ON THE EXTENDED T-SYSTEM OF TYPE C

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Abstract. We continue the study of extended T-systems of quantum affine algebras. We find a sub-system of the extended T-system of the quantum affine algebra $U_q\hat{\mathfrak{g}}$ of type C_3. The sub-system consisting of four systems which are denoted by I, II, III, and IV. Each of the systems I, II, III, IV is closed. The systems I-IV can be used to compute minimal affinizations with weights of the form $\lambda_1\omega_1 + \lambda_2\omega_2 + \lambda_3\omega_3$, where at least one of λ_1, λ_2, λ_3 are zero. Using the systems I-IV, we compute the characters of the restrictions of the minimal affinizations in the systems to $U_q\mathfrak{g}$ and obtain some conjectural decomposition formulas for the restrictions of some minimal affinizations.

1. Introduction

The T-systems are some families of relations in the Grothendieck ring of the category of the finite-dimensional modules of quantum affine algebras (or Yangians), see [K83], [K84], [K87], [KR90], [KNS94], [Nak03], [Her06]. The T-systems are widely applied to representation theory, combinatorics and integrable systems, see the recent survey [KNS11].

The modules in the usual T-systems are the Kirillov-Reshetikhin modules. The Kirillov-Reshetikhin modules are simplest examples of irreducible finite-dimensional modules over quantum affine algebras. Recently, the usual T-systems have been generalized to the so called extended T-systems, see [MY12b], [LM12]. The extended T-systems contains minimal affinizations of quantum affine algebras. The family of minimal affinizations is an important family of irreducible modules which contains the Kirillov-Reshetikhin modules, see [C95], [CP95b], [CP96a], [CP96b]. The minimal affinizations are also interesting from the physical point of view, see Remark 4.2 of [FR92] and [C95].

Minimal affinizations are studied intensively in recently years, see for example, [CMY12], [CG11], [Her07], [LM12], [Mou10], [MF11], [MY12a], [MY12b], [MY12c], [Nao12]. The finite dimensional representations of $U_q\hat{\mathfrak{g}}$ and cluster algebras are closely related, see [IIKKN13a], [IIKKN13b], [HL10], [HL13], [Nak11]. The relations in the extended T-systems are special relations in the cluster algebras. A cluster algebra algorithm for computing q-characters of Kirillov-Reshetikhin modules for any untwisted quantum affine algebra is given in [HL13]. Their method uses the T-systems for Kirillov-Reshetikhin modules. We expect that the extended T-systems can be used to find new algorithms to compute q-characters of more general modules including minimal affinizations.

The extended T-systems of type A, B has been found in [MY12b] and the extended T-system of type G_2 has been found in [LM12]. In [MY12b], it was conjectured that the extended T-systems exist in all types.

In this paper, we continue the study of extended T-systems of quantum affine algebras. Let $\text{Rep}(U_q\hat{\mathfrak{g}})$ denote the Grothendieck ring of the category of finite-dimensional representations of $U_q\hat{\mathfrak{g}}$. The irreducible finite-dimensional modules of quantum affine algebras are parameterized by the highest l-weights or Drinfeld polynomials.

Our method is similar as the method used in [MY12b], [LM12]. Let $a \in \mathbb{C}^\times$ and $\mathcal{T}$ an irreducible $U_q\hat{\mathfrak{g}}$-module such that the zeros of the Drinfeld polynomials of $\mathcal{T}$ belong to $aq^{\mathbb{Z}}$. Following [MY12b], we define the left, right, and bottom modules, denoted by $\mathcal{L}$, $\mathcal{R}$, $\mathcal{B}$ respectively, as the modules whose Drinfeld polynomials has zeros obtained by dropping the rightmost, leftmost, and both left- and rightmost zeros of the union of zeros of the Drinfeld polynomials of the top module $\mathcal{T}$.

The q-character theory and the FM algorithm are important tools to study the representation theory of quantum affine algebras, see [NT98], [FR98], [FM01]. The main tools in this paper are the q-character theory and the FM algorithm.

If the q-character of a module contains only one dominant monomial, then it is called a special module. Let $\mathfrak{g}$ be the simple Lie algebra of type C_3. The FM algorithm applies to special modules. Although in general the minimal affinizations of type C_3 are not special, there are some families of minimal affinizations of type C which are special.

For $k, \ell, m \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}$, let

$$T^{(s)}_{k,\ell,m} = \left(\prod_{i=0}^{k-1} 3_{s+4i}\right)\left(\prod_{i=0}^{\ell-1} 2_{s+4k+2i+1}\right)\left(\prod_{i=0}^{m-1} 1_{s+4k+2\ell+2i+2}\right),$$

$$\tilde{T}^{(s)}_{k,\ell,m} = \left(\prod_{i=0}^{k-1} 1_{s+2i}\right)\left(\prod_{i=0}^{\ell-1} 2_{s+2k+2i+1}\right)\left(\prod_{i=0}^{m-1} 3_{s+2k+2\ell+4i+4}\right).$$

We use $\mathcal{T}$ to denote the highest l-weight module with a highest l-weight T. Then $\mathcal{T}^{(s)}_{k,\ell,m}$ and $\tilde{\mathcal{T}}^{(s)}_{k,\ell,m}$ are all minimal affinizations of type C_3.

We take minimal affinizations $\mathcal{T}^{(s)}_{k,\ell,0}$, $\tilde{\mathcal{T}}^{(s)}_{k,\ell,0}$ as top modules $\mathcal{T}$ respectively. It turns out that the left, right, bottom modules of $\mathcal{T}$ satisfy a relation $[\mathcal{L}][\mathcal{R}] = [\mathcal{T}][\mathcal{B}] + [\mathcal{S}]$. Here $[\cdot]$ denotes the equivalence class of a $U_q\hat{\mathfrak{g}}$-module in $\text{Rep}(U_q\hat{\mathfrak{g}})$ and S is a tensor product of some irreducible modules. The factors of $\mathcal{S}$ are called sources. The factors of the sources can be some modules which are not minimal affinizations. In order to obtain a closed system, we take the sources as top modules and compute new left, right, bottom modules, and sources. We continue this procedure until all modules in the sources are the modules obtained before. Then we obtain the desired closed systems I and III.

The system I contains minimal affinizations $\mathcal{T}^{(s)}_{k,\ell,0}$, $\mathcal{T}^{(s)}_{k,0,m}$, and $\tilde{\mathcal{T}}^{(s)}_{k,0,m}$ for all k, ℓ, $m \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}$. The system III contains minimal affinizations $\tilde{T}^{(s)}_{k,\ell,0}$ for all $k, \ell \in \mathbb{Z}$. We denote by II, IV the dual systems of I, III respectively. The system II contains minimal affinizations which can be obtained from $\tilde{\mathcal{T}}^{(s)}_{0,k,\ell}$, k, ℓ, $m \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}$, by shifting the upper-subscripts. The system IV contains minimal affinizations which can be obtained from $T^{(s)}_{0,k,\ell}$, $k, \ell \in \mathbb{Z}$, $s \in \mathbb{Z}$, by shifting the upper-subscripts. Therefore we find a sub-system of the extended

T-system of the quantum affine algebra $U_q\hat{\mathfrak{g}}$ of type C_3 which is the union of the systems I, II, III, IV and the sub-system can be used to compute minimal affinizations with weights of the form $\lambda_1\omega_1 + \lambda_2\omega_2 + \lambda_3\omega_3$, where at least one of λ_1, λ_2, λ_3 are zero, see Section 3.

We find that the modules in the systems I, III are

$$\begin{aligned} &\mathcal{T}^{(s)}_{k,\ell,0},\ \mathcal{T}^{(s)}_{k,0,m},\ \mathcal{T}^{(s)}_{0,\ell,r},\ \tilde{\mathcal{T}}^{(s)}_{k,0,m},\ \mathcal{S}^{(s)}_{k,\ell},\ \mathcal{R}^{(s)}_{k,2\ell,\ell},\\ &\mathcal{R}^{(s)}_{k,2\ell+1,\ell},\ \mathcal{R}^{(s)}_{k,2\ell+2,\ell},\ \mathcal{U}^{(s)}_{k,\ell},\ \mathcal{V}^{(s)}_{k,\ell},\ \mathcal{P}^{(s)}_{k,\ell},\ \mathcal{O}^{(s)}_{k,\ell},\end{aligned}$$

where $s \in \mathbb{Z}$, $k, \ell, m \in \mathbb{Z}_{\geq 0}$, $r \in \{0, 1, 2\}$. We show that these modules are special. We show that every relation $[\mathcal{L}][\mathcal{R}] = [\mathcal{T}][\mathcal{B}] + [\mathcal{S}]$ in the system holds by comparing the dominant monomials in both sides of $[\mathcal{L}][\mathcal{R}] = [\mathcal{T}][\mathcal{B}] + [\mathcal{S}]$. Moreover, we show that the modules $\mathcal{T} \otimes \mathcal{B}$ and $\mathcal{S}$ in the systems I-IV are irreducible.

The extended T-system is a powerful tool of studying the finite dimensional representations of $U_q\hat{\mathfrak{g}}$. Using the systems I-IV, we compute the characters of the restrictions of the minimal affinizations in the systems to $U_q\mathfrak{g}$ and obtain some conjectural decomposition formulas for the restrictions of some minimal affinizations, see Section 8.

The paper is organized as follows. In Section 2, we give some background material. In Section 3, we describe the system I, II, III, and IV. In Section 4, we prove that the modules in the systems I and III are special. In Section 5, we prove Theorem 3.3 and Theorem 3.12. In Section 6, we prove that the module $\mathcal{T} \otimes \mathcal{B}$ is irreducible for each relation in the systems I, II, III, and IV. In Section 7, we prove Proposition 3.5. In Section 8, we give conjectural character formulas for $\mathrm{Res}(\mathcal{T}^{(s)}_{k,\ell,0})$, $\mathrm{Res}(\mathcal{T}^{(s)}_{k,0,m})$, $\mathrm{Res}(\tilde{\mathcal{T}}^{(s)}_{k,0,m})$, $s \in \mathbb{Z}$, k, ℓ, $m \in \mathbb{Z}_{\geq 0}$.

2. Background

2.1. **Quantum affine algebra.** Let $\mathfrak{g}$ be a complex simple Lie algebra of type C_3 and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Let $I = \{1, 2, 3\}$. We choose simple roots $\alpha_1, \alpha_2, \alpha_3$ and scalar product $(\cdot, \cdot)$ such that

$$(\alpha_1, \alpha_1) = 2,\ (\alpha_2, \alpha_2) = 2,\ (\alpha_3, \alpha_3) = 4,\ (\alpha_1, \alpha_2) = -1,\ (\alpha_1, \alpha_3) = 0,\ (\alpha_2, \alpha_3) = -2.$$

Let $\{\alpha_1^\vee, \alpha_2^\vee, \alpha_3^\vee\}$ and $\{\omega_1, \omega_2, \omega_3\}$ be the sets of simple coroots and fundamental weights respectively. Let $C = (C_{ij})_{i,j\in I}$ denote the Cartan matrix, where $C_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$. Let $r_1 = 1, r_2 = 1, r_3 = 2$, $D = \mathrm{diag}(r_1, r_2, r_3)$ and $B = DC$. Then

$$C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix},\ B = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 4 \end{pmatrix}.$$

Let Q (resp. Q^+) and P (resp. P^+) denote the $\mathbb{Z}$-span (resp. $\mathbb{Z}_{\geq 0}$-span) of the simple roots and fundamental weights respectively. There is a partial order $\leq$ on P such that $\lambda \leq \lambda'$ if and only if $\lambda' - \lambda \in Q^+$.

The quantum affine algebra $U_q\hat{\mathfrak{g}}$ is a $\mathbb{C}(q)$-algebra generated by $x_{i,n}^{\pm}$ $(i \in I, n \in \mathbb{Z})$, $k_i^{\pm 1}$ $(i \in I)$, $h_{i,n}$ $(i \in I, n \in \mathbb{Z}\backslash\{0\})$ and central elements $c^{\pm 1/2}$, subject to certain relations, see [Dri88]. The algebra $U_q\hat{\mathfrak{g}}$ is a Hopf algebra.

Let $U_q\mathfrak{g}$ be the quantized enveloping algebra of $\mathfrak{g}$. The subalgebra of $U_q\hat{\mathfrak{g}}$ generated by $(k_i^{\pm})_{i\in I}, (x_{i,0}^{\pm})_{i\in I}$ is a Hopf subalgebra of $U_q\hat{\mathfrak{g}}$ and is isomorphic as a Hopf algebra to $U_q\mathfrak{g}$. Therefore $U_q\hat{\mathfrak{g}}$-modules restrict to $U_q\mathfrak{g}$-modules. We denote by $\mathrm{Res}(V)$ the restriction of a $U_q\hat{\mathfrak{g}}$-module V to $U_q\mathfrak{g}$.

