

Curvature Couplings in $\mathcal{N} = (2, 2)$ Nonlinear Sigma Models on S^2

Bei Jia, Eric Sharpe

Department of Physics, Virginia Tech, Blacksburg, VA 24061 USA

beijia@vt.edu, ersharpe@vt.edu

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Abstract

Following recent work on GLSM localization, we work out curvature couplings for rigidly supersymmetric nonlinear sigma models with superpotential for general target spaces, describing both ordinary and twisted chiral superfields on round two-sphere worldsheets. We briefly discuss why, unlike four-dimensional theories, there are no constraints on Kahler forms in these theories. We also briefly discuss general issues in topological twists of such theories.

1 Introduction

In recent years, curvature couplings in rigidly supersymmetric nonlinear sigma models on nontrivial spacetime manifolds of various dimensions have been discussed by several groups, see [1, 2, 3, 4, 5, 7, 6, 8, 9, 10, 11, 12, 13, 14] and references contained therein. Often, supersymmetry on the nontrivial spacetime manifold will constrain the target space in some fashion, by *e.g.* requiring the Kähler form on the target space to be cohomologically trivial. Furthermore, supersymmetric localization techniques have been applied to theories of this form to obtain quantum mechanically exact results such as the partition function [15, 16, 17, 18, 19, 20, 21].

Recently these methods were applied to two-dimensional gauged linear sigma models (GLSMs) to derive exact expressions for partition functions [20, 21], which has quickly led to some interesting new computational methods and results for Gromov-Witten invariants [22, 23, 24] and the Seiberg-Witten Kähler potential [25], as well as other insights into older results [26]. As part of that work, the papers [20, 21] worked out curvature couplings in two-dimensional linear sigma models whose target spaces are vector spaces.

In this paper, we return to [20, 21] to re-examine general rigidly supersymmetric nonlinear sigma models with potential and work out curvature couplings, for more general target spaces (and with $U(1)_R$ actions described by more general holomorphic Killing vectors), for both ordinary and twisted chiral supermultiplets, on constant-curvature (round) two-sphere worldsheets, so as to give some insight into the rather complicated linear actions of [20, 21].

We begin in section 2 by working out general nonlinear sigma models with potential for ordinary chiral supermultiplets on round two-spheres. To add a superpotential in a theory of ordinary chiral multiplets, one must extend the flat-space $U(1)_R$ symmetry by a holomorphic Killing vector, which generates *e.g.* a curvature-dependent potential term in the action. We also discuss why two-dimensional theories of this sort do not have constraints on the Kahler form on the target space, unlike typical behavior in four dimensional theories.

In section 3, we perform analogous analyses for twisted chiral multiplets. Here, the curvature couplings have a different form than for ordinary chiral multiplets. For example, although one can extend the $U(1)_R$ of the twisted chiral theory by a holomorphic Killing vector field, that same field can be re-absorbed into the auxiliary field of the multiplet, and so its presence is optional. Moreover, these same couplings naively break a duality between the flat-worldsheet chiral and twisted chiral theories.

We conclude in section 4 with a discussion of topological twists in the presence of such curvature couplings. An appendix further discusses the relationship between ordinary and twisted chiral multiplets, and their topological twists.

2 Ordinary chiral supermultiplets

In this section, we will discuss curvature couplings for ordinary chiral multiplets on an S^2 with a constant curvature metric, the ‘round’ S^2 .

The rigid $\mathcal{N} = (2, 2)$ supersymmetry algebra on S^2 with Euclidean signature is [20, 21]

$$OSp(2|2) \cong SU(2|1),$$

whose bosonic subalgebra is $SU(2) \times U(1)_R$. The $SU(2)$ factor represents the isometries of S^2 , while the $U(1)_R$ factor is the vector-like R-symmetry, which is now contained in the supersymmetry algebra rather than being an outer automorphism of it.

Our spinor notation will follow [21][p. 47], [27]. Spinors are multiplied as

$$\lambda\psi = \lambda^\alpha \varepsilon_{\alpha\beta} \psi^\beta = \psi\lambda,$$

where $\varepsilon_{21} = -\varepsilon_{12} = 1$, $\varepsilon_{11} = -\varepsilon_{22} = 0$. The two-dimensional γ_m matrices are given by the Pauli matrices in local frame coordinates: $\gamma_m = \sigma_m$, where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

with

$$\gamma_3 = -i\sigma_1\sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

With our notation, the explicit form of the supersymmetry algebra $SU(2|1)$ is [21]:

$$\begin{aligned} \{Q_\alpha, \bar{Q}_\beta\} &= \gamma_{\alpha\beta}^m J_m - \frac{1}{2} \varepsilon_{\alpha\beta} R & [R, Q_\alpha] &= Q_\alpha & [R, \bar{Q}_\alpha] &= -\bar{Q}_\alpha \\ [J_m, J_n] &= i\varepsilon_{mnl} J_l & [J_m, Q_\alpha] &= -\frac{1}{2} \gamma_{m\alpha}^\beta Q_\beta & [J_m, \bar{Q}_\alpha] &= -\frac{1}{2} \gamma_{m\alpha}^\beta \bar{Q}_\beta \end{aligned} \quad (1)$$

where Q_α and \bar{Q}_α are the supersymmetry generators, J_m are the generators of the $SU(2)$ isometry of S^2 , and R is the generator of the $U(1)_R$ symmetry.

