Alexander L. Gavrilyuk[*], **Ivan Y. Mogilnykh**[†]

CAMERON – LIEBLER LINE CLASSES IN $\mathrm{PG}(n,4)$, $n \geqslant 3$

May 1, 2014

1 Introduction

A *Cameron – Liebler line class* $\mathcal{L}$ with parameter x is a set of lines of projective geometry $PG(3,q)$ such that each line of $\mathcal{L}$ meets exactly $x(q+1)+q^2-1$ lines of $\mathcal{L}$ and each line that is not from $\mathcal{L}$ meets exactly $x(q+1)$ lines of $\mathcal{L}$ (there are several equivalent definitions of Cameron – Liebler line classes, see Section 2). These classes appeared in connection with an attempt by Cameron and Liebler [1] to classify collineation groups of $PG(n,q)$, $n \geqslant 3$, that have equally many orbits on lines and on points.

The following line classes (and their complementary sets) are examples of Cameron – Liebler line classes:

– the set of all lines in $PG(3,q)$,

– the set of all lines in a given plane of $PG(3,q)$,

– the set of all lines through a point,

– for a non-incident point – hyperplane pair (P,π), the set of all lines through P or in π.

Cameron and Liebler conjectured [1] that, apart from these examples, there are no Cameron – Liebler line classes. The counterexamples were constructed by Drudge [2] (in $PG(3,3)$ with $x=5$), by Bruen and Drudge [3] (for odd q, in $PG(3,q)$ with $x=(q^2+1)/2$), by Govaerts and Penttila [4] (in $PG(3,4)$ with $x=7$), and recently by Rodgers [5] (for some odd q, in $PG(3,q)$ with $x=(q^2-1)/2$). A complete classification of Cameron – Liebler line classes in $PG(3,3)$ was obtained by Drudge [2]. Cameron – Liebler line classes in $PG(3,4)$ were studied by Govaerts and Penttila [4]. However, they left two open cases with parameter $x\in\{6,8\}$.

In this paper, we show the non-existence of Cameron – Liebler line classes with parameter $x\in\{6,8\}$ in $PG(3,4)$ and give a new proof of non-existence of those with parameter $x\in\{4,5\}$ in $PG(3,4)$ previously established in [4]. Further, we prove the uniqueness of Cameron – Liebler line class with $x=7$ in $PG(3,4)$ discovered in [4]. Finally, following the approach by Drudge [12], we obtain a complete classification of Cameron – Liebler line classes in $PG(n,4)$, $n\geqslant 3$ (for the precise definition of those in $PG(n,q)$, see Section 5).

[*] e-mail: alexander.gavriliouk@gmail.com
Institute of Mathematics and Mechanics of Russian Academy of Sciences, Ekaterinburg, Russia.
[†] e-mail: ivmog84@gmail.com
Institute of Mathematics of Russian Academy of Sciences, Novosibirsk, Russia.

In our proof we consider a Cameron – Liebler line class as a subset of the vertex set of the Grassmann graph. Recall that the *Grassmann graph* $G_q(n, e)$ is a graph whose vertex set consists of all e-dimensional subspaces of an n-dimensional vector space over a finite field of order q; two e-subspaces are adjacent if and only if their intersection has dimension $e-1$. In case of $e = 2$ the graph $G_q(n+1, 2)$ can be viewed as a graph on lines in $PG(n, q)$ with two distinct lines adjacent if and only if they meet. There is a natural correspondence between Cameron – Liebler line classes in $PG(3, q)$ and completely regular subsets of the vertex set of $G_q(4, 2)$ with strength 0 and covering radius 1 [6].

The paper is organized as follows. In Section 2, we recall some definitions and certain properties of Cameron – Liebler line classes in $PG(n, q)$, rewrite them in terms of the Grassmann graphs and formulate a new necessary condition. In Section 3, we describe properties of putative Cameron – Liebler line classes in $PG(3, 4)$ and then we obtain a contradiction by simple counting arguments. Section 4 is devoted to the uniqueness of Cameron – Liebler line class with $x = 7$ in $PG(3, 4)$. In Section 5 we obtain a classification of Cameron – Liebler line classes in $PG(n, 4)$, $n \geqslant 3$ as a consequence of results from the previous sections.

2 Properties of Cameron – Liebler line classes

Following Penttila [7], for a point P of $PG(3, q)$, we denote the set of all lines through P by $\mathrm{Star}(P)$, and, for a hyperplane π of $PG(3, q)$, the set of all lines in π by $\mathrm{line}(\pi)$. Both types of line sets will be also reffered to as a *clique* in $PG(3, q)$. For a hyperplane π and a point $P \in \pi$, a *pencil* $\mathrm{pen}(P, \pi)$ is the set of all lines in π through P.

For a set of lines $\mathcal{L}$ in $PG(3, q)$, let $\overline{\mathcal{L}}$ denote a complementary set of lines, and $\chi_{\mathcal{L}}$ denote the characteristic function of $\mathcal{L}$.

Lemma 1. ([7]) *The following conditions are equivalent.*

(1) *There exists an integer x such that, for every line l,*

$$|\{m \in \mathcal{L} \setminus \{l\} : \ m \text{ meets } l\}| = (q+1)x + (q^2-1)\chi_{\mathcal{L}}(l).$$

(2) *There exists an integer x such that, for every incident point-plane pair (P, π),*

$$|\mathrm{Star}(P) \cap \mathcal{L}| + |\mathrm{line}(\pi) \cap \mathcal{L}| = x + (q+1)|\mathrm{pen}(P, \pi) \cap \mathcal{L}|.$$

(3) *There exists an integer x such that, for every pair of skew lines l and m,*

$$|\{n \in \mathcal{L} \setminus \{l, m\} : \ n \text{ meets } l \text{ and } n \text{ meets } m\}| = x + q(\chi_{\mathcal{L}}(l) + \chi_{\mathcal{L}}(m)).$$

A set of lines $\mathcal{L}$ is called a *Cameron – Liebler line class* in $PG(3, q)$ if one of the conditions in Lemma 1 is satisfied. We note that the number x in each of these conditions is the same and is called the *parameter* of the Cameron – Liebler line class. In [1] it is shown that $|\mathcal{L}| = x(q^2+q+1)$ holds so that $x \in \{0, 1, 2, \dots, q^2+1\}$, and $\overline{\mathcal{L}}$ is a Cameron – Liebler line class with parameter q^2+1-x.

