} + 2\alpha,$$

we have

$$\begin{aligned}\left(\frac{1}{2}+2\alpha\right)|G-M_0| &= \left(\frac{1}{2}+2\alpha\right)(n - \lceil 24\beta n\rceil) \geq \left(\frac{1}{2}+2\alpha\right)(n - 24\beta n - 1) \\ &\geq \left(\frac{1}{2}+\alpha\right) n + (\alpha - 12\beta - 48\alpha\beta)\, n - \left(\frac{1}{2}+2\alpha\right) \geq \left(\frac{1}{2}+\alpha\right) n.\end{aligned}$$

Hence, if $C_t \subset G - M_0$ for every $t \in [3, \lceil (1/2+2\alpha)\,|G-M_0|\rceil]$, we see that *(i)* holds. Thus, $C_t \nsubseteq G - M_0$ for some $t \in [3, \lceil (1/2+2\alpha)\,|G-M_0|\rceil]$. Applying Theorem 12 to the graph $G - M_0$ with $\alpha' = 2\alpha$, $\beta' = 0$, it follows that there exists a $M_1 \subset V(G-M_0)$ such that $G - M_0 - M_1 = G_1 \cup G_2$, where G_1 and G_2 are vertex-disjoint graphs satisfying

$$\begin{gathered}|M_1| < 1680\alpha\,(n - |M_0|) < 1800\alpha\,(n-|M_0|)\,, \\ \left(\frac{1}{2} - 1800\alpha\right)(n-|M_0|) < |G_1| \leq |G_2| < \left(\frac{1}{2}+40\alpha\right)(n-|M_0|)\,, \\ \delta(G_1) \geq \frac{3}{7}(n-|M_0|)\,, \qquad \delta(G_2) \geq \frac{3}{7}(n-|M_0|)\,.\end{gathered}$$

Setting

$$V_0 = M_0 \cup M_1, \quad V_1 = V(G_1)\,, \quad V_2 = V(G_2)\,,$$

we first note that $E(G-V_0) \subset V_1^{(2)} \cup V_2^{(2)}$. We shall prove that this selection of V_0, V_1, and V_2 satisfies *(ii)*. To this end we have to derive inequalities (9), (10) and (11). Inequality (9) follows from

$$|V_0| \leq |M_0| + 1800\alpha\,(n-|M_0|) \leq \lceil 24\beta n\rceil + 1800\alpha n < 24\beta n + 1 + 1800\alpha n < 2000\alpha n.$$

Our next goal is to prove (10). Note that

$$\begin{aligned}|G_1| &\geq \left(\frac{1}{2} - 1800\alpha\right)(n-|M_0|) > \left(\frac{1}{2}-1800\alpha\right)(n - 24\beta n - 1) \\ &\geq \frac{n-1}{2} - 12\beta n - 1800\alpha n \geq (200\alpha - 12\beta)\, n - \frac{1}{2} + \left(\frac{1}{2} - 2000\alpha\right) n \\ &\geq \left(\frac{1}{2} - 2000\alpha\right) n \geq \left(\frac{1}{2} - 10\sqrt{\alpha}\right) n \geq \left(\frac{1}{2} - 10\sqrt{\alpha+\beta}\right) n.\end{aligned}$$

Since

$$|G_2| < n - |G_1| \leq \left(\frac{1}{2} + 10\sqrt{\alpha + \beta}\right) n,$$

inequality (10) follows. Finally, (11) follows from

$$\delta\left(G - V_0\right) \geq \frac{3}{7}\left(n - |M_0|\right) = \frac{3}{7}\left(n - \lceil 24\beta n \rceil\right) \geq \frac{3}{7}\left(n - 24\beta n - 1\right) > \frac{2}{5}n.$$

This completes the proof of the theorem if (12) fails. Thus, hereafter, we shall assume that (12) holds. Set $V_0 = M$, $G_0 = G - V_0$, and observe that

$$|V_0| = |M| \leq 20\left(\alpha + 2\beta\right) n < 2000\alpha n, \tag{14}$$

$$\delta\left(G_0\right) \geq \frac{9n}{20} - |M| = \left(\frac{9}{20} - 20\left(\alpha + 2\beta\right)\right) n > \frac{2}{5}n. \tag{15}$$

