Randomization and the American Put

by

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Abstract

While American calls on non-dividend paying stocks may be valued as European, there is no completely explicit exact solution for the values of American puts. We introduce a novel technique called randomization to value American puts and calls on dividend-paying stocks. This technique yields a new semi-explicit approximation for American option values in the Black Scholes model. Numerical results indicate that the approximation is both accurate and computationally efficient.

I Introduction

Closed-form solutions for the value of European-style options have been known since the seminal papers of Black-Scholes[2] and Merton[26]. Since American calls on non-dividend paying stocks are not rationally exercised early, they can be valued in closed form. Unfortunately, the vast majority of listed options are American-style and subject to early exercise. Despite a profusion of research on the subject, no completely satisfactory analytic solution for the value of such options has been found.

The principal difficulty in obtaining an analytic solution arises from the absence of a simple expression for the optimal exercise boundary. An exercise boundary is a time path of critical stock prices at which early exercise occurs. The optimal exercise boundary of an American option is not known ex ante, and must be determined as part of the solution to the valuation problem. Furthermore, it is difficult to analytically approximate American option values using boundary approximations which are consistent with the known short and long time behavior of the exercise boundary.

Previous approaches for determining American option values and exercise boundaries may be classified as either numerical methods or analytic approximations. Numerical methods for valuing American options comprise lattices,[10],[20],[31],[33], finite differences[5][39], and Monte Carlo simulation[14],[38]. Analytic approximations include those based on compound options[7],[12],[15], the quadratic formula[1],[23], exponential exercise boundaries[9],[30], integral formulations,[8], [16],[17], [18], [21], or tight lower and upper bounds[6],[19].

The purpose of this paper is to develop a new analytic approximation for American option values and exercise boundaries based on a novel technique called randomization. In general, randomization describes a three step procedure which can be used to solve a host of problems. The first step is to randomize a parameter by assuming a plausible distribution for it. The second step is to somehow calculate the expected value of the dependent variable in this random parameter setting. This is the difficult step since one does not know the dependent variable in the fixed parameter setting. The final step is to let the variance of the distribution governing the parameter approach zero, holding the mean of the distribution constant at the fixed parameter value.

For standard options, one can randomize the initial stock price, the strike price, the initial time, or the maturity date. In this paper, we randomize the maturity date of an American option and determine the exact solution for its value. The owner of this random maturity American option can exercise at any time up to and including some random maturity date. In order to differentiate random maturity American options from standard American ones, we christen this option as "Canadian". Thus, a Canadian put gives its owner the right to sell an underlying security for a fixed price at any time up to and including its random maturity, while a Canadian call gives the corresponding right to buy. In this paper, the maturity date is determined by the jump times of a standard Poisson process, which are assumed to be independent of the underlying stock price process.

Somewhat surprisingly, randomizing the maturity in this way can *simplify* the formulas for option values. For Canadian options, the simplicity of expression arises in part by taming the behavior of the exercise boundary. The simplest expression arises when the maturity date of a Canadian option is given by the first jump time of a Poisson process. In this case, the memoryless property of the exponential distribution governing the maturity date implies that the exercise boundary is independent of time. As calendar time elapses, a Canadian option gets no closer to its random maturity, and thus its value suffers no time decay. The stationarity in value implies that the exercise boundary is also independent of time. When the underlying security has either no dividends or a constant continuous dividend flow, we can solve explicitly for the critical stock price. In contrast, if the underlying pays continuous proportional dividends, then a nonlinear equation must be solved numerically. As a result, the general formulation leads to semi-explicit valuation formulas.

While Canadian options with exponential maturities lead to simple approximations for American options, the approximation has too much error to be used in practice. To improve the approximation, we instead assume that the time to maturity may be subdivided into n independent exponential sub-periods. Thus, the Canadian option matures at the n-th jump time of a standard Poisson process. The maturity time is thereby gamma distributed with a mean equal to the fixed maturity time of the American option. In this case, the exercise boundary takes the form of a staircase, with the levels being determined by optimizing within each sub-period. The resulting expression for the Canadian option value is a triple sum, involving no special functions other than the natural log.

As the number of random sub-periods becomes large, the variance of the random maturity approaches zero, so that the maturity distribution approaches a Dirac delta function centered at the American option's fixed maturity. Thus, increasing the number of periods increases the accuracy of the solution at the expense of greater computational cost. However, when Richardson extrapolation is used, our numerical results indicate that our Canadian option value converges to the true American option value in a computationally efficient manner.

The structure of this paper is as follows. The next section reviews standard results on the pricing of American puts in the Black-Scholes model. The following section presents our randomization technique in the context of valuing an American put on a non-dividend-paying stock with an exponenential maturity. The subsequent section discusses the more general case of a gamma distributed maturity. The following section discusses the implementation of our formula and compares this implementation with extant approaches in terms of both speed and accuracy. The penultimate section extends the analysis to dividends and American calls. The final section summarizes, while the appendices contain some technical details.

II American Put Valuation in the Black-Scholes Model

In this section, we focus on the valuation of American puts in the Black-Scholes model. We defer the corresponding development for American calls until dividends have been introduced. The Black-Scholes model assumes that over the option's life $[0, T]$, the economy is described by frictionless markets, no arbitrage, a constant riskless rate $r > 0$, no dividends from the underlying stock, and that the underlying spot price process $\{S_t, t \in (0, T)\}$ is a geometric Brownian motion with a constant volatility rate $\sigma > 0$. Let $P(t, S; T)$ denote the value of an American put as a function of the current time t, the current stock price S, and the maturity date T. Figure 1 graphs the value of an American put against the current stock price, holding t and T fixed. The critical stock price $\underline{S}(t), t \in [0, T]$ is defined as the largest price S at which the American put value $P(t, S; T)$ equals its exercise value $K - S$, where K is the strike price. As time evolves, the alive American put value falls, while the exercise value remains constant. The passage of time therefore raises the critical stock price at which exercise occurs. When graphed against time, the critical stock price is a smoothly increasing function termed the *exercise boundary* (see Figure 2).

For quite general stochastic processes, the American put's initial value is given by the solution to an *optimal stopping problem*:

$$P(0, S; T) = \sup_{\tau_s \in [0,T]} E_{0,S}\{e^{-r\tau_s}[K - S_{\tau_s}]^+\}, \tag{1}$$

where τ_s is a stopping time and the expectation is calculated under the risk-neutral probability measure. For initial stock prices above the optimal exercise boundary, the continuity of the stock price process in the Black-Scholes model implies that the optimal stopping time is a first passage time to this boundary. Consequently, the alive

American put may alternatively be valued as:

$$P(0,S;T) = \sup_{B(t);t\in[0,T]} E_{0,S}\{e^{-r(\tau_B\wedge T)}[K - S_{\tau_B\wedge T}]^+ + e^{-rT}[K - S_T]^+ 1(\tau_B \geq T)\}, \qquad S > \underline{S}(0), \tag{2}$$

where τ_B is the first passage time[1] from S to an exercise boundary $B(t), t \in [0,T]$.

McKean[22] showed that an application of Itô's lemma to (1) implies that the alive American put value and exercise boundary jointly solve a *free boundary problem*, consisting of the Black-Scholes partial differential equation (p.d.e.):

$$\frac{\sigma^2}{2} S^2 P_{ss}(t,S;T) + rSP_s(t,S;T) - rP(t,S;T) = P_T(t,S;T), \quad S \in (\underline{S}(t;T), \infty), t \in (0,T), \tag{3}$$

and the following boundary conditions:

$$P(T,S;T) = (K-S)^+, \quad S \in (\underline{S}(T;T), \infty), \quad \text{and} \quad \underline{S}(T;T) = K,$$

$$\lim_{S\uparrow\infty} P(t,S;T) = 0, \qquad \lim_{S\downarrow\underline{S}(t;T)} P(t,S;T) = K - \underline{S}(t;T), \qquad \lim_{S\downarrow\underline{S}(t;T)} P_s(t,S;T) = -1, \qquad t \in (0,T).$$

Unfortunately, there is no known exact and completely explicit solution to either the optimal stopping problem (1) or to the free boundary problem (3). The next section presents a new approach for obtaining approximate solutions to these problems.

III Exponential Maturity Valuation

In order to obtain an approximate solution for the value of an American put and its exercise boundary, we now suppose that the maturity date is random. Let τ denote the random maturity time. We initially assume that τ is exponentially distributed with scale parameter λ:

$$\text{Prob}\{\tau \in dt\} = \lambda e^{-\lambda t} dt.$$

Since the mean of τ is the reciprocal of λ, we set $\lambda = \frac{1}{T}$, so that the mean maturity of the Canadian put is T, the true maturity of the American put. Let $P^{(1)}(S)$ denote the initial value of a Canadian put, which matures at the first jump time of a standard Poisson process with intensity $\lambda = \frac{1}{T}$. We assume that the Poisson process is independent of the stock price process. Furthermore, we assume that the Poisson process is also uncorrelated with

[1] As usual, the first passage time is considered to be infinite if the boundary is never touched.

any market factor. It follows that the risk associated with the randomness of maturity can be diversified away by holding a large portfolio of Canadian options on different stocks. Thus, the Canadian put value can be calculated in a "risk-neutral" fashion.

The analog to (2) for Canadian options is:

$$P^{(1)}(S) = \sup_B E_{0,S}\{e^{-r(\tau_B \wedge \tau)}[K - S_{\tau_B \wedge \tau}]^+\}, S > \underline{S}_1, \tag{4}$$

where $\underline{S}_1$ is the unknown optimal exercise boundary. Note that the supremum is taken only over time-stationary boundaries B rather than functions of time $B(t)$. The memoryless property of the exponential distribution implies that the passage of time has no effect on either the Canadian option value or its optimal exercise boundary. Thus, the time-dependent exercise boundary graphed in Figure 2 becomes flat when maturity is randomized, as indicated in Figure 3. The latter figure shows the optimal flat exercise boundary for a Canadian option with a mean maturity of one year. The graph reflects a realized maturity of 1.23 years, at which time the Canadian option value jumps down to intrinsic value $(K - S)^+$. Thus, one can think of the pent up time decay of the option as being released at the jump time. This release causes the exercise boundary to jump up from $\underline{S}_1$ to K, crudely approximating the behavior of the true exercise boundary graphed in Figure 2.

The expectation in (4) can be evaluated in closed form and the result can be maximized over barriers analytically. Since the details are cumbersome, a perhaps simpler approach is to recognize the following relationship between random and fixed maturity put values:

$$P^{(1)}(S) = \sup_B \lambda \int_0^\infty e^{-\lambda t} D(0, S; t; B) dt, \tag{5}$$

where $D(0, S; T; B)$ is the initial value of a down-and-out put with fixed maturity T, out barrier B, and rebate $K - B$:

$$D(0, S; t; B) = E_{0,S}\{e^{-r(\tau_B \wedge T)}[K - S_{\tau_B \wedge T}]^+\}, S > B,$$

One can immediately oberve that the Canadian put is simply the Laplace-Carson[2] transform of a fixed maturity barrier put, maximized over barriers. Since down-and-out put values satisfy the Black-Scholes p.d.e. (3), one can take the Laplace-Carson transform of both sides of this p.d.e. to obtain the following simpler *ordinary* differential

[2]The Laplace-Carson transform differs from the standard Laplace transform only by the introduction of a constant λ in the kernel. See [36], pgs. 512–517 for the properties of this transform.

equation (o.d.e.):

$$\frac{\sigma^2}{2} S^2 P^{(1)}_{ss}(S) + rSP^{(1)}_s(S) - rP^{(1)}(S) = \lambda[P^{(1)}(S) - (K-S)^+], \quad S > \underline{S}_1, \tag{6}$$

subject to the following boundary conditions:

$$\lim_{S\uparrow\infty} P^{(1)}(S) = 0, \qquad \lim_{S\downarrow\underline{S}_1} P^{(1)}(S) = K - \underline{S}_1, \qquad \lim_{S\downarrow\underline{S}_1} P^{(1)}_s(S) = -1. \tag{7}$$

Using standard techniques for solving o.d.e.'s, the Canadian put value can be decomposed as:

$$P^{(1)}(S) = \begin{cases} p^{(1)}(S) + b^{(1)}(S) & \text{if } S > \underline{S}_0 \equiv K \\ KR - S + c^{(1)}(S) + b^{(1)}(S) & \text{if } S \in (\underline{S}_1, \underline{S}_0) \\ K - S & \text{if } S \leq \underline{S}_1, \end{cases} \tag{8}$$

where $p^{(1)}(S)$ is the value of a European put paying $(K-S)^+$ at the first jump time:

$$p^{(1)}(S) = \left(\frac{S}{K}\right)^{\gamma-\epsilon} (qKR - \hat{q}K), \qquad S > K, \tag{9}$$

with $\gamma \equiv \frac{1}{2} - \frac{r}{\sigma^2}$, $R \equiv \frac{1}{1+rT}$, $\epsilon \equiv \sqrt{\gamma^2 + \frac{2}{R\sigma^2 T}}$, and:

$$p \equiv \frac{\epsilon - \gamma}{2\epsilon}, q \equiv 1 - p, \hat{p} \equiv \frac{\epsilon - \gamma + 1}{2\epsilon}, \text{ and } \hat{q} \equiv 1 - \hat{p}, \tag{10}$$

$b^{(1)}(S)$ is the initial value of interest received below the critical stock price $\underline{S}_1$ until the first jump time:

$$b^{(1)}(S) = \left(\frac{S}{\underline{S}_1}\right)^{\gamma-\epsilon} qKRrT, \tag{11}$$

and finally, $c^{(1)}(S)$ is the initial value of a European call paying $(S-K)^+$ at the first jump time:

$$c^{(1)}(S) = \left(\frac{S}{K}\right)^{\gamma+\epsilon} (\hat{p}K - pKR), \qquad S < K. \tag{12}$$

The first line of our formula (8) represents the randomized version of a decomposition of the American put value into the European put value and the early exercise premium. Note that the European put approximation (9) is simpler than the Black Scholes formula in that it does not use any special functions such as the normal distribution function. On the other hand, (9) holds only for out-of-the-money values ($S > K$). The lack of smoothness in the payoff function implies that Put Call Parity[3] must be used to generate in-the-money values for European puts with random maturity. The second line of our formula (8) reflects this restriction. The third line

[3] Put Call Parity holds so long as the options and a pure discount bond mature at the same jump time.

of (8) sets the Canadian put value to exercise value below the critical stock price $\underline{S}_1$. Figure 4 graphs the value of an exponential maturity Canadian put against the stock price. The function is twice differentiable at the strike price, but only once differentiable at the exercise boundary, as is true for a standard American put.