A graph G defined on the set of lines of $PG(3, q)$ with two distinct lines adjacent if and only if they meet is the well-known Grassmann graph $G_q(4, 2)$. The Grassmann graph $G_q(n, e)$ is distance-transitive and has diameter $\min\{n-e, e\}$. A detailed discussion of the Grassmann graphs is contained in [8]. Here we recall some properties necessary for our partial case (Lemma 2 below, see [8, Chapter 9.3] for its proof).

If X is a subset of the vertex set of $G := G_q(4,2)$ then in order to shorten the notation we write X for the graph induced by G on X. For vertices $v, u \in G$, we define the *neighborhood* $G(v) := \{w \in G|\ w \sim v\}$ of v, the *second neighborhood* $G_2(v) := \{w \in G \setminus \{v\}|\ w \not\sim v\}$, and $G(u,v) := G(u) \cap G(v)$.

For an integer $\alpha \geqslant 1$, the *α-clique extension* of a graph $\overline{H}$ is the graph H obtained from $\overline{H}$ by replacing each vertex $\overline{u} \in \overline{H}$ with a clique U with α vertices, where, for any $\overline{u}, \overline{w} \in \overline{H}$, $u \in U$ and $w \in W$, $\overline{u}$ and $\overline{w}$ are adjacent if and only if u and w are adjacent. By the *$n \times m$-grid*, we mean the Cartesian product of two cliques on n and m vertices.

Lemma 2. *The following holds.*

(1) *For every vertex $v \in G$, the graph $G(v)$ is the q-clique extension of $(q+1)\times(q+1)$-grid (so that G is regular with valency $q(q+1)^2$).*

(2) *For every pair of non-adjacent vertices $u, v \in G$, the graph $G(u,v)$ is the $(q+1) \times (q+1)$-grid.*

It is easily seen that there are exactly two different sets of maximal cliques in G, say Λ_1 and Λ_2, with each maximal clique having the same size $q(q+1)+1$. We define Λ_1 to be the set of all cliques that correspond to line(π), for all hyperplanes π in $PG(3,q)$, and Λ_2 to be the set of all cliques that correspond to Star(P), for all points P in $PG(3,q)$. Further, for every pair of cliques $L, L' \in \Lambda_i$, we have $|L \cap L'| = 1$, while, for every pair of cliques $L \in \Lambda_1$, $L^* \in \Lambda_2$, we have $|L \cap L^*| \in \{0, q+1\}$, and, for every pair of adjacent vertices $u, v \in G$, there is a unique pair of cliques $L \in \Lambda_1$ and $L^* \in \Lambda_2$ such that $u, v \in L \cap L^*$.

Now let $\mathcal{L}$ be a Cameron – Liebler line class in $PG(3,q)$ with parameter x. Consider $\mathcal{L}$ as a subset of vertices of $G_q(4,2)$. Clearly, a partition of the vertex set of G into two parts, $\mathcal{L}$ and $\overline{\mathcal{L}}$, is *equitable* with the following quotient matrix

$$P := \begin{array}{c} \\ \mathcal{L} \\ \overline{\mathcal{L}} \end{array} \begin{array}{c} \begin{array}{cc} \mathcal{L} & \overline{\mathcal{L}} \end{array} \\ \begin{pmatrix} (q+1)x + q^2 - 1 & q(q+1)^2 - (q+1)x - q^2 + 1 \\ (q+1)x & q(q+1)^2 - (q+1)x \end{pmatrix} \end{array},$$

which means that every vertex from a part A has exactly $p_{A,B}$ neighbors in a part B.

Lemma 3. *The following conditions are equivalent.*

(1) *for every vertex $v \in G$,*

$$|G(v) \cap \mathcal{L}| = (q+1)x + (q^2-1)\chi_{\mathcal{L}}(v). \tag{1}$$

(2) *for every pair of cliques $L \in \Lambda_1$, $L^* \in \Lambda_2$ such that $|L \cap L^*| = q+1$,*

$$|L \cap \mathcal{L}| + |L^* \cap \mathcal{L}| = x + (q+1)|L \cap L^* \cap \mathcal{L}|. \tag{2}$$

(3) *for every pair of non-adjacent vertices $u, v \in G$,*

$$|G(u,v) \cap \mathcal{L}| = x + q(\chi_{\mathcal{L}}(u) + \chi_{\mathcal{L}}(v)). \tag{3}$$

(4) *for every pair of adjacent vertices $u, v \in G$,*

$$|G(v) \cap G_2(u) \cap \mathcal{L}| = q(x + q\chi_{\mathcal{L}}(v) - |L \cap L^* \cap \mathcal{L}|), \tag{4}$$

where $L \in \Lambda_1$, $L^* \in \Lambda_2$, and $u, v \in L \cap L^*$.

Proof. The first three statements are equivalent to Lemma 1. Let us show (4). By Eq. (1), for a vertex v, we have $|G(v)\cap\mathcal{L}| = (q+1)x + (q^2-1)\chi_{\mathcal{L}}(v)$, while $G(v)$ contains exactly $|L\cap\mathcal{L}|+|L^*\cap\mathcal{L}|-|L\cap L^*\cap\mathcal{L}|-\chi_{\mathcal{L}}(v)$ vertices from $\mathcal{L}\cap(L\cup L^*\setminus\{v\})\subset G(u)\cup\{u\}$. Therefore,

$$|G(v)\cap G_2(u)\cap\mathcal{L}| = (q+1)x+(q^2-1)\chi_{\mathcal{L}}(v) - (|L\cap\mathcal{L}|+|L^*\cap\mathcal{L}|-|L\cap L^*\cap\mathcal{L}|-\chi_{\mathcal{L}}(v))$$

Taking into account (2), we obtain $|G(v)\cap G_2(u)\cap\mathcal{L}| = q(x+q\chi_{\mathcal{L}}(v)-|L\cap L^*\cap\mathcal{L}|)$.