Case 1: G_0 is bipartite

Write V_1 and V_2 for the vertex classes of G_0, and let say, $|V_1| \leq |V_2|$. We see that $E\left(G_0\right) \subset V_1 \times V_2$, also (9) and (11) hold in view of (14) and (15), so to finish the proof, we need to prove inequalities (10). Since

$$\begin{aligned} e\left(G_0\right) &\geq \left(\frac{1}{4} - \beta\right) n^2 - |V_0|\, n \geq \left(\frac{1}{4} - 20\alpha - 41\beta\right) n^2 \\ &> \left(\frac{1}{4} - \left(10\sqrt{\alpha + \beta}\right)^2\right) n^2, \end{aligned}$$

selecting x so that

$$|V_1| = \left(\frac{1}{2} - x\right)\left(|V_1| + |V_2|\right), \quad |V_2| = \left(\frac{1}{2} + x\right)\left(|V_1| + |V_2|\right),$$

we deduce that

$$\left(\frac{1}{4} - x^2\right) \geq \left(\frac{1}{4} - \left(10\sqrt{\alpha + \beta}\right)^2\right),$$

and,

$$|V_2| = \left(\frac{1}{2} + x\right)\left(|V_1| + |V_2|\right) < \left(\frac{1}{2} + 10\sqrt{\alpha + \beta}\right) n.$$

This inequality implies in turn

$$|V_1| \geq n - |V_2| > \left(\frac{1}{2} - 10\sqrt{\alpha + \beta}\right) n,$$

completing the proof in this case.

Case 2: $\kappa\left(G_0\right) \leq 1$

Let K be a cutset of G_0 with $|K| \leq 1$. Since $\delta\left(G_0 - K\right) > 2n/5 - 1 > n/3$, the graph $G_0 - K$ has exactly two components - G_1 and G_2. Let $V_0 = M \cup K$, $V_1 = V\left(G_1\right)$,

$V_2 = V(G_2)$; assume $|V_1| \leq |V_2|$ and observe that $E(G_0) \subset V_1^{(2)} \cup V_2^{(2)}$. Clearly $|V_0| \leq 20(\alpha + 2\beta)n + 1 \leq 2000\alpha n$, so (9) holds. From

$$\delta(G - V_0) > \frac{9n}{20} - |M| - 1 > \frac{3}{8}n > \frac{n - |V_1|}{2} > \frac{1}{2}|V_2|,$$

we see first, that(11) holds, and second, that G_2 is Hamiltonian. From Theorem 8 it follows that $C_t \subset G_2$ for every $t \in [3, |V_2|]$. This completes the proof of the theorem if

$$|V_2| \geq \left(\frac{1}{2} + 5\sqrt{\alpha + 2\beta}\right)n, \tag{16}$$

since then $|V_2| > (1/2 + \alpha)n$, and so *(i)* holds.

Assume that (16) fails. Then

$$|V_2| < \left(\frac{1}{2} + 5\sqrt{\alpha + 2\beta}\right)n < \left(\frac{1}{2} + 10\sqrt{\alpha + \beta}\right),$$

and so

$$|V_1| > n - |V_2| > \left(\frac{1}{2} - 10\sqrt{\alpha + \beta}\right)n.$$

Thus (10) holds, completing the proof of *(ii)* in this case.

Case 3: G_0 is 2-connected and nonbipartite

In this case we shall show that *(i)* holds. Since $\delta(G_0) > 2n/5$, Dirac's theorem implies that $c(G) \geq 2\delta(G_0) > 4n/5 > \lceil (1/2 + \alpha)n \rceil$. Now, Theorem 8 implies that $C_t \subset G_0$ for all $t \in [3, \lceil (1/2 + \alpha)n \rceil]$, completing the proof. □

4.1 Proof of Theorem 2

We precede the proof of Theorem 2 by a simple lemma whose idea goes back to [6]. The present version of the lemma emerged from recent conversations with Ingo Schiermeyer, Linda Lesniak, and Ralph Faudree, to whom we are grateful. The lemma helped enhance considerably an earlier version of Theorem 2.

