ON QUANTUM ANALOGUE OF THE CALDERO-CHAPOTON FORMULA

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1. Introduction and Main Results

Quantum cluster algebras were introduced by Berenstein and Zelevinsky in [1] as a noncommutative analogue of cluster algebras (see [9]) to lay the groundwork for a study of the canonical basis. A quantum cluster algebra $\mathcal{A}_q(Q)$ of rank n is generated by a (possibly infinite) set of generators called the *cluster variables* inside an ambient skew-field $\mathcal{F}_q$. The goal of this paper is to explicitly compute all cluster variables for a large class of quantum cluster algebras.

We start with rank 2 quantum cluster algebras. Let q be a formal variable. Let $\mathcal{T}_q$ be the 2-dimensional quantum torus, i.e., $\mathcal{T}_q = \mathbb{Z}[q^{\pm 1/2}]\langle X_1^{\pm 1}, X_2^{\pm 1} : X_1X_2 = qX_2X_1\rangle$ and let $\mathcal{F}_q$ be the skew field of fractions of $\mathcal{T}_q$. For $b, c \in \mathbb{Z}_{>0}$ define the quantum cluster algebra $\mathcal{A}_q(b,c)$ to be the $\mathbb{Z}[q^{\pm 1/2}]$-subalgebra of $\mathcal{F}_q$ generated by the cluster variables X_k, $k \in \mathbb{Z}$, defined recursively by

$$X_{m-1}X_{m+1} = \begin{cases} q^{b/2}X_m^b + 1 & \text{if } m \text{ is odd} \\ q^{c/2}X_m^c + 1 & \text{if } m \text{ is even.} \end{cases} \tag{1.1}$$

In what follows we will routinely specialize q to a positive real number, usually the size of a finite field. The quantum Laurent phenomenon ([1, Corollary 5.2]) implies that each X_k, in fact, belongs to the subring $\mathcal{T}_q$ of $\mathcal{F}_q$ and thus $\mathcal{A}_q(b,c)$ is contained in $\mathcal{T}_q$. However, the explicit computation of each X_k as a Laurent polynomial in X_1 and X_2 is a non-trivial task.

We will use the following notation throughout the paper. Define $X^{(a_1,a_2)} \in \mathcal{T}_q$ by the formula $X^{(a_1,a_2)} := q^{-\frac{1}{2}a_1a_2}X_1^{a_1}X_2^{a_2}$. Also define the symmetrized quantum binomial coefficient

$$\begin{bmatrix} n \\ r \end{bmatrix}_q := \frac{(q^n - q^{-n})(q^{n-1} - q^{-n+1})\cdots(q^{n-r+1} - q^{-n+r-1})}{(q^r - q^{-r})\cdots(q - q^{-1})}. \tag{1.2}$$

The following result shows that all cluster variables X_k for $b = c = 2$ are computable combinatorially.

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Proposition 1.1. *For every $n \geq 0$, we have in $A_q(2,2)$:*

$$X_{-n} = X^{(n+2,-n-1)} + \sum_{p+r\leq n} {\begin{bmatrix} n-r \\ p \end{bmatrix}}_q {\begin{bmatrix} n+1-p \\ r \end{bmatrix}}_q X^{(2r-n,2p-n-1)}; \tag{1.3}$$

$$X_{n+3} = X^{(-n-1,n+2)} + \sum_{p+r\leq n} {\begin{bmatrix} n-r \\ p \end{bmatrix}}_q {\begin{bmatrix} n+1-p \\ r \end{bmatrix}}_q X^{(2p-n-1,2r-n)}. \tag{1.4}$$

This result was proved independently in a recent preprint [15], we present our proof for the convenience of the reader. Setting $q = 1$ we recover the Caldero-Chapoton-Zelevinsky formulas for x_k from [6]. This suggests that Proposition 1.1 should have a categorical and quiver-theoretic interpretation in the form of a deformation of the Caldero-Chapoton formula [3, Eq. 14].

In order to state our generalization of Proposition 1.1 to any algebra $\mathcal{A}_q(b,c)$, we need some quiver-theoretic notation. Let $d = \gcd(b,c)$ and let $Q_{b,c}$ be the quiver $\circ_1 \xrightarrow{d} \circ_2$ with d arrows. Let $\mathbb{F}$ be a finite field and $\bar{\mathbb{F}}$ an algebraic closure of $\mathbb{F}$. For any integer $n > 0$ we will denote by $\mathbb{F}_n$ the degree n field extension of $\mathbb{F}$ inside $\bar{\mathbb{F}}$. We define a valued representation V of $Q_{b,c}$ by assigning an $\mathbb{F}_c$-vector space V_1 to the first vertex, an $\mathbb{F}_b$-vector space V_2 to the second vertex, and an $\mathbb{F}_d$-linear map $\varphi_i : V_1 \to V_2$, $i = 1, 2, \ldots, d$, to each arrow.

Let V be a valued representation of $Q_{b,c}$ with dimension vector $[V] = (v_1, v_2)$. For $\mathbf{e} = (e_1, e_2) \in \mathbb{Z}^2_{\geq 0}$, denote by $Gr_{\mathbf{e}}(V)$ the set of all subrepresentations M of V (i.e., $M = (M_1, M_2)$, where M_δ is a subspace of V_δ, $\delta = 1, 2$ such that $\varphi_i(M_1) \subset M_2$ for $i = 1, 2, \ldots, d$) with $[M] = \mathbf{e}$. This is a finite set since V is finite. For each valued representation V of $Q_{b,c}$ we define the element X_V of the quantum torus $\mathcal{T}_{|\mathbb{F}|}$ by

$$X_V = \sum_{\mathbf{e}} |\mathbb{F}|^{-\frac{1}{2}d^V_{\mathbf{e}}} |Gr_{\mathbf{e}}(V)| X^{(-v_1+bv_2-be_2, ce_1-v_2)} \tag{1.5}$$

where $d^V_{\mathbf{e}} = ce_1(v_1 - e_1) - b(ce_1 - e_2)(v_2 - e_2)$. When $b = c$, the valued representations of $Q_{b,b}$ are just the ordinary $\mathbb{F}_b$-representations of $Q_{b,b}$ and, as we will demonstrate below, this formula gives a deformation of the Caldero-Chapoton formula.

Let $C = \begin{pmatrix} 2 & -b \\ -c & 2 \end{pmatrix}$ be a Cartan matrix and let Φ be the associated root system with simple roots $\{\alpha_1, \alpha_2\}$. We will label all negative real roots of Φ by $\mathbb{Z} \setminus \{1, 2\}$ recursively as follows:

$$\alpha_{m-1} + \alpha_{m+1} = \begin{cases} b\alpha_m & \text{if } m \text{ is odd} \\ c\alpha_m & \text{if } m \text{ is even} \end{cases}$$

for $m \in \mathbb{Z} \setminus \{1, 2\}$ with the convention $\alpha_0 = -\alpha_2$, $\alpha_3 = -\alpha_1$.

Then denote by $V_{(m)}$ the unique indecomposable valued representation of $Q_{b,c}$ with dimension vector $-\alpha_m$ (see e.g., [13, Theorem 16]).

Theorem 1.2. *For any $b, c \in \mathbb{Z}^2_{>0}$ and each $m \in \mathbb{Z} \setminus \{1, 2\}$, the m-th cluster variable X_m of $\mathcal{A}_{|\mathbb{F}|}(b,c)$ equals $X_{V_{(m)}}$.*

We will prove Theorem 1.2 in Section 2. In section 3.1 we will illustrate Theorem 1.2 by computing all X_k of finite types, i.e., when $bc \leq 3$.

Now we consider a general class of rank $n \geq 2$ quantum cluster algebras attached to valued quivers on n vertices.

Let Q be a quiver without vertex loops or 2-cycles. Suppose Q has vertices $[1, n]$ and d_{ij} edges from i to j, and let d_i, $i = 1, \ldots, n$ be *valuations* on the vertices. We will call such a quiver a *valued quiver*. Define the matrix $B = B_Q = (b_{ij})$ by

$$b_{ij} = \begin{cases} d_{ij}d_j/gcd(d_i, d_j) & \text{if } i \to j \text{ in } Q \\ -d_{ij}d_j/gcd(d_i, d_j) & \text{if } j \to i \text{ in } Q \\ 0 & \text{otherwise.} \end{cases}$$

For $D = diag(d_1, \ldots, d_n)$ we have DB is skew-symmetric, i.e. $d_i b_{ij} = -d_j b_{ji}$ for all i and j. From the pair (B, D) we can construct a valued quiver Q_B having vertices $[1, n]$ with valuations d_i and, when $b_{ij} > 0$, Q_B has $gcd(|b_{ij}|, |b_{ji}|)$ edges from vertex i to vertex j. This gives a one-to-one correspondence between valued quivers and skew-symmetrizable matrices with symmetrization data. Thus we will freely identify the valued quiver Q with the pair (B, D) and call the pair (B, D) a *valued quiver*. We will call the valued quiver Q *invertible* if B is invertible. If B is not invertible, we replace the valued quiver (B, D) by the invertible valued quiver $\tilde{Q} = (\tilde{B}, D \oplus D)$ with $2n$ vertices as in section 2, we will call Q the principal subquiver of $\tilde{Q}$. Thus we will assume that n is even and valued quiver will always mean invertible valued quiver.

We define valued representations of Q by assigning an $\mathbb{F}_{d_i}$-vector space to each vertex i and an $\mathbb{F}_{gcd(d_i,d_j)}$-linear map to each edge $i \to j$. Denote the category of all finite-dimensional valued representations of Q by *rep* Q. It is well-known (see e.g., [19]) that *rep* Q is a *length category*, i.e. the Grothendieck group $\mathcal{Q}$ of *rep* Q is a free abelian group generated by the classes $\alpha_i = [S_i]$, $i = 1, \ldots, n$ of simple representations associated to the vertices. In particular, the class $[V] \in \mathcal{Q}$ of an object V is naturally identified with the dimension vector of V. The Euler form on $\mathcal{Q}$ is given by bilinearly extending the following formula

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} d_i & \text{if } i = j \\ -\max(0, d_i b_{ij}) & \text{if } i \neq j. \end{cases}$$

Furthermore, abbreviate $\alpha_i^\vee := \frac{1}{d_i}\alpha_i$ and note that $\langle \alpha_i^\vee, \mathbf{e} \rangle$ and $\langle \mathbf{e}, \alpha_i^\vee \rangle$ are integers for all $\mathbf{e} \in \mathcal{Q}$. Then for $\mathbf{e} \in \mathcal{Q}$ define vectors ${}^*\mathbf{e}, \mathbf{e}^* \in \mathbb{Z}^n$ by

$${}^*\mathbf{e} = (\langle \alpha_1^\vee, \mathbf{e} \rangle, \ldots, \langle \alpha_n^\vee, \mathbf{e} \rangle), \ \mathbf{e}^* = (\langle \mathbf{e}, \alpha_1^\vee \rangle, \ldots, \langle \mathbf{e}, \alpha_n^\vee \rangle).$$