The space of Killing spinors on S^2 is four-dimensional [28]; a useful basis of this space consists of two Killing spinors called positive Killing spinors and two Killing spinors called negative Killing spinors [20]. Let’s denote the positive Killing spinors as $\zeta, \bar{\zeta}$, which are independent from each other and satisfy the same Killing spinor equation

$$\begin{aligned} \nabla_m \zeta - \frac{i}{2r} \gamma_m \zeta &= 0, \\ \nabla_m \bar{\zeta} - \frac{i}{2r} \gamma_m \bar{\zeta} &= 0. \end{aligned} \quad (2)$$

Some useful spinor identities include:

$$\begin{aligned} \lambda\psi &= +\psi\lambda, \quad \lambda\gamma_m\psi = \lambda^\alpha(\gamma_m)_\alpha^\beta\psi_\beta = -\psi\gamma_m\lambda, \quad \gamma_m\gamma_n = g_{mn} + i\varepsilon_{mn}\gamma_3, \quad \gamma^m\gamma^n\gamma_m = 0, \\ (\psi_1\psi_2)\psi_3 + (\psi_2\psi_3)\psi_1 + (\psi_3\psi_1)\psi_2 &= 0. \end{aligned}$$

An Euclidean $\mathcal{N} = (2, 2)$ chiral multiplet in two dimensions contains components

$$(\phi^i, \bar{\phi}^{\bar{i}}, \psi^i, \bar{\psi}^{\bar{i}}, F^i, \bar{F}^{\bar{i}}).$$

The chiral fields $\phi^i, \bar{\phi}^{\bar{i}}$ parametrize the target space M , which is a Kähler manifold. As observed in [20, 21], the $U(1)_R$ charges of the chiral fields ϕ^i enter the definition of the Lagrangian of the linear sigma models on S^2 . To construct nonlinear sigma models on a worldsheet S^2 , one needs to use the holomorphic Killing vector $X = X^i\partial_i$ corresponding to the $U(1)_R$ symmetry, which should be interpreted as an isometry on the target space M of the nonlinear sigma model.

The general Lagrangian governing $\mathcal{N} = (2, 2)$ nonlinear sigma models on S^2 is

$$\begin{aligned} \mathcal{L} &= g_{i\bar{j}}\partial_m\phi^i\partial^m\bar{\phi}^{\bar{j}} - ig_{i\bar{j}}\bar{\psi}^{\bar{j}}\gamma^m\nabla_m\psi^i + g_{i\bar{j}}F^i\bar{F}^{\bar{j}} - F^i\left(\frac{1}{2}g_{i\bar{j},\bar{k}}\bar{\psi}^{\bar{j}}\bar{\psi}^{\bar{k}} - W_i\right) \\ &\quad - \bar{F}^{\bar{i}}\left(\frac{1}{2}g_{j\bar{i},k}\psi^j\psi^k - \bar{W}_{\bar{i}}\right) - \frac{1}{2}W_{ij}\psi^i\psi^j - \frac{1}{2}\bar{W}_{\bar{i}\bar{j}}\bar{\psi}^{\bar{i}}\bar{\psi}^{\bar{j}} + \frac{1}{4}g_{i\bar{j},k\bar{\ell}}(\psi^i\psi^k)(\bar{\psi}^{\bar{j}}\bar{\psi}^{\bar{\ell}}) \quad (3) \\ &\quad - \frac{1}{4r^2}g_{i\bar{j}}X^iX^{\bar{j}} + \frac{i}{4r^2}K_iX^i - \frac{i}{4r^2}K_{\bar{i}}X^{\bar{i}} - \frac{i}{2r}g_{i\bar{j}}\bar{\psi}^{\bar{j}}\psi^j\nabla_jX^i, \end{aligned}$$

where¹ r is the radius of S^2 , K is the Kähler potential of the target space M , W is the superpotential, and

$$\begin{aligned} \nabla_m\psi^i &= \tilde{\nabla}_m\psi^i + \Gamma_{jk}^i(\partial_m\phi^j)\psi^k, \\ \nabla_jX^i &= \partial_jX^i + \Gamma_{jk}^iX^k, \end{aligned} \quad (4)$$

with Γ_{jk}^i the Christoffel symbols on the target space M , and $\tilde{\nabla}_m$ denotes the pure worldsheet spin connection covariant derivative. Integrating out the auxiliary fields yields

$$\begin{aligned} \mathcal{L} &= g_{i\bar{j}}\partial_m\phi^i\partial^m\bar{\phi}^{\bar{j}} - ig_{i\bar{j}}\bar{\psi}^{\bar{j}}\gamma^m\nabla_m\psi^i + R_{i\bar{j}k\bar{\ell}}(\psi^i\psi^k)(\bar{\psi}^{\bar{j}}\bar{\psi}^{\bar{\ell}}) \\ &\quad - g^{i\bar{j}}W_i\bar{W}_{\bar{j}} - \frac{1}{2}\nabla_i\partial_jW\psi^i\psi^j - \frac{1}{2}\nabla_{\bar{i}}\partial_{\bar{j}}\bar{W}\bar{\psi}^{\bar{i}}\bar{\psi}^{\bar{j}} \\ &\quad - \frac{1}{4r^2}g_{i\bar{j}}X^iX^{\bar{j}} + \frac{i}{4r^2}K_iX^i - \frac{i}{4r^2}K_{\bar{i}}X^{\bar{i}} - \frac{i}{2r}g_{i\bar{j}}\bar{\psi}^{\bar{j}}\psi^j\nabla_jX^i. \end{aligned} \quad (5)$$

This Lagrangian reduces the the usual $\mathcal{N} = (2, 2)$ nonlinear sigma model on flat two-dimensional space when $r \rightarrow \infty$, as one would expect.

¹ As an aside, terms closely analogous to the curvature-dependent terms in the Lagrangian above have been discussed in two-dimensional theories in a different context in [29].

There are two conditions on the data above. First, the Kähler potential is invariant under the isometry defined by the holomorphic Killing vector X , which means that on each coordinate patch,

$$\mathcal{L}_X K = X^i K_i + X^{\bar{i}} K_{\bar{i}} = 0 \quad (6)$$

and across coordinate patches, $K \mapsto K + f(\phi) + \bar{f}(\bar{\phi})$, the f 's are constrained to obey

$$\sum_i X^i \partial_i f(\phi) = 0. \quad (7)$$

On a Kähler manifold with a holomorphic isometry X , for data on a good open cover, one can always choose K 's, f 's to obey these constraints [30].