Lemma 14 *Let G be a Hamiltonian graph of order $2n$ such that $C_{2n-1} \not\subset G$ and $C_{2n-1} \not\subset \overline{G}$. Then there exists a partition $V(G) = U_1 \cup U_2$ such that $|U_1| = |U_2| = n$ and U_1, U_2 are independent. Moreover, there exists a vertex $u \in V(G)$ such that $G - u = K_{n,n-1}$.*

Proof Assume $v_1, v_2, ..., v_{2n}$ are the vertices of G listed along the Hamiltonian cycle of G. Observe that $(v_1, v_3, ..., v_{2n-1}, v_1)$ and $(v_2, v_4, ..., v_{2n}, v_2)$ are cycles of length n in $\overline{G}$. Our first goal is to show that the sets $U_1 = \{v_1, v_3, ..., v_{2n-1}\}$ and $U_2 = \{v_2, v_4, ..., v_{2n}\}$ are independent. Assume for contradiction that this is not true and let say $v_1 v_{2k+1} \in E(G)$. Then $v_3 v_{2k+2} \notin E(G)$ since otherwise,

$$(v_3, v_4, ..., v_{2k+1}, v_1, v_{2n}, v_{2n-1}..., v_{2k+2})$$

is a cycle of length $2n-1$ in G. Likewise, $v_{2n-1}v_{2k} \notin E(G)$. Then

$$(v_3, v_5, ..., v_{2n-1}, v_{2k}, v_{2k-2}, ..., v_{2k+2})$$

is a cycle of length $2n-1$ in $\overline{G}$, a contradiction. Therefore $\overline{G}[U_1]$ and $\overline{G}[U_2]$ are complete graphs. Since $C_{2n-1} \nsubseteq \overline{G}$, we see that $E_{\overline{G}}(U_1, U_2)$ contains no disjoint edges and therefore is a (possibly empty) star. Taking u to be the center of this star we complete the proof. □

Proof of Theorem 2 Recall that if k is an integer, then for every 2-coloring $E(K_{3k-1}) = E(R) \cup E(B)$, either $C_{2k} \subset R$ or $C_{2k} \subset B$ (e.g., see [6]). Since for $\beta \leq 1/2$, $\lfloor (2-\beta) 2k \rfloor \geq 3k > 3k-1$, we see that the assertion holds immediately for even n. Let n be odd, say $n = 2k+1$, set $V = V\left(K_{\lfloor (2-\beta)(2k+1) \rfloor}\right)$, and assume $E\left(K_{\lfloor (2-\beta)(2k+1) \rfloor}\right) = E(R) \cup E(B)$ is a 2-coloring with $C_{2k+1} \nsubseteq R$ and $C_{2k+1} \nsubseteq B$. From

$$(2-\beta)(2k+1) \geq \left(2 - \frac{k}{2k+1}\right)(2k+1) = 3k+2$$

we see that, up to color, $C_{2k+2} \subset B$. By the assumption of the theorem and Lemma 14 it follows that there exist $W_1, W_2 \subset V$ such that $|W_1| = k$, $|W_2| = k+1$, W_1 and W_2 induce complete graphs in R, and $K_{k,k+1}$ in B. Note that for all $u \in V \backslash (W_1 \cup W_2)$ either $\Gamma_B \cap W_1 = \varnothing$ or $\Gamma_B \cap W_2 = \varnothing$, as otherwise $C_{2k+1} \subset B$. Set

$$\begin{aligned} X_1 &= \{u : u \in V \backslash (W_1 \cup W_2) \text{ and } \Gamma_B \cap W_1 = \varnothing\}, \\ X_2 &= V \backslash (W_1 \cup W_2 \cup U_1), \\ V_1 &= X_1 \cup W_1, \quad V_2 = X_2 \cup W_2, \end{aligned}$$

and note that $X_1 \times W_1 \subset R$ and $X_2 \times W_2 \subset R$. At this stage it is not difficult to check immediately that the assertion of the theorem holds for $k = 2$, so in the sequel we shall assume that $k \geq 3$.