Denote $\Lambda = \Lambda_Q := -DB^{-1}$. By definition Λ is skew-symmetric and satisfies $\Lambda B = -B^T \Lambda = -D$. Write $\Lambda = (\lambda_{ij})$ and let d be the least common multiple of all the denominators of the λ_{ij}'s. Let $\mathcal{T}_{\Lambda_Q,q}$ denote the n-dimensional quantum torus, i.e.

$$\mathcal{T}_{\Lambda_Q,q} = \mathbb{Z}[q^{\pm\frac{1}{2d}}]\langle X_1^{\pm 1}, \ldots, X_n^{\pm 1} | X_i X_j = q^{\lambda_{ij}} X_j X_i \rangle.$$

For each $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ we define a monomial $X^{(\mathbf{a})} \in \mathcal{T}_{\Lambda_Q,q}$ by:

$$X^{(\mathbf{a})} := q^{-\frac{1}{2}\sum_{i<j} \lambda_{ij} a_i a_j} X_1^{a_1} \cdots X_n^{a_n}.$$

For $V \in rep\ Q$ and $\mathbf{e} \in \mathcal{Q}$ define $Gr_{\mathbf{e}}(V)$ to be the set of all subobjects W of V such that $[W] = \mathbf{e}$. Sometimes we will think of this as the set of all short exact sequences $\{0 \to W \subset V \to V/W \to 0 : [W] = \mathbf{e}\}$. Note that $Gr_{\mathbf{e}}(V)$ is finite since V is a finite set. Define the element $X_V \in \mathcal{T}_{\Lambda_Q,|\mathbb{F}|}$ by the formula:

$$X_V = \sum_{\mathbf{e} \in \mathcal{Q}} |\mathbb{F}|^{-\frac{1}{2}\langle \mathbf{e}, [V]-\mathbf{e}\rangle} |Gr_{\mathbf{e}}(V)| X^{(B\mathbf{e} - {}^*[V])}. \tag{1.6}$$

Note that $B\mathbf{e} = {}^*\mathbf{e} - \mathbf{e}^*$. This is equivalent to the following formula for X_V which the reader may find useful:

$$X_V = \sum_{M \subset V} |\mathbb{F}|^{-\frac{1}{2}\langle [M],[V/M]\rangle} X^{(-[M]^* - {}^*[V/M])}.$$

It is easy to see that when $n = 2$ equation (1.6) specializes to equation (1.5). When $d_1 = \cdots = d_n = \delta$, i.e. the quiver is equally valued it is known (see [5, Corollary 4], footnote 5 on page 6 of [17], and the corrected proof in [11]) that for V exceptional and indecomposable $Gr_{\mathbf{e}}(V)$ is the set of $\mathbb{F}_\delta$ points of an algebraic variety of dimension $\langle \mathbf{e}, [V] - \mathbf{e}\rangle/\delta$ and $|Gr_{\mathbf{e}}(V)|$ is given by a positive polynomial in $|\mathbb{F}|^\delta$.

For a sink or source i of Q denote by $\mu_i Q$ the valued quiver $(\mu_i B, D)$ where we change the sign of each entry of B in row i or column i. This is equivalent to reversing all arrows with vertex i as source or target. We will call a sequence of vertices k_1, k_2, …, k_{r+1} in Q *admissible* if the following hold:

- $k_i \neq k_{i+1}$ for each i;
- k_1 is a sink or source in Q;
- for each $1 \leq i \leq r-1$, vertex k_{i+1} is a sink or source in the quiver $\mu_{k_i}\mu_{k_{i-1}} \cdots \mu_{k_1} Q$.

Note that k_{r+1} does not have to be a sink or a source.

Let C denote the $n \times n$ Cartan counterpart to B, i.e. define

$$c_{ij} = \begin{cases} 2 & \text{if } i = j \\ -|b_{ij}| & \text{if } i \neq j. \end{cases}$$

Let Φ denote the root system associated to C, see [14]. We will identify $\mathcal{Q}$ with the root lattice of Φ by taking $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ to be the set of simple roots in Φ. Define simple reflections σ_i in the Weyl group of Φ by setting $\sigma_i(\alpha_j) = \alpha_j - c_{ij}\alpha_i$ and extending linearly (this is the correspondence we will use between the root system Φ and C).

Denote by $\mathcal{A}_q(Q) \subset \mathcal{T}_{\Lambda_Q,q}$ the quantum cluster algebra corresponding to the invertible valued quiver Q (see Section 2 for details).

Remark 1.3. We restrict our attention to invertible valued quivers for simplicity. However the results below hold for any compatible pair $(\Lambda, \tilde{B})$ where Q is the valued quiver associated to the principal part of $\tilde{B}$.

Our main result is the following

Theorem 1.4. *Let V_α be the unique indecomposable object of rep Q with dimension vector α given by $\alpha = \sigma_{k_1}\sigma_{k_2}\cdots\sigma_{k_r}(\alpha_{k_{r+1}})$ where $k_1, k_2, \ldots, k_{r+1}$ is an admissible sequence in Q. Then X_{V_α} is a cluster variable of $\mathcal{A}_{|\mathbb{F}|}(Q)$. Conversely if the quiver Q is acyclic, each cluster variable of $\mathcal{A}_{|\mathbb{F}|}(Q)$ belonging to any acyclic cluster is of the form X_{V_α} for some α as above.*

We will prove Theorem 1.4 in Section 2.

Theorem 1.5. *If the valued quiver Q is a Dynkin diagram, then each cluster variable of $\mathcal{A}_{|\mathbb{F}|}(Q)$ is of the form X_N for some indecomposable $N \in$ rep Q.*

We will prove Theorem 1.5 in Section 2. For any invertible valued quiver Q let $\mathcal{T}_{DB,q} = \mathbb{Z}[q^{\pm\frac{1}{2}}]\langle Z_1^{\pm 1}, \ldots, Z_n^{\pm 1} | Z_i Z_j = q^{d_i b_{ij}} Z_j Z_i\rangle$ be the quantum torus associated to the skew-symmetric matrix DB. It is easy to see that the assignment $Z_i \mapsto X^{(\mathbf{b}^i)}$ defines an embedding of quantum tori which extends to an embedding of skew-fields $j : \mathcal{F}_{DB,q} \to \mathcal{F}_{\Lambda_Q,q}$. We say that an element $X \in \mathcal{F}_{\Lambda_Q,q}$ is *B-compatible* if X can be written as $j(F_X)X^{(\mathbf{g}_X)}$ for some $F_X \in \mathcal{F}_{DB,q}$ and $\mathbf{g}_X \in \mathbb{Z}^n$. We refer to F_X as an F-factor for X thus generalizing the definition of F-polynomial from [22, Theorem 5.3]. It follows from [22, Theorem 5.3] that each cluster monomial X is B-compatible and (under appropriate choice of $\mathbf{g}_X$) the F-factor F_X is actually a polynomial. The following result justifies these definitions.

Proposition 1.6.

(1) *The mutation of B-compatible elements in direction k gives $\mu_k B$-compatible elements.*
(2) *For $V \in$ rep Q, X_V is B-compatible and we have*

$$F_{X_V} = 1 + \sum_{\mathbf{e} \in \mathcal{Q}\setminus\{\mathbf{0}\}} q^{\frac{1}{2}\langle \mathbf{e},\mathbf{e}\rangle} |Gr_{\mathbf{e}}(V)| Z^{(\mathbf{e})}.$$

Remark 1.7. Following up on Remark 1.3, since all our results hold for principal coefficients, Proposition 1.6(2) together with [22, Theorem 5.3(2)] directly let one work with any coefficients.

Corollary 1.8. *Let Q be of finite type. Then for any indecomposable $V \in$ rep Q we have*

$$F_{X_V} \in 1 + \sum_{\mathbf{e} \in \mathbb{Z}^n_{\geq 0}\setminus\{0\}} q^{\frac{1}{2}} \mathbb{Z}[q^{\frac{1}{2}}] Z^{(\mathbf{e})}.$$

Remark 1.9. For simply laced types the positivity of coefficients of F_{X_V} follows from the positivity of grassmannians from [11]. See Section 3.1.3 for positivity in type G_2. Positivity for selected clusters in types B_n and C_n follow from [23, Theorem 6.1].

Based on the above results, we now conjecture a general deformation of the Caldero-Chapoton formula for any quantum cluster algebra with acyclic seed.

Conjecture 1.10. *Let Q be an acyclic valued quiver. Suppose $V \in rep\ Q$ is an indecomposable exceptional valued representation. Then X_V is a cluster variable in $\mathcal{A}_{|\mathbb{F}|}(Q)$.*

In Corollary 2.6 we show that all cluster variables in almost acyclic clusters (clusters which are one mutation away from an acyclic cluster) are of the form X_V for some $V \in rep\ Q$. We present further evidence for this conjecture in section 3.2.

When all valuations of the quiver Q are equal it is known, see [5], that for an indecomposable exceptional representation M we have $|Gr_{\mathbf{e}}(M)|_{|\mathbb{F}|\to 1} = \chi_c(Gr_{\mathbf{e}}(M))$. Thus we see that, when Q is an equally valued quiver, setting $|\mathbb{F}| \to 1$ in Conjecture 1.10 gives the Caldero-Chapoton formula proved in [4].

While completing the final draft of this manuscript we learned from Bernhard Keller that Fan Qin [11] proved Conjecture 1.10 for acyclic equally valued quivers.

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2. Definitions and Notation

We begin this section with a recollection of some of the terminology related to quantum cluster algebras. Let L be a lattice of rank m and $\Lambda : L \times L \to \mathbb{Q}$ a skew-symmetric bilinear form and let d be the least common multiple of all denominators appearing in the image of Λ. Let q be a formal variable and consider the ring of integer Laurent polynomials $\mathbb{Z}[q^{\pm\frac{1}{2d}}]$. Define the *based quantum torus* associated to the pair (L, Λ) to be the $\mathbb{Z}[q^{\pm\frac{1}{2d}}]$-algebra $\mathcal{T}_{\Lambda,q}$ with distinguished $\mathbb{Z}[q^{\pm\frac{1}{2d}}]$-basis $\{X^e : e \in L\}$ with multiplication given by

$$X^e X^f = q^{\Lambda(e,f)/2} X^{e+f}.$$

An easy computation shows that $\mathcal{T}_{\Lambda,q}$ is associative and the basis elements satisfy the following relations:

$$X^e X^f = q^{\Lambda(e,f)} X^f X^e,\ X^0 = 1,\ (X^e)^{-1} = X^{-e}.$$

As the based quantum torus is an Ore domain, it is contained in its skew-field of fractions $\mathcal{F}_{\Lambda,q}$.