The term $\left(\frac{S}{\underline{S}_1}\right)^{\gamma-\epsilon}$ in the early exercise premium $b^{(1)}(S)$ in (11) can be shown to be the risk-neutral probability of touching $\underline{S}_1$ before the Poisson process jumps. The other term $qKRrT$ in the early exercise premium can be shown to be the value of a claim which pays interest on the strike price continuously for all stock prices below the current level, until the Poisson process jumps. Thus, the early exercise premium is given by the value of a claim which pays interest below the boundary until the random maturity. This result holds in the fixed maturity setting as shown previously in [8],[16],[17], [18], and [21]. Its veracity in the random maturity setting is an implication of the uniqueness of Laplace-Carson Transforms.

Imposing value-matching in (8) at the critical stock price $\underline{S}_1$ yields the following balance equation:

$$c^{(1)}(\underline{S}_1) = pKRrT. \tag{13}$$

The left hand side is clearly the randomized value of a European call when the stock price is at the critical stock price. The right hand side represents the randomized value of a claim paying interest on the strike price at all stock prices *above* the current stock price level. The critical stock price is chosen so that the call value just matches the present value of the interest flow received above the boundary. Stationarity in the values involved implies that the exercise boundary remains flat at this level until the jump time.

The simple expression (12) for the European call value implies that the balance equation (13) can be explicitly solved for our first approximation to the exercise boundary $\underline{S}_1$:

$$\underline{S}_1 = K\left(\frac{pRrT}{\hat{p} - Rp}\right)^{\frac{1}{\gamma+\epsilon}}. \tag{14}$$

It is worth pointing out that explicit expressions for the critical stock price are rare. Indeed, we will lose this explicitness once constant proportional dividends are introduced.

For future use, note that substituting (14) into (12) implies that the European call can be valued by a formula similar to that of the the early exercise premium in (11):

$$c^{(1)}(S) = \left(\frac{S}{\underline{S}_1}\right)^{\gamma+\epsilon} pKRrT \equiv A^{(1)}(S).$$

Equations (8) and (14) represent the Canadian versions of (or Laplace-Carson transforms of) the American put and exercise boundary respectively. While these first approximations are simple and explicit, numerical evaluation indicates substantial approximation error. In particular, since the value of an American put is a concave function of its maturity (see Figure 5), randomizing the maturity leads to undervaluation by Jensen's inequality. This error can be reduced by lowering the variance of the distribution governing maturity. Unfortunately, if a random variable with an exponential distribution has mean T, then its variance is T^2. The next section uses a two parameter distribution for maturity, which permits keeping the mean maturity constant at T, while reducing the variance as much as desired. As the variance approaches zero, the result is a de facto inversion of the Laplace-Carson transform (8) of the American put value.

IV Gamma Maturity Valuation

Consider an investor who is faced with the problem of allocating his investable wealth among n different securities. If the security returns are independently and identically distributed (i.i.d.), the variance minimizing allocation is to invest an equal proportion in each security. By the same token, a simple and efficient way to reduce the variance of our option's random maturity is to split it into n i.i.d sub-periods. If we also assume that each of the n periods is exponentially distributed with parameter λ, then the maturity date τ is gamma distributed:

$$\text{Prob}\{\tau \in dt\} = \frac{\lambda^n}{(n-1)!} t^{n-1} e^{-\lambda t} dt.$$

In order that the mean maturity be T, each subperiod must have mean $\triangle \equiv T/n$, which implies $\lambda = 1/\triangle$. By assuming that the maturity is gamma distributed instead of exponentially distributed, the variance is reduced by a factor of $\frac{1}{n}$ to only T^2/n. Figure 6 shows three gamma density functions, with each corresponding to a maturity of mean $T = 1$ year, and with variances of 1, 1/2, and 1/3 respectively. The densities are converging to a Dirac delta function centered at $T = 1$ year.

Let $P^{(n)}(S)$ denote the initial value of a Canadian put option which can be exercised for $(K-S)^+$ at any time up to and including the n-th jump time of a standard Poisson process (with intensity $\lambda = 1/\triangle$). To value this put, we use dynamic programming. Accordingly, suppose that $n-1$ jumps have occurred and that the investor is holding a put maturing at the next jump time of the Poisson process. This valuation problem was solved in the previous section, with the solution $P^{(1)}(S)$ given by (5), except that T must be everywhere replaced by $\triangle \equiv T/n$.

We now back up a random time period and think of $P^{(1)}(S)$ as the random payoff occurring at the end of this random period, provided that no exercise has occurred beforehand. Since exercising yields a payoff of $(K-S)^+$ as usual, the value of the Candian put with two jumps to maturity is:

$$P^{(2)}(S) = \sup_{B>0} E_S\{e^{-r\tau_B}[K-B]^+ 1(\tau_B < \tau_2) + e^{-r\tau_2} P^{(1)}(S_{\tau_2}) 1(\tau_B \geq \tau_2)\}, \qquad S > \underline{S}_2, \tag{15}$$

where τ_2 denotes the length of the second random period prior to maturity and $\underline{S}_2$ denotes the unknown optimal exercise boundary. Once again, the stationarity of the barrier B implies that the expectation in (15) can be evaluated in closed form and the result can be maximized over barriers analytically.

However, a perhaps simpler approach is to work with Laplace-Carson transforms. Proceeding by analogy with the previous section, let $D(S; T-t, B)$ denote the time t value of a down-and-out put with *fixed* maturity T, out barrier B, and which pays a rebate of $K-B$ at the first passage time to B, if this occurs before T, and which pays $P^{(1)}(S_T)$ at T otherwise. Then, $D(S; T-t, B)$ satisfies the Black Scholes p.d.e.:

$$\frac{\sigma^2}{2} S^2 D_{ss}(S; T-t, B) + rSD_s(S; T-t, B) - rD(S; T-t, B) = D_T(S; T-t, B), \quad S \in (B, \infty), t \in (0, T), \tag{16}$$

subject to the terminal condition $D(S; 0, B) = P^{(1)}(S)$ and the boundary conditions:

$$\lim_{S \uparrow \infty} D(S; T-t, B) = 0, \qquad \lim_{S \downarrow B} D(S; T-t, B) = K - B, \qquad t \in (0, T).$$

The value of the Canadian put maturing after two more jumps of the Poisson process is related to this fixed maturity claim by:

$$P^{(2)}(S) = \sup_B \lambda \int_0^\infty e^{-\lambda t} D(S; t, B) dt. \tag{17}$$

Taking Laplace-Carson transforms of both sides of the p.d.e. (16) implies that:

$$\frac{\sigma^2}{2} S^2 P^{(2)}_{ss}(S) + rSP^{(1)}_s(S) - rP^{(2)}(S) = \lambda[P^{(2)}(S) - P^{(1)}(S)], \quad S > \underline{S}_2, \tag{18}$$

subject to the following boundary conditions:

$$\lim_{S \uparrow \infty} P^{(2)}(S) = 0, \qquad \lim_{S \downarrow \underline{S}_2} P^{(2)}(S) = K - \underline{S}_2, \qquad \lim_{S \downarrow \underline{S}_2} P^{(2)}_s(S) = -1. \tag{19}$$

This simpler free boundary problem can be solved analytically. Figure 7 graphs the solution for the Canadian put value against the stock price. Figure 8 shows an example of how this solution was obtained by working backwards through calendar time. The graph shows a realization in which the first jump time occurred at $\tau = 0.53$, while the put matured at the second jump time of 0.93. As indicated, the critical stock price over the earlier of the two periods is below the critical stock price of the later period because the end of period payoff is greater (i.e., $P^{(1)}(S) \geq K - S$). Thus, the optimal exercise boundary over the two remaining periods may be thought of as a staircase, with stair levels $\underline{S}_2$ and $\underline{S}_1$, as graphed in Figure 9.

More generally, let $P^{(m)}(S)$ and $\underline{S}_m$ respectively denote the Canadian put value and exercise boundary stair levels with m random periods to maturity, $m = 0, 1, \ldots, n$, with $P^{(0)}(S) \equiv (K - S)^+$ and $\underline{S}_0 \equiv K$. Then $P^{(m)}(S)$ and $\underline{S}_m$ jointly solve the following sequence of free boundary problems:

$$\frac{\sigma^2}{2} S^2 P^{(m)}_{ss}(S) r S P^{(m)}_{s}(S) - r P^{(m)}(S) = \lambda[P^{(m)}(S) - P^{(m-1)}(S)], \text{ for } S \in (\underline{S}_m, \infty), \tag{20}$$

subject to the boundary conditions:

$$\lim_{S \uparrow \infty} P^{(m)}(S) = 0, \lim_{S \downarrow \underline{S}_m} P^{(m)}(S) = K - \underline{S}_m, \lim_{S \downarrow \underline{S}_m} P^{(m)}_s(S) = -1, \text{ for } m = 1, \ldots, n. \tag{21}$$

Substituting $\lambda \equiv \frac{1}{\triangle}$ on the right side of (20) and comparing with the Black Scholes p.d.e. (3) indicates an alternative interpretation of the approximation induced by our randomization procedure. Our Canadian put value $P^{(m)}(S)$ is also the approximation for $P(T - m\triangle, S; T)$ which arises when time is discretized and the maturity derivative $P_T(t, S; T) \equiv \frac{\partial P}{\partial T}(t, S; T)$ in (3) is replaced with the finite difference $\lambda[\frac{P^{(m)}(S) - P^{(m-1)}(S)}{\triangle}] = \frac{\triangle P^{(m)}(S)}{\triangle}$. Note however that the spatial derivatives are not replaced with their finite differences, in contrast to standard finite difference schemes or the binomial model[4]. The notion of discretizing time while leaving space continuous is known in the numerical methods literature as the *method of horizontal lines* or *Rothe's method* (see Rothe[35] and Rektorys[32]). Its application to free boundary problems has been promulgated in Meyer[27],[28] and in Meyer & van der Hoek[29], who use it to numerically value American options. Goldenberg and Schmidt[13] test this numerical scheme against other approaches and find that it is highly accurate, although slightly slower[5] than some other approaches.

[4]The binomial model uses a forward finite difference for the maturity derivative leading to an explicit scheme. The appearance of a backward difference for the maturity derivative indicates that our randomization procedure may be considered as the limiting case of a fully implicit scheme, where the size of each space step is infinitessimally small. Surprisingly, this implicit scheme has a semi-explicit solution.

[5]However, given the speed of modern computers, they argue that its inherent accuracy makes it the method of choice among those tested.

