Flows on regular semigroups

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Abstract. We study the structure of the flow monoid of a regular semigroup. This arises from the approach of Nambooripad of considering a regular semigroup as a groupoid – a category in which every morphism is invertible. A flow is then a section to the source map in this groupoid, and the monoid structure of the set of all flows is determined in terms of the Green relations on the original semigroup.

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1. Introduction

The use of the algebraic structure of categories in semigroup theory begins with work of Ehresmann and Schein (see [10]) on connections between inverse semigroups and groupoids (here a groupoid is a category in which all morphisms are invertible), considered as equivalent algebraic models of symmetry. This view is fully developed by Lawson in [6], and given a precise formulation by the Ehresmann-Schein-Nambooripad Theorem, which establishes an isomorphism between the categories of inverse semigroups and of inductive groupoids, where an inductive groupoid is a certain kind of ordered groupoid. Nambooripad [8] extended the use of groupoids to the study of regular semigroups, and this has been a fruitful idea, for example in the study of semigroup amalgams [9, 11].

This paper takes some steps in the opposite direction, and is concerned with the interpretation of a category-theoretic structure for regular semigroups. We consider flows on a category, where a flow is simply a set-theoretic section to the source map. The set of all flows is naturally a monoid, and we aim to understand its structure in terms of the original category. Beginning with a regular semigroup, Nambooripad gives a construction of a groupoid whose structure is determined by the Green relations on S. The flow monoid of a regular semigroup is then defined to be the flow monoid of this groupoid, and the structure of the flow monoid can be completely determined, as a direct product of wreath products of $\mathcal{H}$–classes in S and endomorphism monoids of idempotents in the $\mathcal{D}$–classes in S. We discuss the Nambooripad construction in section 3 and the strcuture of the flow monoid in section 4.

The flow monoid originates in the unpublished typescript of Chase [1], where it is called the *incidence* monoid. We have preferred the more picturesque name *flow*, which captures the conceptual connection with vector fields. Chase studies the incidence or flow monoid, and its group of units, in a general category theory context of categories with a fixed object or vertex set X. His main motivation is as follows. If a group Γ acts on X, then $X \times \Gamma$ can be considered as a groupoid, giving a functor from the category of groups acting on X to the category of groupoids with vertex set X. The group of units of the flow monoid then gives a right adjoint to this functor. Chase determines the structure of the flow monoid and its group of units in this general setting, and the structural results presented here can be readily derived from this. However, given the utility of the structural interplay between regular semigroups and groupoids, it seems worthwhile to carry out the investigation of the flow monoid entirely within the particular context of the theory of regular semigroups, as presented in this paper.

The paper [1] contains a wealth of ideas, and further investigation may well reveal other valuable concepts for the category-theoretic approach to regular semigroups.

2. Flows on a category

We consider categories as algebraic structures. Thus a *category* $\mathbb{C}$ consists of a set of vertices X and a set of arrows A, admitting the following structure. There exist *source* and *target* maps $\sigma, \tau : A \to X$, and an identity map $e : X \to A$ such that $(xe)\sigma = x = (xe)\tau$. Whenever $a\tau = b\sigma$ a composite arrow ab is defined, with $(ab)\sigma = a\sigma$ and $(ab)\tau = b\tau$. This composition is to be associative whenever the necessary compositions are defined, and the elements $xe\,(x \in X)$ act as left and right identities whenever this makes sense. We shall identify X with its image Xe, and so identify $\mathbb{C}$ with its set of arrows A. A category is connected if, given any two vertices $x, y \in X$, there exists an arrow $a \in A$ with $\{a\sigma, a\tau\} = \{x, y\}$. More generally, any category $\mathbb{C}$ is the union of its *connected components*, the maximal connected subcategories of $\mathbb{C}$.

A category $\mathbb{C}$ is a *groupoid* if for each $a \in \mathbb{C}$ there exists $a^{-1} \in \mathbb{C}$ such that $a^{-1}a = a\tau$ and $aa^{-1} = a\sigma$. For each vertex $x \in X$, the set $\mathbb{C}(x) = \{a \in \mathbb{C} : a\sigma = x = a\tau\}$ is a subgroup of $\mathbb{C}$, called the *vertex group* at x.

A *flow* on a category $\mathbb{C}$ with vertex set X is a function $\varphi : X \to \mathbb{C}$ that is a section to the source map: that is, for all $x \in X$, $(x\varphi)\sigma = x$. Let $\Phi(\mathbb{C})$ denote the set of all flows on $\mathbb{C}$. It is easy to check that $\Phi(\mathbb{C})$

is a monoid, with composition $*$ defined by

$$x(\varphi * \psi) = x\varphi((x\varphi\tau))\psi$$

and identity $e : x \mapsto x$.