2.2. **Finite-dimensional representations of $U_q\hat{\mathfrak{g}}$ and q-characters.** A $U_q\hat{\mathfrak{g}}$-module is called of type 1 if $c^{\pm 1/2}$ acts as the identity on V and

$$V = \bigoplus_{\lambda \in P} V_\lambda, \; V_\lambda = \{v \in V : k_i v = q^{(\alpha_i, \lambda)} v\}. \tag{2.1}$$

We will only consider finite-dimensional type 1 $U_q\hat{\mathfrak{g}}$-modules in this paper. The decomposition (2.1) of a finite-dimensional $U_q\hat{\mathfrak{g}}$-module V into its $U_q\mathfrak{g}$-weight spaces can be refined as follows, see [FR98]:

$$V = \bigoplus_{\gamma} V_\gamma, \; \gamma = (\gamma_{i,\pm r}^{\pm})_{i\in I, r\in\mathbb{Z}_{\geq 0}}, \; \gamma_{i,\pm r}^{\pm} \in \mathbb{C}, \tag{2.2}$$

where

$$V_\gamma = \{v \in V : \exists k \in \mathbb{N}, \forall i \in I, m \geq 0, (\phi_{i,\pm m}^{\pm} - \gamma_{i,\pm m}^{\pm})^k v = 0\}.$$

If $\dim(V_\gamma) > 0$, then γ is called an *l-weight* of V.

Let $\mathcal{P}$ denote the free Abelian multiplicative group of monomials in infinitely many formal variables $(Y_{i,a})_{i\in I, a\in\mathbb{C}^\times}$. In [FR98], it is shown that there is a bijection γ from from $\mathcal{P}$ to the set of l-weights of finite-dimensional $U_q\hat{\mathfrak{g}}$-modules. Therefore the monomials in $\mathcal{P}$ are also called l-weights. For simplicity, we denote $V_m = V_{\gamma(m)}$.

Let $\mathbb{Z}\mathcal{P} = \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{C}^\times}$. For $\chi \in \mathbb{Z}\mathcal{P}$, we write $m \in \mathcal{P}$ if the coefficient of m in χ is non-zero.

The q-character of a $U_q\hat{\mathfrak{g}}$-module V is given by

$$\chi_q(V) = \sum_{m\in\mathcal{P}} \dim(V_m) m \in \mathbb{Z}\mathcal{P},$$

where $V_m = V_{\gamma(m)}$.

Let $\mathrm{Rep}(U_q\hat{\mathfrak{g}})$ be the Grothendieck ring of finite-dimensional representations of $U_q\hat{\mathfrak{g}}$ and $[V] \in \mathrm{Rep}(U_q\hat{\mathfrak{g}})$ the class of a finite-dimensional $U_q\hat{\mathfrak{g}}$-module V. The q-character map

$$\begin{aligned} \chi_q : \mathrm{Rep}(U_q\hat{\mathfrak{g}}) &\to \mathbb{Z}\mathcal{P}, \\ V &\mapsto \chi_q(V), \end{aligned}$$

is an injective ring homomorphism, see [FR98].

Let $\mathscr{M}(V)$ denote the set of all monomials in $\chi_q(V)$ for any finite-dimensional $U_q\hat{\mathfrak{g}}$-module V. For each $j \in I$, a monomial $m = \prod_{i\in I, a\in\mathbb{C}^\times} Y_{i,a}^{u_{i,a}}$, where $u_{i,a}$ are some integers, is said to be *j-dominant* (resp. *j-anti-dominant*) if and only if $u_{j,a} \geq 0$ (resp. $u_{j,a} \leq 0$)

for all $a \in \mathbb{C}^\times$. A monomial is called *dominant* (resp. *anti-dominant*) if and only if it is j-dominant (resp. j-anti-dominant) for all $j \in I$. Let $\mathcal{P}^+ \subset \mathcal{P}$ denote the set of all dominant monomials.

Let V be a $U_q\hat{\mathfrak{g}}$-module and $m \in \mathscr{M}(V)$ a monomial. Let $v \in V_m$ be a non-zero vector. If

$$x_{i,r}^+ \cdot v = 0,\ \phi_{i,\pm t}^\pm \cdot v = \gamma(m)_{i,\pm t}^\pm v,\ \forall i \in I, r \in \mathbb{Z}, t \in \mathbb{Z}_{\geq 0},$$

then v is called a *highest l-weight vector* with *highest l-weight* $\gamma(m)$. The module V is called a *highest l-weight representation* if $V = U_q\hat{\mathfrak{g}} \cdot v$ for some highest l-weight vector $v \in V$.

In [CP94], [CP95a], it is shown that there is a bijection between the set of isomorphism classes of finite-dimensional irreducible highest l-weight $U_q\hat{\mathfrak{g}}$-modules of type 1. Let $L(m_+)$ denote the irreducible highest l-weight $U_q\hat{\mathfrak{g}}$-module corresponding to $m_+ \in \mathcal{P}^+$. We use $\chi_q(m_+)$ to denote $\chi_q(L(m_+))$ for $m_+ \in \mathcal{P}^+$.

We have the following well-known lemma.

Lemma 2.1. *Let m_1, m_2 be two monomials. Then $L(m_1m_2)$ is a sub-quotient of $L(m_1) \otimes L(m_2)$. In particular, $\mathscr{M}(L(m_1m_2)) \subseteq \mathscr{M}(L(m_1))\mathscr{M}(L(m_2))$.* $\square$

For $b \in \mathbb{C}^\times$, we define the shift of spectral parameter map $\tau_b : \mathbb{Z}\mathcal{P} \to \mathbb{Z}\mathcal{P}$ to be a ring homomorphism sending $Y_{i,a}^{\pm 1}$ to $Y_{i,ab}^{\pm 1}$. Let $m_1, m_2 \in \mathcal{P}^+$. If $\tau_b(m_1) = m_2$, then

$$\tau_b\chi_q(m_1) = \chi_q(m_2). \tag{2.3}$$

A finite-dimensional $U_q\hat{\mathfrak{g}}$-module V is said to be *special* if and only if $\mathscr{M}(V)$ contains exactly one dominant monomial. It is called *anti-special* if and only if $\mathscr{M}(V)$ contains exactly one anti-dominant monomial. It is called *thin* if and only if every l-weight space of V has dimension less or equal to 1. Clearly, if a module is special or anti-special, then it is irreducible.

Define $A_{i,a} \in \mathcal{P}, i \in I, a \in \mathbb{C}^\times$, by

$$\begin{aligned} A_{1,a} &= Y_{1,aq}Y_{1,aq^{-1}}Y_{2,a}^{-1}, \\ A_{2,a} &= Y_{2,aq}Y_{2,aq^{-1}}Y_{1,a}^{-1}Y_{3,a}^{-1}, \\ A_{3,a} &= Y_{3,aq^2}Y_{3,aq^{-2}}Y_{2,aq}^{-1}Y_{2,aq^{-1}}^{-1}. \end{aligned}$$

Let $\mathcal{Q}$ be the subgroup of $\mathcal{P}$ generated by $A_{i,a}, i \in I, a \in \mathbb{C}^\times$. Let $\mathcal{Q}^\pm$ be the monoids generated by $A_{i,a}^{\pm 1}, i \in I, a \in \mathbb{C}^\times$. There is a partial order $\leq$ on $\mathcal{P}$ such that

$$m \leq m' \text{ if and only if } m'm^{-1} \in \mathcal{Q}^+. \tag{2.4}$$

For all $m_+ \in \mathcal{P}^+$, $\mathscr{M}(L(m_+)) \subset m_+\mathcal{Q}^-$, see [FM01].

Let m be a monomial. If for all $a \in \mathbb{C}^\times$ and $i \in I$, we have the property: if the power of $Y_{i,a}$ in m is non-zero and the power of Y_{j,aq^k} in m is zero for all $j \in I, k \in \mathbb{Z}_{>0}$, then the power of $Y_{i,a}$ in m is negative, then the monomial m is called *right negative.* For $i \in I, a \in \mathbb{C}^\times$, $A_{i,a}^{-1}$ is right-negative. A product of right-negative monomials is right-negative. If m is right-negative and $m' \leq m$, then m' is right-negative.

2.3. **Minimal affinizations of $U_q\mathfrak{g}$-modules.** The *minimal affinizations* of type C_3 are classified in [CP95b]. Let $\mathfrak{g}$ be the simple Lie algebra of type C_3 and $V(\lambda)$ the $U_q(\mathfrak{g})$ module of weight λ, where $k, \ell, m \in \mathbb{Z}_{\geq 0}$ and $\lambda = m\omega_1 + \ell\omega_2 + k\omega_3$. A simple $U_q\hat{\mathfrak{g}}$-module $L(m_+)$ is a minimal affinization of $V(\lambda)$ if and only if m_+ is one of the following monomials

$$T^{(s)}_{k,\ell,m} = \left(\prod_{i=0}^{k-1} Y_{3,aq^{s+4i}}\right)\left(\prod_{i=0}^{\ell-1} Y_{2,aq^{s+4k+2i+1}}\right)\left(\prod_{i=0}^{m-1} Y_{1,aq^{s+4k+2\ell+2i+2}}\right),$$

$$\tilde{T}^{(s)}_{k,\ell,m} = \left(\prod_{i=0}^{k-1} Y_{1,aq^{s+2i}}\right)\left(\prod_{i=0}^{\ell-1} Y_{2,aq^{s+2k+2i+1}}\right)\left(\prod_{i=0}^{m-1} Y_{3,aq^{s+2k+2\ell+4i+4}}\right),$$

for some $a \in \mathbb{C}^\times$, see [CP95b]. In particular, when $k = 0$ or $\ell = 0$ or $m = 0$, the minimal affinization $L(m_+)$ is a *Kirillov-Reshetikhin module*.

Let $L(m_+)$ be a Kirillov-Reshetikhin module. It is shown in [Her06] that any non-highest monomial in $\mathscr{M}(L(m_+))$ is right-negative and hence $L(m_+)$ is special.

2.4. **q-characters of $U_q\hat{\mathfrak{sl}}_2$-modules and the FM algorithm.** The q-characters of $U_q\hat{\mathfrak{sl}}_2$-modules are well-understood, see [CP91], [FR98]. We recall the results here.

Let $W^{(a)}_k$ be the irreducible representation $U_q\hat{\mathfrak{sl}}_2$ with highest weight monomial

$$X^{(a)}_k = \prod_{i=0}^{k-1} Y_{aq^{k-2i-1}},$$

where $Y_a = Y_{1,a}$. Then the q-character of $W^{(a)}_k$ is given by

$$\chi_q(W^{(a)}_k) = X^{(a)}_k \sum_{i=0}^{k} \prod_{j=0}^{i-1} A^{-1}_{aq^{k-2j}},$$

where $A_a = Y_{aq^{-1}} Y_{aq}$.

For $a \in \mathbb{C}^\times, k \in \mathbb{Z}_{\geq 1}$, the set $\Sigma^{(a)}_k = \{aq^{k-2i-1}\}_{i=0,\ldots,k-1}$ is called a *string*. Two strings $\Sigma^{(a)}_k$ and $\Sigma^{(a')}_{k'}$ are said to be in *general position* if the union $\Sigma^{(a)}_k \cup \Sigma^{(a')}_{k'}$ is not a string or $\Sigma^{(a)}_k \subset \Sigma^{(a')}_{k'}$ or $\Sigma^{(a')}_{k'} \subset \Sigma^{(a)}_k$.

Let $L(m_+)$ denote the irreducible $U_q\hat{\mathfrak{sl}}_2$-module with highest weight monomial m_+. Let $m_+ \neq 1$, $m_+ \in \mathbb{Z}[Y_a]_{a\in\mathbb{C}^\times}$, be a dominant monomial. Then m_+ can be uniquely written in the form

$$m_+ = \prod_{i=1}^{s}\left(\prod_{b\in\Sigma^{(a_i)}_{k_i}} Y_b\right),$$

up to a permutation, where s is an integer, $\Sigma_{k_i}^{(a_i)}, i = 1, \ldots, s$, are strings which are pairwise in general position and

$$L(m_+) = \bigotimes_{i=1}^{s} W_{k_i}^{(a_i)}, \qquad \chi_q(m_+) = \prod_{i=1}^{s} \chi_q(W_{k_i}^{(a_i)}).$$

For $j \in I$, let

$$\beta_j : \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I; a \in \mathbb{C}^\times} \to \mathbb{Z}[Y_a^{\pm 1}]_{a \in \mathbb{C}^\times}$$

be the ring homomorphism which sends, for all $a \in \mathbb{C}^\times$, $Y_{k,a} \mapsto 1$ for $k \neq j$ and $Y_{j,a} \mapsto Y_a$.