This numerical work suggests that our values obtained from randomization should be even more accurate, since our analytic approach circumvents the error introduced by bounding and discretizing the spatial domain. The accuracy of our formulas may be anticipated *a priori* by noting that as the maturity date T approaches infinity holding the number of periods n fixed, then $\lambda \downarrow 0$ and thus the problem (20) describing the randomized put value approaches that of the perpetual put. As a result, the randomized put solution with any number of jumps remaining will converge to the correct perpetual solution. Conversely, as n gets arbitrarily large with T held fixed, then the finite difference $\frac{\triangle P^{(m)}(S)}{\triangle}$ on the right side of (20) converges to the maturity derivative $P_T(t,S;T)$ in (3). As a result, we conjecture[6]that the solution $(P^{(n)}(S), \underline{S}_n)$ to our Canadian option problem converges to the unknown solution $(P(0,S;T), \underline{S}(0;T))$ of the American problem (1) or (3).

Recall from Section III that our formulas for random maturity option values depended on whether the option was in or out-of-the-money. Similarly, our formula[7] for the Canadian put value, $P^{(n)}(S)$, depends on which interval $(\underline{S}_i, \underline{S}_{i-1})$ contains the current spot price S:

$$P^{(n)}(S) = \begin{cases} p_0^{(n)}(S) + b_1^{(n)}(S) & \text{if } S > \underline{S}_0 \equiv K \\ v_i^{(n)}(S) + b_i^{(n)}(S) + A_i^{(n)}(S;1) & \text{if } S \in (\underline{S}_i, \underline{S}_{i-1}], i = 1, \ldots, n \\ K - S & \text{if } S \leq \underline{S}_n. \end{cases} \quad (22)$$

where $p_0^{(n)}(S)$ is the out-of-the-money[8] value of a European put maturing in n (random-length) periods:

$$p_0^{(n)}(S) = \left(\frac{S}{K}\right)^{\gamma-\epsilon} \sum_{k=0}^{n-1} \frac{\left(2\epsilon \ln\left(\frac{S}{K}\right)\right)^k}{k!} \sum_{l=0}^{n-k-1} \binom{n-1+l}{n-1} [KR^n q^n p^{l+k} - K\hat{q}^n \hat{p}^{l+k}], \qquad S > K, \quad (23)$$

with $\triangle \equiv T/n$, $\gamma \equiv \frac{1}{2} - \frac{r}{\sigma^2}$,

$$R \equiv \frac{1}{1 + r\triangle}, \epsilon \equiv \sqrt{\gamma^2 + \frac{2}{R\sigma^2\triangle}}, \quad (24)$$

$p, q, \hat{p}, \hat{q}$ given in (10), and for $i = 1, \ldots, n$, $v_i^{(n)}(S)$ is the value of a short forward position maturing in $n - i + 1$ periods:

$$v_i^{(n)}(S) = KR^{n-i+1} - S,$$

$b_i^{(n)}(S)$ is the initial value of interest received below the boundary for the first $n - i + 1$ periods:

$$b_i^{(n)}(S) = \sum_{j=1}^{n-i+1} \left(\frac{S}{\underline{S}_{n-j+1}}\right)^{\gamma-\epsilon} \sum_{k=0}^{j-1} \frac{\left(2\epsilon \ln\left(\frac{S}{\underline{S}_{n-j+1}}\right)\right)^k}{k!} \sum_{l=0}^{j-k-1} \binom{j-1+l}{j-1} q^j p^{k+l} R^j K r \triangle, \quad (25)$$

[6]While numerical implementation of our solution will prove to be consistent with this conjectured convergence, a formal proof of convergence remains an open question.

[7]Note that (22) is closely related to the value of a fixed maturity American option when the variance rate is gamma distributed. See Madan and Chang[24] for a closed form solution for European options.

[8]See (61) and (60) in Appendix 2 for the randomized values of European calls and in-the-money European puts respectively.

and finally, $A_i^{(n)}(S;1)$ is the initial value[9] of an out-of-the-money European call less interest paid above the boundary over the complementary period:

$$A_i^{(n)}(S;h) \equiv \sum_{j=h}^{n-i+1} \left(\frac{S}{\underline{S}_{n-j+1}}\right)^{\gamma+\epsilon} \sum_{k=0}^{j-1} \frac{\left(2\epsilon \ln\left(\frac{\underline{S}_{n-j+1}}{S}\right)\right)^k}{k!} \sum_{l=0}^{j-k-1} \binom{j-1+l}{j-1} p^j q^{k+l} R^j K r \triangle. \tag{26}$$

The formula in the first line of (22) reflects the randomized version of the well-known decomposition of the American put value into the value of the corresponding European put and the early exercise premium (see [8], [16], [17], [18]), and [21]). The formula in the second line is the randomized version of a new decomposition of the American put value into the value if forced to sell at a given date prior to expiration, and the premia which arise because exercise can occur before or after this date. Appendix 1 provides an economic justification for this new decomposition in the fixed maturity case[10]. The final line of (22) indicates that the put should be exercised immediately if the stock price S is at or below our approximation for the critical stock price $\underline{S}_n$.

The staircase levels comprising the exercise boundary can be determined by recursive solution of an explicit formula. Continuity at the strike price in each period $m = 1, \ldots, n$ implies $c_1^{(m)}(K) = A_1^{(m)}(K;1)$, which in turn implies the following explicit solution for each critical stock price $\underline{S}_m$:

$$\underline{S}_m = K\left(\frac{pRKr\triangle}{c_1^{(m)}(K) - A_1^{(m)}(K;2)}\right)^{\frac{1}{\gamma+\epsilon}}, \qquad m = 1, \ldots, n, \tag{27}$$

where from (61) in Appendix 2, the at-the-money call value with m periods to maturity simplifies to:

$$c_1^{(m)}(K) = \sum_{l=0}^{m-1} \binom{m-1+l}{m-1} [K\hat{p}^m \hat{q}^l - KR^m p^m q^l], \qquad m = 1, \ldots, n. \tag{28}$$

Since $A^{(m)}$ in (27) depends on $\underline{S}_{m-1}$ to $\underline{S}_1$, the critical stock prices must be solved recursively, with $\underline{S}_1$ given by:

$$\underline{S}_1 = K\left(\frac{pRr\triangle}{\hat{p} - Rp}\right)^{\frac{1}{\gamma+\epsilon}}.$$

V Implementation

Our solution (22) for the Canadian put value $P^{(n)}(S)$ is a triple sum. Clearly, we need the number of periods n to be small in order to achieve computational efficiency. This section describes an ingenious technique called Richardson extrapolation, which can be used to provide accurate answers using at most 3 periods. Richardson extrapolation

[9] This value also accounts for the smoothness at the exercise boundary in every period (see Appendix 1).

[10] Uniqueness of (iterated) Laplace-Carson transforms implies that this decomposition holds in the randomized case as well.

has been used previously to accelerate valuation schemes for American[11] options. Geske and Johnson[12] first used Richardson extrapolation in a financial context to speed up and simplify their compound option valuation model. Breen[4] applied this idea to accelerate the binomial model of Cox, Ross, and Rubinstein[10]. Broadie and Detemple[6] use it to accelerate a hybrid of the binomial and Black-Scholes models. Finally, Huang, Subrahmanyam, and Yu[16] use the approach to accelerate the integral representation of the early exercise premium.

The explicit nature of our solution (22) can be used to show that our approximation is a smooth function $\hat{P}$ of the mean period length $\triangle$, as required by Richardson extrapolation. Given the requisite smoothness, an N point Richardson extrapolation assumes that the approximation is adequately described by the first N terms in a Taylor series expansion about the origin:

$$\hat{P}(\triangle) \approx \sum_{n=0}^{N-1} \frac{\partial^n \hat{P}(0)}{\partial \triangle^n} \frac{\triangle^n}{n!}.$$

The N coefficients $\frac{\partial^n \hat{P}(0)}{\partial \triangle^n}, n = 0, 1, \ldots, N-1$ can be determined by using any N values of $\triangle$ for which $\hat{P}(\triangle)$ is known. The N point Richardson extrapolation is then the first coefficient $\hat{P}(0)$.

For example, a 3 point Richardson extrapolation can be obtained by assuming that our approximation is approximately quadratic in the mean period length:

$$\hat{P}(\triangle) \approx \hat{P}(0) + \hat{P}'(0)\triangle + \frac{1}{2}\hat{P}''(0)\triangle^2.$$

Substituting in $\triangle = T, \triangle = T/2$ and $\triangle = T/3$ leads to 3 equations in the 3 unknowns $\hat{P}(0), \hat{P}'(0)$, and $\hat{P}''(0)$. Inverting the system implies that the 3 point extrapolation is given by:

$$\hat{P}^{1:3}(0) \equiv \frac{1}{2}\hat{P}(T) - 4\hat{P}(T/2) + \frac{9}{2}\hat{P}(T/3). \tag{29}$$

Figures 10 and 11 illustrate the idea behind a 3 point extrapolation. From Marchuk and Shaidurov[25], p. 24, an N point Richardson extrapolation is the following weighted[12] average of N Canadian put values:

$$\hat{P}^{1:N}(0) \equiv \sum_{n=1}^{N} \frac{(-1)^{N-n} n^N}{n!(N-n)!} \hat{P}(T/n). \tag{30}$$

[11] Boyle, Evnine, and Gibbs[3] also use the approach to value multivariate options.

[12] The weights always sum to unity and alternate in sign. In general, higher order approximations involve weights with greater absolute value. As a result, implementing higher order extrapolations on a computer requires double precision to control roundoff error.

It can be shown that the resulting error is of $O(\triangle^N)$. The critical stock price can be obtained by imposing either of the smooth pasting conditions in (21) or[13] by Richardson extrapolation:

$$\underline{S}^{1:N}(0) \equiv \sum_{n=1}^{N} \frac{(-1)^{N-n} n^N}{n!(N-n)!} \underline{S}(T/n).$$

In our implementation of the 3 point Richardson extrapolation (29), we found it useful to modify the weights slightly. To understand the nature of this modification, it is instructive to examine a typical test case: $S = 100, K = 100, T = 1, r = .1$, and $\sigma = .3$. The true value based on the binomial method with 2000 time steps appears to be 8.3378. Table 1 shows that for this test case, extrapolated put values obtained from our approach converge rapidly to this true value, with penny accuracy obtained in only 5 points. In contrast, the unextrapolated values converge very slowly from below. The undervaluation observed in this test case was observed in other cases as well.

The source of this undervaluation is due to the fact that the true American put value is a concave function of maturity (see Figure 5). Jensen's inequality then implies that our approximation obtained by randomizing maturity lies below the true value. Since the variance of our gamma distributed maturity is T^2/n, the undervaluation gets smaller as the time to maturity T approaches zero.

We may mitigate the undervaluation by adjusting the Richardson weights upwards in a manner that depends on the time to maturity. After some numerical experimentation, we settled on the following "fine-tuning" of the 3 point Richardson extrapolation:

$$\hat{P}^{1:3m}(0) \equiv \frac{1}{2}\hat{P}(T) - 4[1 - .0002(5-T)^+]\hat{P}(T/2) + \frac{9}{2}\hat{P}(T/3). \tag{31}$$

Thus, no adjustment is made if the time to maturity exceeds 5 years. For shorter maturities, the middle weight is modified so that the three weights sum to *slightly* more than one. The maximum adjustment to the middle weight is .004, which occurs when $T = 0$. When applied to our test case, the modified value is 8.3332, giving penny accuracy using at most 3 periods.

Broadie and Detemple[6] conduct extensive numerical simulations of a wide array of methods for valuing American options. Their results indicate that our approach dominates most methods in terms of speed and accuracy. Indeed, their results indicate that our above three point extrapolation is on the "efficient frontier",

[13] We prefer the former method when accuracy is important and the latter method when speed matters.

intermediate in terms of speed and accuracy between the quadratic formula and their capped option formulae. We believe that our three point extrapolation given by (31) represents a satisfactory tradeoff between speed and accuracy. Furthermore, our approach displays more flexibility than other efficient approaches, in that speed or accuracy can be emphasized whenever one consideration is paramount, simply by varying the number of Richardson points used. In particular, it appears that arbitrary accuracy can be achieved in contrast to other efficient methods.

VI Extension to Positive Dividends and American Calls

Merton[26] generalized the Black-Scholes analysis to continuously-paid dividends which are either constant or proportional to the price of the underlying. He did not permit a dividend rate which is linear in the spot price, presumably due to the difficulty in generating analytic solutions under this assumption. While we are also unable to deal with a linear dividend rate, this section values Canadian puts explicitly when the dividend payout rate has both a fixed and a proportional component. We also show that our approximation to the put's critical stock price is still given by an explicit formula when dividends are constant, but must be determined numerically when there is a proportional component to the dividend flow. Finally, we develop corresponding results for call options.