Now suppose (4) holds. Summing Eq. (4) over all vertices $u\in G(v)$, we obtain

$$\sum_{u\in G(v)}|G(v)\cap G_2(u)\cap\mathcal{L}| = q^2(q+1)^2x+q^3(q+1)^2\chi_{\mathcal{L}}(v)-q^2\sum_{L\in\Lambda_1,L^*\in\Lambda_2:\ v\in L\cap L^*}|L\cap L^*\cap\mathcal{L}|.$$

Note that

$$\sum_{L\in\Lambda_1,L^*\in\Lambda_2:\ v\in L\cap L^*}|L\cap L^*\cap\mathcal{L}| = |G(v)\cap\mathcal{L}|+(q+1)^2\chi_{\mathcal{L}}(v).$$

On the other hand, for every vertex $u\in G(v)$, $|G(v)\cap G_2(u)|=q^3$ holds, hence

$$\sum_{u\in G(v)}|G(v)\cap G_2(u)\cap\mathcal{L}| = q^3|G(v)\cap\mathcal{L}|,$$

which yields that (1) holds. This proves the lemma. ■

Many of the previous results on Cameron–Liebler line classes were obtained by Drudge's approach. This approach relates the lines of a Cameron–Liebler line class that belong to a clique in $PG(3,q)$ to a blocking set in a projective plane $PG(2,q)$. We recall that a *t-fold blocking set* in $PG(2,q)$ is a set of points that intersects every line in at least t points. A 1-fold blocking set is called just a blocking set, and it is called *trivial* if it contains a line. It is also reasonable to recall that every clique in $PG(3,q)$ and its lines correspond to a projective plane and its points respectively, while pencils in the clique correspond to lines in the plane. In this way, for a clique L, some restrictions on $|L\cap\mathcal{L}|$ may be obtained from the study of blocking sets, [4].

Our main results in the next sections rely on the study of a possible distribution of lines from a Cameron – Liebler line class $\mathcal{L}$ in the set $\cup_{P\in u}\cup_{\pi\ni u}\mathrm{pen}(P,\pi)$ for a line u. In terms of the Grassmann graph G, for a vertex u, we consider the graph $G(u)$, which is the q-clique extension of $(q+1)\times(q+1)$-grid by Lemma 2(1). The intersection of every pair of cliques $L\in\Lambda_1$, $L^*\in\Lambda_2$ that contain u is a q-clique in $G(u)$, and it corresponds to $\mathrm{pen}(P,\pi)\setminus\{u\}$ for some point $P\in u$ and hyperplane $\pi\ni u$. Further, define a square matrix $\mathcal{T}(u)$ of size $q+1$,

$$\mathcal{T}(u) := \begin{pmatrix} t_{1,1} & t_{1,2} & \dots & t_{1,q+1}\\ \dots & \dots & \dots & \dots\\ t_{q+1,1} & t_{q+1,2} & \dots & t_{q+1,q+1}\end{pmatrix},$$

whose elements are equal to $|\mathcal{L}\cap\mathrm{pen(P,\pi)}\setminus\{u\}|$, so that, without the loss of generality, the rows (resp. columns) of $\mathcal{T}(u)$ correspond to points on u (resp. hyperplanes containing u). Clearly, we do not distinguish between matrices obtained by permutations of rows or columns as well as transposition.

For a line u, we call the matrix $\mathcal{T}(u)$ the *pattern* with respect to u (the *pattern* for short). Then a line u *has* pattern $\mathcal{T}(u)$.

Lemma 4. *For every vertex $u \in G$, the pattern $\mathcal{T}(u)$ satisfies the following properties:*

(1) $0 \leqslant t_{ij} \leqslant q$ *for all* $i, j \in \{1, \dots, q+1\}$;

(2) $\sum_{i,j=1}^{q+1} t_{ij} = x(q+1) + \chi_{\mathcal{L}}(u)(q^2-1)$;

(3) $\sum_{j=1}^{q+1} t_{kj} + \sum_{i=1}^{q+1} t_{il} = x + (q+1)(t_{kl} + \chi_{\mathcal{L}}(u))$, *for all* $k, l \in \{1, \dots, q+1\}$;

(4) $\sum_{i,j=1}^{q+1} t_{ij}^2 = (x - \chi_{\mathcal{L}}(u))^2 + q(x - \chi_{\mathcal{L}}(u)) + \chi_{\mathcal{L}}(u)q^2(q+1)$.

Proof. Let u be a vertex of G. Let $L_1, L_2, \dots, L_{q+1}$ ($L^1, L^2, \dots, L^{q+1}$, respectively) be the set of all maximal cliques from Λ_1 (from Λ_2, respectively) containing u. Then, by definition, $t_{ij} := |L_i \cap L^j \cap \mathcal{L} \setminus \{u\}|$ (so that $|L_i \cap \mathcal{L}| = \chi_{\mathcal{L}}(u) + \sum_j t_{ij}$ and $|L^j \cap \mathcal{L}| = \chi_{\mathcal{L}}(u) + \sum_i t_{ij}$). Now Statement (1) is clear, and Statements (2) and (3) follow from Statements (1) and (2) of Lemma 3, respectively.

Let us prove (4). Recall that $|\mathcal{L}| = x(q^2+q+1)$. By Lemma 3(3), this yields that there are exactly

$$(q(\chi_{\mathcal{L}}(u)+1) + x)|G_2(u) \cap \mathcal{L}|$$

edges $\{v, w\}$ such that $v \in G(u) \cap \mathcal{L}$, $w \in G_2(u) \cap \mathcal{L}$, where

$$|G_2(u) \cap \mathcal{L}| = x(q^2+q+1) - x(q+1) - \chi_{\mathcal{L}}(u)(q^2-1) - \chi = q^2(x - \chi_{\mathcal{L}}(u)).$$

On the other hand, by Lemma 3(4), the same number must be equal to

$$\sum_{v \in G(u) \cap \mathcal{L}} |G(v) \cap G_2(u) \cap \mathcal{L}| = \sum_{i,j=1}^{q+1} t_{ij} q(x + q - t_{ij} - \chi_{\mathcal{L}}(u)),$$

which proves the lemma after some straightforward calculations. ∎

Statement (4) of Lemma 4 is a new existence condition for Cameron – Liebler line classes $\mathcal{L}$ in $PG(3, q)$, which follow, in fact, from combining Statements (3) and (4) of Lemma 3.