Let V be a $U_q\hat{\mathfrak{g}}$-module. Then $\beta_i(\chi_q(V))$, $i = 1, 2$, is the q-character of V considered as a $U_{q_i}(\hat{\mathfrak{sl}}_2)$-module. In Section 5 of [FM01], a powerful algorithm is proposed to compute the q-characters of $U_q\hat{\mathfrak{g}}$-modules. The algorithm is called the FM algorithm. The FM algorithm recursively computes the minimal possible q-character which contains m_+ and is consistent when restricted to $U_{q_i}(\hat{\mathfrak{sl}}_2)$, $i \in I$. If a module $L(m_+)$ is special, then the FM algorithm produces the correct q-character $\chi_q(m_+)$, see [FM01].

2.5. **Truncated q-characters.** In this paper, we need to use the concept truncated q-characters, see [HL10], [MY12b]. Given a set of monomials $\mathcal{R} \subset \mathcal{P}$, let $\mathbb{Z}\mathcal{R} \subset \mathbb{Z}\mathcal{P}$ denote the $\mathbb{Z}$-module of formal linear combinations of elements of $\mathcal{R}$ with integer coefficients. Define

$$\text{trunc}_{\mathcal{R}} : \mathcal{P} \to \mathcal{R}; \quad m \mapsto \begin{cases} m & \text{if } m \in \mathcal{R}, \\ 0 & \text{if } m \notin \mathcal{R}, \end{cases}$$

and extend $\text{trunc}_{\mathcal{R}}$ as a $\mathbb{Z}$-module map $\mathbb{Z}\mathcal{P} \to \mathbb{Z}\mathcal{R}$.

Given a subset $U \subset I \times \mathbb{C}^\times$, let $\mathcal{Q}_U$ be the subgroups of $\mathcal{Q}$ generated by $A_{i,a}$ with $(i,a) \in U$. Let $\mathcal{Q}_U^{\pm}$ be the monoid generated by $A_{i,a}^{\pm 1}$ with $(i,a) \in U$. The polynomial $\text{trunc}_{m_+\mathcal{Q}_U^-} \chi_q(m_+)$ is called *the q-character of $L(m_+)$ truncated to U*.

The following theorem can be used to compute some truncated q-characters.

Theorem 2.2 (Theorem 2.1, [MY12b]). *Let $U \subset I \times \mathbb{C}^\times$ and $m_+ \in \mathcal{P}^+$. Suppose that $\mathcal{M} \subset \mathcal{P}$ is a finite set of distinct monomials such that*

(i) $\mathcal{M} \subset m_+\mathcal{Q}_U^-$,
(ii) $\mathcal{P}^+ \cap \mathcal{M} = \{m_+\}$,
(iii) for all $m \in \mathcal{M}$ and all $(i,a) \in U$, if $mA_{i,a}^{-1} \notin \mathcal{M}$, then $mA_{i,a}^{-1}A_{j,b} \notin \mathcal{M}$ unless $(j,b) = (i,a)$,
(iv) for all $m \in \mathcal{M}$ and all $i \in I$, there exists a unique i-dominant monomial $M \in \mathcal{M}$ such that

$$\mathit{trunc}_{\beta_i(M\mathcal{Q}_U^-)} \chi_q(\beta_i(M)) = \sum_{m' \in m\mathcal{Q}_{\{i\} \times \mathbb{C}^\times} \cap \mathcal{M}} \beta_i(m').$$

Then

$$trunc_{m_+\mathcal{Q}_U^-}\,\chi_q(m_+) = \sum_{m\in\mathcal{M}} m.$$

Here $\chi_q(\beta_i(M))$ is the q-character of the irreducible $U_{q_i}(\hat{\mathfrak{sl}}_2)$-module with highest weight monomial $\beta_i(M)$ and $\mathrm{trunc}_{\beta_i(M\mathcal{Q}_U^-)}$ is the polynomial obtained from $\chi_q(\beta_i(M))$ by keeping only the monomials of $\chi_q(\beta_i(M))$ in the set $\beta_i(M\mathcal{Q}_U^-)$.

3. Extended T-system of type C_3

In this section, $\mathfrak{g}$ is the simple Lie algebra of type C_3. We will describe the extended T-system which contains all minimal affinizations of type C_3 whose weights are of the form $\lambda_1\omega_1+\lambda_2\omega_2+\lambda_3\omega_3$, where at least one of $\lambda_1,\lambda_2,\lambda_3$ are 0. The extended T-system consisting of four sub-systems I, II, III, IV. Each of the systems I, II, III, IV is closed.

3.1. **Fundamental q-characters of type C_3.** First we use the FM algorithm to compute the q-characters of the fundamental modules type C_3.

Lemma 3.1. *The the q-characters of the fundamental modules type C_3 are*

$$\chi_q(1_0) = 1_0 + 1_2^{-1}2_1 + 2_3^{-1}3_2 + 2_5 3_6^{-1} + 1_6 2_7^{-1} + 1_8^{-1},$$

$$\begin{aligned}\chi_q(2_0) \;=\;& 2_0 + 1_1 2_2^{-1}3_1 + 1_3^{-1}3_1 + 1_1 2_4 3_5^{-1} + 1_3^{-1}2_2 2_4 3_5^{-1} + 1_1 1_5 2_6^{-1}\\ &+1_3^{-1}1_5 2_2 2_6^{-1} + 1_5 2_4^{-1}2_6^{-1}3_3 + 1_1 1_7^{-1} + 1_3^{-1}1_7^{-1}2_2 + 1_7^{-1}2_4^{-1}3_3\\ &+1_5 3_7^{-1} + 1_7^{-1}2_6 3_7^{-1} + 2_8^{-1},\end{aligned}$$

$$\begin{aligned}\chi_q(3_0) \;=\;& 3_0 + 2_1 2_3 3_4^{-1} + 1_4 2_1 2_5^{-1} + 1_2 1_4 2_3^{-1}2_5^{-1}3_2 + 1_6^{-1}2_1 + 1_2 1_6^{-1}2_3^{-1}3_2\\ &+1_4^{-1}1_6^{-1}3_2 + 1_2 1_4 3_6^{-1} + 1_2 1_6^{-1}2_5 3_6^{-1} + 1_4^{-1}1_6^{-1}2_3 2_5 3_6^{-1} + 1_2 2_7^{-1}\\ &+1_4^{-1}2_3 2_7^{-1} + 2_5^{-1}2_7^{-1}3_4 + 3_8^{-1}. \qquad \square\end{aligned}$$

3.2. **The system I.** For $s\in\mathbb{Z}, k,\ell,m\in\mathbb{Z}_{\geq 0}$, we define the following monomials.

$$T_{k,\ell,m}^{(s)} = \left(\prod_{i=0}^{k-1} 3_{s+4i}\right)\left(\prod_{i=0}^{\ell-1} 2_{s+4k+2i+1}\right)\left(\prod_{i=0}^{m-1} 1_{s+4k+2\ell+2i+2}\right),$$

$$\tilde{T}_{k,\ell,m}^{(s)} = \left(\prod_{i=0}^{k-1} 1_{s+2i}\right)\left(\prod_{i=0}^{\ell-1} 2_{s+2k+2i+1}\right)\left(\prod_{i=0}^{m-1} 3_{s+2k+2\ell+4i+4}\right),$$

$$S_{k,\ell}^{(s)} = \left(\prod_{i=0}^{k-1} 2_{s+2i}\right)\left(\prod_{i=0}^{\ell-1} 2_{s+2k+2i+4}\right),$$

$$R^{(s)}_{k,\ell,m} = \left(\prod_{i=0}^{k-1} 2_{s+2i}\right)\left(\prod_{i=0}^{\ell-1} 1_{s+2k+2i+1}\right)\left(\prod_{i=0}^{m-1} 3_{s+2k+4i+3}\right).$$

Let T be a monomial and $\mathcal{T}$ the highest l-weight module of $U_q\hat{\mathfrak{g}}$ with highest l-weight T. For $k,\ell \in \mathbb{Z}_{\geq 0}, s \in \mathbb{Z}$, we have the following trivial relations:

$$\begin{aligned} &\tilde{\mathcal{T}}^{(s)}_{k,0,0} = \mathcal{T}^{(s-2)}_{0,0,k},\ \tilde{\mathcal{T}}^{(s)}_{0,k,0} = \mathcal{T}^{(s)}_{0,k,0},\ \tilde{\mathcal{T}}^{(s)}_{0,0,k} = \mathcal{T}^{(s+4)}_{k,0,0}, \\ &\mathcal{S}^{(s)}_{k,0} = \mathcal{T}^{(s-1)}_{0,k,0},\ \mathcal{S}^{(s)}_{0,k} = \mathcal{T}^{(s+4)}_{0,k,0}, \\ &\mathcal{R}^{(s)}_{k,\ell,0} = \mathcal{T}^{(s-1)}_{0,k,\ell},\ \mathcal{R}^{(s)}_{k,0,0} = \mathcal{T}^{(s)}_{0,k,0},\ \mathcal{R}^{(s)}_{0,k,0} = \mathcal{T}^{(s-1)}_{0,0,k},\ \mathcal{R}^{(s)}_{0,0,k} = \mathcal{T}^{(s+3)}_{k,0,0}. \end{aligned} \tag{3.1}$$

Theorem 3.2. *The modules $\mathcal{T}^{(s)}_{k,\ell,0}$, $\mathcal{T}^{(s)}_{k,0,m}$, $\mathcal{T}^{(s)}_{0,\ell,r}$, $\tilde{\mathcal{T}}^{(s)}_{k,0,m}$, $\mathcal{S}^{(s)}_{k,\ell}$, $\mathcal{R}^{(s)}_{k,2\ell,\ell}$, $\mathcal{R}^{(s)}_{k,2\ell+1,\ell}$, $\mathcal{R}^{(s)}_{k,2\ell+2,\ell}$ are special. In particular, we can use the FM algorithm to compute these modules.*

We will prove Theorem 3.2 in Section 4.

For $k \in \mathbb{Z}_{\geq 1}$, we have the following relations in $\mathrm{Rep}(U_q\hat{\mathfrak{g}})$ in the usual T-system, see [KR90], [Her06],

$$[\mathcal{T}^{(s)}_{k-1,0,0}][\mathcal{T}^{(s+4)}_{k-1,0,0}] = [\mathcal{T}^{(s)}_{k,0,0}][\mathcal{T}^{(s+4)}_{k-2,0,0}] + [\mathcal{T}^{s+1}_{0,2k-2,0}], \tag{3.2}$$

$$[\mathcal{T}^{(s)}_{0,k-1,0}][\mathcal{T}^{(s+2)}_{0,k-1,0}] = [\mathcal{T}^{(s)}_{0,k,0}][\mathcal{T}^{(s+2)}_{0,k-2,0}] + [\mathcal{T}^{s+1}_{k-1,0,0}][\mathcal{T}^{s+1}_{0,0,\lfloor \frac{k}{2}\rfloor}][\mathcal{T}^{s+3}_{0,0,\lfloor \frac{k-1}{2}\rfloor}], \tag{3.3}$$

$$[\mathcal{T}^{(s)}_{k-1,0,0}][\mathcal{T}^{(s+2)}_{k-1,0,0}] = [\mathcal{T}^{(s)}_{k,0,0}][\mathcal{T}^{(s+2)}_{k-2,0,0}] + [\mathcal{T}^{s+1}_{0,k-1,0}]. \tag{3.4}$$

Let $\sigma : \mathbb{Z}_{\geq 0} \to \{0,1\}$ be the function such that $\sigma(k) = 0$ if k is even and $\sigma(k) = 1$ if k is odd.

Our first main result is the following theorem.