The second condition says that the superpotential W is homogeneous of degree 2 under the vector-like $U(1)_R$ symmetry, meaning

$$2W - iX^i W_i = 0 \quad (8)$$

(up to an additive constant). In particular, if $X = 0$, then the superpotential must vanish (up to a constant).

The Lagrangian above is invariant under the following supersymmetry transformations

$$\begin{aligned} \delta\phi^i &= \zeta\psi^i, \\ \delta\bar{\phi}^{\bar{i}} &= \bar{\psi}^{\bar{i}}\bar{\zeta}, \\ \delta\psi^i &= i\gamma^m \bar{\zeta} \partial_m \phi^i - \frac{i}{2r} \bar{\zeta} X^i + \zeta F^i, \\ \delta\bar{\psi}^{\bar{i}} &= i\gamma^m \zeta \partial_m \bar{\phi}^{\bar{i}} + \frac{i}{2r} \zeta X^{\bar{i}} + \bar{\zeta} \bar{F}^{\bar{i}}, \\ \delta F^i &= i\bar{\zeta} \gamma^m \tilde{\nabla}_m \psi^i + \frac{i}{2r} \bar{\zeta} \psi^j \partial_j X^i, \\ \delta \bar{F}^{\bar{i}} &= i\zeta \gamma^m \tilde{\nabla}_m \bar{\psi}^{\bar{i}} - \frac{i}{2r} \zeta \bar{\psi}^{\bar{j}} \partial_{\bar{j}} X^{\bar{i}}. \end{aligned} \quad (9)$$

(up to total derivatives) provided the Killing spinor equations (2) are satisfied, together with the constraints (6), (8).

In the special case that $X = 0$ (and hence $W = 0$), the Lagrangian and supersymmetry transformations are identical to those on flat space. One can show that the flat-space Lagrangian is invariant under supersymmetry transformations defined by a Killing spinor appropriate for S^2 , not just a constant spinor. Thus, the lagrangian with $X = 0$ is consistent with supersymmetry on both S^2 and \mathbb{R}^2 , as one would expect.

In the linear case, *i.e.* when the target space $M = \mathbb{C}$ with the trivial Kähler potential $K = \bar{\phi}\phi$ and the $U(1)_R$ holomorphic Killing vector $X = -iq\phi \frac{d}{d\phi}$, the Lagrangian (3) reduces to the Lagrangian of the chiral multiplet in [20, 21] (where q is the $U(1)_R$ charge of the chiral

field ϕ). The full gauged linear sigma model in [20, 21] can also be obtained by applying a decoupling gravity procedure analogous to the one in [1, 2] to the coupled theory of (2,2) gauged linear sigma model and (1,1) supergravity model in [31].

From the supersymmetry transformations above, we can get some insight into the constraint on W . Specifically, note that with the Killing spinor condition, the supersymmetry variation of the auxiliary field can be written

$$\delta F^i = i\bar{\zeta}\gamma^m\tilde{\nabla}_m\psi^i - \frac{1}{2}(\nabla_m\bar{\zeta})\gamma^m\psi^j\partial_j X^i$$

If $\partial_j X^i = -2i\delta_j^i$, then δF^i is a total derivative. For example, in the linear case above, this is the statement that when $q = 2$, δF is a total derivative. Ultimately that factor of two is the reason why supersymmetry requires that W be homogeneous of degree two under the action of X .

Now, let us describe the vector $U(1)_R$ symmetry explicitly. Infinitesimally, the action of this global $U(1)_R$ symmetry on the fields is given by

$$\begin{aligned}\delta\phi^i &= \epsilon X^i, \\ \delta\phi^{\bar{i}} &= \epsilon X^{\bar{i}}, \\ \delta\psi^i &= \epsilon(\partial_j X^i - i\delta_j^i)\psi^j, \\ \delta\psi^{\bar{i}} &= \epsilon(\partial_{\bar{j}} X^{\bar{i}} + i\delta_{\bar{j}}^{\bar{i}})\psi^{\bar{j}}, \\ \delta F^i &= \epsilon(\partial_j X^i - 2i\delta_j^i)F^j - \frac{\epsilon}{2}\partial_k\partial_j X^i\psi^j\psi^k, \\ \delta F^{\bar{i}} &= \epsilon(\partial_{\bar{j}} X^{\bar{i}} + 2i\delta_{\bar{j}}^{\bar{i}})F^{\bar{j}} - \frac{\epsilon}{2}\partial_{\bar{k}}\partial_{\bar{j}} X^{\bar{i}}\psi^{\bar{j}}\psi^{\bar{k}}\end{aligned}\tag{10}$$

where ϵ is a real constant parametrizing the global $U(1)_R$. Our Lagrangian (3) is invariant under this symmetry.

We should observe that even for (2,2) supersymmetric theories on \mathbb{R}^2 instead of S^2 , the $U(1)_R$ symmetry sometimes involves an action on bosons, and hence involves a holomorphic Killing vector field X . In fact, the explicit transformations (10) also applies to the usual $\mathcal{N} = (2, 2)$ nonlinear sigma models on \mathbb{R}^2 . In general, a global symmetry of a nonlinear sigma model on any spacetime should act on the bosonic fields as a Killing vector on the target space. We believe that (10) should hold for any two-dimensional nonlinear sigma models with a vector $U(1)_R$ symmetry, regardless of the two-dimensional spacetime they are defined on.

In the case of four-dimensional rigidly supersymmetric theories on spacetimes such as S^4 and AdS_4 , supersymmetry imposes constraints on the theory (see *e.g.* [1, 2, 3, 4]), such as a constraint that the Kähler form on the target space be cohomologically-trivial. In two dimensional theories, on the other hand, we have found no analogous constraint.