A *toric frame* in $\mathcal{F}_{\Lambda,q}$ is a map $M : \mathbb{Z}^m \to \mathcal{F}_{\Lambda,q} \setminus \{0\}$ of the form

$$M(\mathbf{c}) = \varphi(X^{\eta(\mathbf{c})})$$

where φ is an automorphism of $\mathcal{F}_{\Lambda,q}$ and $\eta : \mathbb{Z}^m \to L$ is a lattice isomorphism. The $M(\mathbf{c})$ form a $\mathbb{Z}[q^{\pm\frac{1}{2d}}]$-basis of the based quantum torus $\mathcal{T}_{\Lambda_M,q} := \varphi(\mathcal{T}_{\Lambda,q})$ which is an isomorphic copy of $\mathcal{T}_{\Lambda,q}$ in $\mathcal{F}_{\Lambda,q}$. The following equations hold in $\mathcal{T}_{\Lambda_M,q}$:

$$M(\mathbf{c})M(\mathbf{d}) = q^{\Lambda_M(\mathbf{c},\mathbf{d})/2}M(\mathbf{c}+\mathbf{d}),\ M(\mathbf{c})M(\mathbf{d}) = q^{\Lambda_M(\mathbf{c},\mathbf{d})}M(\mathbf{d})M(\mathbf{c}),$$
$$M(\mathbf{0}) = 1,\ M(\mathbf{c})^{-1} = M(-\mathbf{c}),$$

where Λ_M is the skew-symmetric bilinear form on $\mathbb{Z}^m$ obtained from the lattice isomorphism η. Let Λ_M also denote the skew-symmetric $m \times m$ matrix defined by $\lambda_{ij} = \Lambda_M(\alpha_i, \alpha_j)$ where $\alpha_1, \ldots, \alpha_m$ are the standard basis vectors in $\mathbb{Z}^m$. Given a toric frame M, let $X_i = M(\alpha_i)$. Then we have

$$\mathcal{T}_{\Lambda_M,q} = \mathbb{Z}[q^{\pm\frac{1}{2d}}]\langle X_1^{\pm 1}, \ldots, X_m^{\pm 1} : X_iX_j = q^{\lambda_{ij}}X_jX_i\rangle.$$

Using the relations above we get for $\mathbf{c} \in \mathbb{Z}^m$

$$M(\mathbf{c}) = q^{\frac{1}{2}\sum_{i<j} c_ic_j\lambda_{ji}}X_1^{c_1}X_2^{c_2}\cdots X_m^{c_m} =: X^{(\mathbf{c})}.$$

Let Λ be an $m\times m$ skew-symmetric matrix and let $\tilde{B}$ be any $m\times n$ matrix, $n \leq m$. We call the pair $(\Lambda, \tilde{B})$ *compatible* if $\tilde{B}^T\Lambda = (D|0)$ is an $n \times m$ matrix with D a diagonal integer matrix with positive entries on the diagonal. Throughout this paper we will assume $n = m$ with m even and $\tilde{B}$ is invertible. Otherwise, one may replace $\tilde{B}$ by the invertible $2n \times 2n$ matrix $\begin{pmatrix} B & -I_n \\ I_n & R \end{pmatrix}$, where B is the principal part of $\tilde{B}$ and I_n is the $n \times n$ identity matrix, such that $\tilde{B}^T\Lambda = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} =: D \oplus D$, adjusting the coefficients and Λ if necessary to ensure that R is an integer matrix. By restricting ourselves to mutations in directions $1, \ldots, n$ we will recover our original cluster algebra with principal coefficients.

The pair $(M, \tilde{B})$ is called a *quantum seed* if the pair $(\Lambda_M, \tilde{B})$ is compatible. We now define the mutation of the quantum seed $(M, \tilde{B})$ in direction k for $k \in [1, n]$. Define the $m \times m$ matrix $E = (e_{ij})$ by

$$e_{ij} = \begin{cases} \delta_{ij} & \text{if } j \neq k; \\ -1 & \text{if } i = j = k; \\ [-b_{ik}]_+ & \text{if } i \neq j = k \end{cases}$$

where $[b]_+ = max(0, b)$. Let $\mathbf{c} = (c_1, \ldots, c_m) \in \mathbb{Z}^m$. Define a map $M' : \mathbb{Z}^m \to \mathcal{F}_{\Lambda,q} \setminus \{0\}$ as follows:

$$M'(\mathbf{c}) = \sum_{p\geq 0} \begin{bmatrix} c_k \\ p \end{bmatrix}_{q^{d_k/2}} M(E\mathbf{c} + p\mathbf{b}^k), \tag{2.1}$$

where the vector $\mathbf{b}^k \in \mathbb{Z}^m$ is the kth column of $\tilde{B}$ and $\begin{bmatrix} c_k \\ p \end{bmatrix}_{q^{d_k/2}}$ is given by equation (1.2).

Lemma 2.1. *M' is a well-defined toric frame.*

Proof. The case $c_k \geq 0$ was shown in [1, Section 4]. For $c_k < 0$, this follows from an obvious extension of the argument in [1, Section 4] using the identity $\prod_{r=0}^{n-1} \frac{1}{1+q^r t} = \sum_{r\geq 0} q^{\frac{1}{2}(n-1)r} \begin{bmatrix} -n \\ r \end{bmatrix}_{q^{1/2}} t^r$ and the fact that $M'(-\mathbf{c})^{-1}$ is bar-invariant. $\square$

Equation (2.1) defines a birational isomorphism of based quantum tori $\mu_k : \mathcal{T}_{\Lambda_{M'},q} \to \mathcal{T}_{\Lambda_M,q}$ given by

$$\mu_k(X_i) = \begin{cases} X_i & \text{if } i \neq k; \\ M\left(\sum_{\ell=1}^{m} [-b_{\ell k}]_+ \alpha_\ell\right) + M\left(\sum_{\ell=1}^{m} [b_{\ell k}]_+ \alpha_\ell\right) & \text{if } i = k. \end{cases}$$

This map takes cluster variables in $\mathcal{T}_{\Lambda_{M'},q}$ to cluster variables in $\mathcal{T}_{\Lambda_M,q}$ and hence $\mu_k \mathcal{A}_q(\mu_k Q) = \mathcal{A}_q(Q)$. So applying the isomorphism μ_k is the same as mutating the initial cluster of the cluster algebra in direction k.

Let $\tilde{B}' = \mu_k \tilde{B} = (b'_{ij})$ where

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + sgn(b_{ik})[b_{ik}b_{kj}]_+ & \text{otherwise} \end{cases}$$

where $[b]_+ = max(0, b)$. Then the quantum seed $(M', \tilde{B}')$ is defined to be the mutation of $(M, \tilde{B})$ in direction k, written $\mu_k(M, \tilde{B})$. Since $\tilde{B}$ was invertible, $\mu_k \tilde{B}$ is also invertible. Suppose $\Lambda_M \tilde{B} = -D$. Then $\Lambda_{M'} \mu_k \tilde{B} = -D$ and so $\Lambda_{M'}$ can be recovered from the valued quiver $(\mu_k \tilde{B}, D)$. Thus we will abuse notation further and call the invertible valued quiver $Q = (\tilde{B}, D)$ a seed. We will also denote $\mu_k Q = (\mu_k \tilde{B}, D)$.

A quantum seed $(M', \tilde{B}')$ is *mutation equivalent* to the seed $(M, \tilde{B})$ if there is a sequence of mutations taking one to the other, in this case write $(M', \tilde{B}') \sim (M, \tilde{B})$. Let $\mathcal{X} = \{M'(\alpha_i) : (M', \tilde{B}') \sim (M, \tilde{B}), i \in [1, m]\}$. The elements of $\mathcal{X}$ are called *cluster variables*. The quantum Laurent phenomenon ([1, Corollary 5.2]) states that $\mathcal{X}$ is actually contained in $\mathcal{T}_{\Lambda_M,q}$. Since $\tilde{B}$ is invertible, the data of the compatible pair $(\Lambda_M, \tilde{B})$ is equivalent to the data of the invertible valued quiver $Q = (\tilde{B}, -\Lambda_M \tilde{B})$. Thus we will let $\mathcal{A}_q(Q)$ denote the $\mathbb{Z}[q^{\pm\frac{1}{2d}}]$-subalgebra of $\mathcal{T}_{\Lambda_Q,q} := \mathcal{T}_{\Lambda_M,q}$ generated by $\mathcal{X}$, called the *quantum cluster algebra*.

Let Q be the valued quiver (B, D), where $D = diag(d_1, \ldots, d_m)$. Let $\mathbb{F}$ be a finite field. We will define an $\mathbb{F}$-species A_Q and show that modules over A_Q are the same as representations of Q. Let $\delta_{ij} = gcd(d_i, d_j)$ and $\delta^{ij} = lcm(d_i, d_j)$. Let $\bar{\mathbb{F}}$ be an algebraic closure of $\mathbb{F}$ and define $K_i = \mathbb{F}_{d_i}$, the degree d_i extension of $\mathbb{F}$ in $\bar{\mathbb{F}}$. Also denote $K_{ij} = \mathbb{F}_{d_i} \cap \mathbb{F}_{d_j} = \mathbb{F}_{\delta_{ij}}$ and $K^{ij} = \mathbb{F}_{\delta^{ij}}$.

Define $A_0 = \prod_{i=1}^{m} K_i$ and $A_1 = \bigoplus_{b_{ij}>0} A_{ij}$ where $A_{ij} := \mathbb{F}_{d_i b_{ij}}$. We define a $K_i - K_j$-bimodule structure on A_{ij} by setting $A_{ij} = \mathbb{F}^{r_{ij}} \otimes_{\mathbb{F}} K^{ij} \cong \bigoplus_{k=1}^{r_{ij}} K^{ij}$, where $r_{ij} = \frac{d_i b_{ij}}{\delta^{ij}} = gcd(|b_{ij}|, |b_{ji}|)$. The following easy lemma shows that this gives such a structure.

Lemma 2.2. *$K_i \otimes_{K_{ij}} K_j$ is a field isomorphic to K^{ij}. In particular, K^{ij} is a $K_i - K_j$-bimodule.*

Now define $A_Q = T(A_0, A_1)$ the tensor algebra of A_1 over A_0. A module X over A_Q is given by a K_i-vector space X_i for each vertex i and a K_j-linear map $\theta_{ij} : X_i \otimes_{K_i} A_{ij} \to X_j$ whenever $b_{ij} > 0$, see [13]. A morphism of A_Q-modules $f : X \to Y$ is a collection $\{f_i\}_{i \in [1,m]}$ with $f_i : X_i \to Y_i$ a K_i-linear map, such that $\theta^Y_{ij}(f_i \otimes id) = f_j \theta^X_{ij}$.

Consider the following natural isomorphisms:

$$Hom_{K_j}(X_i \otimes_{K_i} A_{ij}, X_j) \cong \mathbb{F}^{r_{ij}} \otimes_{\mathbb{F}} Hom_{K_j}(X_i \otimes_{K_i} K^{ij}, X_j),$$

$$X_i \otimes_{K_i} K^{ij} = X_i \otimes_{K_i} (K_i \otimes_{K_{ij}} K_j) \cong X_i \otimes_{K_{ij}} K_j,$$

$$Hom_{K_j}(X_i \otimes_{K_{ij}} K_j, X_j) \cong Hom_{K_{ij}}(X_i, X_j).$$

Combining these we obtain a natural isomorphism

(2.2)

$$Hom_{K_j}(X_i \otimes_{K_i} A_{ij}, X_j) \cong \mathbb{F}^{r_{ij}} \otimes_{\mathbb{F}} Hom_{K_j}(X_i \otimes_{K_{ij}} K_j, X_j) \cong \bigoplus_{\ell=1}^{r_{ij}} Hom_{K_{ij}}(X_i, X_j).$$

These natural isomorphisms define inverse equivalences of categories

$$F : mod\ A_Q \leftrightarrow rep\ Q : F^{-1}.$$

Indeed one can easily check that the commuting squares defining morphisms of modules and representations are compatible under the natural isomorphism 2.2. Thus the categories *mod* A_Q and *rep* Q are equivalent and we can apply all results concerning $\mathbb{F}$-species to valued representations.