Theorem 3.3. *For $k,\ell,m \in \mathbb{Z}_{\geq 1}$, $r \in \{1,2\}$, we have the following relations in* $\mathrm{Rep}(U_q\hat{\mathfrak{g}})$.

$$[\mathcal{T}^{(s)}_{k,\ell-1,0}][\mathcal{T}^{(s+4)}_{k-1,\ell,0}] = [\mathcal{T}^{(s)}_{k,\ell,0}][\mathcal{T}^{(s+4)}_{k-1,\ell-1,0}] + [\mathcal{R}^{(s+1)}_{2k-1,\ell,\lfloor \frac{\ell}{2}\rfloor}][\mathcal{T}^{(s+4k+4)}_{\lfloor \frac{\ell-1}{2}\rfloor,0,0}], \tag{3.5}$$

$$[\mathcal{T}^{(s)}_{k,0,m-1}][\mathcal{T}^{(s+4)}_{k-1,0,m}] = [\mathcal{T}^{(s)}_{k,0,m}][\mathcal{T}^{(s+4)}_{k-1,0,m-1}] + [\mathcal{S}^{(s+1)}_{2k-1,m-1}], \tag{3.6}$$

$$[\mathcal{S}^{(s)}_{k,\ell-1}][\mathcal{S}^{(s+2)}_{k-1,\ell}] = [\mathcal{S}^{(s)}_{k,\ell}][\mathcal{S}^{(s+2)}_{k-1,\ell-1}] + [\tilde{\mathcal{T}}^{(s+1)}_{k,0,\lfloor \frac{\ell}{2}\rfloor}][\mathcal{T}^{(s+2\sigma(k)+1)}_{\lfloor \frac{k}{2}\rfloor,0,\ell}][\mathcal{T}^{(s+2k+7)}_{\lfloor \frac{\ell-1}{2}\rfloor,0,0}][\mathcal{T}^{(s+2\sigma(k+1)+1)}_{\lfloor \frac{k-1}{2}\rfloor,0,0}], \tag{3.7}$$

$$[\tilde{\mathcal{T}}^{(s)}_{k,0,m-1}][\tilde{\mathcal{T}}^{(s+2)}_{k-1,0,m}] = [\tilde{\mathcal{T}}^{(s)}_{k,0,m}][\tilde{\mathcal{T}}^{(s+2)}_{k-1,0,m-1}] + [\mathcal{S}^{(s+1)}_{k-1,2m-1}], \tag{3.8}$$

$$[\mathcal{T}^{(s)}_{0,\ell,r-1}][\mathcal{T}^{(s+2)}_{0,\ell-1,r}] = [\mathcal{T}^{(s)}_{0,\ell,r}][\mathcal{T}^{(s+2)}_{0,\ell-1,r-1}] + [\mathcal{T}^{(s-1)}_{0,0,\ell-1}][\mathcal{T}^{(s+2\sigma(\ell)+2)}_{\lfloor \frac{\ell}{2}\rfloor,r-1,0}][\mathcal{T}^{(s+2\sigma(\ell+1)+2)}_{\lfloor \frac{\ell+1}{2}\rfloor,0,0}], \tag{3.9}$$

$$[\mathcal{R}^{(s)}_{k,2\ell,\ell-1}][\mathcal{R}^{(s+2)}_{k-1,2\ell,\ell}] = [\mathcal{R}^{(s)}_{k,2\ell,\ell}][\mathcal{R}^{(s+2)}_{k-1,2\ell,\ell-1}] + [\mathcal{T}^{(s-1)}_{0,0,k+2\ell}][\mathcal{T}^{(s+2\sigma(k+1)+1)}_{\lfloor \frac{k-1}{2} \rfloor,0,2\ell}][\mathcal{T}^{(s+2\sigma(k)+1)}_{\lfloor \frac{k}{2} \rfloor,2\ell-1,0}], \tag{3.10}$$

$$[\mathcal{R}^{(s)}_{k,2\ell,\ell}][\mathcal{R}^{(s+2)}_{k-1,2\ell+1,\ell}] = [\mathcal{R}^{(s)}_{k,2\ell+1,\ell}][\mathcal{R}^{(s+2)}_{k-1,2\ell,\ell}] + [\tilde{\mathcal{T}}^{(s+1)}_{k-1,0,\ell}][\mathcal{T}^{(s+2\sigma(k+1)+1)}_{\lfloor \frac{(k+1)}{2} \rfloor+\ell,0,0}][\mathcal{T}^{(s+2\sigma(k)+1)}_{\lfloor \frac{k}{2} \rfloor,2\ell,0}], \tag{3.11}$$

$$[\mathcal{R}^{(s)}_{k,2\ell+1,\ell-1}][\mathcal{R}^{(s+2)}_{k-1,2\ell+2,\ell}] = [\mathcal{R}^{(s)}_{k,2\ell+2,\ell}][\mathcal{R}^{(s+2)}_{k-1,2\ell+1,\ell}] + [\tilde{\mathcal{T}}^{(s+1)}_{k-1,0,\ell}][\mathcal{T}^{(s+2\sigma(k+1)+1)}_{\lfloor \frac{k+1}{2} \rfloor+\ell,0,0}][\mathcal{T}^{(s+2\sigma(k)+1)}_{\lfloor \frac{k}{2} \rfloor,2\ell+1,0}], \tag{3.12}$$

$$[\mathcal{R}^{(s)}_{0,2\ell+i,\ell}] = [\mathcal{T}^{(s-2)}_{0,0,2\ell+i}][\mathcal{T}^{(s+2)}_{\ell,0,0}], \ i = 0,1,2. \tag{3.13}$$

We will prove Theorem 3.3 in Section 5.

From the relations (3.10), (3.11), and (3.12), we see that if $\mathcal{R}^{(s)}_{k,\ell,0}$ is a module in the relations of Theorem 3.3 for some k, ℓ, s, then $\ell \in \{0, 1, 2\}$. Therefore we need the relations for the modules $\mathcal{R}^{(s)}_{k,\ell,0} = \mathcal{T}^{(s-1)}_{0,k,\ell}$, $\ell \in \{0, 1, 2\}$, in order to obtain a closed system. These relations are (3.9).

Let us use I to denote the system consisting of the relations in Theorem 3.3 and the relations (3.1), (3.2)-(3.4). Then the system I is closed.

All relations except (3.13) in Theorem 3.3 are written in the form $[\mathcal{L}][\mathcal{R}] = [\mathcal{T}][\mathcal{B}] + [\mathcal{S}]$, where $\mathcal{L}, \mathcal{R}, \mathcal{T}, \mathcal{B}$ are irreducible modules which are called *left, right, top* and *bottom modules* and $\mathcal{S}$ is a tensor product of some irreducible modules. The factors of $\mathcal{S}$ are called *sources*, see [MY12b], [LM12]. Moreover, we have the following theorem.

Theorem 3.4. *For each relation in Theorem 3.3, all summands on the right hand side, $\mathcal{T} \otimes \mathcal{B}$ and $\mathcal{S}$, are irreducible.*

We will prove Theorem 3.4 in Section 6.

The system I can be used to compute the q-characters of the modules in the system. We have the following proposition.

Proposition 3.5. *One can compute the q-characters of $\mathcal{T}^{(s)}_{k,\ell,0}$, $\mathcal{T}^{(s)}_{k,0,m}$, $\mathcal{T}^{(s)}_{0,\ell,r}$, $\tilde{\mathcal{T}}^{(s)}_{k,0,m}$, $\mathcal{S}^{(s)}_{k,\ell}$, $\mathcal{R}^{(s)}_{k,2\ell+j,\ell}$, $j = 0, 1, 2$, $s \in \mathbb{Z}$, $k, \ell, m \in \mathbb{Z}_{\geq 0}$, recursively, from $\chi_q(1_0), \chi_q(2_0)$, and $\chi_q(3_0)$, by using the relations in the system I.*

We will prove Proposition 3.5 in Section 7.

3.3. **The system II.** We will describe the system II which is dual to the system I. Let $\bar{T}^{(s)}_{k,\ell,m}, \bar{\tilde{T}}^{(s)}_{k,\ell,m}, \bar{S}^{(s)}_{k,\ell}, \bar{R}^{(s)}_{k,\ell,m}$ be the monomials obtained from $T^{(s)}_{k,\ell,m}$, $\tilde{T}^{(s)}_{k,\ell,m}$, $S^{(s)}_{k,\ell}$, $R^{(s)}_{k,\ell,m}$ by replacing i_a with i_{-a}, $i = 1, 2, 3$. Namely,

$$\bar{T}^{(s)}_{k,\ell,m} = \left(\prod_{i=0}^{m-1} 1_{-s-4k-2\ell-2i-2} \right) \left(\prod_{i=0}^{\ell-1} 2_{-s-4k-2i-1} \right) \left(\prod_{i=0}^{k-1} 3_{-s-4i} \right),$$

$$\bar{\tilde{T}}^{(s)}_{k,\ell,m} = \left(\prod_{i=0}^{m-1} 3_{-s-2k-2\ell-4i-4}\right)\left(\prod_{i=0}^{\ell-1} 2_{-s-2k-2i-1}\right)\left(\prod_{i=0}^{k-1} 1_{-s-2i}\right),$$

$$\bar{S}^{(s)}_{k,\ell} = \left(\prod_{i=0}^{\ell-1} 2_{-s-2k-2i-4}\right)\left(\prod_{i=0}^{k-1} 2_{-s-2i}\right),$$

$$\bar{R}^{(s)}_{k,\ell,m} = \left(\prod_{i=0}^{m-1} 3_{-s-2k-4i-3}\right)\left(\prod_{i=0}^{\ell-1} 1_{-s-2k-2i-1}\right)\left(\prod_{i=0}^{k-1} 2_{-s-2i}\right).$$

For $k,\ell \in \mathbb{Z}_{\geq 0}, s \in \mathbb{Z}$, we have the following trivial relations

$$\begin{aligned}
&\bar{\tilde{\mathcal{T}}}^{(s)}_{k,0,0} = \mathcal{T}^{(-s-2k)}_{0,0,k},\ \bar{\tilde{\mathcal{T}}}^{(s)}_{0,k,0} = \mathcal{T}^{(-s-2k)}_{0,k,0},\ \bar{\tilde{\mathcal{T}}}^{(s)}_{0,0,k} = \mathcal{T}^{(-s-4k)}_{k,0,0},\\
&\bar{\mathcal{S}}^{(s)}_{0,k} = \bar{\mathcal{S}}^{(s+4)}_{k,0} = \mathcal{T}^{(-s-2k-3)}_{0,k,0},\\
&\bar{\mathcal{R}}^{(s)}_{k,\ell,0} = \bar{\mathcal{T}}^{(s-1)}_{0,k,\ell},\ \bar{\mathcal{R}}^{(s)}_{k,0,0} = \mathcal{T}^{(-s-2k+1)}_{0,k,0},\ \bar{\mathcal{R}}^{(s)}_{0,k,0} = \mathcal{T}^{(-s-2k-1)}_{0,0,k},\ \bar{\mathcal{R}}^{(s)}_{0,0,k} = \mathcal{T}^{(-s-4k+1)}_{k,0,0}.
\end{aligned} \tag{3.14}$$

Theorem 3.6. *The modules* $\bar{\mathcal{T}}^{(s)}_{k,\ell,0}, \bar{\mathcal{T}}^{(s)}_{k,0,m}, \bar{\mathcal{T}}^{(s)}_{0,\ell,r}, \bar{\tilde{\mathcal{T}}}^{(s)}_{k,0,m}, \bar{\mathcal{S}}^{(s)}_{k,\ell}, \bar{\mathcal{R}}^{(s)}_{k,2\ell+j,\ell},\ k,l,m \in \mathbb{Z}_{\geq 0}, r \in \{1,2\}, j \in \{0,1,2\}$*, are anti-special.*

Proof. This theorem can be proved using the dual arguments of the proof of Theorem 3.2. ☐

Lemma 3.7. *Let* $\iota : \mathbb{Z}\mathcal{P} \to \mathbb{Z}\mathcal{P}$ *be a ring homomorphism such that* $Y_{i,aq^s} \mapsto Y^{-1}_{i,aq^{8-s}}$, $i = 1,2,3$*, for all* $a \in \mathbb{C}^\times, s \in \mathbb{Z}$*. Let* m_+ *be one of the monomials*

$$T^{(s)}_{k,\ell,0},\ T^{(s)}_{k,0,m},\ T^{(s)}_{0,\ell,r},\ \tilde{T}^{(s)}_{k,0,m},\ S^{(s)}_{k,\ell},\ R^{(s)}_{k,2\ell+j,\ell},\ j = 0,1,2.$$

Then $\chi_q(\bar{m_+}) = \iota(\chi_q(m_+))$.