Mechanically, one way to understand this lack of constraints on two-dimensional theories is to think of a two-dimensional theory as a dimensional reduction of a four-dimensional theory on $\mathbf{R} \times \Sigma_3$ (for Σ_3 a three-manifold), or $\mathbf{R}^2 \times \Sigma_2$ (for Σ_2 a two-manifold). Such four dimensional theories were unconstrained by supersymmetry; constraints only existed in four dimensions when all four spacetime directions were ‘wrapped up’ nontrivially in the topology, when none were flat. Another more abstract way to think about this in the context of the decoupling procedure is as follows². In four dimensional supergravities, the Fayet-Iliopoulos parameter³, the curvature of the Bagger-Witten line bundle, and so forth are weighted by inverse factors of the four-dimensional Planck mass. The decoupling limit of [1, 2] involves sending the Planck mass to infinity, which necessarily truncates those terms, and leaves one with a rigidly supersymmetric theory in which Fayet-Iliopoulos parameters vanish and Kähler forms are exact. By contrast, in two dimensions, the “Planck mass” is dimensionless. Hence, the procedure of decoupling gravity in two dimensions is a formal way of obtaining rigid supersymmetric theories from supergravity theories, with no further constraints on the target space geometry. Thus, one should not be surprised to find no constraints on the target space geometry in two dimensional cases.

3 Twisted chiral supermultiplets

In two dimensions, there is another $\mathcal{N} = (2, 2)$ supermultiplet known as the twisted chiral multiplet. The field content of a twisted chiral multiplet is the same as that of an ordinary chiral multiplet:

$$(\rho, \bar{\rho}, \chi, \bar{\chi}, G, \bar{G}).$$

The fields $\rho^i, \bar{\rho}^{\bar{i}}$ are bosons describing maps into a target space \tilde{M} , which is required⁴ to be a Kähler manifold. The fields G, \bar{G} are auxiliary fields.

Although the flat-worldsheet action of a twisted chiral multiplet is identical to that of an ordinary chiral multiplet, the curvature couplings to a superpotential are of a very different form.

The fields $\chi, \bar{\chi}$ are Dirac spinors. In a twisted chiral multiplet, their components mix holomorphic and antiholomorphic target space indices:

$$\chi = \begin{pmatrix} \chi_-^i \\ \chi_+^{\bar{i}} \end{pmatrix}, \quad \bar{\chi} = \begin{pmatrix} \bar{\chi}_-^{\bar{i}} \\ \bar{\chi}_+^i \end{pmatrix} \quad (11)$$

² We would like to thank I. Melnikov for making this observation.

³See [32, 33, 34, 35] for a recent discussion of the Fayet-Iliopoulos parameter in supergravity, and how old obstruction issues summarized in *e.g.* [36] can be circumvented.

⁴ We do not attempt to consider H -flux backgrounds in this paper.

In this section, we will work with Killing spinors $\zeta, \bar{\zeta}$ obeying

$$\begin{aligned}\nabla_m \zeta &= \frac{i}{2r} \gamma_m \zeta, \\ \nabla_m \bar{\zeta} &= -\frac{i}{2r} \gamma_m \bar{\zeta},\end{aligned}\tag{12}$$

a slightly different convention than we used for ordinary chiral multiplets.

Although the flat-worldsheet action of a twisted chiral multiplet is identical to that of an ordinary chiral multiplet, the curvature couplings to a superpotential are of a very different form. In the Killing spinor convention (12), the most general $\mathcal{N} = (2, 2)$ Lagrangian for twisted chiral multiplets on a round S^2 is

$$\begin{aligned}\mathcal{L}_T &= g_{i\bar{j}} \partial_m \rho^i \partial^m \bar{\rho}^{\bar{j}} + 2i g_{i\bar{j}} \bar{\chi}_-^{\bar{j}} \nabla_{\bar{z}} \chi_-^i + 2i g_{i\bar{j}} \chi_+^{\bar{j}} \nabla_z \bar{\chi}_+^i + g_{i\bar{j}} G^i \bar{G}^{\bar{j}} \\ &\quad - G^i \left(i g_{i\bar{j}, \bar{k}} \bar{\chi}_-^{\bar{j}} \chi_+^{\bar{k}} - \mathcal{W}_i \right) - i \mathcal{W}_{i\bar{j}} \chi_-^i \bar{\chi}_+^{\bar{j}} \\ &\quad - \bar{G}^{\bar{i}} \left(i g_{i\bar{j}, k} \chi_-^j \bar{\chi}_+^k - \bar{\mathcal{W}}_{\bar{i}} \right) - i \bar{\mathcal{W}}_{\bar{i}\bar{j}} \bar{\chi}_-^{\bar{i}} \chi_+^{\bar{j}} \\ &\quad + g_{i\bar{j}, k\bar{l}} \bar{\chi}_+^i \chi_+^{\bar{j}} \chi_-^k \bar{\chi}_-^{\bar{l}} \\ &\quad + \frac{i}{r} \mathcal{W} - \frac{i}{r} \bar{\mathcal{W}},\end{aligned}\tag{13}$$

where, as in the case of ordinary chiral multiplets, $g_{i\bar{j}}$ is the Kähler metric on \tilde{M} , and \mathcal{W} is the twisted superpotential. (For notational simplicity, we have chosen to write the spinors in the lagrangian above in chiral components.) Notice that the twisted superpotential \mathcal{W} is coupled to the curvature of S^2 , unlike the superpotential W of the ordinary chiral multiplets. Furthermore, unlike ordinary chiral multiplets, no holomorphic Killing vector is needed to define the superpotential. (It is straightforward to check that a curvature coupling of this form is incompatible with the supersymmetry of the ordinary chiral multiplets.) This Lagrangian generalizes⁵ that given in [26][equ'ns (4.2), (4.5)] for flat target spaces. It also reduces to the flat \mathbb{R}^2 lagrangian when $r \rightarrow \infty$, as expected (compare appendix A).