EXAMPLES

1. If M is a monoid, regarded as a category with a single vertex, then $\Phi(M) = M$. However, we shall see that a *regular* monoid can be considered in another way as a groupoid, and that the structure of the flow monoid of this groupoid is of greater interest. However, if M is a group, then the two constructions coincide.

2. For any set X, the *simplicial* groupoid ΔX on X is the set $X \times X$ with partial composition $(u, v)(v, w) = (u, w)$. Let φ be a flow on X. Then $x\varphi = (x, y)$ for some $y \in X$, and φ is completely determined by the mapping $x \mapsto y$. Hence $\Phi(\Delta X) = \mathsf{End}(X)$, the monoid of all functions $X \to X$.

3. Let a group Γ act on a set X. The groupoid $X \rtimes \Gamma$ has vertex set x and arrow set $X \times \Gamma$, with $(x, \gamma)\sigma = x$ and $(x, \gamma)\tau = x\gamma$. A flow φ on $X \rtimes \Gamma$ is identified with a function $f : X \to G$ so that $x\varphi = (x, xf)$.

The simplicial groupoid of example 2 above may be generalised by the introduction of a group G. On the set $X \times G \times X$ we impose the partial composition $(u, g, v)(v, h, w) = (u, gh, w)$. This is easily seen to be a groupoid. Moreover, every connected groupoid $\mathbb{G}$ arises in this way, with the group G isomorphic to a vertex group $\mathbb{G}(x)$. See [3, 6] for details.

A flow φ on a category $\mathbb{C}$ determines an endomorphism π_φ of the set of vertices X, defined by $x\pi_\varphi = x\varphi\tau$. The mapping $\varphi \mapsto \varphi\tau$ gives a monoid homomorphism $\pi : \Phi(\mathbb{C}) \to \mathsf{End}(X)$. If $\mathbb{C}$ is connected, then π is surjective: more generally, if the connected components of $X \subseteq \mathbb{C}$ are $X_i\,, i \in I$ then the image of π is the monoid direct product $\prod_{i \in I} \mathsf{End}(X_i)$.

Following [3], we call a category $\mathbb{C}$ *unicursal* if, given two vertices x, y of $\mathbb{C}$, there is at most one arrow $a \in \mathbb{C}$ with $a\sigma = x$ and $a\tau = y$. A unicursal category with vertex set X embeds in the simplicial groupoid ΔX.

If a, b are arrows in a category $\mathbb{C}$, write $a \sim b$ if $a\sigma = b\sigma$ and $a\tau = b\tau$. Then $\sim$ is an equivalence relation on the set of arrows of $\mathbb{C}$, and $\bar{\mathbb{C}} = \mathbb{C}/\sim$ is a unicursal category with vertex set X. An arrow in $\bar{\mathbb{C}}$ is an equivalence class of arrows in $\mathbb{C}$.

PROPOSITION 2.1. *Let $\mathbb{C}$ be a category with vertex set X.*

1. *If $\mathbb{C}$ is the union of connected components $\mathbb{C}_i\,, i \in I$, then $\Phi(\mathbb{C})$ is isomorphic to the direct product $\prod_{i\in I} \Phi(\mathbb{C}_i)$.*

2. *If $\mathbb{C}$ and $\mathbb{C}'$ are categories and $F : \mathbb{C} \to \mathbb{C}'$ is a functor that is bijective on the vertex sets, then F induces a monoid homomorphism $\Phi(\mathbb{C}) \to \Phi(\mathbb{C}')$.*

3. *The monoid homorphism $\Phi(\mathbb{C}) \to \Phi(\bar{\mathbb{C}})$ induced by the functor $\mathbb{C} \to \bar{\mathbb{C}}$ is precisely $\pi : \Phi(\mathbb{C}) \to \prod_{i\in I} \mathsf{End}(X_i)$.*

PROPOSITION 2.2. *A flow φ is a unit in $\Phi(\mathbb{C})$ if and only if, for all $x \in X$, $x\varphi$ is an isomorphism in $\mathbb{C}$ and the map $\pi_\varphi : X \to X$ is a bijection.*