We thus assume that the underlying stock pays dividends continuously until the fixed maturity T. To obtain a truly fixed component ϕ of this dividend flow, we follow Roll[34] in assuming that this component has been escrowed out of the stock price. In other words, the time t stock price S_t decomposes into:

$$S_t = \frac{\phi}{r}[1 - e^{-r(T-t)}] + s_t, \qquad t \in [0, T], \tag{32}$$

where the first term is the present value at t of the constant flow ϕ until T, and the residual s_t is the *stripped price*, reflecting the stripping off of the fixed component of the dividend flow from the stock price. We assume that the risk-neutralized process for the stripped price $\{s_t, t \in [0, T]\}$ is the following geometric Brownian motion:

$$s_t = s \exp\left[\left(r - \delta - \frac{\sigma^2}{2}\right) t + \sigma W_t\right], \qquad t \in [0, T], \tag{33}$$

where $\{W_t, t \in [0, T]\}$ is a standard Brownian motion, and from (32), the initial value is:

$$s = S - \frac{\phi}{r}[1 - e^{-rT}]. \tag{34}$$

Thus, the dollar dividend rate d_t has both a fixed and a proportional component:

$$d_t = \phi + \delta s_t, \qquad t \in [0, T]. \tag{35}$$

The parameter ϕ captures the stickiness of dividends in the short run, while δ captures the tendency for dividends to increase with stock prices in the long run. If $\delta = 0$, then ϕ is the constant dividend rate, while if $\phi = 0$, then δ is the constant dividend *yield*, since $s_t = S_t$ from (32).

VI-A Positive Dividends and American Puts

We generalize the previous analysis by letting $P(t, s; T)$ denote the value of an American put as a function of the current time t, the current *stripped* price s, and the maturity date T. We also define the *critical stripped price* $\underline{s}(t)$ as the largest stripped price s at which the American put value $P(t, s; T)$ equals its exercise value $K - s - \frac{\phi}{r}[1 - e^{-r(T-t)}]$, for $t \in [0, T]$. From (32), the critical stock price $\underline{S}(t)$ is now defined by:

$$\underline{S}(t) \equiv \frac{\phi}{r}[1 - e^{-r(T-t)}] + \underline{s}(t), \qquad t \in [0, T]. \tag{36}$$

In the random maturity setting, the underlying stock pays dividends continuously until the opton matures. Recalling that $R \equiv \frac{1}{1+r\triangle}$ is the discount factor over a single period of random length, the random maturity analog of (34) is:

$$s = S - \phi\triangle(R + R^2 + \ldots + R^n) = S - \frac{\phi}{r}R(1 - R^n). \tag{37}$$

We define $P^{(m)}(s)$ as our approximation for the American put value when m random periods remain, $m = 1, \ldots, n$. Our approximation for the critical stripped price, $\underline{s}_m$, is the largest s satisfying $P^{(m)}(s) = K - s - \frac{\phi}{r}R(1 - R^m)$, $m = 1, \ldots, n$.

The values of European options maturing in n random-length periods are:

$$p^{(n)}(s) = \begin{cases} \left(\frac{s}{K}\right)^{\gamma-\epsilon} \sum_{k=0}^{n-1} \frac{\left(2\epsilon \ln\left(\frac{s}{K}\right)\right)^k}{k!} \sum_{l=0}^{n-k-1} \binom{n-1+l}{n-1} [KR^n q^n p^{k+l} - KD^n \hat{q}^n \hat{p}^{k+l}] & \text{if } s > K \\ KR^n - sD^n + c^{(n)}(S) & \text{if } s \le K \end{cases} \tag{38}$$

$$c^{(n)}(s) = \begin{cases} sD^n - KR^n + p^{(n)}(s) & \text{if } s > K \\ \left(\frac{s}{K}\right)^{\gamma+\epsilon} \sum_{k=0}^{n-1} \frac{\left(2\epsilon \ln\left(\frac{K}{s}\right)\right)^k}{k!} \sum_{l=0}^{n-k-1} \binom{n-1+l}{n-1} [KD^n \hat{p}^n \hat{q}^{k+l} - KR^n p^n q^{k+l}] & \text{if } s \le K, \end{cases} \tag{39}$$

where now $\gamma \equiv \frac{1}{2} - \frac{r-\delta}{\sigma^2}$, $R, \epsilon, p, q, \hat{p}, \hat{q}$, are again given by (24) and (10), while:

$$D \equiv \frac{1}{1 + \delta\triangle}. \tag{40}$$

For $\delta = 0$ and $\phi \ge rK$, American puts are not rationally exercised early. Consequently, the Canadian put value $P^{(n)}(s)$ is given by (38) in this case. For $\delta > 0$ or $\phi < rK$, the Canadian put value decomposes as:

$$P^{(n)}(s) = \begin{cases} p_0^{(n)}(s) + b_1^{(n)}(s) & \text{if } s > \underline{s}_0 \equiv K \\ v_i^{(n)}(s) + b_i^{(n)}(s) + A_i^{(n)}(s;1) & \text{if } s \in (\underline{s}_i, \underline{s}_{i-1}], i = 1, \ldots, n \\ K - S & \text{if } s \leq \underline{s}_n, \end{cases} \tag{41}$$

where for $i = 1, \ldots, n$, $v_i^{(n)}(s)$ is the initial value of a short forward position maturing in $n - i + 1$ periods:

$$v_i^{(n)}(s) = KR^{n-i+1} - sD^{n-i+1} - \phi R \frac{R^{n-i+1} - R^n}{1 - R}, \tag{42}$$

$b_i^{(n)}(s)$ is the initial value of the interest less dividends (net interest) received when below the boundary for the first $n - i + 1$ periods:

$$b_i^{(n)}(s) = \sum_{j=1}^{n-i+1} \left(\frac{s}{\underline{s}_{n-j+1}}\right)^{\gamma-\epsilon} \sum_{k=0}^{j-1} \frac{\left(2\epsilon \ln\left(\frac{s}{\underline{s}_{n-j+1}}\right)\right)^k}{k!} \sum_{l=0}^{j-k-1} \binom{j-1+l}{j-1} [q^j p^{k+l} R^j (Kr - \phi) - \hat{q}^j \hat{p}^{k+l} D^j \underline{s}_{n-j+1}\delta] \triangle, \tag{43}$$

while $A_i^{(n)}(s;1)$ represents the initial value of a European call less the net interest paid above the boundary over the complementary period, after accounting for the smoothness at the exercise boundary in every period:

$$A_i^{(n)}(s;h) = \sum_{j=h}^{n-i+1} \left(\frac{s}{\underline{s}_{n-j+1}}\right)^{\gamma+\epsilon} \sum_{k=0}^{j-1} \frac{\left(2\epsilon \ln\left(\frac{\underline{s}_{n-j+1}}{s}\right)\right)^k}{k!} \sum_{l=0}^{j-k-1} \binom{j-1+l}{j-1} [p^j q^{k+l} R^j (Kr - \phi) - \hat{p}^j \hat{q}^{k+l} D^j \underline{s}_{n-j+1}\delta] \triangle.$$

Continuity in s at the strike price in each period $m = 1, \ldots, n$ again implies $c_1^{(m)}(K) = A_1^{(m)}(K;1)$, which in turn implies that each critical stripped price $\underline{s}_m$ implicitly solves:

$$c_1^{(m)}(K) - A_1^{(m)}(K;2) = \left(\frac{K}{\underline{s}_m}\right)^{\gamma+\epsilon} [pR(Kr - \phi) - \hat{p}D\underline{s}_m\delta]\triangle, \qquad m = 1, \ldots, n, \tag{44}$$

where from (39), the at-the-money call value on the left hand side (LHS) of (44) simplifies to:

$$c_1^{(m)}(K) = \sum_{l=0}^{m-1} \binom{m-1+l}{m-1} [KD^m \hat{p}^m \hat{q}^l - KR^m p^m q^l] \qquad m = 1, \ldots, n. \tag{45}$$

It is straightforward to recursively solve (44) numerically for each critical stripped price $\underline{s}_m$, since $\underline{s}_m$ does not appear on the LHS. Setting $\delta = 0$ in (44) implies the following semi-explicit solution for the critical stripped prices when the dividend rate is constant at ϕ:

$$\underline{s}_m = K \left(\frac{pR(Kr - \phi)\triangle}{c_1^{(m)}(K) - A_1^{(m)}(K;2)}\right)^{\frac{1}{\gamma+\epsilon}}, \qquad m = 1, \ldots, n, \tag{46}$$

where the call value $c_1^{(m)}(K)$ is now given by (28). This solution is a good initial guess when numerically solving (44). From (37), the critical stock price $\underline{S}_m$ is determined by:

$$\underline{S}_m = \frac{\phi}{r} R(1 - R^m) + \underline{s}_m, \qquad m = 1, \ldots, n, \tag{47}$$

where $\underline{s}_m$ is given by (46) when $\delta = 0$ and solves (44) otherwise.

VI-B Positive Dividends and American Calls

Let $C(t, s; T)$ denote the value of a standard American call as a function of the current time t, the current *stripped* price s, and the fixed maturity date T. We also define the call's *critical stripped price* $\bar{s}(t)$ as the lowest stripped price s at which the American call value $C(t, s; T)$ equals its exercise value $s + \frac{\phi}{r}[1 - e^{-r(T-t)}] - K$, for $t \in [0, T]$. From (32), the call's critical stock price $\bar{S}(t)$ is defined by:

$$\bar{S}(t) \equiv \frac{\phi}{r}[1 - e^{-r(T-t)}] + \bar{s}(t), \qquad t \in [0, T]. \tag{48}$$

$C^{(m)}(s)$ denotes our approximation for the American call value when m random periods remain, $m = 1, \ldots, n$. Our approximation for the critical stripped price, $\bar{s}_m$, is the smallest s satisfying $C^{(m)}(s) = s + \phi \triangle R \frac{1-R^m}{1-R} - K$, $m = 1, \ldots, n$. For $\delta = 0$ and $\phi \leq rK$, American calls are not rationally exercised early. Consequently, the Canadian call value $C^{(n)}(s)$ is given by (39) in this case. For $\delta > 0$ or $\phi > rK$, the value of the Canadian call decomposes as:

$$C^{(n)}(s) = \begin{cases} S - K & \text{if } s \geq \bar{s}_n \\ -v_i^{(n)}(s) + \alpha_i^{(n)}(s) + B_i^{(n)}(s; 1) & \text{if } s \in [\bar{s}_{i-1}, \bar{s}_i), i = 1, \ldots, n \\ c_0^{(n)}(s) + \alpha_1^{(n)}(s) & \text{if } s < \bar{s}_0 \equiv K \end{cases} \tag{49}$$

where for $i = 1, \ldots, n$, $-v_i^{(n)}(s)$ is the initial value of a long[14] forward position maturing in $n - i + 1$ periods,

$$\alpha_i^{(n)}(s) = \sum_{j=1}^{n-i+1} \left(\frac{s}{\bar{s}_{n-j+1}} \right)^{\gamma+\epsilon} \sum_{k=0}^{j-1} \frac{\left(2\epsilon \ln\left(\frac{\bar{s}_{n-j+1}}{s}\right)\right)^k}{k!} \sum_{l=0}^{j-k-1} \binom{j-1+l}{j-1} [\hat{p}^j \hat{q}^{k+l} D^j \bar{s}_{n-j+1} \delta - p^j q^{k+l} R^j (Kr - \phi)] \triangle$$

is the initial value of the dividends less interest (net dividends) received when above the boundary for the first $n - i + 1$ periods, while:

$$B_i^{(n)}(s; h) = \sum_{j=h}^{n-i+1} \left(\frac{s}{\bar{s}_{n-j+1}} \right)^{\gamma-\epsilon} \sum_{k=0}^{j-1} \frac{\left(2\epsilon \ln\left(\frac{s}{\bar{s}_{n-j+1}}\right)\right)^k}{k!} \sum_{l=0}^{j-k-1} \binom{j-1+l}{j-1} [\hat{q}^j \hat{p}^{k+l} D^j \bar{s}_{n-j+1} \delta - q^j p^{k+l} R^j (Kr - \phi)] \triangle.$$

[14] See (42) for $v_i^{(n)}(s)$.

$B_i^{(n)}(s;1)$ represents the initial value of a European put less the net dividends paid below the boundary over the complementary period, after accounting for the smoothness at the exercise boundary in every period. Appendix 1 provides an economic justification for the decompositions in (49).