Proof. The lemma is proved by using similar arguments of the proof of Lemma 7.3 of [LM12]. ☐

The modules $\bar{\mathcal{T}}^{(s)}_{k,\ell,0}, \bar{\mathcal{T}}^{(s)}_{k,0,m}, \bar{\mathcal{T}}^{(s)}_{0,\ell,r}, \bar{\tilde{\mathcal{T}}}^{(s)}_{k,0,m}, \bar{\mathcal{S}}^{(s)}_{k,\ell}, \bar{\mathcal{R}}^{(s)}_{k,2\ell+j,\ell}$ satisfy the same relations as in Theorem 3.3 but the roles of left and right modules are exchanged.

Theorem 3.8. *For* $k,\ell,m \in \mathbb{Z}_{\geq 1}$, $r \in \{1,2\}$*, we have the following relations in* $\mathrm{Rep}(U_q\hat{\mathfrak{g}})$.

$$[\bar{\mathcal{T}}^{(s+4)}_{k-1,\ell,0}][\bar{\mathcal{T}}^{(s)}_{k,\ell-1,0}] = [\bar{\mathcal{T}}^{(s)}_{k,\ell,0}][\bar{\mathcal{T}}^{(s+4)}_{k-1,\ell-1,0}] + [\bar{\mathcal{R}}^{(s+1)}_{2k-1,\ell,\lfloor \frac{\ell}{2} \rfloor}][\bar{\mathcal{T}}^{(s+4k+4)}_{\lfloor \frac{\ell-1}{2} \rfloor,0,0}], \tag{3.15}$$

$$[\bar{\mathcal{T}}^{(s+4)}_{k-1,0,m}][\bar{\mathcal{T}}^{(s)}_{k,0,m-1}] = [\bar{\mathcal{T}}^{(s)}_{k,0,m}][\bar{\mathcal{T}}^{(s+4)}_{k-1,0,m-1}] + [\bar{\mathcal{S}}^{(s+1)}_{2k-1,m-1}], \tag{3.16}$$

$$[\bar{\mathcal{S}}^{(s+2)}_{k-1,\ell}][\bar{\mathcal{S}}^{(s)}_{k,\ell-1}] = [\bar{\mathcal{S}}^{(s)}_{k,\ell}][\bar{\mathcal{S}}^{(s+2)}_{k-1,\ell-1}] + [\bar{\bar{\mathcal{T}}}^{(s+1)}_{k,0,\lfloor\frac{\ell}{2}\rfloor}][\bar{\mathcal{T}}^{(s+2\sigma(k)+1)}_{\lfloor\frac{k}{2}\rfloor,0,\ell}][\bar{\mathcal{T}}^{(s+2k+7)}_{\lfloor\frac{\ell-1}{2}\rfloor,0,0}][\bar{\mathcal{T}}^{(s+2\sigma(k+1)+1)}_{\lfloor\frac{k-1}{2}\rfloor,0,0}], \tag{3.17}$$

$$[\bar{\bar{\mathcal{T}}}^{(s+2)}_{k-1,0,m}][\bar{\bar{\mathcal{T}}}^{(s)}_{k,0,m-1}] = [\bar{\bar{\mathcal{T}}}^{(s)}_{k,0,m}][\bar{\bar{\mathcal{T}}}^{(s+2)}_{k-1,0,m-1}] + [\bar{\mathcal{S}}^{(s+1)}_{k-1,2m-1}], \tag{3.18}$$

$$[\bar{\mathcal{T}}^{(s+2)}_{0,\ell-1,r}][\bar{\mathcal{T}}^{(s)}_{0,\ell,r-1}] = [\bar{\mathcal{T}}^{(s)}_{0,\ell,r}][\bar{\mathcal{T}}^{(s+2)}_{0,\ell-1,r-1}] + [\bar{\mathcal{T}}^{(s-1)}_{0,0,\ell-1}][\bar{\mathcal{T}}^{(s+2\sigma(\ell)+2)}_{\lfloor\frac{\ell}{2}\rfloor,r-1,0}][\bar{\mathcal{T}}^{(s+2\sigma(\ell+1)+2)}_{\lfloor\frac{\ell+1}{2}\rfloor,0,0}], \tag{3.19}$$

$$[\bar{\mathcal{R}}^{(s+2)}_{k-1,2\ell,\ell}][\bar{\mathcal{R}}^{(s)}_{k,2\ell,\ell-1}] = [\bar{\mathcal{R}}^{(s)}_{k,2\ell,\ell}][\bar{\mathcal{R}}^{(s+2)}_{k-1,2\ell,\ell-1}] + [\bar{\mathcal{T}}^{(s-1)}_{0,0,k+2\ell}][\bar{\mathcal{T}}^{(s+2\sigma(k+1)+1)}_{\lfloor\frac{k-1}{2}\rfloor,0,2\ell}][\bar{\mathcal{T}}^{(s+2\sigma(k)+1)}_{\lfloor\frac{k}{2}\rfloor,2\ell-1,0}], \tag{3.20}$$

$$[\bar{\mathcal{R}}^{(s+2)}_{k-1,2\ell+1,\ell}][\bar{\mathcal{R}}^{(s)}_{k,2\ell,\ell}] = [\bar{\mathcal{R}}^{(s)}_{k,2\ell+1,\ell}][\bar{\mathcal{R}}^{(s+2)}_{k-1,2\ell,\ell}] + [\bar{\bar{\mathcal{T}}}^{(s+1)}_{k-1,0,\ell}][\bar{\mathcal{T}}^{(s+2\sigma(k+1)+1)}_{\lfloor\frac{(k+1)}{2}\rfloor+\ell,0,0}][\bar{\mathcal{T}}^{(s+2\sigma(k)+1)}_{\lfloor\frac{k}{2}\rfloor,2\ell,0}], \tag{3.21}$$

$$[\bar{\mathcal{R}}^{(s+2)}_{k-1,2\ell+2,\ell}][\bar{\mathcal{R}}^{(s)}_{k,2\ell+1,\ell-1}] = [\bar{\mathcal{R}}^{(s)}_{k,2\ell+2,\ell}][\bar{\mathcal{R}}^{(s+2)}_{k-1,2\ell+1,\ell}] + [\bar{\bar{\mathcal{T}}}^{(s+1)}_{k-1,0,\ell}][\bar{\mathcal{T}}^{(s+2\sigma(k+1)+1)}_{\lfloor\frac{k+1}{2}\rfloor+\ell,0,0}][\bar{\mathcal{T}}^{(s+2\sigma(k)+1)}_{\lfloor\frac{k}{2}\rfloor,2\ell+1,0}], \tag{3.22}$$

$$[\bar{\mathcal{R}}^{(s)}_{0,2\ell+i,\ell}] = [\bar{\mathcal{T}}^{(s-2)}_{0,0,2\ell+i}][\bar{\mathcal{T}}^{(s+2)}_{\ell,0,0}], \ i = 0, 1, 2. \tag{3.23}$$

Moreover, the modules corresponding to each summand on the right hand side of the above relations are all irreducible.

Proof. The theorem follows from the relations in Theorem 3.3, Theorem 3.4, and Lemma 3.7. □

Let us use II to denote the system consisting of the relations in Theorem 3.8 and the relations (3.14), (3.2)-(3.4). Then the system II is closed.

Since $\bar{\mathcal{T}}^{(s)}_{k,\ell,0} = \tilde{\mathcal{T}}^{(-s-4k-2\ell)}_{0,\ell,k}$, the system II contains minimal affinizations $\tilde{\mathcal{T}}^{(-s-4k-2\ell)}_{0,\ell,k}$, $k, \ell \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}$. Using the shift defined in (2.3), we can obtain minimal affinizations $\tilde{\mathcal{T}}^{(s)}_{0,\ell,k}$ from $\tilde{\mathcal{T}}^{(-s-4k-2\ell)}_{0,\ell,k}$, $k, \ell \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}$.

The following proposition is similar to Proposition 3.5.

Proposition 3.9. *One can compute the q-characters of*

$$\bar{\mathcal{T}}^{(s)}_{k,\ell,0}, \bar{\mathcal{T}}^{(s)}_{k,0,m}, \bar{\mathcal{T}}^{(s)}_{0,\ell,r}, \bar{\bar{\mathcal{T}}}^{(s)}_{k,0,m}, \bar{\mathcal{S}}^{(s)}_{k,\ell}, \bar{\mathcal{R}}^{(s)}_{k,2\ell+j,\ell}, \ s \in \mathbb{Z}, \ k, \ell, m \in \mathbb{Z}_{\geq 0}, r \in \{1, 2\},$$

recursively, from $\chi_q(1_0), \chi_q(2_0), \chi_q(3_0)$ by using the relations in the system II. □

3.4. **The system III.** For $s \in \mathbb{Z}, k, \ell \in \mathbb{Z}_{\geq 0}$, we define the following monomials.

$$U^{(s)}_{k,\ell} = \left(\prod_{i=0}^{k-1} 2_{s+2i}\right)\left(\prod_{i=0}^{\ell-1} 3_{s+2k+2i+1}\right),$$

$$V^{(s)}_{k,\ell} = \left(\prod_{i=0}^{k-1} 3_{s+4i}\right)\left(\prod_{i=0}^{\ell-1} 3_{s+4k+4i+2}\right),$$

$$P^{(s)}_{k,\ell} = \left(\prod_{i=0}^{k-1} 3_{s+4i}\right) 2_{4k+1}\left(\prod_{i=0}^{\ell-1} 3_{s+4k+4i+6}\right),$$

$$O^{(s)}_{k,\ell} = \left(\prod_{i=0}^{k-1} 2_{s+2i}\right) 1_{2k+1}1_{2k+3}\left(\prod_{i=0}^{\ell-1} 2_{s+2k+2i+6}\right).$$

For $k \in \mathbb{Z}_{\geq 0}, s \in \mathbb{Z}$, we have the following trivial relations

$$\begin{aligned} &\mathcal{U}^{(s)}_{k,0} = \mathcal{T}^{(s-1)}_{0,k,0},\ \mathcal{V}^{(s)}_{k,0} = \mathcal{T}^{(s)}_{k,0,0},\ \mathcal{V}^{(s)}_{0,k} = \mathcal{T}^{(s+2)}_{k,0,0}, \\ &\mathcal{O}^{(s)}_{k,0} = \mathcal{T}^{(s-1)}_{0,k,2},\ \mathcal{O}^{(s)}_{0,k} = \tilde{\mathcal{T}}^{(s+6)}_{2,k,0},\ \mathcal{P}^{(s)}_{k,0} = \mathcal{T}^{(s)}_{k,1,0},\ \mathcal{P}^{(s)}_{0,k} = \tilde{\mathcal{T}}^{(s)}_{0,1,k}. \end{aligned} \tag{3.24}$$

Theorem 3.10 ([Her07])**.** *The modules* $\tilde{\mathcal{T}}^{(s)}_{k,\ell,0}$*,* $s \in \mathbb{Z}$*,* $k, \ell \in \mathbb{Z}_{\geq 0}$*, are special.*

Theorem 3.11. *The modules* $\mathcal{U}^{(s)}_{k,\ell}$*,* $\mathcal{V}^{(s)}_{k,\ell}$*,* $\mathcal{P}^{(s)}_{k,\ell}$*,* $\mathcal{O}^{(s)}_{k,\ell}$*,* $s \in \mathbb{Z}$*,* $k, \ell \in \mathbb{Z}_{\geq 0}$*, are special.*

We will prove Theorem 3.11 in Section 4.

Our second main result is the following theorem.