Let $\mathcal{Q}$ denote the Grothendieck group of *rep* Q. For a valued representation V denote by $[V] \in \mathcal{Q}$ the isomorphism class of V. Clearly, $[V] = \sum_{i \in Q} (\dim_{K_i} V_i) \alpha_i$ where $\alpha_i = [S_i]$. For objects $V, W \in rep\ Q$ define the Euler form

$$\langle V, W \rangle = \dim_{\mathbb{F}} \mathrm{Hom}(V, W) - \dim_{\mathbb{F}} \mathrm{Ext}^1(V, W),$$

where Hom and Ext are computed in *rep* Q. It is known, see [19] for example, that $\langle V, W \rangle$ only depends on the classes of V and W in $\mathcal{Q}$.

Let $V \in rep\ Q$ and define $Gr_{\mathbf{e}}(V)$ to be the set of subrepresentations W of V with $[W] = \mathbf{e}$. For $\mathbf{e} \in \mathcal{Q}$ define vectors ${}^*\mathbf{e}, \mathbf{e}^* \in \mathbb{Z}^n$ as in the introduction. Define $X_V \in \mathcal{T}_{\Lambda_Q, |\mathbb{F}|}$ by

$$X_V = \sum_{\mathbf{e}} |\mathbb{F}|^{-\frac{1}{2}\langle \mathbf{e}, [V] - \mathbf{e} \rangle} |Gr_{\mathbf{e}}(V)| X^{(B\mathbf{e} - {}^*[V])}. \qquad (2.3)$$

Let *rep* $Q\langle i \rangle$ denote the full subcategory of *rep* Q of all representations of Q which do not contain S_i as a direct summand. In [7] it is shown that the reflection functors

$$\mathbb{S}_i^- : rep\ Q \leftrightarrow rep\ \mu_i Q : \mathbb{S}_i^+$$

restrict to inverse equivalences of categories

$$\mathbb{S}_i^- : rep\ Q\langle i\rangle \leftrightarrow rep\ \mu_i Q\langle i\rangle : \mathbb{S}_i^+.$$

Since it will be clear from context which to use we will drop the $^{\pm}$ and simply denote both functors by $\mathbb{S}_i$. See [7] for precise definitions of these functors. We will use the following result proved in [7].

Lemma 2.3. [7, Proposition 2.1] *For $X \in rep\ Q\langle i\rangle$ we have $[\mathbb{S}_i X] = \sigma_i([X])$ and $\mathbb{S}_i^2 X = X$.*

Our main tool will be the following powerful result proved in section 5.

Theorem 2.4. *For any $N \in rep\ Q\langle i\rangle$, $\mu_i X_N = X_{\mathbb{S}_i N}$. In particular, if X_N is a cluster variable in $\mathcal{A}_q(Q)$ then $X_{\mathbb{S}_i N}$ is a cluster variable in $\mathcal{A}_q(\mu_i Q)$.*

In order to state our main theorem we introduce some new notation for describing a cluster variable. Define $X^Q_{[a_0]} := X^{(\alpha_{a_0})}$ in $\mathcal{A}_q(Q)$, so the ordered tuple $(X_{[1]}, \ldots, X_{[n]})$ forms the initial cluster of $\mathcal{A}_q(Q)$. Now recursively define

$$X^Q_{[a_0;a_1,a_2,\ldots,a_r]} = \mu_{a_r} X^{\mu_{a_r}Q}_{[a_0;a_1,a_2,\ldots,a_{r-1}]}$$

where $X^{\mu_{a_r}Q}_{[a_0;a_1,a_2,\ldots,a_{r-1}]}$ is in $\mathcal{A}_q(\mu_{a_r}Q)$ and the birational isomorphism μ_{a_r} pulls it back to $\mathcal{A}_q(Q)$. Alternatively one could start with the initial ordered seed $(\mathbf{X}, Q)$ and mutate in direction a_1, then a_2, etc. to obtain the ordered seed $\mu_{a_r}\cdots\mu_{a_1}(\mathbf{X}, Q)$ then $X^Q_{[a_0;a_1,a_2,\ldots,a_r]}$ is the ${a_0}^{th}$ cluster variable in this seed. When it is clear from context we will drop the Q from the notation.

Here are some simple observations that follow from this notation:

(1) If $a_i = a_{i+1}$ for some $i > 0$, then $X^Q_{[a_0;a_1,a_2,\ldots,a_r]} = X^Q_{[a_0;a_1,\ldots,a_{i-1},a_{i+2},\ldots,a_r]}$.
(2) If $a_0 \neq a_r$, then $X^Q_{[a_0;a_1,a_2,\ldots,a_r]} = X^Q_{[a_0;a_1,a_2,\ldots,a_{r-1}]}$.
(3) If we mutate the seed $(X^Q_{[1;a_1,\ldots,a_r]}, \ldots, X^Q_{[n;a_1,\ldots,a_r]}, Q')$ in direction t we get the new seed $(X^Q_{[1;a_1,\ldots,a_r,t]}, \ldots, X^Q_{[n;a_1,\ldots,a_r,t]}, \mu_t Q')$.
(4) If we start with $X^Q_{[a_0;a_1,a_2,\ldots,a_{r-1}]}$ and mutate the *initial* seed in direction t then we get $X^{\mu_t Q}_{[a_0;t,a_1,a_2,\ldots,a_{r-1}]}$.

The content of Theorem 1.4 is contained in the following Theorem and Corollary.

Theorem 2.5. *Suppose k_1, k_2, $\ldots$, k_{r+1} is an admissible sequence of vertices in Q. Let $M \in rep\ Q$ be the unique indecomposable representation of Q with $[M] = \sigma_{k_1}\sigma_{k_2}\cdots\sigma_{k_r}(\alpha_{k_{r+1}})$. Then in the cluster algebra $\mathcal{A}_{|\mathbb{F}|}(Q)$ we have $X^Q_{[k_{r+1};k_1,k_2,\ldots,k_{r+1}]} = X_M$.*

We prove this in section 5. We will call a quiver Q *almost acyclic* if there exists i so that the valued quiver $\mu_i Q$ is acyclic.

Corollary 2.6. *Suppose the valued quiver Q is acyclic. Then each cluster variable of $\mathcal{A}_{|\mathbb{F}|}(Q)$ in an almost acyclic cluster is of the form X_N for some indecomposable $N \in rep\ Q$.*

Proof. In [4, Corollary 4], the authors show that all acyclic clusters are connected by sink and source mutations. The result follows from Theorem 2.5. □

Theorem 1.2 immediately follows.

Proof of Theorem 1.2. For rank 2 cluster algebras, the valued quiver associated to any cluster is acyclic. So the result follows from Corollary 2.6. □

We also get Theorem 1.5.

Proof of Theorem 1.5. We will use the concepts from [7]. Let $k_1, k_2, \ldots, k_n$ be an admissible ordering of Q and $C = \mathbb{S}_{k_1}\mathbb{S}_{k_2}\cdots\mathbb{S}_{k_n}$ be the corresponding Coxeter functor. Let $P_{k_t} = \mathbb{S}_{k_1}\mathbb{S}_{k_2}\cdots\mathbb{S}_{k_{t-1}}S_{k_t}$ where $S_{k_t} \in rep\ \mu_{k_t}\mu_{k_{t+1}}\cdots\mu_{k_{t_n}}Q$ is the simple representation associated to vertex k_t. By [7, Propositions 1.9 and 2.6], every indecomposable representation of Q is of the form $C^r P_{k_t}$ for some t and $1 \leq r \leq a_t$ where a_t is the largest integer for which all such $C^r P_{k_t}$ are nonzero.

There is a one-to-one correspondence between cluster variables of $\mathcal{A}_{|\mathbb{F}|}(Q)$ and positive roots in the root system Φ_Q and thus a one-to-one correspondence between cluster variables and indecomposable representations of Q. From Theorem 2.5 and the definition of P_{k_t} we see that $X_{P_{k_t}}$ is a cluster variable in $\mathcal{A}_{|\mathbb{F}|}(Q)$.

The result follows from the proof of Theorem 2.5 if $X_{P_{k_t}}$. □

Proof of Proposition 1.6. Write $\underline{\mu_k} : \mathcal{F}_{D\mu_k B,q} \to \mathcal{F}_{DB,q}$ for the mutation in direction k from [12, Section 3.3] and $j' : \mathcal{F}_{D\mu_k B,q} \to \mathcal{F}_{\Lambda_{\mu_k Q},q}$. The following Lemma follows from the definitions.

Lemma 2.7. *$\mu_k j'(F') = j\underline{\mu_k}(F')$ for any $F' \in \mathcal{F}_{D\mu_k B,q}$.*

This says that the mutation of the $\mu_k B$-compatible element $j'(F')$ is B-compatible. Now equation (2.1) says that the mutation of any monomial in $\mathcal{F}_{\Lambda_{\mu_k Q},q}$ is B-compatible. Combining the last two statements completes the proof of Proposition 1.6(1). Proposition 1.6(2) follows from an easy computation using the fact that $\langle \alpha_i, \alpha_j \rangle = \Lambda(\mathbf{b}^i, {}^*\alpha_j)$. □

Proof of Corollary 1.8. Note that for finite types the Cartan counterpart of B is always positive definite and so for any $\mathbf{e} \in \mathcal{Q} \setminus 0$ we have $\langle \mathbf{e}, \mathbf{e} \rangle > 0$. So this follows from Proposition 1.6(2), [23, Theorem 6.1], [11], and Section 3.1.3. □

3. Examples

Throughout this section we will let $\mathbb{F}$ be the finite field with q elements. In what follows each cluster variable $X_{[a_0;a_1,a_2,\ldots,a_r]}$ will have $a_0 = a_r$ so we will drop a_0 from the notation.

3.1. Finite Type Rank 2 Quantum Cluster Algebras.

In what follows, we abbreviate $X^{(a_1,a_2)} := q^{-\frac{1}{2}a_1 a_2 \lambda_{12}} X_1^{a_1} X_2^{a_2}$, without loss of generality we may assume $\lambda_{12} = 1$. Also note that there is no significance to the numbering of the indecomposable representations.

3.1.1. *Type A_2.* We begin with the valued quiver $Q=(B,D)$ where $B=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $D=diag(1,1)$. The Cartan counterpart of B is $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$.

Lemma 3.1 ([1]). *The quantum cluster algebra $\mathcal{A}_q(Q)$ is of finite type and we have the following formulas for the cluster variables in terms of the initial cluster (X_1, X_2):*

$$\begin{aligned}
X_3 &:= X_{[1]} = X^{(-1,1)} + X^{(-1,0)} \\
X_4 &:= X_{[1,2]} = X^{(-1,0)} + X^{(0,-1)} + X^{(-1,-1)} \\
X_5 &:= X_{[1,2,1]} = X^{(1,-1)} + X^{(0,-1)} \\
X_6 &:= X_{[1,2,1,2]} = X^{(1,0)} = X_1 \\
X_7 &:= X_{[1,2,1,2,1]} = X^{(0,1)} = X_2.
\end{aligned}$$

The indecomposable valued representations of Q are $S_1=\mathbb{F}\to 0$ (dimension vector α_1), $S_2=0\to\mathbb{F}$ (dimension vector $\alpha_2=s_1s_2(\alpha_1)$), and $I_1=\mathbb{F}\xrightarrow{id}\mathbb{F}$ (dimension vector $\alpha_1+\alpha_2=s_1(\alpha_2)$).