Proof. Since the function $\varphi \mapsto \pi_\varphi$ is a monoid map $\Phi(\mathbb{C}) \to \mathsf{End}(X)$, it follows that if φ is a unit in $\Phi(\mathbb{C})$ then π_φ is bijective. Moreover, $x\varphi$ must be an isomorphism in $\mathbb{C}$, for if $\bar{\varphi}$ is the inverse of φ in $\Phi(\mathbb{C})$ then $(x\pi_\varphi)\bar{\varphi}$ is the inverse of $x\varphi$ in $\mathbb{C}$, for

$$\begin{aligned} x = xe &= x(\varphi * \bar{\varphi}) = x\varphi(x\pi_\varphi)\bar{\varphi}, \\ x\varphi\tau = (x\varphi\tau)e &= x\pi_\varphi e = x\pi_\varphi(\bar{\varphi} * \varphi) \\ &= (x\pi_\varphi)\bar{\varphi}(x\pi_\varphi\pi_{\bar{\varphi}})\varphi \\ &= (x\pi_\varphi)\bar{\varphi}(x\varphi). \end{aligned}$$

Conversely, if π_φ is bijective and φ takes values in the isomorphisms of $\mathbb{C}$ then φ is a unit in $\Phi(\mathbb{C})$, with inverse given by $x\bar{\varphi} = ((x\pi_\varphi^{-1})\varphi)^{-1}$.

Examples

1. If M is a monoid, regarded as a category with a single vertex, then $\Phi^*(M) = M^*$. In particular, if G is a group then $\Phi(G) = \Phi^*(G) = G$.

2. $\Phi^*(\Delta X) = \Sigma_X$, the symmetric group on X.

Idempotents in $\Phi(\mathbb{C})$ can be characterised in a similar way. but we content ourselves with the following observation. If α is an arrow in $\mathbb{C}$ with $\alpha\sigma = x$, we define the flow ε_α by $x\varepsilon_\alpha = \alpha$ and, if $y \neq x$, $y\varepsilon_\alpha = y$.

PROPOSITION 2.3. *If $\alpha\sigma \neq \alpha\tau$ then ε_α is an idempotent in $\Phi(\mathbb{C})$.*

3. Groupoids from regular semigroups

Let S be a regular semigroup. Then given $a \in S$ there exists $b \in S$ with $aba = a$ and $bab = b$. The elements a, b are called *inverses* of one another. In general, an element $a \in S$ can have many inverses. However, if each element $a \in S$ has a *unique* inverse a^{-1} then S is called an *inverse* semigroup. An inverse semigroup may also be defined as a regular semigroup in which the idempotents commute, see [4].

To any regular semigroup S we associate a groupoid $\mathbb{S}$. The vertex set of $\mathbb{S}$ is the set $E(S)$ of idempotents of S, and an arrow in $\mathbb{S}$ is a pair (a, b) of inverse elements, with source $(a, b)\sigma = ab$ and target $(a, b)\tau = ba$. The composition of arrows is given by $(a, b)(c, d) = (ac, db)$ whenever $ba = cd$, the identity arrow at $e \in E(S)$ is (e, e), and an arrow (a, b) has inverse arrow (b, a). We shall describe the structure of $\mathbb{S}$ in terms of the Green relations $\mathcal{L}, \mathcal{R}, \mathcal{H}$ and $\mathcal{D}$ on S. If $\mathcal{F}$ is one of the Green relations, we denote the $\mathcal{F}$–class of $a \in S$ by F_a. In particular, if $e \in S$ is an idempotent, then H_e is a subgroup of S with identity e, the *local subgroup* at e. The proof of the following proposition uses well-known computations for the Green relations on a regular semigroup (see [2], section 2.3 for example), but interpreted here in the context of the structure of $\mathbb{S}$.

PROPOSITION 3.1. *Let e and f be idempotents in the regular semigroup S.*

(a) There exists an arrow (a, b) with $(a, b)\sigma = e$ and if and only if $e\mathcal{R}a$. Likewise, there exists an arrow (c, d) with $(c, d)\tau = f$ if and only if $c\mathcal{L}f$.

(b) The vertex set of the connected component of e is $E(S) \cap D_e$, and an arrow (a, b) lies in the connected component of e if and only if $a\mathcal{D}e$ and $e\mathcal{D}b$.

(c) The vertex group of $\mathbb{S}$ at e is isomorphic to the local subgroup H_e.

Proof. (a) Let $(a, b) \in \mathbb{S}$, so that $aba = a$ and $bab = b$. Now $(a, b)\sigma = ab$ and if $e = ab$ it is easy to see that $e\mathcal{R}a$. Conversely, if $a\mathcal{R}e$ then there exists $y \in S$ such that $e = ay$ and yay is an inverse of a. Then $(a, yay)\sigma = ayay = e^2 = e$. The other part of (a) is proved similarly.