The fermions couple to the following bundles:

$$\begin{aligned}\bar{\chi}_+^i &\in \Gamma_{C^\infty} \left(K_\Sigma^{1/2} \otimes \rho^* T^{1,0} \tilde{M} \right), & \chi_-^i &\in \Gamma_{C^\infty} \left(\bar{K}_\Sigma^{1/2} \otimes \left(\rho^* T^{0,1} \tilde{M} \right)^* \right), \\ \chi_+^{\bar{i}} &\in \Gamma_{C^\infty} \left(K_\Sigma^{1/2} \otimes \left(\rho^* T^{1,0} \tilde{M} \right)^* \right), & \bar{\chi}_-^{\bar{i}} &\in \Gamma_{C^\infty} \left(\bar{K}_\Sigma^{1/2} \otimes \rho^* T^{0,1} \tilde{M} \right).\end{aligned}$$

where Σ is the worldsheet (here, S^2), K_Σ and \bar{K}_Σ are the holomorphic and antiholomorphic canonical bundles.

⁵ We have absorbed the weight Δ in [26] in field redefinitions; later in this section we shall give an alternative form of the lagrangian in which that weight reappears, in terms of a vector Y .

The above Lagrangian is invariant under the following supersymmetry transformations:

$$\begin{aligned}
\delta\rho^i &= i\bar{\zeta}_+\chi_-^i + i\zeta_-\bar{\chi}_+^i, \\
\delta\bar{\rho}^{\bar{i}} &= i\bar{\zeta}_-\chi_+^{\bar{i}} + i\zeta_+\bar{\chi}_-^{\bar{i}}, \\
\delta\bar{\chi}_+^i &= -2\bar{\zeta}_-\bar{\partial}\rho^i - \bar{\zeta}_+G^i, \\
\delta\chi_-^i &= -2\zeta_+\partial\rho^i + \zeta_-G^i, \\
\delta\bar{\chi}_+^{\bar{i}} &= -2\bar{\zeta}_-\bar{\partial}\bar{\rho}^{\bar{i}} - \bar{\zeta}_+\bar{G}^{\bar{i}}, \\
\delta\bar{\chi}_-^{\bar{i}} &= -2\zeta_+\partial\bar{\rho}^{\bar{i}} + \zeta_-\bar{G}^{\bar{i}}, \\
\delta G^i &= 2i(\zeta_+\tilde{\nabla}_z\bar{\chi}_+^i - \bar{\zeta}_-\tilde{\nabla}_{\bar{z}}\chi_-^i), \\
\delta\bar{G}^{\bar{i}} &= 2i(\bar{\zeta}_+\tilde{\nabla}_z\chi_+^{\bar{i}} - \zeta_-\tilde{\nabla}_{\bar{z}}\bar{\chi}_-^{\bar{i}}),
\end{aligned} \tag{14}$$

provided that the Killing spinor equations (12) are satisfied.

In order to check supersymmetry, it is useful to write down the Killing spinor equations (12) in chiral components:

$$\begin{aligned}
\nabla_z\zeta_- &= 0, & \nabla_{\bar{z}}\zeta_- &= \frac{i}{2r}\zeta_+, \\
\nabla_z\zeta_+ &= \frac{i}{2r}\zeta_-, & \nabla_{\bar{z}}\zeta_+ &= 0, \\
\nabla_z\bar{\zeta}_- &= 0, & \nabla_{\bar{z}}\bar{\zeta}_- &= -\frac{i}{2r}\bar{\zeta}_+, \\
\nabla_z\bar{\zeta}_+ &= -\frac{i}{2r}\bar{\zeta}_-, & \nabla_{\bar{z}}\bar{\zeta}_+ &= 0.
\end{aligned}$$

Note that the supersymmetry transformation of the auxiliary field above is not a total derivative, even though the theory is supersymmetric and contains a superpotential, which is very different from the behavior of ordinary chiral multiplets. We will shortly describe how one can couple a holomorphic Killing vector field, which could be used to make δG^i a total derivative, but unlike the case of ordinary chiral multiplets, it is not necessary in order to couple a superpotential. One suspects that this may be linked to the existence of superfield representations on spheres, but we will not speculate further in that direction.

For $\mathcal{N} = (2, 2)$ nonlinear sigma models on \mathbb{R}^2 , the Lagrangian of twisted chiral multiplets can be obtained by a simple “twist” from the Lagrangian of ordinary chiral multiplets, just by dualizing the tangent bundle TM to T^*M on the right-movers, as elaborated in appendix A. However, notice that for a nonzero superpotential, the above Lagrangian for twisted chiral fields on S^2 cannot be obtained from the Lagrangian of chiral fields on S^2 , given in equation (3), in a similar fashion, because of different couplings to the curvature of S^2 .

To compare to the lagrangian for twisted chiral multiplets given in [26], one performs a slight field redefinition. Specifically, redefine G^i to be $G^i + (i/r)Y^i$ for Y a vector field.

Then, the lagrangian becomes

$$\begin{aligned}
\mathcal{L}_T = & g_{i\bar{j}} \partial_m \rho^i \partial^m \bar{\rho}^{\bar{j}} + 2i g_{i\bar{j}} \bar{\chi}_-^{\bar{j}} \nabla_{\bar{z}} \chi_-^i + 2i g_{i\bar{j}} \chi_+^{\bar{j}} \nabla_z \bar{\chi}_+^i + g_{i\bar{j}} \left(G^i + \frac{i}{r} Y^i \right) \left(\bar{G}^{\bar{j}} - \frac{i}{r} Y^{\bar{j}} \right) \\
& - \left(G^i + \frac{i}{r} Y^i \right) \left(i g_{i\bar{j},\bar{k}} \bar{\chi}_-^{\bar{j}} \bar{\chi}_+^{\bar{k}} - \mathcal{W}_i \right) - i \mathcal{W}_{ij} \chi_-^i \bar{\chi}_+^{\bar{j}} \\
& - \left(\bar{G}^{\bar{i}} - \frac{i}{r} Y^{\bar{i}} \right) \left(i g_{\bar{i}j,k} \chi_-^j \bar{\chi}_+^k - \bar{\mathcal{W}}_{\bar{i}} \right) - i \bar{\mathcal{W}}_{\bar{i}\bar{j}} \bar{\chi}_-^{\bar{i}} \chi_+^{\bar{j}} \\
& + g_{i\bar{j},k\bar{l}} \bar{\chi}_+^i \chi_+^{\bar{j}} \chi_-^k \bar{\chi}_-^{\bar{l}} \\
& + \frac{i}{r} \mathcal{W} - \frac{i}{r} \bar{\mathcal{W}},
\end{aligned} \tag{15}$$