If there exist two disjoint edges $v_1u_1, u_2v_2 \in E_R(V_1, V_2)$, then $C_t \subset R$ for any odd $t \in [7, 2k+1]$. Hence, $E_R(V_1, V_2)$ is a (possibly empty) star; let u be its center or any other vertex if $E_R(V_1, V_2)$ is empty; set $U_1 = V_1 \backslash \{u\}$, $U_2 = V_2 \backslash \{u\}$. Then $U_1 \times U_2 \subset E(B)$ and hence, $E_B(U_1) = E_B(U_2) = \varnothing$, as otherwise $C_{2k+1} \subset B$. To prove inequalities (1), we shall assume that $|U_1| \leq |U_2|$. This implies that $|U_2| \leq 2k$, and so

$$|U_1| = \lfloor (2-\beta)(2k+1) \rfloor - 2k - 1 > (2-\beta)(2k+1) - 2k - 2 = (1-\beta)(2k+1) - 1,$$

completing the proof. □

4.2 Proof of Theorem 3

For convenience we shall rephrase the Theorem 3 in terms of 3-colorings of K_{2n-1}.

Theorem 15 *Let the edges of* K_{2n-1} *be* 3*-colored, i.e.,* $E(K_{2n-1})$ *be partitioned as* $E(K_{2n-1}) = E(R) \cup E(B) \cup E(Y)$, *where* R, B, *and* Y *are graphs with* $V(R) = V(B) = V(Y) = [2n-1]$. *Let the minimum degree* $\delta(R \cup B)$ *satisfies* $\delta(R \cup B) > (2 - 10^{-6})\, n$. *Then, if* n *is sufficiently large, either* $C_t \subset R$ *for all* $t \in [3, n]$ *or* $C_t \subset B$ *for all* $t \in [3, n]$.

Proof Set for brevity

$$\begin{aligned} c &= 10^{-6}, \\ \beta &= c/8 = 10^{-6}/8, \qquad (17) \\ \alpha &= 25\beta = 10^{-4}/32, \qquad (18) \end{aligned}$$

and assume, without loss of generality, that $e(R) \geq e(B)$. Hence, from

$$e(R) + e(B) \geq \left(2 - 10^{-6}\right)(2n-1)\, n \geq \left(\frac{1}{2} - 2\beta\right)(2n-1)^2,$$

we see that

$$e(R) \geq \left(\frac{1}{4} - \beta\right)(2n-1)^2.$$

According to Theorem 13, one of the following conditions hold:

(i) $C_t \subset R$ for every $t \in [3, \lceil (1/2 + \alpha)(2n-1) \rceil]$;

(ii) there exists a partition $[2n-1] = V_0 \cup V_1 \cup V_2$ such that

$$\begin{gathered} |V_0| < 2000\alpha\,(2n-1), \\ \left(\frac{1}{2} - 10\sqrt{\alpha+\beta}\right)(2n-1) < |V_1| \leq |V_2| < \left(\frac{1}{2} + 10\sqrt{\alpha+\beta}\right)(2n-1), \end{gathered}$$

and either

$$E(R - V_0) \subset V_1^{(2)} \cup V_2^{(2)} \quad \text{or} \quad E(R - V_0) \subset V_1 \times V_2.$$

If *(i)* holds, there is nothing to prove, so we shall assume that *(ii)* holds. Then, in view of (18), (17), and

$$\begin{aligned} 2000\alpha &= \frac{2000 \cdot 25}{8 \cdot 10^6} = \frac{1}{16 \cdot 10^2} = \frac{1}{160}, \\ 10\sqrt{\alpha+\beta} &= 10\sqrt{\frac{26}{8 \cdot 10^6}} < \frac{1}{50}, \end{aligned}$$

we find that

$$\begin{gathered} |V_0| < \frac{1}{160}(2n-1), \\ \frac{12}{25}(2n-1) < |V_1| \leq |V_2| < \frac{13}{25}(2n-1). \end{gathered}$$

Assume $E(R-V_0) \subset V_1^{(2)} \cup V_2^{(2)}$. We shall prove that, then $E(B-V_0) \subset V_1 \times V_2$. We clearly have

$$\begin{aligned}\delta(B-V_0) &\geq |V_1| - \Delta(Y) \geq \frac{12}{25}(2n-1) - ((2n-2) - \delta(R\cup B)) \\ &> \frac{12}{25}(2n-1) + 2 - cn > \left(\frac{12}{25} - c\right)(2n-1) \geq \frac{1}{2}\left(\frac{13}{25}(2n-1)\right) + 1.\end{aligned}$$