(b) If there exists an arrow (a, b) with $(a, b)\sigma = e$ and $(a, b)\tau = f$ then $e\mathcal{R}a\mathcal{L}f$ and hence $e\mathcal{D}f$. Conversely, suppose that $e\mathcal{D}f$, with $e\mathcal{R}a\mathcal{L}f$ for some $a \in S$. Then we have $ea = a = af$ and for some $x, y \in S$,

$e = ax$ and $f = ya$. Set $b = fxe$. Then $ab = afxe = axe = e^2 = e$ and $ba = fba = yaba = yea = ya = f$. It is now easy to see that b is an inverse of a, and so (a, b) is an arrow in $\mathbb{S}$ with $(a, b)\sigma = e$ and $(a, b)\tau = f$.

Now if (a, b) is an arrow in the connected component of e, we have $a\mathcal{R}ab\mathcal{D}e$ and $b\mathcal{R}ba\mathcal{D}e$. Hence $a\mathcal{D}e\mathcal{D}b$. Conversely, suppose that $a\mathcal{D}e$. Since every inverse of a is in D_a we have $e\mathcal{D}b$ also. Now $e\mathcal{D}a\mathcal{R}ab$ so that $e\mathcal{D}ab$ and ab is in the connected component of e.

(c) The vertex group at e is $\mathbb{S}_e = \{(a, b) : aba = a, bab = b, ab = e = ba\}$. By part (a), if $(a, b) \in \mathbb{S}_e$ then $a \in H_e$ so that we can map $\mathbb{S}_e \to H_e$ by $(a, b) \mapsto a$ and this is certainly a group homomorphism. Moreover, if $(a, b), (a, b') \in \mathbb{S}_e$ then

$$b = bab = babab = ebe = b'abab' = b'ab' = b'$$

so that the map is injective. Finally, since H_e is a subgroup of S with identity e, if $a \in H_e$ then H_e contains a unique inverse a' of a with $aa' = e = a'a$ and $(a, a') \in \mathbb{S}_e$ maps to a.

COROLLARY 3.2. *The groupoid $\mathbb{S}$ is unicursal if and only if S is combinatorial.*

We can neatly characterise when a simplicial groupoid arises from the Nambooripad constrcution.

PROPOSITION 3.3. *Let S be a regular semigroup with associated groupoid $\mathbb{S}$. Then the following are equivalent:*

(a) $\mathbb{S}$ is the simplicial groupoid on ΔS on S,

(b) S is a rectangular band.

Proof. Suppose that G is the simplicial groupoid on S. Then $V(\mathbb{S}) = E(S) = S$, so that S is a band. Moreover, any two elements of S are connnected in $\mathbb{S}$ by a unique arrow: hence given $a, b \in S$ there exists unique $x, y \in S$ with $a = xy$ and $yx = b$. Therefore

$$aba = xy^2x^2y = xyxy = (xy)^2 = xy = a$$

and it follows that S is a rectangular band.

Conversely, if S is a rectangular band, then $S = E(S) = V(\mathbb{S})$, and since each pair of elements are inverse, the set of arrows in $\mathbb{S}$ is $S \times S$. We claim that, given $a, b \in S$ there exists a unique arrow (x, y) from a to b: that is a unique solution (x, y) to the equations $a = xy$ and

$yx = b$. A solution certainly exists: take $x = ab, y = ba$. Then if (x', y') is another solution, $x = xyx = xy^2x = ab = x'y'y'x' = x'y'x' = x'$ and similarly $y = y'$.

4. Flows on regular semigroups

A *flow on a regular semigroup* S is now defined to be a flow on the groupoid $\mathbb{S}$. Hence if $\varphi \in \Phi(S)$ then $x\varphi = (x\varphi_1, x\varphi_2)$ for some pair of functions $\varphi_1, \varphi_2 : E(S) \to S$ with the property that, for all $x \in E(S)$, $x\varphi_1$ and $x\varphi_2$ are inverses, and $(x\varphi_1)(x\varphi_2) = x$. Note that in $\mathbb{S}$, we have $x\varphi\tau = (x\varphi_2)(x\varphi_1)$.