Theorem 3.12. *For* $s \in \mathbb{Z}$*,* $k, \ell \in \mathbb{Z}_{\geq 1}$*,* $p \in \mathbb{Z}_{\geq 2}$*, we have the following relations in* $\mathrm{Rep}(U_q\hat{\mathfrak{g}})$*.*

$$[\tilde{\mathcal{T}}^{(s)}_{k,\ell-1,0}][\tilde{\mathcal{T}}^{(s+2)}_{k-1,\ell,0}] = [\tilde{\mathcal{T}}^{(s)}_{k,\ell,0}][\tilde{\mathcal{T}}^{(s+2)}_{k-1,\ell-1,0}] + [\tilde{\mathcal{T}}^{(s+2k+2)}_{\ell-1,0,0}][\mathcal{U}^{(s+1)}_{k-1,\ell}], \tag{3.25}$$

$$[\mathcal{U}^{(s)}_{r,\ell}] = [\tilde{\mathcal{T}}^{(s-1)}_{0,r,\lfloor\frac{\ell}{2}\rfloor}][\mathcal{T}^{(s+2r+1)}_{\lfloor\frac{\ell+1}{2}\rfloor,0,0}], \quad r = 0, 1, \tag{3.26}$$

$$[\mathcal{U}^{(s)}_{p,\ell-1}][\mathcal{U}^{(s+2)}_{p-1,\ell}] = [\mathcal{U}^{(s)}_{p,\ell}][\mathcal{U}^{(s+2)}_{p-1,\ell-1}] + \begin{cases} [\tilde{\mathcal{T}}^{(s+1)}_{p,\ell-1,0}][\mathcal{T}^{(s+2\sigma(p)+1)}_{\lfloor\frac{p}{2}\rfloor+\frac{\ell}{2},0,0}][\mathcal{V}^{(s+2\sigma(p+1)+1)}_{\lfloor\frac{p-1}{2}\rfloor,\frac{\ell}{2}}], & \text{if } \ell \text{ is even}, \\ [\tilde{\mathcal{T}}^{(s+1)}_{p,\ell-1,0}][\mathcal{T}^{(s+2\sigma(p+1)+1)}_{\lfloor\frac{p+1}{2}\rfloor+\frac{\ell-1}{2},0,0}][\mathcal{P}^{(s+2\sigma(p)+1)}_{\lfloor\frac{p-2}{2}\rfloor,\frac{\ell-1}{2}}], & \text{if } \ell \text{ is odd}, \end{cases} \tag{3.27}$$

$$[\mathcal{V}^{(s)}_{k,\ell-1}][\mathcal{V}^{(s+4)}_{k-1,\ell}] = [\mathcal{V}^{(s)}_{k,\ell}][\mathcal{V}^{(s+4)}_{k-1,\ell-1}] + [\mathcal{O}^{(s+1)}_{2(k-1),2(\ell-1)}], \tag{3.28}$$

$$[\mathcal{O}^{(s)}_{k,\ell-1}][\mathcal{O}^{(s+2)}_{k-1,\ell}] = [\mathcal{O}^{(s)}_{k,\ell}][\mathcal{O}^{(s+2)}_{k-1,\ell-1}] + [\mathcal{P}^{(s+\sigma(k)+1)}_{\lfloor\frac{k}{2}\rfloor,\lfloor\frac{\ell}{2}\rfloor}][\mathcal{V}^{(s+\sigma(k+1)+1)}_{\lfloor\frac{k+1}{2}\rfloor,\lfloor\frac{\ell+1}{2}\rfloor}][\tilde{\mathcal{T}}^{(s+1)}_{k-1,0,0}][\tilde{\mathcal{T}}^{(s+2k+7)}_{\ell-1,0,0}], \tag{3.29}$$

$$[\mathcal{P}^{(s)}_{k,\ell-1}][\mathcal{P}^{(s+2)}_{k-1,\ell}] = [\mathcal{P}^{(s)}_{k,\ell}][\mathcal{P}^{(s+2)}_{k-1,\ell-1}] + [\mathcal{O}^{(s+1)}_{2k-1,2\ell-1}]. \tag{3.30}$$

We will prove Theorem 3.12 in Section 5.

Note that the relations for the modules $\mathcal{O}^{(s)}_{0,k} = \tilde{\mathcal{T}}^{(s+6)}_{2,k,0}$ are contained in (3.25). The modules $\mathcal{T}^{(s-1)}_{0,k,2}$, $\mathcal{T}^{(s)}_{k,1,0}$ in (3.24) can be computed by using the relations (3.5, $\ell = 1$), (3.9), (3.11, $\ell = 0$) in the system I. The modules $\tilde{\mathcal{T}}^{(s)}_{0,1,k}$ in (3.24) can be computed by using the relations (3.15, $\ell = 1$) and (3.21, $\ell = 0$) in the system II.

Let us use III to denote the system consisting of the relations in Theorem 3.12, the relations (3.5, $\ell = 1$), (3.9), (3.11, $\ell = 0$) in the system I, the relations (3.15, $\ell = 1$), (3.21, $\ell = 0$) in the system II, and the relations (3.24), (3.2)-(3.4). Then the system III is closed.

All relations except (3.26) in Theorem 3.12 are written in the form $[\mathcal{L}][\mathcal{R}] = [\mathcal{T}][\mathcal{B}]+[\mathcal{S}]$. We have the following theorem.

Theorem 3.13. *For each relation in Theorem 3.12, all summands on the right hand side, $\mathcal{T} \otimes \mathcal{B}$ and $\mathcal{S}$, are irreducible.*

We will prove Theorem 3.13 in Section 6.

The system III can be used to compute the q-characters of the modules in the system III. The following proposition can be proved by similar arguments as the proof of Proposition 3.5.

Proposition 3.14. *One can compute the q-characters of $\mathcal{U}^{(s)}_{k,\ell}$, $\mathcal{V}^{(s)}_{k,\ell}$, $\mathcal{P}^{(s)}_{k,\ell}$, $\mathcal{O}^{(s)}_{k,\ell}$, $s \in \mathbb{Z}$, $k, \ell \in \mathbb{Z}_{\geq 0}$, recursively, from $\chi_q(1_0), \chi_q(2_0)$, and $\chi_q(3_0)$ by using the relations in the system III.* □

3.5. **The system IV.** We will describe the system IV which is dual to the system III. Let $\bar{U}^{(s)}_{k,\ell}, \bar{V}^{(s)}_{k,\ell}, \bar{P}^{(s)}_{k,\ell}, \bar{O}^{(s)}_{k,\ell}$ be the monomials obtained from $U^{(s)}_{k,\ell}$, $V^{(s)}_{k,\ell}$, $P^{(s)}_{k,\ell}$, $O^{(s)}_{k,\ell}$ by replacing i_a with i_{-a}, $i = 1, 2, 3$. Namely,

$$\bar{U}^{(s)}_{k,\ell} = \left(\prod_{i=0}^{\ell-1} 3_{-s-2k-2i-1}\right)\left(\prod_{i=0}^{k-1} 2_{-s-2i}\right),$$

$$\bar{V}^{(s)}_{k,\ell} = \left(\prod_{i=0}^{\ell-1} 3_{-s-4k-4i-2}\right)\left(\prod_{i=0}^{k-1} 3_{-s-4i}\right),$$

$$\bar{P}^{(s)}_{k,\ell} = \left(\prod_{i=0}^{\ell-1} 3_{-s-4k-4i-6}\right) 2_{-4k-1} \left(\prod_{i=0}^{k-1} 3_{-s-4i}\right),$$

$$\bar{O}^{(s)}_{k,\ell,m} = \left(\prod_{i=0}^{\ell-1} 2_{-s-2k-2i-6}\right) 1_{-2k-1}1_{-2k-3} \left(\prod_{i=0}^{k-1} 2_{-s-2i}\right).$$

For $k \in \mathbb{Z}_{\geq 0}, s \in \mathbb{Z}$, we have the following trivial relations

$$\begin{aligned} &\bar{\mathcal{U}}^{(s)}_{k,0} = \bar{\mathcal{T}}^{(s-1)}_{0,k,0},\ \bar{\mathcal{V}}^{(s)}_{k,0} = \bar{\mathcal{T}}^{(s)}_{k,0,0},\ \bar{\mathcal{V}}^{(s)}_{0,k} = \bar{\mathcal{T}}^{(s+2)}_{k,0,0}, \\ &\bar{\mathcal{O}}^{(s)}_{k,0} = \bar{\mathcal{T}}^{(s-1)}_{0,k,2},\ \bar{\mathcal{O}}^{(s)}_{0,k} = \bar{\bar{\mathcal{T}}}^{(s+6)}_{2,k,0}, \\ &\bar{\mathcal{P}}^{(s)}_{k,0} = \bar{\mathcal{T}}^{(s)}_{k,1,0},\ \bar{\mathcal{P}}^{(s)}_{0,k} = \bar{\bar{\mathcal{T}}}^{(s)}_{0,1,k} = \mathcal{T}^{(-s-4k-2)}_{k,1,0}. \end{aligned} \tag{3.31}$$

The following theorem is proved by using the dual arguments of the proof of Theorem 3.11.

Theorem 3.15. *The modules $\bar{\bar{\mathcal{T}}}^{(s)}_{k,\ell,0}$, $\bar{\mathcal{U}}^{(s)}_{k,\ell}$, $\bar{\mathcal{V}}^{(s)}_{k,\ell}$, $\bar{\mathcal{P}}^{(s)}_{k,\ell}$, $\bar{\mathcal{O}}^{(s)}_{k,\ell}$, $s \in \mathbb{Z}$, $k, \ell \in \mathbb{Z}_{\geq 0}$, are anti-special.* □

Lemma 3.16. *Let $\iota : \mathbb{Z}\mathcal{P} \to \mathbb{Z}\mathcal{P}$ be a homomorphism of rings such that $Y_{i,aq^s} \mapsto Y^{-1}_{i,aq^{8-s}}$, $i = 1, 2, 3$, for all $a \in \mathbb{C}^\times, s \in \mathbb{Z}$. Let m_+ be one of the monomials*

$$\tilde{T}^{(s)}_{k,\ell,0},\ U^{(s)}_{k,\ell},\ V^{(s)}_{k,\ell},\ P^{(s)}_{k,\ell},\ O^{(s)}_{k,\ell},\ s \in \mathbb{Z}, k, \ell \in \mathbb{Z}_{\geq 0},$$

Then $\chi_q(\bar{m_+}) = \iota(\chi_q(m_+))$.

Proof. The lemma is proved by using similar arguments of the proof of Lemma 7.3 of [LM12]. □

The modules $\bar{\bar{\mathcal{T}}}^{(s)}_{k,\ell,0}, \bar{\mathcal{U}}^{(s)}_{k,\ell}, \bar{\mathcal{V}}^{(s)}_{k,\ell}, \bar{\mathcal{P}}^{(s)}_{k,\ell}, \bar{\mathcal{O}}^{(s)}_{k,\ell}$ satisfy the same relations as in Theorem 3.12 but the roles of left and right modules are exchanged. More precisely, we have the following theorem.

Theorem 3.17. *For $s \in \mathbb{Z}$, $k, \ell \in \mathbb{Z}_{\geq 1}$, $p \in \mathbb{Z}_{\geq 2}$, we have the following relations in* $\operatorname{Rep}(U_q\hat{\mathfrak{g}})$.

$$[\bar{\bar{\mathcal{T}}}^{(s+2)}_{k-1,\ell,0}][\bar{\bar{\mathcal{T}}}^{(s)}_{k,\ell-1,0}] = [\bar{\bar{\mathcal{T}}}^{(s)}_{k,\ell,0}][\bar{\bar{\mathcal{T}}}^{(s+2)}_{k-1,\ell-1,0}] + [\bar{\bar{\mathcal{T}}}^{(s+2k+2)}_{\ell-1,0,0}][\bar{\mathcal{U}}^{(s+1)}_{k-1,\ell}], \tag{3.32}$$

$$[\bar{\mathcal{U}}^{(s)}_{r,\ell}] = [\bar{\bar{\mathcal{T}}}^{(s-1)}_{0,r,\lfloor \frac{\ell}{2} \rfloor}][\bar{\mathcal{T}}^{(s+2r+1)}_{\lfloor \frac{\ell+1}{2} \rfloor,0,0}], \quad r = 0, 1, \tag{3.33}$$

$$[\bar{\mathcal{U}}^{(s+2)}_{p-1,\ell}][\bar{\mathcal{U}}^{(s)}_{p,\ell-1}] = [\bar{\mathcal{U}}^{(s)}_{p,\ell}][\bar{\mathcal{U}}^{(s+2)}_{p-1,\ell-1}] + \begin{cases} [\bar{\bar{\mathcal{T}}}^{(s+1)}_{p,\ell-1,0}][\bar{\mathcal{T}}^{(s+2\sigma(p)+1)}_{\lfloor \frac{p}{2} \rfloor + \frac{\ell}{2},0,0}][\bar{\mathcal{V}}^{(s+2\sigma(p+1)+1)}_{\lfloor \frac{p-1}{2} \rfloor, \frac{\ell}{2}}], & \text{if } \ell \text{ is even}, \\ [\bar{\bar{\mathcal{T}}}^{(s+1)}_{p,\ell-1,0}][\bar{\mathcal{T}}^{(s+2\sigma(p+1)+1)}_{\lfloor \frac{p+1}{2} \rfloor + \frac{\ell-1}{2},0,0}][\bar{\mathcal{P}}^{(s+2\sigma(p)+1)}_{\lfloor \frac{p-2}{2} \rfloor, \frac{\ell-1}{2}}], & \text{if } \ell \text{ is odd}, \end{cases} \tag{3.34}$$

$$[\bar{\mathcal{V}}^{(s+4)}_{k-1,\ell}][\bar{\mathcal{V}}^{(s)}_{k,\ell-1}] = [\bar{\mathcal{V}}^{(s)}_{k,\ell}][\bar{\mathcal{V}}^{(s+4)}_{k-1,\ell-1}] + [\bar{\mathcal{O}}^{(s+1)}_{2(k-1),2(\ell-1)}], \tag{3.35}$$

$$[\bar{\mathcal{O}}^{(s+2)}_{k-1,\ell}][\bar{\mathcal{O}}^{(s)}_{k,\ell-1}] = [\bar{\mathcal{O}}^{(s)}_{k,\ell}][\bar{\mathcal{O}}^{(s+2)}_{k-1,\ell-1}] + [\bar{\mathcal{P}}^{(s+\sigma(k)+1)}_{\lfloor \frac{k}{2} \rfloor, \lfloor \frac{\ell}{2} \rfloor}][\bar{\mathcal{V}}^{(s+\sigma(k+1)+1)}_{\lfloor \frac{k+1}{2} \rfloor, \lfloor \frac{\ell+1}{2} \rfloor}][\bar{\bar{\mathcal{T}}}^{(s+1)}_{k-1,0,0}][\bar{\bar{\mathcal{T}}}^{(s+2k+7)}_{\ell-1,0,0}], \tag{3.36}$$

$$[\bar{\mathcal{P}}^{(s+2)}_{k-1,\ell}][\bar{\mathcal{P}}^{(s)}_{k,\ell-1}] = [\bar{\mathcal{P}}^{(s)}_{k,\ell}][\bar{\mathcal{P}}^{(s+2)}_{k-1,\ell-1}] + [\bar{\mathcal{O}}^{(s+1)}_{2k-1,2\ell-1}]. \tag{3.37}$$

Moreover, the modules corresponding to each summand on the right hand side of the above relations are all irreducible.