The representation S_1 has unique subrepresentations with dimension vectors α_1 and 0. So we get:

$$X_{S_1} = X^{(-\alpha_1+\alpha_2)} + X^{(-\alpha_1)} = X_3.$$

The representation I_1 has unique subrepresentations with dimension vectors $\alpha_1+\alpha_2$, 0 and α_2. So we get:

$$X_{I_1} = X^{(-\alpha_1)} + X^{(-\alpha_2)} + X^{(-\alpha_1-\alpha_2)} = X_4.$$

The representation S_2 has unique subrepresentations with dimension vectors 0 and α_2. So we get:

$$X_{S_2} = X^{(\alpha_1-\alpha_2)} + X^{(-\alpha_2)} = X_5.$$

3.1.2. *Type C_2.* (We get type B_2 by dualizing all representations.) We begin with the valued quiver $Q=(B,D)$ where $B=\begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}$ and $D=diag(1,2)$. Define $K_1=\mathbb{F}=:k$ and $K_2=\mathbb{F}_2=:K$. The Cartan counterpart of B is $\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$.

Lemma 3.2 ([1]). *The quantum cluster algebra $\mathcal{A}_q(Q)$ is of finite type and we have the following formulas for the cluster variables in terms of the initial cluster (X_1, X_2):*

$$\begin{aligned}
X_3 &:= X_{[1]} = X^{(-1,1)} + X^{(-1,0)}\\
X_4 &:= X_{[1,2]} = X^{(-2,1)} + (q^{1/2} + q^{-1/2})X^{(-2,0)} + X^{(0,-1)} + X^{(-2,-1)}\\
X_5 &:= X_{[1,2,1]} = X^{(-1,0)} + X^{(1,-1)} + X^{(-1,-1)}\\
X_6 &:= X_{[1,2,1,2]} = X^{(2,-1)} + X^{(0,-1)}\\
X_7 &:= X_{[1,2,1,2,1]} = X^{(1,0)} = X_1\\
X_8 &:= X_{[1,2,1,2,1,2]} = X^{(0,1)} = X_2.
\end{aligned}$$

The indecomposable valued representations of Q are $S_1 = k \to 0$ (dimension vector α_1), $S_2 = 0 \to K$ (dimension vector $\alpha_2 = s_1 s_2 s_1(\alpha_2)$), $I_1 = k \xrightarrow{\iota} K$ (dimension vector $\alpha_1 + \alpha_2 = s_1 s_2(\alpha_1)$) where ι is the inclusion map, and $I_2 = k^2 \xrightarrow{\sigma} K$ (dimension vector $2\alpha_1 + \alpha_2 = s_1(\alpha_2)$) where σ identifies K as a 2-dimensional vector space over k.

The representation S_1 has unique subrepresentations with dimension vectors α_1 and 0. So we get:

$$X_{S_1} = X^{(-\alpha_1+\alpha_2)} + X^{(-\alpha_1)} = X_3.$$

The representation I_2 has unique subrepresentations with dimension vectors $2\alpha_1+\alpha_2$, 0 and α_2, and it has $1 + q$ subrepresentations with dimension vector $\alpha_1 + \alpha_2$. So we get:

$$X_{I_1} = X^{(-2\alpha_1+\alpha_2)} + X^{(-\alpha_2)} + X^{(-2\alpha_1-\alpha_2)} + (q^{1/2} + q^{-1/2})X^{(-2\alpha_1)} = X_4.$$

The representation I_1 has unique subrepresentations with dimension vectors $\alpha_1+\alpha_2$, 0 and α_2. So we get:

$$X_{I_1} = X^{(-\alpha_1)} + X^{(\alpha_1-\alpha_2)} + X^{(-\alpha_1-\alpha_2)} = X_5.$$

The representation S_2 has unique subrepresentations with dimension vectors 0 and α_2. So we get:

$$X_{S_2} = X^{(2\alpha_1-\alpha_2)} + X^{(-\alpha_2)} = X_6.$$

3.1.3. *Type G_2.* We begin with the valued quiver $Q = (B, D)$ where $B = \begin{pmatrix} 0 & 3 \\ -1 & 0 \end{pmatrix}$ and $D = diag(1, 3)$. Define $K_1 = \mathbb{F} =: k$ and $K_2 = \mathbb{F}_3 =: K$. The Cartan counterpart of B is $\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$.

Lemma 3.3 ([1]). *The quantum cluster algebra $\mathcal{A}_q(Q)$ is of finite type and we have the following formulas for the cluster variables in terms of the initial cluster (X_1, X_2):*

$$\begin{aligned}
X_3 &:= X_{[1]} = X^{(-1,1)} + X^{(-1,0)} \\
X_4 &:= X_{[1,2]} = X^{(-3,2)} + (q+1+q^{-1})X^{(-3,1)} + (q+1+q^{-1})X^{(-3,0)} \\
&\quad + X^{(0,-1)} + X^{(-3,-1)} \\
X_5 &:= X_{[1,2,1]} = X^{(-2,1)} + (q^{1/2}+q^{-1/2})X^{(-2,0)} + X^{(1,-1)} + X^{(-2,-1)} \\
X_6 &:= X_{[1,2,1,2]} = X^{(-3,1)} + (q+1+q^{-1})X^{(-3,0)} + (q+1+q^{-1})X^{(0,-1)} \\
&\quad + (q+1+q^{-1})X^{(-3,-1)} + X^{(3,-2)} + (q^{3/2}+q^{-3/2})X^{(0,-2)} + X^{(-3,-2)} \\
X_7 &:= X_{[1,2,1,2,1]} = X^{(-1,0)} + X^{(2,-1)} + X^{(-1,-1)} \\
X_8 &:= X_{[1,2,1,2,1,2]} = X^{(3,-1)} + X^{(0,-1)} \\
X_9 &:= X_{[1,2,1,2,1,2,1]} = X^{(1,0)} = X_1 \\
X_{10} &:= X_{1,2,1,2,1,2,1,2]} = X^{(0,1)} = X_2.
\end{aligned}$$

The indecomposable valued representations of Q are $S_1 = k \to 0$ (dimension vector α_1), $S_2 = 0 \to K$ (dimension vector $\alpha_2 = s_1s_2s_1s_2s_1(\alpha_2)$), $I_1 = k \xrightarrow{\iota} K$ (dimension vector $\alpha_1+\alpha_2 = s_1s_2s_1s_2(\alpha_1)$) where ι is the inclusion map, $I_2 = k^2 \xrightarrow{\sigma} K$ (dimension vector $2\alpha_1+\alpha_2 = s_1s_2(\alpha_1)$) where σ identifies a 2-dimensional k-subspace of K, $I_3 = k^3 \xrightarrow{\tau} K$ (dimension vector $3\alpha_1+\alpha_2 = s_1(\alpha_2)$) where τ identifies K as a 3-dimensional vector space over k, and $I_4 = k^3 \xrightarrow{\omega} K^2$ (dimension vector $3\alpha_1 + 2\alpha_2 = s_1s_2s_1(\alpha_2)$) where $\omega = (\iota_1, \iota_2, \Delta_\iota)$ with ι_i the inclusion map to the i^{th} factor of K^2 and Δ_ι the diagonal inclusion map.

The representation S_1 has unique subrepresentations with dimension vectors α_1 and 0. So we get:

$$X_{S_1} = X^{(-\alpha_1+\alpha_2)} + X^{(-\alpha_1)} = X_3.$$

The representation I_3 has unique subrepresentations with dimension vectors $3\alpha_1+\alpha_2$, 0 and α_2, and it has $1+q+q^2$ subrepresentations with dimension vectors $2\alpha_1+\alpha_2$ and $\alpha_1+\alpha_2$. So we get:

$$\begin{aligned}
X_{I_3} &= X^{(-3\alpha_1+2\alpha_2)} + X^{(-\alpha_2)} + X^{(-3\alpha_1-\alpha_2)} + (q+1+q^{-1})X^{(-3\alpha_1+\alpha_2)} \\
&\quad + (q+1+q^{-1})X^{(-3\alpha_1)} = X_4.
\end{aligned}$$

The representation I_2 has unique subrepresentations with dimension vectors $2\alpha_1+\alpha_2$, 0 and α_2, and it has $1+q$ subrepresentations with dimension vector $\alpha_1+\alpha_2$. So we get:

$$X_{I_2} = X^{(-2\alpha_1+\alpha_2)} + X^{(\alpha_1-\alpha_2)} + X^{(-2\alpha_1-\alpha_2)} + (q^{1/2}+q^{-1/2})X^{(-2\alpha_1)} = X_5.$$

The representation I_4 has unique subrepresentations with dimension vectors $3\alpha_1 + 2\alpha_2$, 0 and $2\alpha_2$, it has $1+q+q^2$ subrepresentations with dimension vectors $2\alpha_1+2\alpha_2$,

$\alpha_1 + 2\alpha_2$, and $\alpha_1 + \alpha_2$ (choosing a 1-dimensional k-subspace of k^3 forces the 1-dimensional K-subspace of K^2), and it has $1+q^3$ subrepresentations with dimension vector α_2. So we get:

$$\begin{aligned} X_{I_4} &= X^{(-3\alpha_1+\alpha_2)} + X^{(3\alpha_1-2\alpha_2)} + X^{(-3\alpha_1-2\alpha_2)} \\ &+ (q+1+q^{-1})X^{(-3\alpha_1)} + (q+1+q^{-1})X^{(-3\alpha_1-\alpha_2)} \\ &+ (q+1+q^{-1})X^{(-\alpha_2)} + (q^{3/2}+q^{-3/2})X^{(-2\alpha_2)} = X_6. \end{aligned}$$

The representation I_1 has unique subrepresentations with dimension vectors $\alpha_1+\alpha_2$, 0 and α_2 so we get:

$$X_{I_1} = X^{(-\alpha_1)} + X^{(2\alpha_1-\alpha_2)} + X^{(-\alpha_1-\alpha_2)} = X_7.$$

The representation S_2 has unique subrepresentations with dimension vectors 0 and α_2 so we get:

$$X_{S_2} = X^{(3\alpha_1-\alpha_2)} + X^{(-\alpha_2)} = X_8.$$

3.2. **A Rank 4 Example.** In this section we will work in the quantum cluster algebra $\mathcal{A}_q(Q)$ where $Q = (B, D)$ is the acyclic valued quiver with

$$B = \begin{pmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{pmatrix}$$

and $D = diag(1,1,1,1)$. One can easily compute the following cluster variables of $\mathcal{A}_q(Q)$ each living in a *cyclic* cluster:

$$\begin{aligned} X_{[2;2]} &= X^{(0,-1,2,0)} + X^{(2,-1,0,0)} \\ X_{[3;2,3]} &= X^{(0,-2,3,2)} + (q^{1/2}+q^{-1/2})X^{(2,-2,1,2)} + X^{(4,-2,-1,2)} + X^{(4,0,-1,0)} \\ X_{[4;2,3,4]} &= X^{(0,-4,6,3)} + (q^{3/2}+q^{1/2}+q^{-1/2}+q^{-3/2})X^{(2,-4,4,3)} + (q^2+q+2+q^{-1}+q^{-2})X^{(4,-4,2,3)} \\ &+ (q^{3/2}+q^{1/2}+q^{-1/2}+q^{-3/2})X^{(6,-4,0,3)} + X^{(8,-4,-2,3)} + (q^{1/2}+q^{-1/2})X^{(4,-2,2,1)} \\ &+ (q+2+q^{-1})X^{(6,-2,0,1)} + (q^{1/2}+q^{-1/2})X^{(8,-2,-2,1)} + X^{(8,0,0,-1)} + X^{(8,0,-2,-1)} \end{aligned}$$

We verify Conjecture 1.10 for these cluster variables.