PROPOSITION 4.1. *The flow monoid $\Phi(S)$ of a regular semigroup S is a regular monoid.*

Proof. Let $\varphi \in \Phi(S)$ with $x\varphi = (x\varphi_1, x\varphi_2)$ as above. Let $Y \subseteq E(S)$ be a transversal to the partition of $E(S)$ induced by the endomorphism π_φ. We define a flow ψ on S as follows. If $x \in E(S)$ and $x = y\pi_\varphi$ for some (necessarily unique) $y \in Y$ then set $x\psi = (y\varphi_2, y\varphi_1)$. If $x \in E(S)$ is not in the image of π_φ then set $x\psi = (x, x)$. It is then easy to check that $\varphi = \varphi\psi\varphi$, so that φ is a regular element of $\Phi(S)$.

If S is an *inverse* semigroup, then we can identify an arrow (a, a^{-1}) in the groupoid $\mathbb{S}$ with $a \in S$. Then $a \in \mathbb{S}$ has source $a\sigma = aa^{-1}$ and target $a\tau = a^{-1}a$. Elements $a, b \in \mathbb{S}$ have composite $ab \in \mathbb{S}$ if $a^{-1}a = bb^{-1}$, and the composite is otherwise undefined. The description of the flow monoid then takes the simpler form:

$$\Phi(S) = \{\varphi : E(S) \to S : (x\varphi)(x\varphi)^{-1} = x\}.$$

However, $\Phi(S)$ need not be an inverse monoid:

PROPOSITION 4.2. *Let S be a regular semigroup. Then $\Phi(S)$ is an inverse monoid if and only if S is the disjoint union of its local subgroups,*

$$S = \sqcup_{e \in E(S)} H_e.$$

Proof. If $S = \sqcup_{e \in E(S)} H_e$ then by proposition 2.1 (a), the flow monoid $\Phi(S)$ is the direct product $\prod_{e \in E(S)} H_e$ and hence is a group. For the converse, we note that if $\alpha \in \mathbb{S}$ has $\alpha\sigma \neq \alpha\tau$, then by proposition 2.3, the flows ε_α and $\varepsilon_{\alpha^{-1}}$ are non-commuting idempotents, whence $\Phi(S)$

is not an inverse monoid. Hence if $\Phi(S)$ is inverse, each $\mathcal{D}$–class in S contains a unique idempotent. It follows that S is inverse, that $\mathcal{H} = \mathcal{D}$, and that $S = \sqcup_{e \in E(S)} H_e$.

From any groupoid $\mathbb{G}$ we may construct an inverse semigroup G^0. The underlying set of G^0 is $\mathbb{G} \cup \{0\}$: the composition in $\mathbb{G}$ is used where it is defined, and products undefined in $\mathbb{G}$ are set equal to 0. The inverse semigroup G^0 is said to arise from $\mathbb{G}$ by adjoining a zero, and it is a primitive inverse semigroup with zero. Indeed, all such semigroups arise in this way:

THEOREM 4.3. *(see [6], Theorem 3.3.4.) Let S be an inverse semigroup with zero. Then S is primitive if and only if it is isomorphic to a groupoid with a zero adjoined.*

So from a regular semigroup S we obtain the primitive inverse semigroup with zero $\mathbb{S}^0$ where

$$\mathbb{S}^0 = \{(a, b) : aba = a, bab = b\} \cup \{0\}$$

which admits the product

$$\begin{aligned}(a, b)(c, d) &= (ac, db) \text{ if } ba = cd \\ &= 0 \text{ otherwise}\end{aligned}$$

and in which (a, b) has the unique inverse $(a, b)^{-1} = (b, a)$.

PROPOSITION 4.4. *The flow monoid of a regular semigroup S is isomorphic to the flow monoid of the primitive inverse semigroup $\mathbb{S}^0$.*

Proof. For any groupoid $\mathbb{G}$ we let $\mathbb{G}^\bullet$ denote the groupoid obtained from $\mathbb{G}$ by adding an additional isolated vertex 0. Then the groupoid associated to the primitive inverse semigroup with zero $\mathbb{G}^0$ is equal to $\mathbb{G}^\bullet$. Putting $\mathbb{G} = \mathbb{S}$ we deduce that $\Phi(\mathbb{S}^0) = \Phi(\mathbb{S}^\bullet)$. However, a flow on $\mathbb{S}^\bullet$ is uniquely determined by a flow on $\mathbb{S}$, since any $\varphi \in \Phi(\mathbb{S}^\bullet)$ has $\varphi(0) = 0$. Hence $\Phi(\mathbb{S}^\bullet)$ and $\Phi(\mathbb{S})$ are isomorphic monoids, and by definition the latter is $\Phi(S)$.