Continuity in s at the strike price in each period $m = 1, \ldots, n$ implies that the critical stripped price $\bar{s}_m$ implicitly solves:

$$p_0^{(m)}(K) - B_1^{(m)}(K;2) = \left(\frac{K}{\bar{s}_m}\right)^{\gamma-\epsilon} [\hat{q}D\bar{s}_m\delta - qR(Kr - \phi)]\triangle, \qquad m = 1, \ldots, n, \tag{50}$$

where from (38), the at-the-money put value on the LHS of (50) simplifies to:

$$p_0^{(m)}(K) = \sum_{l=0}^{m-1} \binom{m-1+l}{m-1} [KR^m q^m p^l - KD^m \hat{q}^m \hat{p}^l] \qquad m = 1, \ldots, n. \tag{51}$$

It is straightforward to solve (50) numerically for the critical stripped price $\bar{s}_m$, since it does not appear on the LHS. If $\phi = rK$, then (50) yields the following explicit solution for the critical stripped price:

$$\underline{s}_m = K \left(\frac{K\hat{q}D\delta\triangle}{p_0^{(m)}(K) - B_1^{(m)}(K;2)}\right)^{\frac{1}{\gamma-\epsilon-1}}, \qquad m = 1, \ldots, n. \tag{52}$$

This solution is a good initial guess when numerically solving (50). From (37), the call's critical stock price $\bar{S}_m$ is determined by:

$$\bar{S}_m = \frac{\phi}{r}R(1 - R^m) + \bar{s}_m, \qquad m = 1, \ldots, n, \tag{53}$$

where $\bar{s}_m$ is given by (52) when $\phi = rK$ and solves (50) otherwise. Appendix 2 collects all the formulas needed to implement European and American puts and calls when the underlying has a continuous payout with a fixed component ϕ and a proportional component δ.

VII Summary

We implemented a new approach to valuing American options, which is fast, accurate, and flexible. The approach is to value "Canadian" options, which mature by definition at the n-th jump time of a standard Poisson process. Between jump times, the memoryless property of the exponential distribution implies that the option value and exercise boundary are time-stationary. In contrast, at jump times, the option value jumps down and the exercise boundary jumps up. The local time-stationarity yields semi-explicit solutions for the option value and critical stock price, while the jump behavior roughly captures the global behavior of these values. As we let the number of jump times approach infinity, keeping the mean maturity fixed, our numerical results indicate that the Canadian option value appears to converge smoothly from below to the true American option value. Richardson extrapolation is used to dramatically enhance convergence, with a modified 3 point scheme performing particularly well in numerical tests.

References

[1] Barone-Adesi G. and R. Whaley, 1987, Efficient Analytic Approximation of American Option Values, *The Journal of Finance*, 42, 301–320.

[2] Black, F. and M. Scholes, 1973, The Pricing of Options and Corporate Liabilities, *The Journal of Political Economy*, 81, 637–659.

[3] Boyle, P., J. Evnine and S. Gibbs, 1989, Numerical Evaluation of Multivariate Contingent Claims, *Review of Financial Studies*, 2, 2, 241-250.

[4] Breen, R., 1991, The Accelerated Binomial Option Pricing Model, *Journal of Financial and Quantitative Analysis*, 26, 2, 153–164.

[5] Brennan, M. and E. Schwartz, 1977, The Valuation of American Put Options, *Journal of Finance*, 32, 449–462.

[6] Broadie, M. and J. Detemple, 1995, American Option Valuation: New Bounds, Approximations, and a Comparison of Existing Methods, *Review of Financial Studies*, forthcoming.

[7] Bunch, D. and H. Johnson, 1992, A Simple and Numerically Efficient Valuation Method for American Puts Using a Modified Geske-Johnson Approach, *Journal of Finance*, 47, 2, 809–816.

[8] Carr, P., R. Jarrow, and R. Myneni, 1992, Alternative Characterizations of American Put Options, *Mathematical Finance*, 87–106.

[9] Chesney, M., 1989, Pricing American Currency Options: An Analytical Approach, HEC working paper.

[10] Cox, J., S. Ross, and M. Rubinstein, 1979, Option Pricing: A Simplified Approach, *Journal of Financial Economics*, 7, 229–263.

[11] Duffie, D., 1988, An Extension of the Black-Scholes Model of Security Valuation, *Journal of Economic Theory* 46, 1, 194–204.

[12] Geske, R. and H. Johnson, 1984, The American Put Option Valued Analytically, *Journal of Finance*, 39, 1511–1524.

[13] Goldenberg D. and R. Schmidt, 1995, Estimating the Early Exercise Boundary and Pricing American Options, Rensselaer Polytechnic Institute working paper.

[14] Grant, D., G. Vora, and D. Weeks, 1993, Path-Dependent Options: Extending the Monte Carlo Simulation Approach, University of New Mexico working paper.

[15] Ho T., R. Stapleton, and M. Subrahmanyam, 1994, A Simple Technique for the Valuation and Hedging of American Options, *The Journal of Derivatives*, Fall 1994, 52–66.

[16] Huang, J, M. Subrahmanyam, and G. Yu, 1996, "Pricing and Hedging American Options: A Recursive Integration Method", *Review of Financial Studies*, 9, 1, 277-300.

[17] Jacka, S.D., 1991, Optimal Stopping and the American Put, *Mathematical Finance*, 1, 2, 1–14.

[18] Jamshidian, F., 1989, Formulas for American Options, Merrill Lynch Capital Markets working paper.

[19] Johnson, H., 1983, An Analytic Approximation for the American Put Price, *Journal of Financial and Quantitative Analysis*, 18, 1, 141–148.

[20] Kamrad, B. and P. Ritchken, 1991, Multinomial Approximating Models for Options with k State Variables, Case Western Reserve University working paper.

[21] Kim, I. J., 1990, The Analytic Valuation of American Puts, *Review of Financial Studies*, 3, 4, 547–572.

[22] McKean, Jr., H. P., 1965, Appendix: A Free Boundary Problem for the Heat Equation Arising from a Problem in Mathematical Economics, *Industrial Management Review*, 6, 32–39.

[23] MacMillan, L., 1986, Analytic Approximation for the American Put Option, *Advances in Futures and Options Research,* Volume 1, 119-139.

[24] Madan, D., and E. Chang, Volatility Smiles, Skewness Premia and Risk Metrics: Applications of a Four Parameter Closed Form Generalization of Geometric Brownian Motion to the Pricing of Options, Univesity of Maryland working paper.

[25] Marchuk, G. and V. Shaidurov, 1983, *Difference Methods and Their Extrapolations,* Springer Verlag, NY.

[26] Merton, R. C., 1973, Theory of Rational Option Pricing, *Bell Journal of Economics and Management Science*, 4, 141–183.

[27] Meyer, G. H., 1970, On a Free Boundary Problem for Linear Ordinary Differential Equations and the One-Phase Stephan Problem, *Numerical Mathematics*, 16, 248–67.

[28] Meyer, G. H., 1979, One-Dimensional Free Boundary Problems *SIAM Review*, 19, 17–34.

[29] Meyer, G. H. and J. van der Hoek, 1994, The Evaluation of American Options with the Method of Lines, Georgia Institute of Technology working paper.

[30] Omberg, E., 1987, The Valuation of American Puts with Exponential Exercise Policies, *Advances in Futures and Options Research,* Volume 2, 117–142.

[31] Parkinson, M., 1977, Option Pricing: The American Put, *Journal of Business*, 50, 21–36.

[32] Rektorys, K., 1982, *The Method of Discretization in Time and Partial Differential Equations*, D. Reidel Publishing Co., Dordrecht-Boston, MA.

[33] Rendleman R., and B. Bartter, 1979, Two-State Option Pricing, *Journal of Finance*, 34, 1093-1110.

[34] Roll, R., 1977, An Analytic Valuation Formula for Unprotected American Call Options on Stocks with Known Dividends, *Journal of Financial Economics*, 5, 251–58.

[35] Rothe, E., 1930, Zweidimensionale parabolische Randwertaufgaben als Grenzfall eindimensinaler Randwert-taufgaben, *Math. Ann.* 102.

[36] Rubinstein, I., and L. Rubinstein, 1993, Partial Differential Equations in Classical Mathematical Physics, Cambridge University Press.

[37] Samuelson, P. A., 1965, Rational Theory of Warrant Pricing, *Industrial Management Review*, 6, 13–31.

[38] Tilley, J., 1992, Valuing American Options in a Path Simulation Model, Morgan Stanley working paper.

[39] Wilmott, P., J. Dewynne, and S. Howison, 1993, *Option Pricing: Mathematical Models and Computation*, Oxford Financial Press, Oxford, U.K.

Appendix 1

This appendix provides an economic justification for the following representation of an American put:

$$P^{(n)}(S) = \begin{cases} p_0^{(n)}(S) + b_1^{(n)}(S) & \text{if } S > \underline{S}_0 \equiv K \\ v_i^{(n)}(S) + b_i^{(n)}(S) + a_i^{(n)}(S) & \text{if } S \in (\underline{S}_i, \underline{S}_{i-1}], i = 1, \ldots, n \\ K - S & \text{if } S \le \underline{S}_n. \end{cases} \quad (54)$$

where for $i = 1, \ldots, n$:

$$v_i^{(n)}(S) = KR^{n-i+1} - S,$$

is the value of a short forward position maturing in $n - i + 1$ periods,

$$b_i(n)_i(S) = \sum_{j=1}^{n-i+1} \left(\frac{S}{\underline{S}_{n-j+1}} \right)^{\gamma-\epsilon} \sum_{k=0}^{j-1} \frac{\left(2\epsilon \ln \left(\frac{S}{\underline{S}_{n-j+1}} \right) \right)^k}{k!} \sum_{l=0}^{j-k-1} \binom{j-1+l}{j-1} q^j p^{k+l} R^j K r \triangle$$

is the initial value of interest received below the boundary for the first $n - i + 1$ periods, while:

$$a_i^{(n)}(S) = c_1^{(n)}(S) - \sum_{j=n-i+2}^{n} \left(\frac{S}{\underline{S}_{n-j+1}} \right)^{\gamma+\epsilon} \sum_{k=0}^{j-1} \frac{\left(2\epsilon \ln \left(\frac{\underline{S}_{n-j+1}}{S} \right) \right)^k}{k!} \sum_{l=0}^{j-k-1} \binom{j-1+l}{j-1} p^j q^{k+l} R^j K r \triangle$$

is the initial value of an out-of-the-money European call less interest paid above the boundary over the complementary period.

The formula in the first line of (22) for the out-of-the-money value of an American put reflects the randomized version of the well-known decomposition into the value of the corresponding European put and the early exercise premium (see [21], [17], and [8]). The formula (22) for the in-the-money value of an alive American put reflects the randomized version of a new decomposition into the value if forced to sell the underlying at a given date $T_x \in [0, T]$, and the premia which arise because exercise can occur before or after this date. This decomposition may be understood from the following trading strategy, which applies to fixed maturity (standard) American options.

The objective of the strategy is to convert the initial purchase of an American put into the final payoff of a European *call*. Suppose that at $t = 0$ an investor buys an alive American put maturing at T. In the period $t \in (0, T_x)$, the holder of the put exercises it each time the stock price crosses the exercise boundary from above and repurchases the put each time the stock price crosses from below. In order that the transitions self-finance, the investor keeps K dollars in an interest earning bank account and is short one share in the exercise region for $t \in [0, T_x]$. Thus, when the stock price is in this region during this period, the investor withdraws interest continuously so as to keep his bank balance flat at K. Consequently, if the stock price is still in the exercise region at T_x, the investor can liquidate the position for a net payoff $K - S_{T_x}$, matching that of a short forward position expiring at T_x. However, if the stock price is in the continuation region at T_x, obtaining this payoff requires that the investor borrow the delivery price K and purchase one share at T_x. For stock prices in the continuation region during the period $t \in (T_x, T)$, the investor continuously pays interest on these borrowings so as to keep his liability constant at K. As a result, his overall position in this region during this period consists of the alive American put, the constant liability of K, and long one share. During this period, the investor liquidates this position by exercising the put each time the stock price crosses the exercise boundary from above and re-enters the position each time the stock price crosses from below. The optimality of the exercise boundary again ensures that these transitions are self-financing. At expiration, the position is worthless if the stock price finishes below the strike and pays the difference between the stock price and the strike price otherwise.

Equating the initial cost of the strategy to the present value of the cash flows resulting from it gives the following fundamental decomposition of the value of an alive American put:

$$P(0, S; T) = v(0, S; T_x) + b(0, S; 0, T_x) + a(0, S; T_x, T), \qquad S > \underline{S}(0), \tag{55}$$

where the first term is given by:

$$v(t, S; T_x) = Ke^{-r(T_x - t)} - S,$$

and letting $1(A)$ denote the indicator function of the event A, and $N(\cdot)$ denote the standard normal distribution function, the last two terms in (55) are given by:

$$b(t, S; t, T_x) = E_{S,t} \int_t^{T_x} e^{-r(u-t)} rK1(S_u < \underline{S}(u))du = rK \int_t^{T_x} e^{-r(u-t)} N(-d_2(\underline{S}(u), u - t))du,$$

where $d_2(K, t) = \frac{\ln(S/K) + (r - \sigma^2/2)t}{\sigma\sqrt{t}}$ and:

$$\begin{aligned} a(t, S; T_x, T) &= E_{S,t} e^{-r(T-t)}(S_T - K)^+ - E_{S,t} \int_{T_x}^{T} e^{-r(u-t)} rK1(S_u > \underline{S}(u))du \\ &= SN(d_1(K, T - t)) - Ke^{-r(T-t)} N(d_2(K, T - t)) - rK \int_{T_x}^{T} e^{-r(u-t)} N(d_2(\underline{S}(u), u - t))du, \end{aligned}$$

where $d_1(K, t) = d_2(K, t) + \sigma\sqrt{t}$.