In general, a pattern w.r.t. a line is determined by any pair of its row and column. Hence, all the possible patterns w.r.t. a line may be obtained by enumerating all the row-column pairs and checking the properties (1)–(4) of Lemma 4. Sometimes it is enough to show the non-existence of a putative Cameron – Liebler line class in the sense that the set of possible patterns turns out to be empty, see Section 3.

The following remark may be useful in the study of structure of a Cameron – Liebler line class (although we do not involve it in Section 4, we think that it could be used in an alternative proof of the uniqueness of Cameron – Liebler line class with $x = 7$ in $PG(3, 4)$).

Let u, v be a pair of vertices of G at distance 2. Consider the subgraph $G(u, v)$, and recall that, by Lemma 2, it is the $(q+1) \times (q+1)$-grid. Let $L_1, L_2, \dots, L_{q+1}$ ($L^1, L^2, \dots, L^{q+1}$, respectively) be the set of all maximal cliques from Λ_1 (from Λ_2, respectively) containing u, and $K_1, K_2, \dots, K_{q+1}$ ($K^1, K^2, \dots, K^{q+1}$, respectively) be the set of all maximal cliques from Λ_1 (from Λ_2, respectively) containing v.

Let the cliques $L_1, L_2, \dots, L_{q+1}$, $L^1, L^2, \dots, L^{q+1}$, $K_1, K_2, \dots, K_{q+1}$, $K^1, K^2, \dots, K^{q+1}$ be ordered so that each of the maximal $(q+1)$-cliques of $G(u, v)$ is the intersection of the type $L_i \cap K^i$ or $L^j \cap K_j$, $i, j \in \{1, \dots, q+1\}$.

Set $m_i := |L_i \cap K^i \cap \mathcal{L}|$, $n_j := |L^j \cap K_j \cap \mathcal{L}|$, $i, j \in \{1, \ldots, q+1\}$. Then, by Lemma 3, m_i, n_j are non-negative integers, each of them is at most $q+1$, and the following holds:

$$|L_i \cap \mathcal{L}| + |K^i \cap \mathcal{L}| = x + (q+1)m_i, \ \ |L^j \cap \mathcal{L}| + |K_j \cap \mathcal{L}| = x + (q+1)n_j,$$

$$\sum m_i = \sum n_j = x + q(\chi_{\mathcal{L}}(u) + \chi_{\mathcal{L}}(v)) = |G(u,v) \cap \mathcal{L}|.$$

We note that the set $G(u,v)$ can be naturally associated with a $(q+1) \times (q+1)$-matrix M, whose entry equals 1 or 0 whenever the corresponding vertex of $G(u,v)$ belongs to $\mathcal{L}$ or $\overline{\mathcal{L}}$. We may assume that the ith row of M contains exactly m_i ones, while the jth column of M contains exactly n_j ones.

The question of existence of such a matrix M is quite well-studied and leads us to the so-called *Ryser* classes of $(0,1)$-matrices with given row and column sums, the non-emptiness of which can be easily settled, see [9, Theorem 1.2.8].

3 Non-existence of Cameron – Liebler line classes with $x \in \{4,5,6,8\}$ in $PG(3,4)$

In this section, we show that there are no Cameron – Liebler line classes in $PG(3,4)$ with parameter $x \in \{6,8\}$ or, equivalently, the Grassmann graph $G_4(4,2)$ does not admit equitable partitions with quotient matrices

$$\begin{pmatrix} 45 & 55 \\ 30 & 70 \end{pmatrix}, \text{ and } \begin{pmatrix} 55 & 45 \\ 40 & 60 \end{pmatrix}.$$

The non-existence of Cameron – Liebler line classes with parameter $x \in \{4,5\}$ was shown in [4, Theorem 1.3]. Our arguments can also cover these cases.

Theorem 1. *There are no Cameron – Liebler line classes with parameter $x \in \{4,5,6,8\}$ in $PG(3,4)$.*

Proof. Let $G := G_4(4,2)$ and $\mathcal{L}$ be a Cameron – Liebler line class with parameter 6 in $PG(3,4)$. First of all, we need the following lemma [4, Theorem 1.3(3)].

Lemma 5. *If $\mathcal{L}$ is a Cameron – Liebler line class with parameter 6 in $PG(3,4)$, then every maximal clique of G intersects $\mathcal{L}$ in 3 mod 5 vertices. Moreover, for each $\alpha \in \{3,8,13,18\}$, there exists a maximal clique containing exactly α vertices of $\mathcal{L}$.*

We follow the notation used in the proof of Lemma 4. Let u be a vertex from $\overline{\mathcal{L}}$. By Lemma 5, without loss of generality, we may assume that $|L_1 \cap \mathcal{L}| = \sum_{j=1}^{5} t_{1j} = 3$, and consider the following three possibilities for t_{1j}, $j = 1, \ldots, 5$:

- $t_{11} = 3$, $t_{12} = \cdots = t_{15} = 0$,
- $t_{11} = 2$, $t_{12} = 1$, $t_{13} = t_{14} = t_{15} = 0$,
- $t_{11} = t_{12} = t_{13} = 1$, $t_{14} = t_{15} = 0$,

and the following three possibilities for $|L_k \cap \mathcal{L}| = \sum_{j=1}^{5} t_{kj}$, $k = 1, \ldots, 5$:

- $|L_k \cap \mathcal{L}| = 3$, $k = 1, \ldots, 4$, and $|L_5 \cap \mathcal{L}| = 18$,
- $|L_k \cap \mathcal{L}| = 3$, $k = 1, 2, 3$, and $|L_4 \cap \mathcal{L}| = 8$, $|L_5 \cap \mathcal{L}| = 13$,
- $|L_k \cap \mathcal{L}| = 3$, $k = 1, 2$, and $|L_k \cap \mathcal{L}| = 8$, $k = 3, 4, 5$.