Lemma 11 implies that $C_t \subset B - V_0$ for all even $t \in [4, 2(2\delta(B-V_0) - |V_1| - 1)]$. Moreover, if $E(B(V_1)) \cup E(B(V_2)) \neq \varnothing$, then obviously $C_t \subset B - V_0$ for all odd $t \in [3, 2(2\delta(B-V_0) - |V_1|)]$. Since

$$\begin{aligned}2(2\delta(B-V_0) - |V_1| - 1) &\geq 2\left(2\left(\frac{12}{25} - c\right)(2n-1) - \frac{12}{25}(2n-1) - 1\right) \qquad (19) \\ &\geq \left(\left(\frac{24}{25} - 4c\right)(2n-1) - 1\right) \geq n,\end{aligned}$$

the proof is completed. Hence, $E(B(V_1)) \cup E(B(V_2)) = \varnothing$, implying that $E(B-V_0) \subset V_1 \times V_2$.

Now, suppose that there exists $u \in V_0$ such that $\Gamma_B(u) \cap V_1 \neq \varnothing$ and $\Gamma_B(u) \cap V_2 \neq \varnothing$. Select $x \in \Gamma_B(u) \cap V_1$, $y \in \Gamma_B(u) \cap V_2$ and note that Lemma 11 implies that $B - V_0$ contains an xy-path of length t for every odd $t \in [3, 2(2\delta(B-V_0) - |V_1| - 1)]$. In view of (19), $C_t \subset B - V_0$ for all odd $t \in [3, n]$, completing the proof. Therefore, for every $u \in V_0$, either $\Gamma_B(u) \cap V_1 = \varnothing$, or $\Gamma_B(u) \cap V_2 = \varnothing$. Set

$$W_1 = \{u : u \in V_0, \Gamma_B(u) \cap V_1 = \varnothing\}, \quad W_2 = V_0 \backslash W_1.$$

and let say $|W_1 \cup V_1| \geq |W_2 \cup V_2|$. This implies $|W_1 \cup V_1| \geq n$. Let $t \in [3, n]$. If $t \leq |V_1|$, then $C_t \subset R(V_1) \subset R$. If $t > |V_1|$, select $W_1' \subset W_1$ so that $|W_1' \cup V_1| = t$. Note that $V_1 \times W_1' \subset E(R) \cup E(Y)$ and so for every $u \in W_1' \cup V_1$ we have

$$\begin{aligned}|\Gamma_R(u) \cap (W_1' \cup V_1)| &\geq \min\{(|V_1| - \Delta(Y)), |V_1| - 1\} \\ &\geq \frac{12}{25}(2n-1) - 10^{-6}n - 1 \geq \frac{1}{2}n > \frac{1}{2}t.\end{aligned}$$

Therefore, $R(W_1' \cup V_1)$ is Hamiltonian, i.e., $C_t \subset R$, completing the proof. □

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References

[1] B. Bollobás, *Extremal Graph Theory*, Academic Press, 1978.

[2] B. Bollobás, *Modern graph theory,* Graduate Texts in Mathematics, **184**, Springer-Verlag, New York (1998), xiv+394 pp.

[3] J. A. Bondy, Basic graph theory: paths and circuits, *Handbook of combinatorics*, Vol. 1, pp. 3–110, Elsevier, Amsterdam, 1995.

[4] S. Brandt, R. Faudree, and W. Goddard, Weakly pancyclic graphs, J. Graph Theory 27 (1998), 141–176.

[5] P. Erdős and T. Gallai, On maximal paths and circuits of graphs, *Acta Math. Acad. Sci. Hungar* 10 (1959), 337–356.

[6] R.J. Faudree and R.H. Schelp, All Ramsey numbers for cycles in graphs, Discrete Math. 8 (1974), 313–329.

[7] G. Fan, X. Lv, and P. Wang, Cycles in 2-connected graphs, Journal of Comb. Theory, Series B 92 (2004), 379–394.

[8] V. Nikiforov, C.C. Rousseau, and R.H. Schelp, Book Ramsey numbers and Quasi-randomness, Comb. Prob. Comp. 14 (2005), 851-860.

[9] D.R. Woodall, Maximal circuits of graphs I, Acta Math. Acad. Sci. Hungar. 28 (1976) 77–80.