Proof. The theorem follows from the relations in Theorem 3.12, Theorem 3.13, and Lemma 3.16. □

Note that the relations for the modules $\bar{\mathcal{O}}^{(s)}_{0,k} = \bar{\bar{\mathcal{T}}}^{(s+6)}_{2,k,0}$ are contained in (3.32). The modules $\bar{\mathcal{T}}^{(s-1)}_{0,k,2}$, $\bar{\mathcal{T}}^{(s)}_{k,1,0}$ in (3.31) can be computed by using the relations (3.15, $\ell = 1$), (3.19), (3.21, $\ell = 0$) in the system II. The modules $\mathcal{T}^{(-s-4k-2)}_{k,1,0}$ in (3.31) can be computed by using the relations (3.5, $\ell = 1$) and (3.11, $\ell = 0$) in the system I.

Let us use IV to denote the system consisting of the relations in Theorem 3.17, the relations (3.15, $\ell = 1$), (3.19), (3.21, $\ell = 0$) in the system II, the relations (3.5, $\ell = 1$), (3.11, $\ell = 0$) in the system I, and the relations (3.24), (3.2)-(3.4). Then the system IV is closed.

Since $\bar{\bar{\mathcal{T}}}^{(s)}_{k,\ell,0} = \mathcal{T}^{(-s-2k-2\ell)}_{0,\ell,k}$, the system II contains minimal affinizations $\mathcal{T}^{(-s-2k-2\ell)}_{0,\ell,k}$, $k, \ell \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}$. Using the shift defined in (2.3), we can obtain minimal affinizations $\mathcal{T}^{(s)}_{0,\ell,k}$ from $\mathcal{T}^{(-s-2k-2\ell)}_{0,\ell,k}$, $k, \ell \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}$.

The following proposition is similar to Proposition 3.5.

Proposition 3.18. *One can compute the q-characters of*

$$\bar{\bar{\mathcal{T}}}^{(s)}_{k,\ell,0},\ \bar{\mathcal{U}}^{(s)}_{k,\ell},\ \bar{\mathcal{V}}^{(s)}_{k,\ell},\ \bar{\mathcal{P}}^{(s)}_{k,\ell},\ \bar{\mathcal{O}}^{(s)}_{k,\ell},\ s \in \mathbb{Z},\ k, \ell \in \mathbb{Z}_{\geq 0}, r \in \{1, 2\},$$

recursively, from $\chi_q(1_0), \chi_q(2_0), \chi_q(3_0)$ by using the relations in the system IV. □

4. Special modules

In this section, we prove Theorem 3.2 and Theorem 3.11. Namely, we will prove that for $s \in \mathbb{Z}$, $k, \ell, m \in \mathbb{Z}_{\geq 0}$, $r \in \{0, 1, 2\}$, the modules

$$\begin{aligned} &\mathcal{T}^{(s)}_{k,\ell,0},\ \mathcal{T}^{(s)}_{k,0,m},\ \tilde{\mathcal{T}}^{(s)}_{k,0,m},\ \mathcal{T}^{(s)}_{0,\ell,r},\ \mathcal{S}^{(s)}_{k,\ell},\ \mathcal{R}^{(s)}_{k,2\ell,\ell}, \\ &\mathcal{R}^{(s)}_{k,2\ell+1,\ell},\ \mathcal{R}^{(s)}_{k,2\ell+2,\ell},\ \mathcal{U}^{(s)}_{k,\ell},\ \mathcal{V}^{(s)}_{k,\ell},\ \mathcal{P}^{(s)}_{k,\ell},\ \mathcal{O}^{(s)}_{k,\ell}, \end{aligned} \tag{4.1}$$

are special. Since the modules

$$\begin{aligned} &\mathcal{T}^{(s)}_{k,0,0},\ \mathcal{T}^{(s)}_{0,k,0},\ \mathcal{T}^{(s)}_{0,0,k},\ \tilde{\mathcal{T}}^{(s)}_{k,0,0},\ \tilde{\mathcal{T}}^{(s)}_{0,0,k},\ \mathcal{S}^{(s)}_{0,k}, \\ &\mathcal{S}^{(s)}_{k,0},\ \mathcal{R}^{(s)}_{k,0,0},\ \mathcal{R}^{(s)}_{0,1,0},\ \mathcal{R}^{(s)}_{0,2,0},\ U^{(s)}_{k,0},\ V^{(s)}_{k,0},\ V^{(s)}_{0,k}, \end{aligned}$$

are Kirillov-Reshetikhin modules, they are special. In the following, we will prove that the other modules in (4.1) are special.

4.1. **The case of $\mathcal{T}^{(s)}_{k,\ell,0}$.** Let $m_+ = T^{(s)}_{k,\ell,0}$. Without loss of generality, we may assume that $s = 0$. Then

$$m_+ = (3_0 3_4 \cdots 3_{4k-4})(2_{4k+1} 2_{4k+3} \cdots 2_{4k+2\ell-1}).$$

Let

$$U = I \times \{aq^s : s \in \mathbb{Z}, s \leq 4k + 2\ell - 1\}.$$

Since all monomials in $\mathscr{M}(\chi_q(m_+) - \operatorname{trunc}_{m_+\mathcal{Q}_U^-} \chi_q(m_+))$ are right-negative, it is sufficient to show that $\operatorname{trunc}_{m_+\mathcal{Q}_U^-} \chi_q(m_+)$ is special.

Let $\mathcal{M}$ be the finite set consisting of the following monomials

$$m_0 = m_+,\ m_1 = m_0 A^{-1}_{3,4k-2},\ m_2 = m_1 A^{-1}_{3,4k-6}, \ldots, m_k = m_{k-1} A^{-1}_{3,2}.$$

It is easy to see that $\mathcal{M}$ satisfies the conditions in Theorem 2.2. Therefore

$$\operatorname{trunc}_{m_+\mathcal{Q}_U^-} \chi_q(m_+) = \sum_{m \in \mathcal{M}} m$$

and hence $\operatorname{trunc}_{m_+\mathcal{Q}_U^-} \chi_q(m_+)$ is special.

4.2. **The case of $\mathcal{T}^{(s)}_{k,0,m}$.** Let $m_+ = T^{(s)}_{k,0,m}$. Without loss of generality, we may assume that $s = 0$. Then

$$m_+ = (3_0 3_4 \cdots 3_{4k-4})(1_{4k+2} 1_{4k+4} \cdots 1_{4k+2\ell}).$$

Case 1. $k = 1$. In this case,

$$m_+ = 3_0(1_6 1_{4k+4} \cdots 1_{2\ell+4}).$$

Let

$$U = I \times \{aq^s : s \in \mathbb{Z}, s \leq 2\ell + 4\}.$$

Let $\mathcal{M}$ be the finite set consisting of the following monomials

$$m_0 = m_+,\ m_1 = m_0 A^{-1}_{3,2},\ m_2 = m_1 A^{-1}_{2,4},\ m_3 = m_2 A^{-1}_{2,2},\ m_4 = m_3 A^{-1}_{3,4}.$$

By Theorem 2.2,

$$\operatorname{trunc}_{m_+\mathcal{Q}_U^-} \chi_q(m_+) = \sum_{m \in \mathcal{M}} m$$

and hence $\operatorname{trunc}_{m_+\mathcal{Q}_U^-} \chi_q(m_+)$ is special. Therefore $\mathcal{T}^{(s)}_{1,0,m}$ is special.

Case 2. $k > 1$. We embed $L(m_+)$ into two different tensor products. Since each factor in the tensor product is special, we can use the FM algorithm to compute the q-characters of the factors. We classify the dominant monomials in the first tensor product and prove that the only dominant monomial in the first tensor product which occurs in the second tensor product is m_+. Hence $L(m_+)$ is special.

The first tensor product is $L(m_1') \otimes L(m_2')$, where

$$m_1' = 3_0 3_4 \cdots 3_{4k-8},\ m_2' = 3_{4k-4} 1_{4k+2} 1_{4k+4} \cdots 1_{4k+2\ell}.$$

In Case 1, we have shown that $L(m_2')$ is special. Therefore the FM algorithm works for $L(m_2')$. We will use the FM algorithm to compute $\chi_q(L(m_1')), \chi_q(L(m_2'))$ and classify all dominant monomials in $\chi_q(L(m_1'))\chi_q(L(m_2'))$. Let $m = m_1m_2$ be a dominant monomial, where $m_i \in \mathscr{M}(L(m_i'))$, $i = 1, 2$.

Suppose that $m_2 \neq m_2'$. If m_2 is right-negative, then m is a right negative monomial and therefore m is not dominant. This is a contradiction. Hence m_2 is not right-negative. By Case 1, m_2 is one of the following monomials

$$\begin{aligned}
\bar{m}_1 &= m_2' A_{3,4k-2}^{-1} = 3_{4k}^{-1} 2_{4k-3} 2_{4k-1} 1_{4k+2} 1_{4k+4} \cdots 1_{4k+2\ell},\\
\bar{m}_2 &= \bar{m}_1 A_{2,4k}^{-1} = 2_{4k-3} 2_{4k+1}^{-1} 1_{4k} 1_{4k+2} 1_{4k+4} \cdots 1_{4k+2\ell},\\
\bar{m}_3 &= \bar{m}_2 A_{2,4k-2}^{-1} = 2_{4k-1}^{-1} 2_{4k+1}^{-1} 3_{4k-2} 1_{4k-2} 1_{4k} 1_{4k+2} 1_{4k+4} \cdots 1_{4k+2\ell},\\
\bar{m}_4 &= \bar{m}_3 A_{3,4k}^{-1} = 3_{4k+2}^{-1} 1_{4k-2} 1_{4k} 1_{4k+2} 1_{4k+4} \cdots 1_{4k+2\ell}.
\end{aligned}$$

By Lemma 3.1, 3_{4k}^{-1} cannot be canceled by any monomial in $\chi_q(m_1')$. Therefore $m = m_1m_2$ ($m_1 \in \chi_q(m_1')$) is not dominant. This is a contradiction. Hence $m_2 \neq \bar{m}_1$. Similarly, m_2 cannot be $\bar{m}_i, i = 2, 3, 4$. This is a contradiction. Therefore $m_2 = m_2'$.