The first term in (55) is the put value arising if exercise were forced to occur at T_x, and is recognized as the value of a short forward position maturing at T_x. The second term captures the value of the ability to exercise the put before T_x and is given by the value of the interest which can be earned on the strike price until T_x, while the stock price is below the boundary[15]. Similarly, the third term reflects the ability to exercise after T_x (or not at all) and is given by the value of a European call less the value of the interest flow above the boundary after T_x. Note that the simpler decomposition of the alive American put value into the corresponding European put value and the early exercise premium is obtained by setting $T_x = T$. In contrast, the random maturity analog (22) of (55) is generated by setting T_x to the time T_s which solves $\hat{\underline{S}}(T_s) = S$, where $\hat{\underline{S}}$ is the staircase approximation of the exercise boundary (see Figure 9) and S is the current stock price, assumed to be in the interval $(\hat{\underline{S}}(0), \hat{\underline{S}}(T))$.

The structure of our random maturity approximation (22) for the American put value can be simplified. By imposing smoothness[16] at the critical stock price in every period, the premium for allowing exercise after T_s can be re-written as:

$$a_i^{(n)}(S) = A_i^{(n)}(S;1), \text{ where:}$$

$$A_i^{(n)}(S;h) \equiv \sum_{j=h}^{n-i+1} \left(\frac{S}{\underline{S}_{n-j+1}} \right)^{\gamma+\epsilon} \sum_{k=0}^{j-1} \frac{\left(2\epsilon \ln\left(\frac{\underline{S}_{n-j+1}}{S}\right)\right)^k}{k!} \sum_{l=0}^{j-k-1} \binom{j-1+l}{j-1} p^j q^{k+l} R^j K r \triangle. \tag{56}$$

We used $A_i^{(n)}(S;1)$ instead of $a_i^{(n)}(S)$ to simplify the presentation and implementation of our results, since its structure is simpler and very similar to that of $b_i^{(n)}(S)$.

Given our assumption on dividends, the alive American put value again decomposes into three terms:

$$P(0,s;T) = v(0,s;T_x) + b(0,s;0,T_x) + a(0,s;T_x,T), \qquad s > \underline{s}(0). \tag{57}$$

The value if forced to sell at T_x is still the value of a short forward position:

$$v(0,s;T_x) = e^{-rT_x} E_{0,s}(K - S_{T_x}) = Ke^{-rT_x} - se^{-\delta T_x} - \frac{\phi}{r}[e^{-rT_x} - e^{-rT}]. \tag{58}$$

The premium for allowing the sale to occur prior to T_x is now the value of the interest *less dividends* (i.e. net interest) received below the boundary before T_x:

$$b(0,s;0,T_x) \quad = \quad E_{0,s} \int_0^{T_x} e^{-rt}[rK - \phi - \delta s_t] 1(s_t < \underline{s}(t)) dt$$

[15] Note that the boundary still reflects a maturity date of T.

[16] It can be shown that in any sub-period, our solution for the American put is twice differentiable for all stock prices strictly above the critical price. Furthermore, at the critical stock price, our solution is continuous and differentiable. It is not twice differentiable at the critical price, as is true of the unknown solution to the original free boundary problem (3).

$$= \int_0^{T_x} [(rK-\phi)e^{-rt}N(-d_2(\underline{s}(t),t)) - \delta s e^{-\delta t}N(-d_1(\underline{s}(t),t))]dt,$$

where:

$$d_2(\underline{s}(t),t) \equiv \frac{\ln(s/\underline{s}(t)) + (r-\delta-\sigma^2/2)t}{\sigma\sqrt{t}}, \qquad d_1(\underline{s}(t),t) \equiv d_2(\underline{s}(t),t) + \sigma\sqrt{t}.$$

Finally, the premium for allowing the sale to occur after T_x or never is now the value of the *net* interest received above the boundary after T_x:

$$\begin{aligned} a(0,s;T_x,T) &= E_{0,s}e^{-rT}(S_T-K)^+ - E_{0,s}\int_{T_x}^{T} e^{-rt}[rK-\phi-\delta s_t]1(s_t \geq \underline{s}(t))dt \\ &= se^{-\delta T}N(d_1(K,T)) - Ke^{-rT}N(d_2(K,T)) - \int_{T_x}^{T}[(rK-\phi)e^{-rt}N(d_2(\underline{s}(t),t)) - \delta s e^{-\delta t}N(d_1(\underline{s}(t),t))]dt. \end{aligned}$$

Using a strategy mirroring the one underlying the decomposition of an American put, the alive American call can be shown to decompose into three terms:

$$C(0,s;T) = -v(0,s;T_x) + \alpha(0,s;0,T_x) + \beta(0,s;T_x,T), \qquad s < \underline{s}(0). \tag{59}$$

The first term is the value if forced to buy at T_x and is given by the value of a *long*[17] forward position maturing at T_x. The premium for allowing the purchase to occur prior to T_x is now the value of the dividends less interest (i.e. net dividends) received *above* the boundary before T_x:

$$\begin{aligned} \alpha(0,s;0,T_x) &= E_{0,s}\int_0^{T_x} e^{-rt}[\phi+\delta s_t - rK]1(s_t > \bar{s}(t))dt \\ &= \int_0^{T_x}[\delta s e^{-\delta t}N(d_1(\bar{s}(t),t)) - (rK-\phi)e^{-rt}N(d_2(\bar{s}(t),t))]dt. \end{aligned}$$

Finally, the premium for allowing the purchase to occur after T_x or never is now the value of the net *dividends* received below the boundary after T_x:

$$\begin{aligned} &\beta(0,s;T_x,T) \\ &= E_{0,s}e^{-rT}(K-S_T)^+ - E_{0,s}\int_{T_x}^{T} e^{-rt}[\phi+\delta s_t - rK]1(s_t < \bar{s}(t))dt \\ &= Ke^{-rT}N(-d_2(K,T)) - se^{-\delta T}N(-d_1(K,T)) - \int_{T_x}^{T}[\delta s e^{-\delta t}N(-d_1(\bar{s}(t),t)) - (rK-\phi)e^{-rt}N(-d_2(\bar{s}(t),t))]dt. \end{aligned}$$

[17] See (58) for $v(0,s;T_x)$.

Appendix 2

This appendix collects all the formulas needed to claculate random maturity values of European and American puts and calls when the underlying has a continuous payout with a fixed component ϕ and a proportional component δ. Letting $s = S - \frac{\phi}{r}[1 - e^{-rT}]$, the N-point Richardson extrapolation of the randomized European put formula is:

$$p^{1:N}(s) \equiv \sum_{n=1}^{N} \frac{(-1)^{N-n} n^N}{n!(N-n)!} p^{(n)}(s),$$

where:

$$p^{(n)}(s) = \begin{cases} \left(\frac{s}{K}\right)^{\gamma-\epsilon} \sum_{k=0}^{n-1} \frac{\left(2\epsilon \ln\left(\frac{s}{K}\right)\right)^k}{k!} \sum_{l=0}^{n-k-1} \binom{n-1+l}{n-1} [KR^n q^n p^{k+l} - KD^n \hat{q}^n \hat{p}^{k+l}] & \text{if } s > K \\ KR^n - sD^n + c^{(n)}(s) & \text{if } s \leq K, \end{cases} \quad (60)$$

and where:

$$\gamma \equiv \frac{1}{2} - \frac{r-\delta}{\sigma^2}, \triangle \equiv \frac{T}{n}, R \equiv \frac{1}{1+r\triangle}, D \equiv \frac{1}{1+\delta\triangle}, \epsilon \equiv \sqrt{\gamma^2 + \frac{2}{R\sigma^2\triangle}}, p \equiv \frac{\epsilon-\gamma}{2\epsilon}, q \equiv 1-p, \hat{p} \equiv \frac{\epsilon-\gamma+1}{2\epsilon}, \text{ and } \hat{q} \equiv 1-\hat{p}.$$

The N-point Richardson extrapolation of the Canadian put formula is:

$$P^{1:N}(s) \equiv \sum_{n=1}^{N} \frac{(-1)^{N-n} n^N}{n!(N-n)!} P^{(n)}(s).$$

where:

$$P^{(n)}(s) = \begin{cases} p_0^{(n)}(s) + b_1^{(n)}(s) & \text{if } s > \underline{s}_0 \equiv K \\ v_i^{(n)}(s) + b_i^{(n)}(s) + A_i^{(n)}(s;1) & \text{if } s \in (\underline{s}_i, \underline{s}_{i-1}], i = 1, \ldots, n \\ K - S & \text{if } s \leq \underline{s}_n, \end{cases}$$

where for $i = 1, \ldots, n$, $v_i^{(n)}(s) = KR^{n-i+1} - sD^{n-i+1} - \frac{\phi}{r} R(R^{n-i+1} - R^n)$,

$$b_i^{(n)}(s) = \sum_{j=1}^{n-i+1} \left(\frac{s}{\underline{s}_{n-j+1}}\right)^{\gamma-\epsilon} \sum_{k=0}^{j-1} \frac{\left(2\epsilon \ln\left(\frac{s}{\underline{s}_{n-j+1}}\right)\right)^k}{k!} \sum_{l=0}^{j-k-1} \binom{j-1+l}{j-1} [q^j p^{k+l} R^j (Kr - \phi) - \hat{q}^j \hat{p}^{k+l} D^j \underline{s}_{n-j+1} \delta] \triangle,$$

$$A_i^{(n)}(s;h) = \sum_{j=h}^{n-i+1} \left(\frac{s}{\underline{s}_{n-j+1}}\right)^{\gamma+\epsilon} \sum_{k=0}^{j-1} \frac{\left(2\epsilon \ln\left(\frac{\underline{s}_{n-j+1}}{s}\right)\right)^k}{k!} \sum_{l=0}^{j-k-1} \binom{j-1+l}{j-1} [p^j q^{k+l} R^j (Kr - \phi) - \hat{p}^j \hat{q}^{k+l} D^j \underline{s}_{n-j+1} \delta] \triangle.$$

If $\delta = 0$, the critical stripped prices are given by $\underline{s}_m = K \left(\frac{pR(Kr-\phi)\triangle}{c_1^{(m)}(K) - A_1^{(m)}(K;2)}\right)^{\frac{1}{\gamma+\epsilon}}, \qquad m = 1, \ldots, n.$

If $\delta > 0$, the critical stripped prices solve:

$$\sum_{l=0}^{m-1} \binom{m-1+l}{m-1} [KD^m \hat{p}^m \hat{q}^l - KR^m p^m q^l] - A_1^{(m)}(K;2) = \left(\frac{K}{\underline{s}_m}\right)^{\gamma+\epsilon} [pR(Kr-\phi) - \hat{p}D\underline{s}_m \delta] \triangle, m = 1, \ldots, n.$$

The N-point Richardson extrapolation of the put's critical stock price is $\underline{S}^{1:N} \equiv \frac{\phi}{r}[1 - e^{-rT}] + \sum_{n=1}^{N} \frac{(-1)^{N-n} n^N}{n!(N-n)!} \underline{s}_n$.