with, assuming Y^i is chosen holomorphic, supersymmetry transformations

$$\begin{aligned}
\delta \rho^i &= i \bar{\zeta}_+ \chi_-^i + i \zeta_- \bar{\chi}_+^i, \\
\delta \bar{\rho}^{\bar{i}} &= i \bar{\zeta}_- \bar{\chi}_+^{\bar{i}} + i \zeta_+ \bar{\chi}_-^{\bar{i}}, \\
\delta \bar{\chi}_+^i &= -2 \bar{\zeta}_- \bar{\partial} \rho^i - \bar{\zeta}_+ \left(G^i + \frac{i}{r} Y^i \right), \\
\delta \chi_-^i &= -2 \zeta_+ \partial \rho^i + \zeta_- \left(G^i + \frac{i}{r} Y^i \right), \\
\delta \bar{\chi}_+^{\bar{i}} &= -2 \zeta_- \bar{\partial} \bar{\rho}^{\bar{i}} - \zeta_+ \left(\bar{G}^{\bar{i}} - \frac{i}{r} Y^{\bar{i}} \right), \\
\delta \bar{\chi}_-^{\bar{i}} &= -2 \bar{\zeta}_+ \partial \bar{\rho}^{\bar{i}} + \bar{\zeta}_- \left(\bar{G}^{\bar{i}} - \frac{i}{r} Y^{\bar{i}} \right), \\
\delta G^i &= 2i (\zeta_+ \tilde{\nabla}_z \bar{\chi}_+^i - \bar{\zeta}_- \tilde{\nabla}_{\bar{z}} \chi_-^i) - \frac{i}{r} (i \bar{\zeta}_+ \chi_-^j + i \zeta_- \bar{\chi}_+^j) \partial_j Y^i, \\
\delta \bar{G}^{\bar{i}} &= 2i (\bar{\zeta}_+ \tilde{\nabla}_z \bar{\chi}_+^{\bar{i}} - \zeta_- \tilde{\nabla}_{\bar{z}} \bar{\chi}_-^{\bar{i}}) + \frac{i}{r} (i \bar{\zeta}_- \bar{\chi}_+^{\bar{j}} + i \zeta_+ \bar{\chi}_-^{\bar{j}}) \partial_{\bar{j}} Y^{\bar{i}},
\end{aligned} \tag{16}$$

provided that the Killing spinor equations (12) are satisfied. To recover [26][equ'ns (4.2), (4.5)] on flat target spaces, take $Y^i = -i \Delta \rho^i$.

4 Topological twists

When the superpotential vanishes (and, for ordinary chiral multiplets, $X = 0$), there are no curvature couplings, and the theories admit the same topological twists discussed in *e.g.* [37].

When the superpotential is nonzero, this story is more complicated. Sometimes those curvature couplings are incompatible with the topological twist (which can sometimes be

alleviated by twisting bosons as in *e.g.* [38, 39]). However, a more fundamental problem is that even if one can consistently twist the theory, the curvature terms induced by the superpotential break the BRST symmetry, in the sense that the action is no longer BRST closed. Those terms are compatible with supersymmetry only so long as the supersymmetry parameters ζ , $\bar{\zeta}$ obey the Killing spinor equation, and serve to ‘mop up’ the curvature-dependent terms that result from using the Killing spinor equations. Since in a topological field theory the BRST transformations are parametrized by a scalar, the Killing spinor equations are not relevant, and so the curvature-dependent terms in the action are extraneous, breaking the BRST symmetry.

However, just because we cannot always twist, does not imply that topological field theories do not exist. For example, the papers [38, 39] describe examples of two-dimensional topological field theories with nonzero superpotential, on two-spheres and other two-dimensional worldsheets. These topological field theories were *not* obtained by topological twisting of a two-dimensional theory with curvature couplings. Instead, they were obtained by twisting a flat-worldsheet theory. The result continues to make sense on two-spheres and other two-dimensional worldsheets because the BRST transformations are parametrized by a scalar; the action neither needs nor contains curvature-dependent terms.

Thus, to summarize, if the superpotential vanishes (and $X = 0$), then topological twistings exist. If the superpotential is nonzero, there still exist topological field theories, but they are not obtained by twisting an action that includes curvature-dependent terms.

5 Discussion

In this note we have discussed curvature couplings in general nonlinear sigma models with potential, for both ordinary and twisted chiral multiplets, following a program initiated in [1, 2]. In the special case of target spaces that are vector spaces, such couplings have been previously discussed in *e.g.* [20, 21]; our purpose here was to generalize such couplings to general target spaces (and with $U(1)_R$ actions described by general holomorphic Killing vectors), so as to give some degree of insight into the structure of their actions. We have also discussed general issues surrounding existence of topological twists in this context.

It would be interesting to understand localization in these more general theories, and *e.g.* compare the results of localization in A-twisted Landau-Ginzburg models of the form considered in [38] to corresponding Landau-Ginzburg models with curvature interactions as described here. The A-twisted Landau-Ginzburg models realized Mathai-Quillen classes which gave a mathematical understanding of the behavior of renormalization group flow, as a type of Thom class phenomenon. (See also [40] for closely analogous behaviors in elliptic genera.) It would be interesting to see if something analogous arises in localization computations.