Lemma 3.4. *Conjecture 1.10 holds for $X_{[2;2]}$.*

Proof. By Lemma 5.6 we have $X_{[2;2]} = X_{S_2}$. □

Let I_1 be the unique indecomposable representation of Q with dimension vector $\sigma_2(\alpha_3) = 2\alpha_2 + \alpha_3$.

Lemma 3.5. *Conjecture 1.10 holds for $X_{[3;2,3]}$.*

Proof. The following table shows how each term of X_{I_1} arises, in particular we see that $X_{[3;2,3]} = X_{I_1}$:

$\mathbf{e}$	$d_{\mathbf{e}}^{I_1}$	$\lvert Gr_{\mathbf{e}}(I_1)\rvert$	${}^*\mathbf{e} - \mathbf{e}^* - {}^*(2\alpha_2 + \alpha_3)$
$2\alpha_2 + \alpha_3$	0	1	$(0, -2, 3, 2)$
$\alpha_2 + \alpha_3$	1	$\binom{2}{1}_q$	$(2, -2, 1, 2)$
α_3	0	1	$(4, -2, -1, 2)$
0	0	1	$(4, 0, -1, 0)$

□

Let I_2 be the unique indecomposable representation of Q with dimension vector $\sigma_2\sigma_3(\alpha_4) = 4\alpha_2 + 2\alpha_3 + \alpha_4$.

Lemma 3.6. *Conjecture 1.10 holds for* $X_{[4;2,3,4]}$.

Proof. The following table shows how each term of X_{I_2} arises, in particular we see that $X_{[4;2,3,4]} = X_{I_2}$:

$\mathbf{e}$	$d_{\mathbf{e}}^{I_2}$	$\lvert Gr_{\mathbf{e}}(I_2)\rvert$	${}^*\mathbf{e} - \mathbf{e}^* - {}^*(4\alpha_2 + 2\alpha_3 + \alpha_4)$
$4\alpha_2 + 2\alpha_3 + \alpha_4$	0	1	$(0, -4, 6, 3)$
$3\alpha_2 + 2\alpha_3 + \alpha_4$	3	$\binom{4}{3}_q$	$(2, -4, 4, 3)$
$2\alpha_2 + 2\alpha_3 + \alpha_4$	4	$\binom{4}{2}_q$	$(4, -4, 2, 3)$
$\alpha_2 + 2\alpha_3 + \alpha_4$	3	$\binom{4}{1}_q$	$(6, -4, 0, 3)$
$2\alpha_3 + \alpha_4$	0	1	$(8, -4, -2, 3)$
$2\alpha_2 + \alpha_3 + \alpha_4$	1	$\binom{2}{1}_q$	$(4, -2, 2, 1)$
$\alpha_2 + \alpha_3 + \alpha_4$	2	$\binom{2}{1}_q\binom{2}{1}_{q^2}$	$(6, -2, 0, 1)$
$\alpha_3 + \alpha_4$	1	$\binom{2}{1}_q$	$(8, -2, -2, 1)$
α_4	0	1	$(8, 0, -2, -1)$
0	0	1	$(8, 0, 0, -1)$

□

Remark 3.7. The above computations are easily generalized to any linearly ordered rank 4 valued quiver.

3.3. **Type A_n.** In what follows we will work with ordinary quivers. These can be considered as valued quivers by assigning valuation 1 to each vertex.

In [9], Fomin and Zelevinsky show that the cluster algebras of type A_n can be recovered from triangulations of the $(n+3)$-gon: the clusters are in one-to-one correspondence with the triangulations. In [18], Schiffler gives a combinatorial description of the expansion of an arbitrary cluster variable in terms of paths in a triangulation. In this section we show that this combinatorial description carries over to quantum cluster variables.

We begin by recalling some notions from [18]. Let P be an $(n + 3)$-gon, with vertices labeled $v_0, v_1, \ldots, v_{n+2}$. A diagonal $D_{a,b}$ in P is a line segment connecting two non-adjacent vertices a and b. Two diagonals cross if they intersect in the interior of P and a triangulation of P is a maximal set of non-crossing diagonals. Note that each triangulation contains exactly n diagonals, label them $T_1, T_2, \dots, T_n$ (these correspond to cluster variables), and $n+3$ boundary edges, label them $T_{n+1}, \dots, T_{2n+3}$ (these correspond to coefficients). We construct a skew-symmetric $n \times n$ matrix B_T from a triangulation T as follows:

- For each i, $b_{ii} = 0$.
- Suppose diagonals $T_i \neq T_j$ bound the same triangle in T. Then $b_{ij} = 1$ (respectively $b_{ij} = -1$) if the sense of rotation from T_i to T_j is counterclockwise (respectively clockwise).
- If T_i and T_j do not bound the same triangle in T, then $b_{ij} = 0$.

Note that this process can be reversed: starting from a matrix B we can construct a triangulation of P.

Let T be a triangulation of P and let $D_{a,b}$ be a diagonal of P.

Definition 3.8. A T-path ρ from a to b is a sequence

$$\rho = (a_0, a_1, \dots a_{\ell(\rho)} : i_1, i_2, \dots, i_{\ell(\rho)})$$

such that

(T1) $a = a_0, a_1, \dots, a_{\ell(\rho)} = b$ are vertices of P
(T2) $i_k \in \{1, 2, \dots, 2n+3\}$ such that T_{i_k} connects the vertices $a_{i_k - 1}$ and a_{i_k} for each k
(T3) $i_j \neq i_k$ if $j \neq k$
(T4) $\ell(\rho)$ is odd
(T5) T_{i_k} crosses $D_{a,b}$ if k is even
(T6) If $j < k$ and both T_{i_j} and T_{i_k} cross $D_{a,b}$, then the crossing point of T_{i_j} with $D_{a,b}$ is closer to the vertex a than the crossing point of T_{i_k} with $D_{a,b}$.

Definition 3.9. Let $\mathcal{P}_T(a, b)$ denote the set of all T-paths from a to b.

Let Q be the valued quiver $(B, diag(1, \dots, 1))$ and $\mathcal{A}_q(Q)$ be the associated quantum cluster algebra.

Let $\{\alpha_i\}$ be the standard set of generators in $\mathbb{Z}^n$. For a T-path ρ define

$$\overline{\rho} = \sum_{k \text{ odd}} \alpha_{i_k} - \sum_{k \text{ even}} \alpha_{i_k}.$$

We have the following quantum analogue of the main theorem of [18].

Theorem 3.10. *Let a and b be two non-adjacent vertices of P, let $D = D_{a,b}$ be the diagonal of P connecting a and b, and let X_D be the corresponding cluster variable in $\mathcal{A}_q(Q)$. Then*

$$X_D = \sum_{\rho \in \mathcal{P}_T(a,b)} X^{(\bar{\rho})}.$$

Proof. This is an obvious adaptation of the proof of [18, Theorem 1.2]. □

Remark 3.11. Let Q be the quiver associated to the matrix B. By Corollary 1.4 we know that there exists $V_D \in rep\ Q$ so that $X_D = X_{V_D}$. This correspondence between elements of $\mathcal{P}_T(a,b)$ and subrepresentations of V_D can easily be made explicit in the case of a linearly ordered quiver or an alternating quiver of type A_n.

4. Proof of Proposition 1.1

In this section we will again let $\mathbb{F}$ be the finite field with q elements.

Let $\mathcal{A} = \mathcal{A}_q(Q)$ where $Q = (B, D)$ with $B = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$ and $D = diag(2,2)$. This is the Kronecker quiver $\circ \xrightarrow{2} \circ$, with two arrows, and $K_1 = K_2 = \mathbb{F}_2$. $\mathcal{A}$ is a subring of the skew-field

$$\mathbb{Q}(q^{1/2})\langle X_1, X_2 : X_1X_2 = qX_2X_1\rangle$$

generated by the elements X_m for $m \in \mathbb{Z}$ satisfying the recurrence relations

$$X_{m-1}X_{m+1} = qX_m^2 + 1 \quad (m \in \mathbb{Z}) \ . \tag{4.1}$$

We use the results of [21] and Theorem 1.2 to prove closed formulas for the Laurent polynomial expressions of the elements X_m.

The Kronecker quiver has two classes of non-regular indecomposable representations: the preprojective representations P_n with dimension vector $(n, n+1)$ and the postinjective representations I_n with dimension vector $(n+1, n)$ (note that the arrows of our Kronecker quiver are reversed from those in [21] and we are using degree 2 field extensions of $\mathbb{F}_q$). The preprojectives and postinjectives are uniquely determined (up to isomorphism) by their dimension vectors. Define for $n, k \in \mathbb{Z}$, $k \geq 0$, quantum binomial coefficients $\binom{n}{k}_q = \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-r+1}-1)}{(q^r-1)\cdots(q-1)}$ and take $\binom{t}{0}_q = 1$ for any $t \in \mathbb{Z}$. We have the following theorem proved in [21]:

Theorem 4.1. [21, Theorem 4.1, 4.3] *For $n \geq 0$,*

1. $|Gr_{(a,b)}(P_n)| = \binom{n+1-a}{n+1-b}_{q^2}\binom{b-1}{a}_{q^2}$
2. $|Gr_{(a,b)}(I_n)| = \binom{n-a}{n-b}_{q^2}\binom{b+1}{a}_{q^2}$.

Notice that for $a \geq b$ there are no subrepresentations of P_n with dimension vector (a,b) unless $a = 0$ and $b = 0$. For $\mathbf{e} = (0,0)$ we get $|Gr_{\mathbf{e}}(P_n)| = 1$ and ${}^*\mathbf{e} - \mathbf{e}^* - {}^*[P_n] = (n+2, -n-1)$. So $\mathbf{e} = (0,0)$ gives the floating term $X^{(n+2,-n-1)}$ in equation (1.3). All remaining dimension vectors of subrepresentations of P_n are of the form (a,b) with $a < b \leq n+1$. Set $p = a$ and $r = n+1-b$, so we get $r + p = a + n + 1 - b \leq n$.