Similarly, letting $s = S - \frac{\phi}{r}[1 - e^{-rT}]$, the N-point Richardson extrapolation of the randomized European call formula is:

$$c^{1:N}(s) \equiv \sum_{n=1}^{N} \frac{(-1)^{N-n} n^N}{n!(N-n)!} c^{(n)}(s),$$

where:

$$c^{(n)}(s) = \begin{cases} sD^n - KR^n + p^{(n)}(s) & \text{if } s > K \\ \left(\frac{s}{K}\right)^{\gamma+\epsilon} \sum_{k=0}^{n-1} \frac{\left(2\epsilon \ln\left(\frac{K}{s}\right)\right)^k}{k!} \sum_{l=0}^{n-k-1} \binom{n-1+l}{n-1} [KD^n \hat{p}^n \hat{q}^{k+l} - KR^n p^n q^{k+l}] & \text{if } s \leq K, \end{cases} \tag{61}$$

and where again:

$$\gamma \equiv \frac{1}{2} - \frac{r-\delta}{\sigma^2}, \triangle \equiv \frac{T}{n}, R \equiv \frac{1}{1+r\triangle}, D \equiv \frac{1}{1+\delta\triangle}, \epsilon \equiv \sqrt{\gamma^2 + \frac{2}{R\sigma^2\triangle}}, p \equiv \frac{\epsilon-\gamma}{2\epsilon}, q \equiv 1-p, \hat{p} \equiv \frac{\epsilon-\gamma+1}{2\epsilon}, \text{ and } \hat{q} \equiv 1-\hat{p}.$$

The N-point Richardson extrapolation of the Canadian call formula is:

$$C^{(n)}(s) = \begin{cases} S - K & \text{if } s \geq \bar{s}_n \\ -v_i^{(n)}(s) + \alpha_i^{(n)}(s) + B_i^{(n)}(s;1) & \text{if } s \in [\bar{s}_{i-1}, \bar{s}_i), i = 1, \ldots, n \\ c_0^{(n)}(s) + \alpha_1^{(n)}(s) & \text{if } s < \bar{s}_0 \equiv K \end{cases}$$

where for $i = 1, \ldots, n$,

$$-v_i^{(n)}(s) sD^{n-i+1} + \frac{\phi}{r} R(R^{n-i+1} - R^n) - KR^{n-i+1},$$

$$\alpha_i^{(n)}(s) = \sum_{j=1}^{n-i+1} \left(\frac{s}{\bar{s}_{n-j+1}}\right)^{\gamma+\epsilon} \sum_{k=0}^{j-1} \frac{\left(2\epsilon \ln\left(\frac{\bar{s}_{n-j+1}}{s}\right)\right)^k}{k!} \sum_{l=0}^{j-k-1} \binom{j-1+l}{j-1} [\hat{p}^j \hat{q}^{k+l} D^j \bar{s}_{n-j+1}\delta - p^j q^{k+l} R^j (Kr - \phi)]\triangle,$$

$$B_i^{(n)}(s;h) = \sum_{j=h}^{n-i+1} \left(\frac{s}{\bar{s}_{n-j+1}}\right)^{\gamma-\epsilon} \sum_{k=0}^{j-1} \frac{\left(2\epsilon \ln\left(\frac{s}{\bar{s}_{n-j+1}}\right)\right)^k}{k!} \sum_{l=0}^{j-k-1} \binom{j-1+l}{j-1} [\hat{q}^j \hat{p}^{k+l} D^j \bar{s}_{n-j+1}\delta - q^j p^{k+l} R^j (Kr - \phi)]\triangle.$$

If $\phi = rK$, the critical stripped price $\bar{s}_m$ is given by:

$$\underline{s}_m = K \left(\frac{K\hat{q}D\delta\triangle}{p_0^{(m)}(K) - B_1^{(m)}(K;2)}\right)^{\frac{1}{\gamma-\epsilon-1}}, \qquad m = 1, \ldots, n.$$

Otherwise, the critical stripped price $\bar{s}_m$ implicitly solves:

$$\sum_{l=0}^{m-1} \binom{m-1+l}{m-1} [KR^m q^m p^l - KD^m \hat{q}^m \hat{p}^l] - B_1^{(m)}(K;2) = \left(\frac{K}{\bar{s}_m}\right)^{\gamma-\epsilon} [\hat{q}D\bar{s}_m\delta - qR(Kr - \phi)]\triangle, m = 1, \ldots, n.$$

The N-point Richardson extrapolation of the call's critical stock price is $\bar{S}^{1:N}(T) \equiv \frac{\phi}{r}[1 - e^{-rT}] + \sum_{n=1}^{N} \frac{(-1)^{N-n} n^N}{n!(N-n)!} \bar{s}_n$.

Table 1: Convergence of Canadian Put Value to American without and with Richardson Extrapolation

S = 100, K = 100, T = 1, r = 0.1, δ = 0, σ = 0.3

Number of Steps n or Points N	Unextrapolated Put Value $P^{(n)}$	Extrapolated Put Value $P^{1:N}$
1	7.0405	7.0405
2	7.6175	8.1946
3	7.8353	8.3089
4	7.9505	8.3257
5	8.0220	8.3311
6	8.0709	8.3333
7	8.1065	8.3345
8	8.1335	8.3353
9	8.1548	8.3358
10	8.1720	8.3362
11	8.1862	8.3365
12	8.1981	8.3367
13	8.2082	8.3369
14	8.2169	8.3370
15	8.2246	8.3371