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A Twisted chiral superfields without curvature couplings

As twisted chiral multiplets are discussed less commonly than ordinary chiral multiplets, it will be useful to review a few of their basic properties. (See also [41][section 12.2] for another comparison of ordinary and twisted chiral multiplets.) Given twisted chiral multiplets of the form $(\rho^i, \bar{\chi}_+, \chi_-^i)$, where the ρ are bosons, the Lagrangian for a theory containing only twisted chiral multiplets on \mathbb{R}^2 is of the form

$$\begin{aligned}
\mathcal{L}_T = & g_{i\bar{j}} \partial_m \rho^i \partial^m \bar{\rho}^{\bar{j}} + 2i g_{i\bar{j}} \bar{\chi}_-^{\bar{j}} \nabla_{\bar{z}} \chi_-^i + 2i g_{i\bar{j}} \chi_+^{\bar{j}} \nabla_z \bar{\chi}_+^i + g_{i\bar{j}} G^i \bar{G}^{\bar{j}} \\
& - G^i \left(g_{i\bar{j}, \bar{k}} \bar{\chi}_-^{\bar{j}} \chi_+^{\bar{k}} - \mathcal{W}_i \right) - i \mathcal{W}_{i\bar{j}} \chi_-^i \bar{\chi}_+^{\bar{j}} \\
& - \bar{G}^{\bar{i}} \left(g_{\bar{i}j, k} \chi_+^j \bar{\chi}_+^k - \bar{\mathcal{W}}_{\bar{i}} \right) - i \bar{\mathcal{W}}_{\bar{i}j} \bar{\chi}_-^{\bar{i}} \chi_+^j \\
& - g_{i\bar{j}, k\bar{l}} \bar{\chi}_+^i \chi_+^{\bar{j}} \chi_-^k \bar{\chi}_-^{\bar{l}},
\end{aligned} \tag{17}$$

with supersymmetry transformations

$$\begin{aligned}
\delta \rho^i &= i \bar{\zeta}_+ \chi_-^i + i \zeta_- \bar{\chi}_+^i, \\
\delta \bar{\rho}^{\bar{i}} &= i \bar{\zeta}_- \chi_+^{\bar{i}} + i \zeta_+ \bar{\chi}_-^{\bar{i}}, \\
\delta \bar{\chi}_+^i &= -2 \bar{\zeta}_- \bar{\partial} \rho^i - \bar{\zeta}_+ G^i, \\
\delta \chi_-^i &= -2 \zeta_+ \partial \rho^i + \zeta_- G^i, \\
\delta \chi_+^{\bar{i}} &= -2 \zeta_- \bar{\partial} \bar{\rho}^{\bar{i}} - \zeta_+ \bar{G}^{\bar{i}}, \\
\delta \bar{\chi}_-^{\bar{i}} &= -2 \bar{\zeta}_+ \partial \bar{\rho}^{\bar{i}} + \bar{\zeta}_- \bar{G}^{\bar{i}}, \\
\delta G^i &= 2i (\zeta_+ \tilde{\nabla}_z \bar{\chi}_+^i - \bar{\zeta}_- \tilde{\nabla}_{\bar{z}} \chi_-^i), \\
\delta \bar{G}^{\bar{i}} &= 2i (\bar{\zeta}_+ \tilde{\nabla}_z \chi_+^{\bar{i}} - \zeta_- \tilde{\nabla}_{\bar{z}} \bar{\chi}_-^{\bar{i}}).
\end{aligned} \tag{18}$$

They can be obtained by taking the limit of $r \rightarrow \infty$ in the corresponding theory on S^2 discussed in section 3. If we integrate out the auxiliary fields $G^i, \bar{G}^{\bar{i}}$, the lagrangian takes the simpler form

$$\begin{aligned}
\mathcal{L}_T = & g_{i\bar{j}} \partial_m \rho^i \partial^m \bar{\rho}^{\bar{j}} + 2i g_{i\bar{j}} \bar{\chi}_-^{\bar{j}} \nabla_{\bar{z}} \chi_-^i + 2i g_{i\bar{j}} \chi_+^{\bar{j}} \nabla_z \bar{\chi}_+^i + R_{i\bar{j}k\bar{l}} \bar{\chi}_+^i \chi_+^{\bar{j}} \chi_-^k \bar{\chi}_-^{\bar{l}} \\
& - g^{i\bar{j}} \mathcal{W}_i \bar{\mathcal{W}}_{\bar{j}} - i \chi_-^i \bar{\chi}_+^{\bar{j}} D_i \partial_{\bar{j}} \mathcal{W} - i \bar{\chi}_-^{\bar{i}} \chi_+^{\bar{j}} D_{\bar{i}} \partial_j \bar{\mathcal{W}}
\end{aligned} \tag{19}$$

with supersymmetry transformations

$$\begin{aligned}
\delta\rho^i &= i\bar{\zeta}_+\chi_-^i + i\zeta_-\bar{\chi}_+^i, \\
\delta\bar{\rho}^{\bar{i}} &= i\bar{\zeta}_-\chi_+^{\bar{i}} + i\zeta_+\bar{\chi}_-^{\bar{i}}, \\
\delta\bar{\chi}_+^i &= -2\bar{\zeta}_-\bar{\partial}\rho^i + \bar{\zeta}_+ \left(\Gamma_{jk}^i \chi_-^j \bar{\chi}_+^k + g^{i\bar{j}} \bar{\mathcal{W}}_{\bar{j}} \right), \\
\delta\chi_-^i &= -2\zeta_+\partial\rho^i + \zeta_- \left(\Gamma_{jk}^i \chi_-^j \bar{\chi}_+^k + g^{i\bar{j}} \bar{\mathcal{W}}_{\bar{j}} \right), \\
\delta\chi_+^{\bar{i}} &= -2\zeta_-\bar{\partial}\bar{\rho}^{\bar{i}} + \zeta_+ \left(\Gamma_{\bar{j}\bar{k}}^{\bar{i}} \bar{\chi}_-^{\bar{j}} \chi_+^{\bar{k}} + g^{\bar{i}j} \mathcal{W}_j \right), \\
\delta\bar{\chi}_-^{\bar{i}} &= -2\bar{\zeta}_+\partial\bar{\rho}^{\bar{i}} + \bar{\zeta}_- \left(\Gamma_{\bar{j}\bar{k}}^{\bar{i}} \bar{\chi}_-^{\bar{j}} \chi_+^{\bar{k}} + g^{\bar{i}j} \mathcal{W}_j \right).
\end{aligned} \tag{20}$$

For contrast, compare the Lagrangian for ordinary chiral multiplets $(\phi^i, \psi_+^i, \psi_-^i)$ given in [37][equ'n (2.4)]. Modulo signs and irrelevant factors, the lagrangian for a theory of purely twisted chiral multiplets is the same as the action for a theory of purely ordinary chiral multiplets – the difference between the two lies in the supersymmetry transformations.