Given the values of t_{1j}, $j = 1, \ldots, 5$, and $|L_k \cap \mathcal{L}|$, $k = 1, \ldots, 5$, Lemma 4(3) allows us to determine the remaining elements of pattern $\mathcal{T}(u) := (t_{ij})_{5\times 5}$. Indeed, the numbers $|L^l \cap \mathcal{L}| = \sum_{i=1}^{5} t_{il}$, $l = 1, \ldots, 5$, are calculated from Lemma 4(3) when $k = 1$:

$$\sum_{j=1}^{5} t_{1j} + \sum_{i=1}^{5} t_{il} = 6 + 5t_{1l},$$

and, further, for every pair of indices $k, l \in \{1, \ldots, 5\}$, the value of t_{kl} is determined by $\sum_{j=1}^{5} t_{kj}$ and $\sum_{i=1}^{5} t_{il}$.

Taking into account Statements (1), (2), and (3) of Lemma 4, we obtain only 6 admissible variants for $\mathcal{T}(u)$. However, two pairs of them are equivalent under the action of automorphism of G that interchanges Λ_1 and Λ_2 (see [8, Theorem 9.3.1]). Therefore, we have the following candidates for $\mathcal{T}(u)$:

$$T_1 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 4 & 4 & 4 & 3 & 3 \end{pmatrix}, T_2 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 3 & 2 & 1 & 1 & 1 \\ 4 & 3 & 2 & 2 & 2 \end{pmatrix}, T_3 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 1 & 1 \\ 3 & 3 & 3 & 2 & 2 \end{pmatrix},$$

$$\text{and } T_4 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 \end{pmatrix}.$$

Now, by Lemma 4(4), we have

$$\sum_{i,j=1}^{5} t_{ij}^2 = x(q + x) = 60.$$

However, the left-hand side of the last equality turns out to be equal to 78 for T_1, 68 for T_2, 58 for T_3, and 48 for T_4, a contradiction. This shows Theorem 1 for $x = 6$.

The cases $x = 4$ and $x = 8$ can be drawn in exactly the same manner (with more candidates for $\mathcal{T}(u)$ so we omit the details).

For $x = 5$, the matrix

$$\begin{pmatrix} 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is the only pattern w.r.t. $u \in \overline{\mathcal{L}}$ admissible by Lemma 4, while the matrix

$$\begin{pmatrix} 4 & 4 & 2 & 2 & 2 \\ 4 & 4 & 2 & 2 & 2 \\ 2 & 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 \end{pmatrix}$$

is the only pattern w.r.t. $v \in \mathcal{L}$.

Further, taking into account $\mathcal{T}(u)$, we see that there exists a maximal clique L of G such that $|L \cap \mathcal{L}| = 10$, which contradicts $\mathcal{T}(v)$. This completes the proof of Theorem 1. ∎

4 Uniqueness of Cameron – Liebler line classes with $x = 7$ in $PG(3,4)$

Throughout the section we use the projective geometry notions. Let us recall a construction of Cameron – Liebler line class with $x = 7$ due to Govaerts and Penttila [4]. We recall that a *hyperoval* in a projective plane of order q is a set of $q+2$ points, no 3 of which are collinear.

Let P be a point of $PG(3,4)$ and π be a plane not containing P. Let O be a hyperoval in π and C be the set of lines incident to the points of O and P. Then C, all 2-secants of C and all lines in π external to O form a Cameron – Liebler line class with parameter $x = 7$.

In this section we show the uniqueness of Cameron – Liebler line class with $x = 7$ in $PG(3,4)$ in the sense that any such line class arises in that way. The proof essentially relies on the analysis of the patterns admissible by Lemma 4.

Theorem 2. *A Cameron – Liebler line class in $PG(3,4)$ with $x = 7$ is unique.*

The result is proven in several lemmas. In the remainder of this section, let $\mathcal{L}$ be a Cameron – Liebler line class with $x = 7$ in $PG(3,4)$.

Lemma 6. *A line from $\mathcal{L}$ has one of the following patterns:*

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \end{pmatrix}, \qquad (\mathcal{L}.1)$$

$$\begin{pmatrix} 4 & 4 & 2 & 3 & 2 \\ 4 & 4 & 2 & 3 & 2 \\ 3 & 3 & 1 & 2 & 1 \\ 2 & 2 & 0 & 1 & 0 \\ 2 & 2 & 0 & 1 & 0 \end{pmatrix}, \qquad (\mathcal{L}.2)$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 & 4 \end{pmatrix}, \qquad (\mathcal{L}.3)$$

while a line from $\overline{\mathcal{L}}$ has one of the following patterns:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 4 & 3 & 3 & 3 & 3 \\ 2 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \end{pmatrix}, \qquad (\overline{\mathcal{L}}.1)$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 3 & 2 & 2 & 2 & 2 \\ 3 & 2 & 2 & 2 & 2 \\ 3 & 2 & 2 & 2 & 2 \end{pmatrix}. \qquad (\overline{\mathcal{L}}.2)$$

Proof. The exhaustive enumeration of 5×5-matrices admissible by Lemma 4. ■

Lemma 7. *Let L be a clique, such that $x < |C \cap \mathcal{L}| \leqslant x + q$. Then $C \cap \mathcal{L}$ forms a blocking set in the plane, corresponding to L. If there exist no Cameron–Liebler line classes with parameter $x - 1$, then the blocking set is nontrivial.*

Proof. Lemma 4.1 (1) [4]. ■

Lemma 8. *A line from $\mathcal{L}$ cannot have pattern ($\mathcal{L}$.3).*

Proof. On the contrary, for a line that has pattern ($\mathcal{L}$.3), we may relate any column of ($\mathcal{L}$.3) to a projective plane with 11 points corresponding to lines from $\mathcal{L}$. By Lemma 7 and Theorem 1, we see that these points form a non-trivial blocking set. However, it is easily seen from the last row of the pattern ($\mathcal{L}$.3) that the blocking set contains a line, a contradiction. ■

Lemma 9. *There exists a line that has pattern ($\mathcal{L}$.1).*

Proof. It follows from the first row of patterns ($\overline{\mathcal{L}}$.1) and ($\overline{\mathcal{L}}$.2) that there exists a clique in $PG(3,4)$ containing a line v from $\mathcal{L}$ and 20 lines from $\overline{\mathcal{L}}$. Clearly, the line v cannot have pattern ($\mathcal{L}$.2). The lemma is proved. ■

Lemma 10. *There exist six lines $l_1, \ldots, l_6$ from $\mathcal{L}$ such that:*

(1) *there is a point P such that* $\text{Star}(P) \cap \mathcal{L} = \{l_1, \ldots, l_6\}$.