THEOREM 4.5. *Let S be a regular semigroup. Let $D \subset E(S)$ be a set of idempotent representatives for the $\mathcal{D}$–classes in S. Then the flow monoid of S is the direct product of monoid wreath products*

$$\Phi(S) = \prod_{d \in D} H_d \wr \mathsf{End}(D_d \cap E(S)).$$

Proof. By Proposition 3.1 (b), the connected components of $\mathbb{S}$ are indexed by D, and so by Proposition 2.1, $\Phi(S) \cong \prod_{d\in D} \Phi(\mathbb{S}_d)$, where $\mathbb{S}_d$ is the connected component of $\mathbb{S}$ containing $d \in D$.

Now $\mathbb{S}_d$ is a connected groupoid with vertex set $D_d \cap E(S)$ and with vertex groups isomorphic to H_d, by Proposition 3.1 (c). It follows that S_d is isomorphic to the groupoid $[D_d \cap E(S)] \times H_d \times [D_d \cap E(S)]$ (see section 1) and that $\Phi(S_d) \cong \Phi([D_d \cap E(S)] \times H_d \times [D_d \cap E(S)])$. Now a flow on $[D_d \cap E(S)] \times H_d \times [D_d \cap E(S)]$ is a function $\varphi : D_d \cap E(S) \to [D_d \cap E(S)] \times H_d \times [D_d \cap E(S)]$ such that $x\varphi = (x, x\varphi_1, x\pi_\varphi)$, where $\varphi_1 : D_d \cap E(S) \to H_d$ and π_φ is an endomorphism of $D_d \cap E(S)$. Hence φ is determined by the pair of functions (φ_1, π_φ) and we have a bijection of sets

$$\Phi(S_d) \cong H_d^{D_d \cap E(S)} \times \mathsf{End}(D_d \cap E(S)).$$

To check the monoid operation on the right-hand side, we compute

$$\begin{aligned} x(\varphi * \psi) &= x\varphi(x\pi_\varphi)\psi \\ &= (x, x\varphi_1, x\pi_\varphi)(x\pi_\varphi, x\pi_\varphi\psi_1, (x\pi_\varphi)\pi_\psi) \\ &= (x, (x\varphi_1)(x\pi_\varphi\psi_1), (x\pi_\varphi)\pi_\psi) \end{aligned}$$

Therefore,we obtain a bijection of monoids if $H_d^{D_d \cap E(S)} \times \mathsf{End}(D_d \cap E(S))$ is equipped with the monoid operation

$$(\varphi_1, \alpha)(\psi_1, \beta) = ((-)\varphi_1((-)\alpha\psi_1), (-)\alpha\beta)$$

and this is exactly the monoid wreath product $H_d \wr \mathsf{End}(D_d \cap E(S))$ (see [5]).

COROLLARY 4.6. *Let S be a regular semigroup. Then $\Phi(S)$ is commutative if and only if S is a disjoint union of abelian groups.*

Proof. Certainly if S is an abelian group, then $\Phi(S) = S$ is commutative. On the other hand, if $\Phi(S)$ is commutative, then each set $D_d \cap E(S)$ is a singleton and it follows that the connected components of $\mathbb{S}$ are the vertex groups $\mathbb{S}(d)$. By Proposition 3.1 (c), $\Phi(S) = \prod_{d\in D} H_d$, and since $\Phi(S)$ is commutative, each H_d is abelian.

COROLLARY 4.7. *Let S be a bisimple regular semigroup. Then the ideals of $\Phi(S)$ are in one-to-one correspondence with the ideals of the full transformation semigroup $\mathsf{End}(E(S))$.*

Proof. Fix an idempotent $e \in S$. Since S has a single $\mathcal{D}$–class, we have $\Phi(S) = H_e \wr \mathsf{End}(E(S))$. Since H_e is a group, it follows that an ideal $L \triangleleft \Phi(S)$ has the form $L = H_e \wr N$ for some ideal $N \triangleleft \mathsf{End}(S)$.

If $E(S)$ is finite of order r then the ideals of $\mathsf{End}(E(S))$ are all principal and are indexed by the set $\{\ 1, 2, \ldots r\}$. If $E(S)$ is countably infinite then the principal ideals of $\mathsf{End}(E(S))$ are indexed by $\mathbb{N} \cup \{\infty\}$ and $\mathsf{End}(E(S))$ has a unique non-principal ideal. See [2] section 2.2 for details.

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