Note that if one were to dualize the tangent bundle TM to T^*M on the right-movers, the effect would be to convert the twisted chiral multiplets back into chiral multiplets. This is a special case of a duality in (0,2) theories discussed in *e.g.* [42].

Now, let us specialize to the case $\mathcal{W} = 0$, in which case there are no curvature couplings on any worldsheet, and consider topological twists. It is straightforward to topologically twist the theory of twisted chiral multiplets; however, the results are identical to the topological field theories obtained from twisting theories of ordinary chiral multiplets. Specifically, the A-twist of twisted chiral multiplets is equivalent to the B-twist of ordinary chiral multiplets, and conversely, as one would expect from ideas related to aspects of (2,2) algebras in two dimensions.

It will be useful to make that correspondence explicit. Adapting the conventions of [37], the A twist is defined by making the supersymmetry transformation parameters $\zeta_+, \bar{\zeta}_-$ into (Grassmann-valued) scalars, parametrizing the BRST transformations. The same twist here would result in fermions coupling to bundles as

$$\begin{aligned}
\bar{\chi}_+^i &\in \Gamma_{C^\infty} \left(K_\Sigma \otimes \rho^* T^{1,0} \tilde{M} \right), & \chi_-^i &\in \Gamma_{C^\infty} \left(\bar{K}_\Sigma \otimes \left(\rho^* T^{0,1} \tilde{M} \right)^* \right), \\
\chi_+^{\bar{i}} &\in \Gamma_{C^\infty} \left(\left(\rho^* T^{1,0} \tilde{M} \right)^* \right), & \bar{\chi}_-^{\bar{i}} &\in \Gamma_{C^\infty} \left(\rho^* T^{0,1} \tilde{M} \right).
\end{aligned}$$

where Σ is the worldsheet, K_Σ and \bar{K}_Σ are the holomorphic and antiholomorphic canonical

bundles, and BRST transformations

$$\begin{aligned}
\delta\rho^i &= 0, \\
\delta\rho^{\bar{i}} &= i\bar{\zeta}_-\chi_+^{\bar{i}} + i\zeta_+\bar{\chi}_-^{\bar{i}}, \\
\delta\bar{\chi}_+^i &= -2\bar{\zeta}_-\bar{\partial}\rho^i, \\
\delta\chi_-^i &= -2\zeta_+\partial\rho^i, \\
\delta\bar{\chi}_+^{\bar{i}} &= \zeta_+\Gamma_{\bar{j}\bar{k}}^{\bar{i}}\bar{\chi}_-^{\bar{j}}\bar{\chi}_+^{\bar{k}}, \\
\delta\bar{\chi}_-^{\bar{i}} &= \bar{\zeta}_-\Gamma_{\bar{j}\bar{k}}^{\bar{i}}\bar{\chi}_-^{\bar{j}}\bar{\chi}_+^{\bar{k}}.
\end{aligned}$$

This structure, the A-twisted theory of twisted chiral multiplets, can easily be seen to be equivalent to the B-twisted theory of ordinary chiral multiplets described in [37], and as such is only well-defined in the special case that K_M^2 is trivial [42].

For completeness, let us examine the opposite case. In the B-twisted theory of ordinary chiral multiplets described in [37], the supersymmetry transformation parameters $\bar{\zeta}_+$, $\bar{\zeta}_-$ are (Grassmann-valued) scalars. This twist results in fermions coupling to bundles as

$$\begin{aligned}
\bar{\chi}_+^i &\in \Gamma_{C^\infty}\left(K_\Sigma \otimes \rho^*T^{1,0}\tilde{M}\right), & \chi_-^i &\in \Gamma_{C^\infty}\left(\left(\rho^*T^{0,1}\tilde{M}\right)^*\right), \\
\chi_+^{\bar{i}} &\in \Gamma_{C^\infty}\left(\left(\rho^*T^{1,0}\tilde{M}\right)^*\right), & \bar{\chi}_-^{\bar{i}} &\in \Gamma_{C^\infty}\left(\bar{K}_\Sigma \otimes \rho^*T^{0,1}\tilde{M}\right).
\end{aligned}$$

and the BRST transformations are given by

$$\begin{aligned}
\delta\rho^i &= i\bar{\zeta}_+\chi_-^i, \\
\delta\rho^{\bar{i}} &= i\bar{\zeta}_-\chi_+^{\bar{i}}, \\
\delta\bar{\chi}_+^i &= -2\bar{\zeta}_-\bar{\partial}\rho^i + \bar{\zeta}_+\Gamma_{jk}^i\chi_-^j\bar{\chi}_+^k, \\
\delta\chi_-^i &= 0, \\
\delta\chi_+^{\bar{i}} &= 0, \\
\delta\bar{\chi}_-^{\bar{i}} &= -2\bar{\zeta}_+\partial\rho^{\bar{i}} + \bar{\zeta}_-\Gamma_{\bar{j}\bar{k}}^{\bar{i}}\bar{\chi}_-^{\bar{j}}\bar{\chi}_+^{\bar{k}}.
\end{aligned}$$

This structure, the B-twisted theory of twisted chiral multiplets, can easily be seen to be equivalent to the A-twisted theory of ordinary chiral multiplets described in [37].

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