(2) *each of the lines $l_1, \ldots, l_6$ has pattern ($\mathcal{L}$.1).*

(3) *any plane contains either* 0 *or* 2 *lines from $l_1, \ldots, l_6$.*

Proof. Let a line l_1 have pattern ($\mathcal{L}$.1). The pattern ($\mathcal{L}$.1) has the all-ones row. Therefore, up to the duality in $PG(3,q)$, there exists a point P such that $\text{Star}(P) \cap \mathcal{L}$ consists of six lines $l_1, \ldots, l_6$, and any plane containing l_1 has exactly one line among $l_2, \ldots, l_6$. This proves (1).

Suppose that the pattern w.r.t. the line l_2 is ($\mathcal{L}$.2). Then one of the 3rd or 5th columns or of the last two rows of ($\mathcal{L}$.2) corresponds to a clique in $PG(3,4)$ that contains the lines $l_1, \ldots, l_6$. This yields, without loss of generality, that there exist a plane containing l_2, l_3, l_4, a plane containing l_2, l_5, l_6, and a plane containing l_2, l_1. Let us consider the line l_3. The pattern w.r.t. l_3 is ($\mathcal{L}$.2), since there is a pencil containing three lines of $\mathcal{L}$ (namely, l_2, l_3, l_4). Therefore there exists one more plane, containing l_3 and a pair of lines chosen from $\{l_1, l_5, l_6\}$, a contradiction. This gives (2).

Finally, Statement (3) follows from (1) and (2). The lemma is proved. ■

Note that ($\mathcal{L}$.1) has the all-zero row. This means that a line with pattern ($\mathcal{L}$.1) has a unique point incident to the only line of $\mathcal{L}$. We call such a point *poor*.

Lemma 11. *The following holds.*

(1) *There exists a plane π that contains all the poor points of lines $l_1, \dots, l_6$.*

(2) *The poor points of lines $l_1, \dots, l_6$ form a hyperoval O in π.*

(3) *The 2-secants of the set of poor points are the only lines from $\overline{\mathcal{L}} \cap \mathrm{line}(\pi)$.*

Proof. Let π be a plane containing three poor points of the lines l_1, l_2, l_3, and a point R of l_6. Suppose R is not poor. Since $R \neq P$ (recall that $\mathrm{Star}(P) \cap \mathcal{L} = \{l_1, \dots, l_6\}$), the point R corresponds to the last three rows of ($\mathcal{L}$.1), which is the pattern w.r.t. l_6. Hence, $|\mathrm{Star}(R) \cap \mathcal{L}| = 16$.

Now any line from $\mathrm{Star}(R)$ has pattern ($\mathcal{L}$.1), ($\mathcal{L}$.2), or ($\overline{\mathcal{L}}$.1) (as only these patterns admit a clique in $PG(3,4)$ with 16 lines from $\mathcal{L}$). Further, it follows from those patterns that any pencil in a clique in $PG(3,4)$ with 16 lines from $\mathcal{L}$ contains at least 3 lines from $\mathcal{L}$. In other words, a pencil $\mathrm{pen}(R, \pi')$ contains at least 3 lines from $\mathcal{L}$ for any plane $\pi' \ni R$.

Let us show that there exists a line $l' \in \mathrm{pen}(R, \pi)$ that is incident to at least two poor points of l_1, l_2, l_3. On the contrary, suppose that each of the lines in $\mathrm{pen}(R, \pi)$ is incident to at most one poor point of l_1, l_2, l_3. Then R is incident to at least three lines from $\overline{\mathcal{L}}$ (these are the lines, each of them is incident to a poor point of l_1, l_2, l_3) and to at least three lines of $\mathcal{L}$ as $|\mathrm{pen}(R, \pi) \cap \mathcal{L}| \geqslant 3$, a contradiction. Therefore, the line l' exists.

However, clearly, l' has pattern ($\overline{\mathcal{L}}$.2), which is impossible, and Statement (1) follows.

Statement (2) follows directly from Lemma 10(3).

Any 2-secant of the set O of poor points of $l_1, \dots, l_6$ has pattern ($\overline{\mathcal{L}}$.2), and the plane π corresponds to one of the last four columns of ($\overline{\mathcal{L}}$.2). Hence, π contains exactly 6 lines from $\mathcal{L}$ and $15 = \binom{6}{2}$ lines from $\overline{\mathcal{L}}$, each of them is a 2-secant of O. The lemma is proved. ■

Lemma 12. *Any 2-secant of lines $l_1, \dots, l_6$ external to poor points belongs to $\mathcal{L}$.*

Proof. By Lemma 1, a line of $\mathcal{L}$ meets exactly 50 lines of $\mathcal{L}$. By Lemma 10(3), a plane $\pi' \ni l_1$ contains exactly one more line of $l_2, \dots, l_6$, for example, l_2. Now the line l_1 meets 10 lines of $\overline{\mathcal{L}} \cap \mathrm{line}(\pi')$: the 2-secant of poor points of l_1, l_2, the six 1-secants of poor points of l_1, l_2, and the three lines from $\mathrm{Star}(P) \cap \mathrm{line}(\pi') \setminus \{l_1, l_2\}$. Therefore, the 9 remaining lines of π' (i.e., 2-secants of l_1 and l_2 external to their poor points) belong to $\overline{\mathcal{L}}$. The lemma is proved. ■

Now, by Lemmas 6–12, the Cameron – Liebler line class $\mathcal{L}$ consists of the six lines $l_1, \dots, l_6$ with common point P, the six lines from π external to the poor points of $l_1, \dots, l_6$ that form a hyperoval O in π, and $9 \times \binom{6}{2} = 135$ lines that are 2-secants of pairs l_i, l_j, $i, j \in \{1, \dots, 6\}$. Theorem 2 is proved.