For $\mathbf{e} = (a, b)$ we get the summand

$$
\begin{aligned}
& q^{-\frac{1}{2}d_{\mathbf{e}}^{P_n}} |Gr_{(a,b)}(P_n)| X^{(n+2-2b,2a-n-1)} \\
&= q^{(a-b)(n+1-b)} \binom{n+1-a}{n+1-b}_{q^2} q^{(a-b+1)a} \binom{b-1}{a}_{q^2} X^{(n+2-2b,2a-n-1)} \\
&= \begin{bmatrix} n+1-a \\ n+1-b \end{bmatrix}_q \begin{bmatrix} b-1 \\ a \end{bmatrix}_q X^{(n+2-2b,2a-n-1)} \\
&= \begin{bmatrix} n+1-p \\ r \end{bmatrix}_q \begin{bmatrix} n-r \\ p \end{bmatrix}_q X^{(2r-n,2p-n-1)}.
\end{aligned}
$$

Now applying Theorem 1.2 we get for $n \in \mathbb{Z}_{\geq 0}$

$$
\begin{aligned}
X_{-n} = X_{P_n} &= \sum_{\mathbf{e}} q^{-\frac{1}{2}d_{\mathbf{e}}^{P_n}} |Gr_{\mathbf{e}}(P_n)| X^{(n+2-2e_2,2e_1-n-1)} \\
&= X^{(n+2,-n-1)} + \sum_{p+r \leq n} \begin{bmatrix} n-r \\ p \end{bmatrix}_q \begin{bmatrix} n+1-p \\ r \end{bmatrix}_q X^{(2r-n,2p-n-1)}.
\end{aligned}
\tag{4.2}
$$

Notice that for $a > b$ there are no subrepresentations of I_n with dimension vector (a, b) unless $a = n + 1$ and $b = n$. For $\mathbf{e} = (n + 1, n)$ we get $|Gr_{\mathbf{e}}(I_n)| = 1$ and ${}^*\mathbf{e} - \mathbf{e}^* - {}^*[I_n] = (-n-1, n+2)$. So $\mathbf{e} = (n+1, n)$ gives the floating term $X^{(-n-1,n+2)}$ in equation (1.4). All remaining dimension vectors of subrepresentations of I_n are of the form (a, b) with $a \leq b \leq n$. Set $r = a$ and $p = n-b$, so we get $r+p = a+n-b \leq n$. For $\mathbf{e} = (a, b)$ we get the summand

$$
\begin{aligned}
& q^{-\frac{1}{2}d_{\mathbf{e}}^{I_n}} |Gr_{(a,b)}(I_n)| X^{(n-1-2b,2a-n)} \\
&= q^{(a-b)(n-b)} \binom{n-a}{n-b}_{q^2} q^{(a-b-1)a} \binom{b+1}{a}_{q^2} X^{(n-1-2b,2a-n)} \\
&= \begin{bmatrix} n-a \\ n-b \end{bmatrix}_q \begin{bmatrix} b+1 \\ a \end{bmatrix}_q X^{(n-1-2b,2a-n)} \\
&= \begin{bmatrix} n-r \\ p \end{bmatrix}_q \begin{bmatrix} n+1-p \\ r \end{bmatrix}_q X^{(2p-n-1,2r-n)}.
\end{aligned}
$$

Again applying Theorem 1.2 we get for $n \in \mathbb{Z}_{\geq 0}$

$$
\begin{aligned}
X_{n+3} = X_{I_n} &= \sum_{\mathbf{e}} q^{-\frac{1}{2}d_{\mathbf{e}}^{I_n}} |Gr_{\mathbf{e}}(I_n)| X^{(n-1-2e_2,2e_1-n)} \\
&= X^{(-n-1,n+2)} + \sum_{p+r \leq n} \begin{bmatrix} n-r \\ p \end{bmatrix}_q \begin{bmatrix} n+1-p \\ r \end{bmatrix}_q X^{(2p-n-1,2r-n)}.
\end{aligned}
\tag{4.3}
$$

It is known for general q that the coefficients of X_{-n} and X_{n+3} in $\mathcal{A}_q(2, 2)$ written in terms of the initial cluster $\{X_1, X_2\}$ are given by polynomials in q. Now equations (4.2) and (4.3) are valid for infinitely many values of q, thus Proposition 1.1 holds for any q.

5. Proof of Theorem 2.5

Let $\mathbb{F}$ be the finite field with q elements. Suppose that vertex i is a source in the valued quiver $Q = (B, D)$. Note that the valued quiver $\mu_i Q$ is obtained from Q by reversing all arrows at vertex i so that i is a sink in $\mu_i Q$ and that the valuations of $\mu_i Q$ equal those of Q. Let $\mathcal{Q}$ denote the Grothendieck group of *rep* Q. Recall that we identify $\mathcal{Q}$ with the root system associated to the Cartan counterpart of B with simple roots $\{\alpha_i\}$ where $\alpha_i = [S_i] \in \mathcal{Q}$. We will abuse notation and also denote by $\mathcal{Q}$ the Grothendieck group of *rep* $\mu_i Q$ with $\alpha_i = [S'_i]$. We also denote by σ_i the simple reflection associated to α_i in the Weyl group of $\mathcal{Q}$.

Let *rep* $Q\langle i\rangle$ denote the full subcategory of *rep* Q of all representations of Q which do not contain S_i as a direct summand. In [7] it is shown that the reflection functors

$$\mathbb{S}_i^- : rep\ Q \leftrightarrow rep\ \mu_i Q : \mathbb{S}_i^+$$

restrict to inverse equivalences of categories

$$\mathbb{S}_i^- : rep\ Q\langle i\rangle \leftrightarrow rep\ \mu_i Q\langle i\rangle : \mathbb{S}_i^+.$$

Since it will be clear from context which to use we will drop the $^\pm$ and simply denote both functors by $\mathbb{S}_i$. See [7] for precise definitions of these functors. We will use the following result proved in [7].

Lemma 5.1. [7, Proposition 2.1] *For $X \in rep\ Q\langle i\rangle$ we have $[\mathbb{S}_i X] = \sigma_i([X])$ and $\mathbb{S}_i^2 X = X$.*

We now prove a recursion for the Grassmannians of *rep* Q, denoted Gr^Q, in terms of the Grassmannians of *rep* $Q\langle i\rangle$, denoted $Gr^{Q\langle i\rangle}$. We will use the following convention for Grassmannians of $V \in rep\ Q\langle i\rangle$:

$$Gr_{\mathbf{e}}^{Q\langle i\rangle}(V) = \{0 \to W \subset V \to V/W \to 0 : [W] = \mathbf{e}; W, V/W \in rep\ Q\langle i\rangle\}.$$

Theorem 5.2. *Let $M \in rep\ \mu_i Q\langle i\rangle$ with $[M] = \mathbf{m}$ and $\mathbf{e} \in \mathcal{Q}$ with $\mathbf{e} \le \mathbf{m}$. Then we have*

$$Gr_{\mathbf{e}}^{\mu_i Q}(M) = \coprod_{c\ge 0} \mathbb{F}^{d_i c(\sigma_i(\mathbf{e})_i + c)} \times Gr_{c\alpha_i}^{Q}((m_i - \sigma_i(\mathbf{e})_i - e_i)S_i) \times Gr_{\sigma_i(\mathbf{e})+c\alpha_i}^{Q}(\mathbb{S}_i M)$$

where $\sigma_i(\mathbf{e})_j = (1-\delta_{ij})e_j + \delta_{ij}(\sum_{\ell=1}^{n} e_\ell [b_{i\ell}]_+ - e_i)$.

Proof. The main content of the proof is contained in the following lemmas.

Lemma 5.3. *For $M \in rep\ \mu_i Q\langle i\rangle$ with $[M] = \mathbf{m}$ and $\mathbf{e} \in \mathcal{Q}$ with $\mathbf{e} \le \mathbf{m}$ we have*

$$Gr_{\mathbf{e}}^{\mu_i Q}(M) = \coprod_{a\ge 0} Gr_{a\alpha_i}^{\mu_i Q}((m_i - e_i + a)S'_i) \times Gr_{\mathbf{e}-a\alpha_i}^{\mu_i Q\langle i\rangle}(M)$$

.

Proof. Consider the map $\zeta : Gr_{\mathbf{e}}^{\mu_i Q}(M) \to \coprod_{a\ge 0} Gr_{\mathbf{e}-a\alpha_i}^{\mu_i Q\langle i\rangle}(M)$ given by $N \oplus aS'_i \mapsto N$. This map is clearly surjective. Suppose $f : N \hookrightarrow M$ is an element of $Gr_{\mathbf{e}-a\alpha_i}^{\mu_i Q\langle i\rangle}(M)$.

The fibers of ζ are given by $\zeta^{-1}(N) = \{(f,g) : N \oplus aS'_i \hookrightarrow M\} = \{g : aS'_i \hookrightarrow M/N\} = Gr^{\mu_i Q}_{a\alpha_i}((m_i - e_i + a)S'_i)$. The result follows. □

Lemma 5.4. *For $M \in rep\ Q\langle i\rangle$ with $[M] = \mathbf{m}$ and $\mathbf{e} \in \mathcal{Q}$ with $\mathbf{e} \leq \mathbf{m}$ we have*

$$Gr^Q_{\mathbf{e}}(M) = \coprod_{d\geq 0} Gr^Q_{d\alpha_i}((e_i + d)S_i) \times Gr^{Q\langle i\rangle}_{\mathbf{e}+d\alpha_i}(M)$$

.

Proof. Let Q^* denote the quiver obtained from Q by reversing all the arrows. Note that vertex i is a sink in Q^*. We will use the same notation for the linear duality functor $?^* = Hom(?, \mathbb{F}) : rep\ Q \to rep\ Q^*$. The following equalities are immediate:

$$\begin{aligned}
Gr^Q_{\mathbf{e}}(M) = Gr^{Q^*}_{\mathbf{m}-\mathbf{e}}(M^*) &= \coprod_{d\geq 0} Gr^{Q^*}_{d\alpha_i}((e_i + d)S^*_i) \times Gr^{Q^*\langle i\rangle}_{\mathbf{m}-\mathbf{e}-d\alpha_i}(M^*) \\
&= \coprod_{d\geq 0} Gr^Q_{e_i\alpha_i}((e_i + d)S_i) \times Gr^{Q\langle i\rangle}_{\mathbf{e}+d\alpha_i}(M) \\
&= \coprod_{d\geq 0} Gr^Q_{d\alpha_i}((e_i + d)S_i) \times Gr^{Q\langle i\rangle}_{\mathbf{e}+d\alpha_i}(M).
\end{aligned}$$

where the second equality follows from Lemma 5.3. □

The following result is well-known.