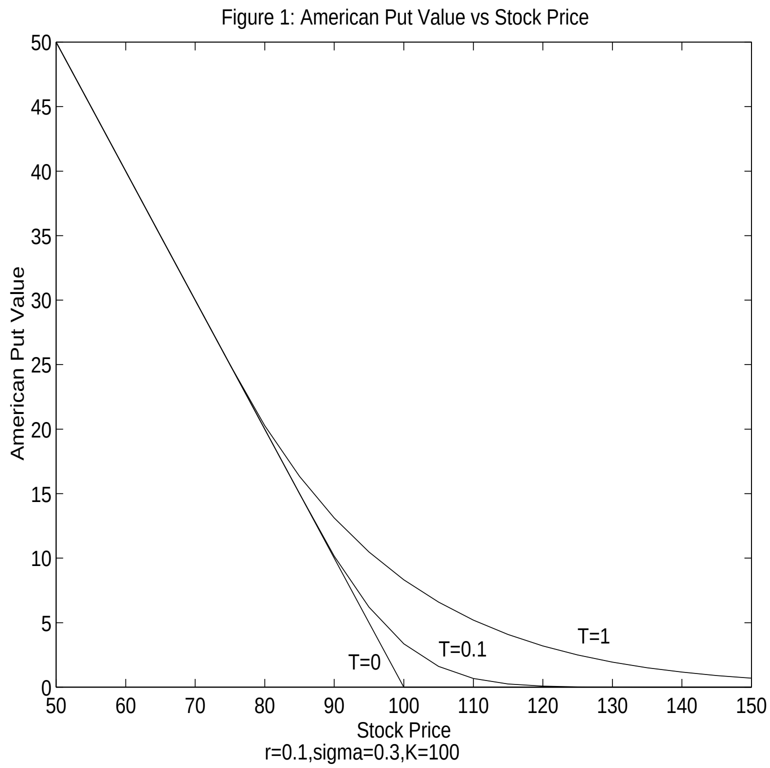

Figure 1: American Put Value vs. Stock Price

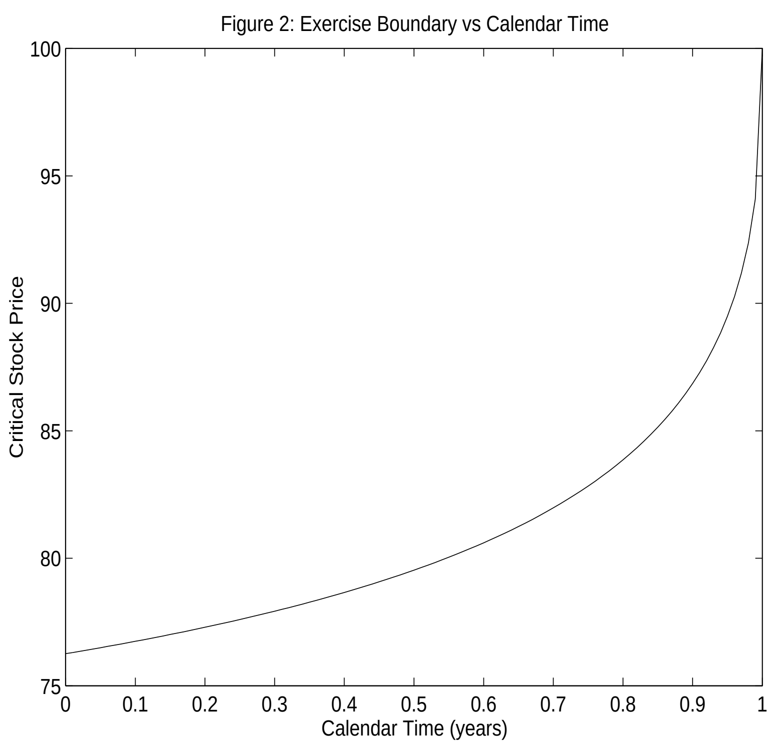

Figure 2: Exercise Boundary vs. Calendar Time

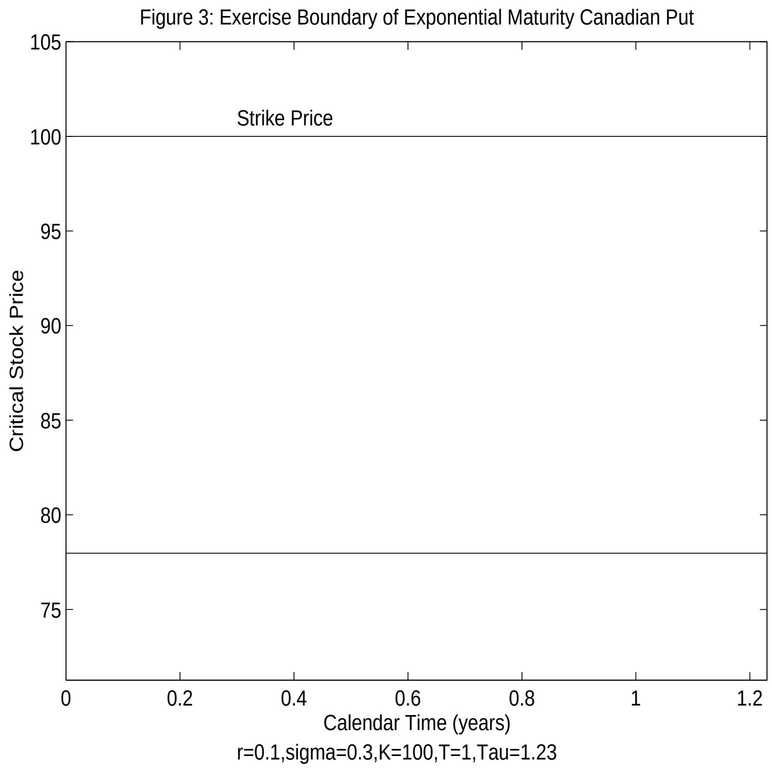

Figure 3: Exercise Boundary of Exponential Maturity Canadian Put

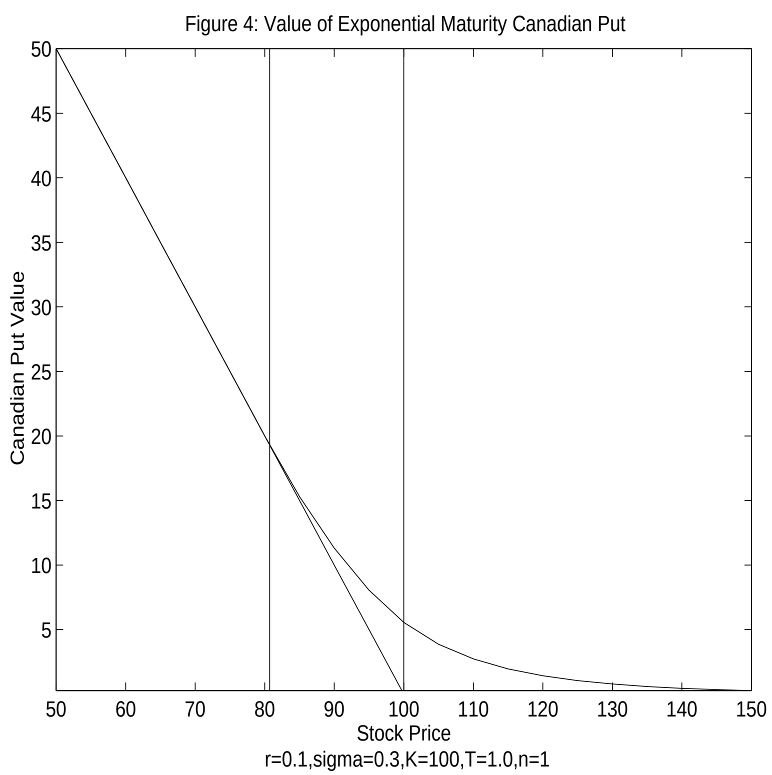

Figure 4: Value of Exponential Maturity Canadian Put

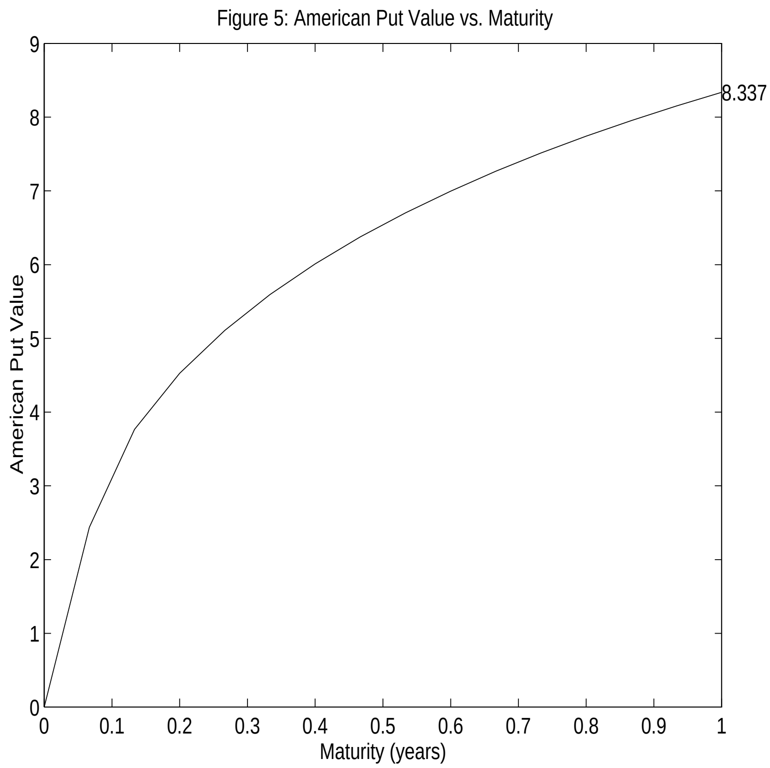

Figure 5: American Put Value vs. Maturity

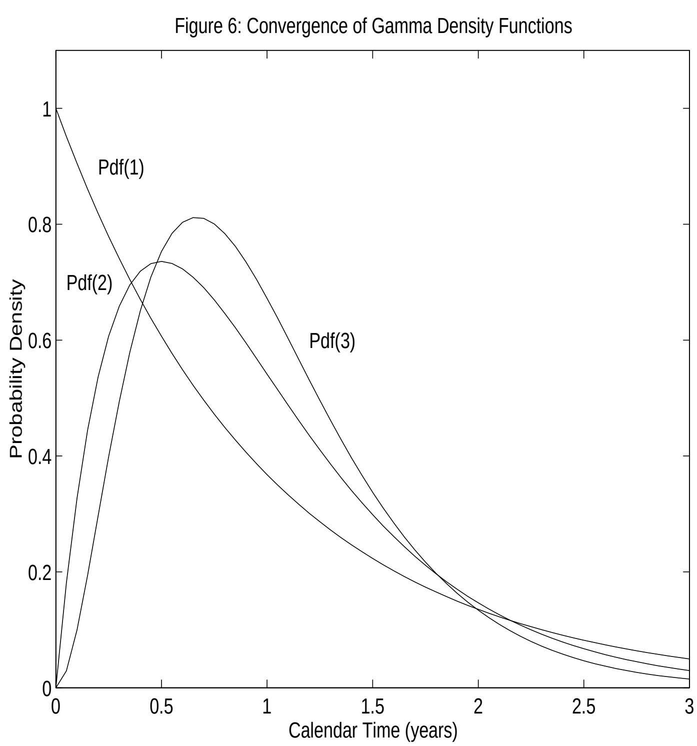

Figure 6: Convergence of Gamma Density Functions

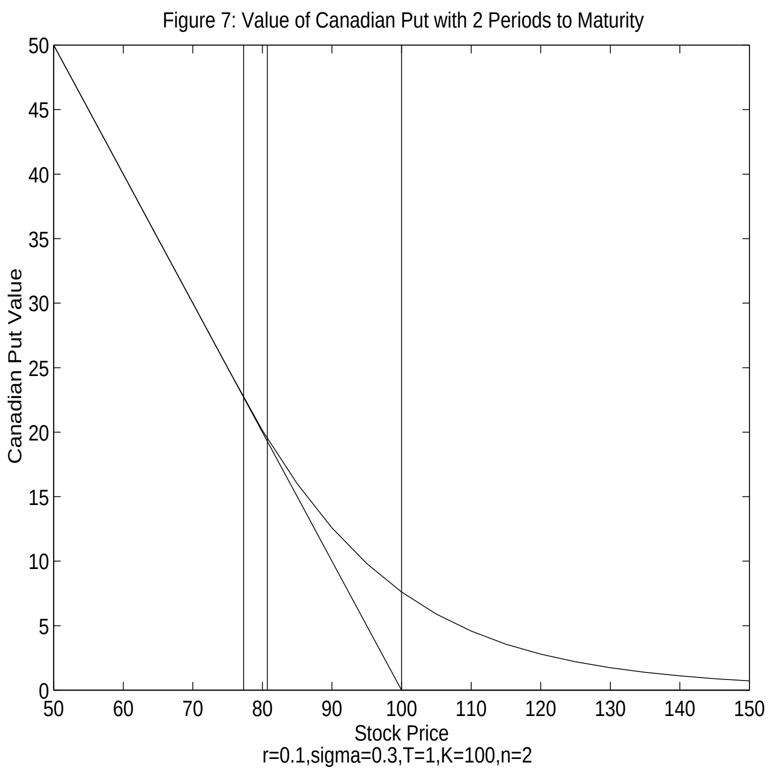

Figure 7: Value of Canadian Put with 2 Periods to Maturity

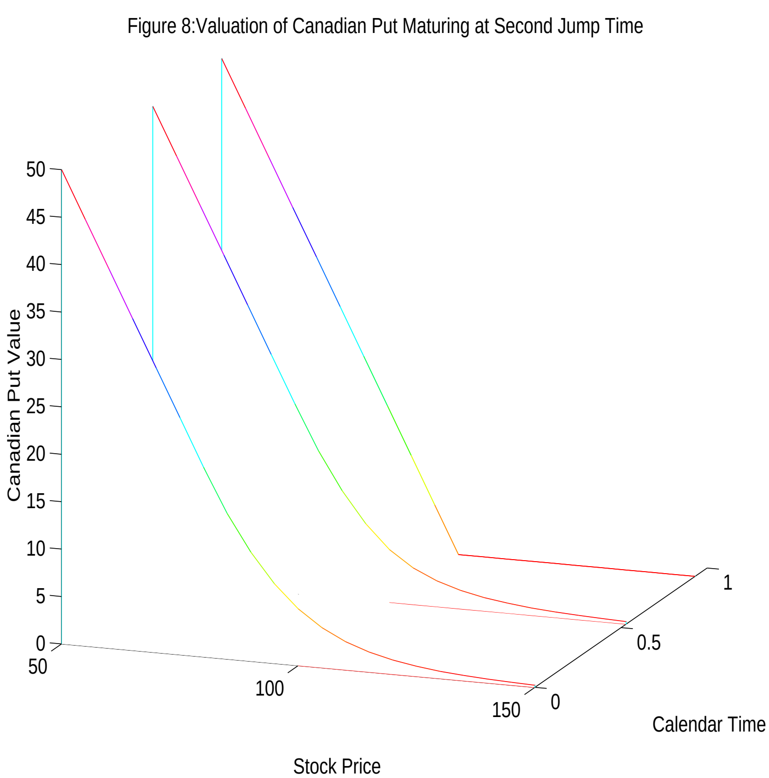

Figure 8: Valuation of Canadian Put Maturing at Second Jump Time

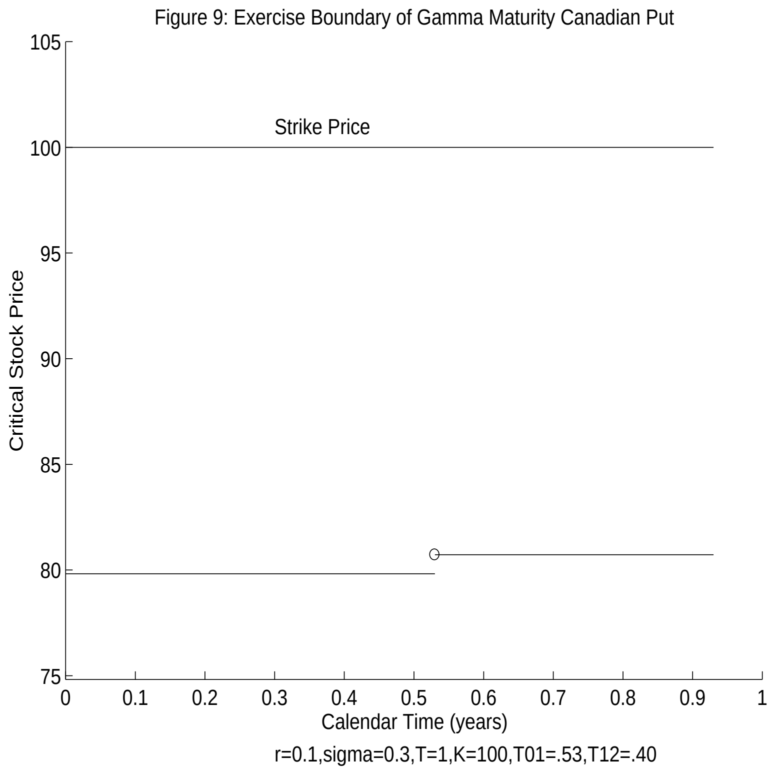

Figure 9: Exercise Boundary of Gamma Maturity Canadian Put

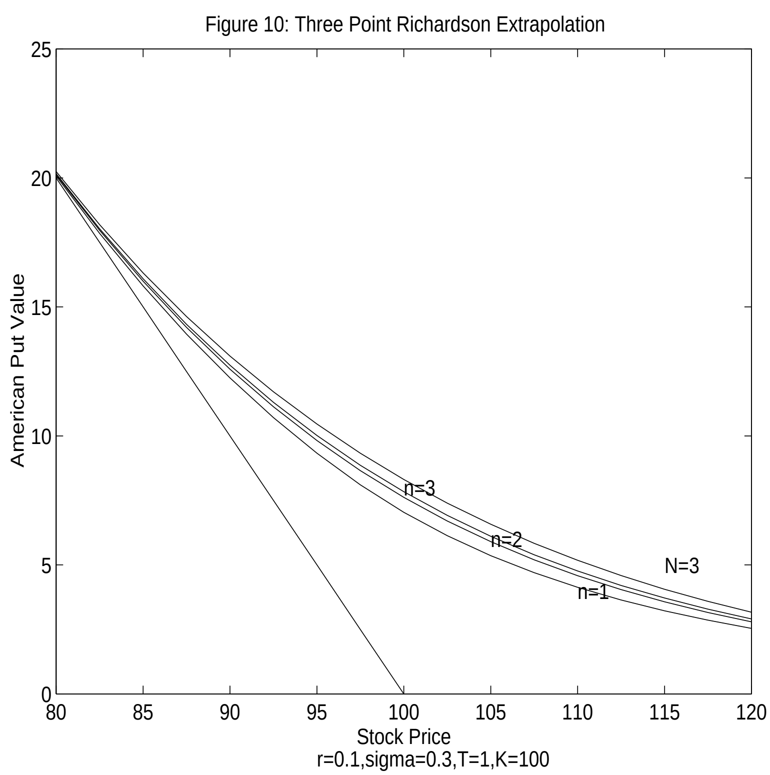

Figure 10: Three Point Richardson Exrapolation

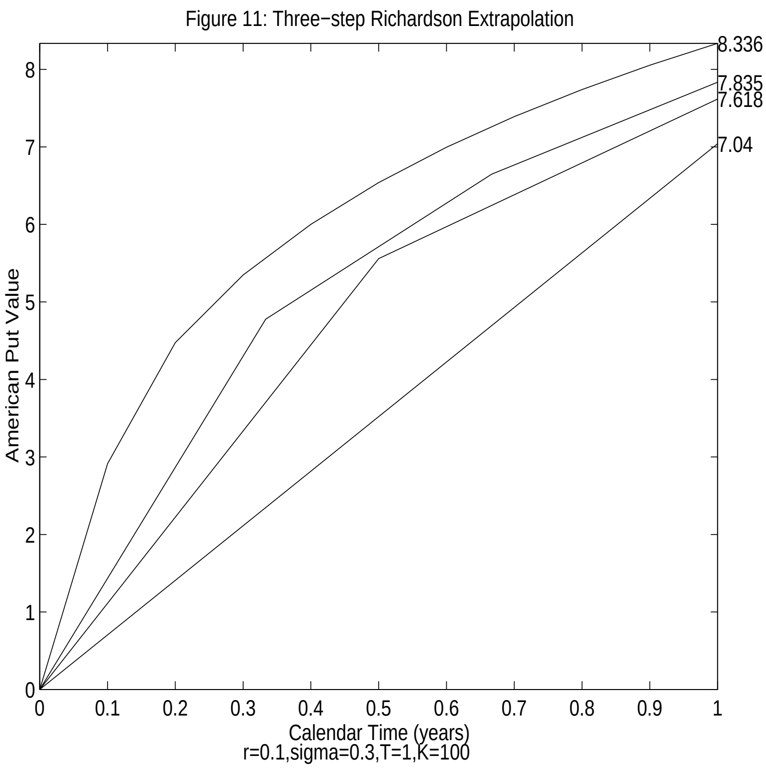

Figure 11: Valuation of Canadian Put Maturing at Second Jump Time