5 Cameron – Liebler line classes in $PG(n, 4)$

The results of previous sections complete the classification of Cameron – Liebler line classes in $PG(3,4)$. The notion of Cameron – Liebler line class in $PG(3,q)$ can be naturally generalized to that in $PG(n,q)$, see [12]. For a subspace X of $PG(n,q)$, we denote the set of all lines in X by $\mathrm{line}(X)$.

Lemma 13. ([12, *Theorem* 3.2]) *Let $\mathcal{L}$ be a non-empty set of lines in $PG(n,q)$ with characteristic function $\chi_{\mathcal{L}}$. Then the following are equivalent.*

(1) *There exists x such that, for any flag (P,X) with X an i-dimensional subspace of $PG(n,q)$, we have*

$$|\mathrm{Star}(P)\cap\mathcal{L}| + \frac{\theta_{n-2}}{\theta_{i-1}\theta_{i-2}}|\mathrm{line}(X)\cap\mathcal{L}| = x + \frac{\theta_{n-2}}{\theta_{i-2}}|\mathrm{pen}(P,X)\cap\mathcal{L}|,$$

where $\theta_d := q^d + \ldots + 1$.

(2) *There exists x such that, for every line l,*

$$|\{m \in \mathcal{L}\setminus\{l\} : \ m \text{ meets } l\}| = (q+1)x + (q^{n-1}+\ldots+q^2-1)\chi_{\mathcal{L}}(l).$$

In addition, if n is odd:

(3) *There exists x such that $|\mathcal{S}\cap\mathcal{L}| = x$ for any line-spread $\mathcal{S}$.*

A set $\mathcal{L}$ of lines of $PG(n,q)$ satisfying the above conditions is called a *Cameron – Liebler line class* in $PG(n,q)$. In case of $n = 3$ this definition coincides with that from Section 2. The following are examples of Cameron – Liebler line classes:

– the set of all lines in $PG(n,q)$,

– the set of all lines contained in a hyperplane of $PG(n,q)$,

– the set of all lines through a point,

– for a non-incident point – hyperplane pair (P,H), the set of all lines through P or in H.

Note that the complement to a Cameron – Liebler line class with parameter x is a Cameron – Liebler line class with parameter $\frac{q^n+\ldots+1}{q+1} - x$.

Drudge [12] formulated the following generalized conjecture:

The only Cameron – Liebler line classes in $PG(n,q)$ are those listed above and their complementary sets.

In his Ph.D. Thesis [12], Drudge proved this conjecture for the case $q = 3$. His approach essentially relies on the classification of Cameron – Liebler line classes in $PG(3,3)$, see [2], and, probably, can be applied to an arbitrary q once a classification of those in $PG(3,q)$ has been obtained.

In this section, we prove the conjecture for $q = 4$. For the convenience of the reader, we give all the lemmas due to the Drudge approach (for the proofs we refer to his Ph.D. Thesis). In what follows, let $\mathcal{L}$ be a Cameron – Liebler line class in $PG(n,q)$, $n \geqslant 4$, X be a three-dimensional subspace of $PG(n,q)$.

Lemma 14. ([12, *Theorem* 6.1]) *The line class* $\mathrm{line}(X)\cap\mathcal{L}$ *is a Cameron – Liebler line class in X.*

The intersection of $\mathcal{L}$ with X can, in some cases, determine $\mathcal{L}$.

Lemma 15. ([12, *Lemma* 6.1]) *If* $\mathrm{line}(X)\cap\mathcal{L}$ *consists of all the lines of X on some point $P \in X$, i.e.,* $\mathrm{line}(X)\cap\mathcal{L} = \mathrm{Star}(P)\cap\mathrm{line}(X)$, *then* $\mathcal{L} = \mathrm{Star}(P)$.

The lack of point – plane duality in $PG(n,q)$, $n > 3$, does not allow to state immediately the similar lemma when $\mathrm{line}(X)\cap\mathcal{L} = \mathrm{line}(\pi)$ for some plane π of X.

Lemma 16. ([12, *Lemma* 6.3]) *If* $\mathrm{line}(X)\cap\mathcal{L}$ *is a Cameron – Liebler line class with parameter* $x = 2$ *(in* $PG(3,q)$*) then* $\mathcal{L}$ *consists of all lines on a point plus all lines in a hyperplane, for a non-icident point – hyperplane pair in* $PG(n,q)$*.*

A key step in the Drudge approach is a proof that a counterexample to the Cameron – Liebler conjecture on line classes in $PG(3,q)$ cannot occur as the intersection of a three-dimensional subspace of $PG(n,q)$ with a Cameron – Liebler line class of $PG(n,q)$.

Lemma 17. *For* $q = 4$*, the set* $\mathrm{line}(X)\cap\mathcal{L}$ *cannot be a Cameron – Liebler line class in* $PG(3,4)$ *with parameter* $x = 7$*.*

Proof. By taking complements if necessary, we may assume that the set $\mathrm{line}(X)\cap\mathcal{L}$ is a Cameron – Liebler line class in $PG(3,4)$ with parameter $x = 7$, which is unique by Theorem 2. It follows from Lemma 6 that there exists an incident point–plane pair (P,π) in X such that $|\mathrm{line}(\pi)\cap\mathcal{L}| = 11$, $|\mathrm{pen}(P,\pi)\cap\mathcal{L}| = 1$ and $|\mathrm{pen}(P,X)\cap\mathcal{L}| = 1$ (we called such point *poor* in the previous section).

Let us show that $\mathrm{Star}(P)\cap\mathcal{L}$ consists of the only line, which belongs to π. Clearly, π and a line through P are contained in some three-dimensional subspace Y of $PG(n,4)$. By Lemma 14, we see that $\mathrm{line}(Y)\cap\mathcal{L}$ is a Cameron – Liebler line class in Y, which intersects the plane $\pi\subset Y$ in 11 points. However, the Cameron – Liebler line class in $PG(3,4)$ with parameter $x = 7$ is the only line class that intersects a plane in 11 lines, see Lemma 6. Therefore $\mathrm{line}(Y)\cap\mathcal{L}$ is the Cameron – Liebler line class with parameter $x = 7$, and, hence, $\mathrm{pen}(P,Y)\cap\mathcal{L} = \mathrm{pen}(P,X)\cap\mathcal{L}$. This yields that $\mathrm{Star}(P)\cap\mathcal{L} = \mathrm{pen}(P,X)\cap\mathcal{L}$, and every line from $\mathrm{Star}(P)\setminus\pi$ does not belong to $\mathcal{L}$.