Lemma 5.5. *Let $\mathbb{F}$ be a field, $V, W \in Vect_{\mathbb{F}}$, and $\ell \in \mathbb{Z}_{>0}$. Then*

$$Gr_\ell(V \oplus W) = \coprod_{a+b=\ell} \mathbb{F}^{a(w-b)} \times Gr_a(V) \times Gr_b(W).$$

Putting the preceding three lemmas together we get our recursion. Let $a = c + d$ and consider the following:

$$\begin{aligned}
&\coprod_{c\geq 0} \mathbb{F}^{d_i c(\sigma_i(\mathbf{e})_i + c)} \times Gr^Q_{c\alpha_i}((m_i - \sigma_i(\mathbf{e})_i - e_i)S_i) \times Gr^Q_{\sigma_i(\mathbf{e})+c\alpha_i}(\mathbb{S}_i M) \\
&= \coprod_{c\geq 0}\coprod_{d\geq 0} \mathbb{F}^{d_i c(\sigma_i(\mathbf{e})_i + c)} \times Gr^Q_{c\alpha_i}((m_i - \sigma_i(\mathbf{e})_i - e_i)S_i) \times Gr^Q_{d\alpha_i}((\sigma_i(\mathbf{e})_i + (c+d)S_i) \\
&\qquad \times Gr^{Q\langle i\rangle}_{\sigma_i(\mathbf{e})+(c+d)\alpha_i}(\mathbb{S}_i M) \\
&= \coprod_{a\geq 0}\coprod_{c\geq 0} \mathbb{F}^{d_i c(\sigma_i(\mathbf{e})_i + c)} \times Gr^Q_{c\alpha_i}((m_i - \sigma_i(\mathbf{e})_i - e_i)S_i) \times Gr^Q_{(a-c)\alpha_i}((\sigma_i(\mathbf{e})_i + aS_i) \\
&\qquad \times Gr^{Q\langle i\rangle}_{\sigma_i(\mathbf{e})+a\alpha_i}(\mathbb{S}_i M) \\
&= \coprod_{a\geq 0} Gr^Q_{a\alpha_i}((m_i - e_i + a)S_i) \times Gr^{Q\langle i\rangle}_{\sigma_i(\mathbf{e})+a\alpha_i}(\mathbb{S}_i M) \\
&= \coprod_{a\geq 0} Gr^{\mu_i Q}_{a\alpha_i}((m_i - e_i + a)S'_i) \times Gr^{\mu_i Q\langle i\rangle}_{\mathbf{e}-a\alpha_i}(M) \\
&= Gr^{\mu_i Q}_{\mathbf{e}}(M).
\end{aligned}$$

To see the second to last equality note that each of $Gr^Q_{a\alpha_i}((m_i - e_i + a)S_i)$ and $Gr^{\mu_i Q}_{a\alpha_i}((m_i - e_i + a)S'_i)$ is just the classical Grassmannian of vector subspaces $Gr_a(\mathbb{F}^{m_i-e_i+a}_{d_i})$. Also note that $Gr^{Q\langle i\rangle}_{\sigma_i(\mathbf{e})+a\alpha_i}(\mathbb{S}_i M) = Gr^{\mu_i Q\langle i\rangle}_{\mathbf{e}-a\alpha_i}(M)$ under the equivalence $\mathbb{S}_i$. □

We now show that the recursion on the Grassmannians just obtained matches the recursion in the quantum cluster algebra obtained by mutating the initial cluster, this will prove Theorem 2.4. To simplify notation we will use $A^Q_{\mathbf{e}}(M) = q^{-\frac{1}{2}\langle \mathbf{e},\mathbf{m}-\mathbf{e}\rangle}|Gr^Q_{\mathbf{e}}(M)|$. We compute the normalized size of the sets in Theorem 5.2 to get:

$$A^{\mu_i Q}_{\mathbf{e}}(M) = \sum_{c\geq 0} \begin{bmatrix} m_i - \sigma_i(\mathbf{e})_i - e_i \\ c \end{bmatrix}_{q^{d_i/2}} A^Q_{\sigma_i(\mathbf{e})+c\alpha_i}(\mathbb{S}_i M).$$

Suppose $N \in rep\ Q\langle i\rangle$. First we expand X_N via the formula (2.3) to get an element of $\mathcal{A}_q(Q)$. Then we mutate the initial cluster in direction i to get an element of the quantum cluster algebra $\mathcal{A}_q(\mu_i Q)$ which turns out to be $X_{\mathbb{S}_i N}$. This result holds regardless of whether or not X_N is a cluster variable.

This will immediately imply that the same property holds when i is a sink in Q and $M \in rep\ Q\langle i\rangle$. Indeed, assume the result holds when i is a source. If we begin with $X_{\mathbb{S}_i M}$ in $\mathcal{A}_q(\mu_i Q)$ and mutate the initial cluster in direction i we will get $X_{\mathbb{S}_i\mathbb{S}_i M} = X_M$ in the quantum cluster algebra $\mathcal{A}_q(Q)$. But the mutation of clusters is involutive so starting with X_M and mutating the initial cluster in direction i gives $X_{\mathbb{S}_i M}$.

We expand X_N in terms of the initial seed $(\{X_1,\dots,X'_i,\dots,X_n\},Q)$:

$$X_N = \sum_{\mathbf{e}} q^{-\frac{1}{2}\langle \mathbf{e},\mathbf{n}-\mathbf{e}\rangle}|Gr^Q_{\mathbf{e}}(N)|X^{({}^*\mathbf{e}-\mathbf{e}^*-{}^*\mathbf{n})} = \sum_{\mathbf{e}} A^Q_{\mathbf{e}}(N)X^{({}^*\mathbf{e}-\mathbf{e}^*-{}^*\mathbf{n})}.$$

We apply equation (2.1) with $\mathbf{c} = {}^*\mathbf{e} - \mathbf{e}^* - {}^*\mathbf{n}$ to get X_N in terms of the seed $(\{X_1,\dots,X_i,\dots,X_n\},\mu_i Q)$.

$$\sum_{\mathbf{e}} A^Q_{\mathbf{e}}(N)X^{({}^*\mathbf{e}-\mathbf{e}^*-{}^*\mathbf{n})} = \sum_{\mathbf{e}}\sum_{r\geq 0} \begin{bmatrix} c_i \\ r \end{bmatrix}_{q^{d_i/2}} A^Q_{\mathbf{e}}(N)X^{\left(\sum\limits_{\ell=1}^{n}((1-\delta_{i\ell})c_\ell+(c_i-r)[-b_{\ell i}]_+ - c_i\delta_{i\ell})\alpha_\ell\right)}$$

Now we substitute $\mathbf{f} = \sigma_i(\mathbf{e})+r\alpha_i = \sum\limits_{\ell=1}^{n}(1-\delta_{i\ell})e_\ell\alpha_\ell+\delta_{i\ell}\left(\sum_{m=1}^n e_m[b_{im}]_+ - e_i + r\right)\alpha_i$ and $\sigma_i(\mathbf{n}) = \sum\limits_{\ell=1}^{n}(1-\delta_{i\ell})n_\ell\alpha_\ell + \delta_{i\ell}\left(\sum_{m=1}^n n_m[b_{im}]_+ - n_i\right)\alpha_i$ then simplify to get:

$$\sum_{\mathbf{f}}\sum_{r\geq 0}\begin{bmatrix}\sigma_i(\mathbf{n})_i-\sigma_i(\mathbf{f})_i-e_i\\ r\end{bmatrix}_{q^{d_i/2}} A^{Q}_{\sigma_i(\mathbf{f})+r\alpha_i}(\mathbb{S}_i\mathbb{S}_iN)\times$$
$$\times X^{\left(\sum\limits_{\ell=1}^{n}(\sum_{m=1}^{n}(f_m[-b'_{\ell m}]_+ + (\sigma_i(\mathbf{n})_m-f_m)[b'_{\ell m}]_+)-\sigma_i(\mathbf{n})_\ell)\alpha_\ell\right)}$$
$$=\sum_{\mathbf{f}} A^{\mathbb{S}_iN}_{\mathbf{f}} X^{({}^*\mathbf{f}-\mathbf{f}^*-{}^*\sigma_i(\mathbf{n}))} = X_{\mathbb{S}_iN}.$$

This completes the proof of Theorem 2.4. We now are ready to prove Theorem 2.5.

Lemma 5.6. *Let Q be a valued quiver. Inside the quantum cluster algebra $\mathcal{A}_{|\mathbb{F}|}(Q)$, we have $X^{Q}_{[k;k]} = X_{S_k}$ where S_k is the simple representation associated to vertex k in Q.*

Proof. First note that S_k has only two subrepresentations 0 and S_k. So we have

$$X_{S_k} = X^{({}^*\mathbf{0}-\mathbf{0}^*-{}^*\alpha_k)} + X^{({}^*\alpha_k-\alpha_k^*-{}^*\alpha_k)} = X^{\left(-\alpha_k+\sum\limits_{\ell=1}^{n}[b_{\ell k}]_+\alpha_\ell\right)} + X^{\left(-\alpha_k+\sum\limits_{\ell=1}^{n}[-b_{\ell k}]_+\alpha_\ell\right)}.$$

But the last expression is just the exchange relation defining $X_{[k;k]}$. □

Suppose the seed $(\mathbf{X}, Q)$, can be transformed into the seed $(\mathbf{X}', Q')$, by a sequence of mutations in directions $k_1, k_2, \ldots, k_{r+1}$ such that the corresponding sequence of vertices is admissible in Q, i.e. $Q' = \mu_{k_{r+1}}\mu_{k_r}\cdots\mu_{k_1}Q$.

We start with the cluster variable $X'_{k_{r+1}}$ in $\mathcal{A}_q(Q')$. This is the cluster variable $X^{\mu_{k_{r+1}}Q'}_{[k_{r+1};k_{r+1}]}$ in $\mathcal{A}_q(\mu_{k_{r+1}}Q')$. By Lemma 5.6 we can write $X_{[k_{r+1};k_{r+1}]} = X_{S_{k_{r+1}}}$ for $S_{k_{r+1}} \in rep\ \mu_{k_{r+1}}Q'$. Now assume inside the quantum cluster algebra $\mathcal{A}_q(\mu_{k_{i+1}}\cdots\mu_{k_{r+1}}Q')$ that we have

$$X_{[k_{r+1};k_{i+1},\ldots,k_{r+1}]} = X_{\mathbb{S}_{k_{i+1}}\cdots\mathbb{S}_{k_r}(S_{k_{r+1}})}$$

for some $i \in [1, r]$, where $\mathbb{S}_{k_{i+1}}\cdots\mathbb{S}_{k_r}(S_{k_{r+1}}) \in rep\ \mu_{k_{i+1}}\cdots\mu_{k_{r+1}}Q'$. Notice that this representation is indecomposable and, since the sequence of vertices was admissible, it does not contain S_{k_i} as a direct summand. Thus mutating the initial cluster in direction k_i gives

$$X_{[k_{r+1};k_i,k_{i+1},\ldots,k_{r+1}]} = X_{\mathbb{S}_{k_i}\mathbb{S}_{k_{i+1}}\cdots\mathbb{S}_{k_r}(S_{k_{r+1}})}$$

in the quantum cluster algebra $\mathcal{A}_q(\mu_{k_i}\mu_{k_{i+1}}\cdots\mu_{k_{r+1}}Q')$. So by induction and the fact that mutations are involutive we have inside the quantum cluster algebra $\mathcal{A}_q(\mu_{k_1}\cdots\mu_{k_{r+1}}Q') = \mathcal{A}_q(Q)$ the equality

$$X_{[k_{r+1};k_1,k_2,\ldots,k_{r+1}]} = X_{\mathbb{S}_{k_1}\cdots\mathbb{S}_{k_r}(S_{k_{r+1}})}$$

where $\mathbb{S}_{k_1}\cdots\mathbb{S}_{k_r}(S_{k_{r+1}}) \in rep\ \mu_{k_1}\cdots\mu_{k_{r+1}}Q' = rep\ Q$. This completes the proof.

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