Further, since $n \geqslant 4$, there exists a three-dimensional subspace Y on P such that all lines of Y on P are not in $\mathcal{L}$ (note that $\pi\not\subset Y$ in this case). By Lemma 14 and Theorem 2, we see that a line class $\mathrm{line}(Y)\setminus\mathcal{L}$, which contains at least all the lines of Y on P, is a Cameron – Liebler line class in Y and its parameter $x\neq 7$. By Theorem 1, this implies that $\mathrm{line}(Y)\setminus\mathcal{L}$ can only be one of the following line classes: $\mathrm{pen}(P,Y)$, $\mathrm{line}(Y)$, $\mathrm{pen}(P,Y)\cup\mathrm{line}(\pi')$ (where $P\notin\pi'$), or $\mathrm{line}(Y)\setminus\mathrm{line}(\pi')$ for some plane $\pi'\neq\pi$. Therefore $|\mathrm{line}(Y)\cap\mathcal{L}|$ equals $(17-1)\cdot 21$, 0, $(17-2)\cdot 21$, or $(17-1)\cdot 21$, respectively, whereas $|\mathrm{pen}(P,Y)\cap\mathcal{L}|$ equals 0.

However, recall that $|\mathrm{Star}(P)\cap\mathcal{L}| = 1$, $|\mathrm{line}(X)\cap\mathcal{L}| = 7\cdot(4^2+4+1) = 147$, and $|\mathrm{pen}(P,X)\cap\mathcal{L}| = 1$. Now applying Lemma 13 (1) to the subspaces X and Y gives

$$|\mathrm{Star}(P)\cap\mathcal{L}| + \frac{\theta_{n-2}}{\theta_2\theta_1}|\mathrm{line}(X)\cap\mathcal{L}| = x + \frac{\theta_{n-2}}{\theta_1}|\mathrm{pen}(P,X)\cap\mathcal{L}|,$$

$$|\mathrm{Star}(P)\cap\mathcal{L}| + \frac{\theta_{n-2}}{\theta_2\theta_1}|\mathrm{line}(Y)\cap\mathcal{L}| = x + \frac{\theta_{n-2}}{\theta_1}|\mathrm{pen}(P,Y)\cap\mathcal{L}|,$$

which is impossible. The lemma is proved. ∎

Theorem 3. *For* $q = 4$*, if a Cameron – Liebler line class* $\mathcal{L}$ *is not of the type* $\mathrm{Star}(P)$ *for a point* P *or* $\mathrm{Star}(P)\cup\mathrm{line}(H)$ *for a non-incident point – plane pair* (P,H)*, or the complement of a set of one of these two types then* $\mathcal{L}$ *is of the type* $\mathrm{line}(H)$*, for a hyperplane* H*, or its complement. In other words, a Cameron – Liebler line class in* $PG(n,4)$*,* $n\geqslant 4$*, is one of those listed above or their complementary sets.*

Proof. It follows from Theorem 1 and Lemmas 15, 16, 17 that a Cameron – Liebler line class $\mathcal{L}$ satisfying the hypothesis of the theorem intersects X in one of the following two ways:

(1) All lines of X, or no lines of X, or

(2) All lines in a plane of X, or all lines not in a plane of X.

Further, Lemma 6.5 and Theorem 6.3 from [12] show that such a line class is of the type line(H), for a hyperplane H, or its complement (in fact, these lemma and theorem are proved under a slightly different assumption on Cameron – Liebler line classes in $PG(3, q)$, but it does not matter in our case due to Lemma 17). The theorem is proved. ∎

We close this paper with several remarks.

Suppose we are given the set $G(v) \cap \mathcal{L}$ for some vertex v. Then we are able to "reconstruct" the whole set $\mathcal{L}$ by using, for instance, Lemma 3(3). Actually, the same idea was exploited in the study of completely regular codes in the Johnson graphs [10], and it led us to the study of those in the Grassmann graphs.

Recently Bamberg [11] announced the non-existence of Cameron – Liebler line classes with parameter $x \in \{6, 8\}$ in $PG(3, 4)$ as a negative result of a computer search.

We are grateful to Frédéric Vanhove for the useful references.

References

[1] P.J. Cameron, R.A. Liebler, "Tactical decompositions and orbits of projective groups", *Linear Algebra Applied* 46, 1982. P. 91–102.

[2] K. Drudge, "On a conjecture of Cameron and Liebler", *European Journal of Combinatorics* 20(4), 1999. P. 263–269.

[3] A.A. Bruen, K. Drudge, "The construction of Cameron – Liebler line classes in $PG(3, q)$", *Finite Fields Appl.* 5, 1999. P. 35–45.

[4] P. Govaerts, T. Penttila, "Cameron–Liebler line classes in $PG(3, 4)$", *Bull. Belg. Math. Soc.* 12, 2005. P. 793–804.

[5] M. Rodgers, "Cameron – Liebler line classes", *Designs, Codes and Cryptography*, 2011, DOI: 10.1007/s10623-011-9581-2

[6] F. Vanhove, "Incidence geometry from an algebraic graph theory point of view", University of Ghent, PhD Thesis, 2011.

[7] T. Penttila, "Cameron – Liebler line classes in $PG(3, q)$", *Geometriae Dedicata* 37, 1991. P. 245–252.

[8] A.E. Brouwer, A.M. Cohen, A. Neumaier, "Distance regular graphs", Berlin, Springer-Verl. 1989.

[9] V.N. Sachkov, V.E. Tarakanov, "Combinatorics of nonnegative matrices", AMS, 2002.

[10] A. Gavrilyuk, S. Goryainov, "On perfect 2-colorings of Johnson Graphs $J(v, 3)$", *Journal of Combinatorial Designs*, 2012.

[11] J. Bamberg, "There is no Cameron – Liebler line class of $PG(3, 4)$ with parameter 6", URL: http://symomega.wordpress.com

[12] K. Drudge, "Extremal sets in projective and polar spaces", University of Western Ontario, Ph.D. Thesis, 1998.