If $m_1 \neq m_1'$, then m_1 is right negative. Since m is dominant, each factor with a negative power in m_1 needs to be canceled by a factor in m_2'. By Lemma 3.1, the only factor in m_2' which can be canceled is 3_{4k-4}. We have $\mathcal{M}(L(m_1')) \subset \mathcal{M}(\chi_q(3_0 3_4 \cdots 3_{4k-12}))\chi_q(L(3_{4k-8})))$. Only monomials in $\chi_q(L(3_{4k-8}))$ can cancel 3_{4k-4}. The only monomial in $\chi_q(L(3_{4k-8}))$ which can cancel 3_{4k-4} is $3_{4k-4}^{-1} 2_{4k-7} 2_{4k-5}$. Therefore m_1 is in the set

$$\mathcal{M}(\chi_q(3_0 3_4 \cdots 3_{4k-12})) 3_{4k-4}^{-1} 2_{4k-7} 2_{4k-5}.$$

If $m_1 = (3_0 3_4 \cdots 3_{4k-12}) 3_{4k-4}^{-1} 2_{4k-7} 2_{4k-5}$, then

$$m = m_1 m_2 = 3_0 3_4 \cdots 3_{4k-12} 2_{4k-7} 2_{4k-5} 1_{4k+2} 1_{4k+4} \cdots 1_{4k+2\ell} \tag{4.2}$$

is dominant. Suppose that

$$m_1 \neq (3_0 3_4 \cdots 3_{4k-12}) 3_{4k-4}^{-1} 2_{4k-7} 2_{4k-5}.$$

Then $m_1 = n_1 3_{4k-4}^{-1} 2_{4k-7} 2_{4k-5}$, where n_1 is a non-highest monomial in $\chi_q(3_0 3_4 \cdots 3_{4k-12})$. Since n_1 is right negative, 2_{4k-7} or 2_{4k-5} should cancel a factor of n_1 with a negative power. It is easy to see that there exists either a factor 2_{4k-9}^2 or 2_{4k-7}^2 in a monomial in $\chi_q(3_0 3_4 \cdots 3_{4k-12}) 3_{4k-4}^{-1} 2_{4k-7} 2_{4k-5}$ by using the FM algorithm. Therefore we need a factor 2_{4k-9} or 2_{4k-7} in a monomial in $\chi_q(3_0 3_4 \cdots 3_{4k-12})$. We have

$$\chi_q(3_0 3_4 \cdots 3_{4k-12}) 3_{4k-4}^{-1} 2_{4k-7} 2_{4k-5} \subseteq \chi_q(3_0 3_4 \cdots 3_{4k-16}) \chi_q(3_{4k-12}) 3_{4k-4}^{-1} 2_{4k-7} 2_{4k-5}.$$

The factors 2_{4k-9} and 2_{4k-7} can only come from the monomials in $\chi_q(3_{4k-12})$. By Lemma 3.1, any monomial in $\chi_q(3_{4k-12})$ does not have a factor 2_{4k-9}. The only monomial in $\chi_q(3_{4k-12})$ which contains a factor 2_{4k-7} is $1_{4k-10} 1_{4k-6}^{-1} 2_{4k-7} 3_{4k-6}^{-1}$. Therefore m_1 is in the set

$$\begin{aligned}
&\mathcal{M}(\chi_q(3_0 3_4 \cdots 3_{4k-16})) 1_{4k-10} 1_{4k-6}^{-1} 2_{4k-7} 3_{4k-6}^{-1} 3_{4k-4}^{-1} 2_{4k-7} 2_{4k-5}\\
= \ &\mathcal{M}(\chi_q(3_0 3_4 \cdots 3_{4k-16})) 1_{4k-10} 1_{4k-6}^{-1} 3_{4k-6}^{-1} 3_{4k-4}^{-1} 2_{4k-7}^2 2_{4k-5}.
\end{aligned}$$

Since $m = m_1 m_2$ is dominant, $1_{4k-6}^{-1}3_{4k-6}^{-1}$ should be canceled by some monomial in $\chi_q(3_0 3_4 \cdots 3_{4k-16})$. But by Lemma 3.1, $1_{4k-6}^{-1}3_{4k-6}^{-1}$ cannot be canceled by any monomial in $\chi_q(3_0 3_4 \cdots 3_{4k-16})$. This is a contradiction.

Therefore the only dominant monomials in $\chi_q(L(m_1'))\chi_q(L(m_2'))$ are m_+ and (4.2).

The second tensor product is $L(m_1'') \otimes L(m_2'')$, where

$$m_1'' = 3_0 3_4 \cdots 3_{4k-4},\ m_2'' = 1_{4k+2} 1_{4k+4} \cdots 1_{4k+2\ell}.$$

The monomial (4.2) is

$$n = m_+ A_{3,4k-2}^{-1}. \tag{4.3}$$

Since $A_{i,a}, i \in I, a \in \mathbb{C}^\times$ are algebraically independent, the expression (4.3) of n of the form $m_+ \prod_{i\in I, a\in \mathbb{C}^\times} A_{i,a}^{-v_{i,a}}$, where $v_{i,a}$ are some integers, is unique. Suppose that the monomial n is in $\chi_q(L(m_1''))\chi_q(L(m_2''))$. Then $n = n_1 n_2$, where $n_i \in \mathscr{M}(L(m_i'')), i = 1, 2$. By the expression (4.3), we have $n_2 = m_2''$ and $n_1 = m_1'' A_{3,4k-2}^{-1}$. By the FM algorithm, the monomial $m_1'' A_{3,4k-2}^{-1}$ is not in $\mathscr{M}(L(m_1''))$. This contradicts the fact that $n_1 \in \mathscr{M}(L(m_1''))$. Therefore n is not in $\chi_q(L(m_1''))\chi_q(L(m_2''))$.

4.3. **The case of $\tilde{\mathcal{T}}_{k,0,m}^{(s)}$.** Let $m_+ = \tilde{T}_{k,0,m}^{(s)}$ with $k, m \in \mathbb{Z}_{\geq 0}$. Without loss of generality, we may assume that $s = 0$. Then

$$m_+ = (1_0 1_2 \cdots 1_{2k-2})(3_{2k+4} 3_{2k+8} \cdots 3_{2k+4m})$$

Case 1. $k = 1$. In this case,

$$m_+ = 1_0(3_6 3_{10} \cdots 3_{4m+2}).$$

Let

$$U = I \times \{aq^s : s \in \mathbb{Z}, s \leq 4m + 2\}.$$

Let $\mathcal{M}$ be the finite set consisting of the following monomials

$$m_0 = m_+,\ m_1 = m_0 A_{1,1}^{-1},\ m_2 = m_1 A_{2,2}^{-1}.$$

By Theorem 2.2,

$$\mathrm{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+) = \sum_{m \in \mathcal{M}} m$$

and hence $\mathrm{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+)$ is special. Therefore $\tilde{\mathcal{T}}_{1,0,m}^{(s)}$ is special.

Case 2. $k > 1$. Let

$$m_1' = 1_0 1_2 \cdots 1_{2k-4},\ m_2' = 1_{2k-2} 3_{2k+4} 3_{2k+8} \cdots 3_{2k+4m},$$
$$m_1'' = 1_0 1_2 \cdots 1_{2k-2},\ m_2'' = 3_{2k+4} 3_{2k+8} \cdots 3_{2k+4m}.$$

Then $\mathscr{M}(L(m_+)) \subset \mathscr{M}(\chi_q(m_1')\chi_q(m_2')) \cap \mathscr{M}(\chi_q(m_1'')\chi_q(m_2''))$.

By using similar arguments as the case of $\mathcal{T}_{k,0,m}^{(s)}$, we show that the only possible dominant monomials in $\chi_q(m_1')\chi_q(m_2')$ are m_+ and

$$n_1 = 1_0 1_2 \cdots 1_{2k-6} 2_{2k-3} 3_{2k+4} 3_{2k+8} \cdots 3_{2k+4m} = m_+ A_{1,2k-3}^{-1}.$$

Moreover, n_1 is not in $\chi_q(m_1'')\chi_q(m_2'')$. Therefore the only dominant monomial in $\chi_q(m_+)$ is m_+.

4.4. **The case of $\mathcal{T}^{(s)}_{0,\ell,r}$.** Let $m_+ = T^{(s)}_{0,\ell,r}$ with $\ell \in \mathbb{Z}_{\geq 0}, r \in \{1,2\}$. Without loss of generality, we may assume that $s = 0$. Then

$$m_+ = (2_1 2_3 \cdots 2_{2k-1}) 1_{2k+2}$$

or

$$m_+ = (2_1 2_3 \cdots 2_{2k-1}) 1_{2k+2} 1_{2k+4}.$$

Case 1. $m_+ = (2_1 2_3 \cdots 2_{2k-1}) 1_{2k+2}$. Let

$$U = I \times \{aq^s : s \in \mathbb{Z}, s \leq 2k+2\}.$$

Let $\mathcal{M}$ be the finite set consisting of the following monomials

$$m_0 = m_+,\ m_1 = m_0 A^{-1}_{2,2k},\ m_2 = m_1 A^{-1}_{2,2k-2}, \ldots, m_k = m_{k-1} A^{-1}_{2,2}.$$

By Theorem 2.2,

$$\mathrm{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+) = \sum_{m \in \mathcal{M}} m$$

and hence $\mathrm{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+)$ is special. Therefore $\mathcal{T}^{(s)}_{0,\ell,r}$, $r = 1$, is special.

Case 2. $m_+ = (2_1 2_3 \cdots 2_{2k-1}) 1_{2k+2} 1_{2k+4}$. Let $U = I \times \{aq^s : s \in \mathbb{Z}, s \leq 2k+4\}$. It is sufficient to show that $\mathrm{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+)$ is special.

Let $\mathcal{M}$ be the finite set consisting of the following monomials

$$m_0 = m_+,\ m_1 = m_0 A^{-1}_{2,2k},\ m_2 = m_1 A^{-1}_{2,2k-2}, \ldots, m_k = m_{k-1} A^{-1}_{2,2},$$
$$n_1 = m_1 A^{-1}_{3,2k+2},\ n_2 = n_1 A^{-1}_{2,2k-2},\ n_3 = n_2 A^{-1}_{2,2k-4}, \ldots, n_k = n_{k-1} A^{-1}_{2,2}.$$

It is clear that $\mathcal{M}$ satisfies the conditions in Theorem 2.2. Therefore

$$\mathrm{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+) = \sum_{m \in \mathcal{M}} m$$

and $\mathrm{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+)$ is special.

4.5. **The case of $\mathcal{S}^{(s)}_{k,\ell}$.** Let $m_+ = S^{(s)}_{k,\ell}$ with $k, \ell \in \mathbb{Z}_{\geq 0}$. Without loss of generality, we may assume that $s = 0$. Then

$$m_+ = (2_0 2_2 \cdots 2_{2k-2})(2_{2k+4} 2_{2k+6} \cdots 2_{2k+2\ell+2}).$$

Case 1. $k = 1$. In this case,

$$m_+ = 2_0 (2_6 2_8 \cdots 2_{2\ell+4}).$$

Let

$$U = I \times \{aq^s : s \in \mathbb{Z}, s \leq 2\ell+4\}.$$

Let $\mathcal{M}$ be the finite set consisting of the following monomials

$$m_0 = m_+,\ m_1 = m_0 A^{-1}_{2,2},\ m_2 = m_1 A^{-1}_{1,2},\ m_3 = m_1 A^{-1}_{3,3},\ m_4 = m_3 A^{-1}_{1,2}.$$

By Theorem 2.2,

$$\mathrm{trunc}_{m_+\mathcal{Q}_U^-}\,\chi_q(m_+) = \sum_{m\in\mathcal{M}} m$$

and hence $\mathrm{trunc}_{m_+\mathcal{Q}_U^-}\,\chi_q(m_+)$ is special. Therefore $\mathcal{S}_{1,\ell}^{(s)}$ is special.

Case 2. $k>1$. Let

$$\begin{aligned} m_1' &= 2_0 2_2 \cdots 2_{2k-4},\ m_2' = 2_{2k-2}2_{2k+4}2_{2k+6}\cdots 2_{2k+2\ell+2},\\ m_1'' &= 2_0 2_2 \cdots 2_{2k-2},\ m_2'' = 2_{2k+4}2_{2k+6}\cdots 2_{2k+2\ell+2}. \end{aligned}$$

Then $\mathscr{M}(L(m_+)) \subset \mathscr{M}(\chi_q(m_1')\chi_q(m_2')) \cap \mathscr{M}(\chi_q(m_1'')\chi_q(m_2''))$.

By using similar arguments as the case of $\mathcal{T}_{k,0,m}^{(s)}$, we show that the only possible dominant monomials in $\chi_q(m_1')\chi_q(m_2')$ are m_+ and

$$\begin{aligned} n_1 &= 2_0 2_2 \cdots 2_{2k-6}1_{2k-3}3_{2k-3}2_{2k+4}2_{2k+6}\cdots 2_{2k+2\ell+2} = m_+A_{2,2k-3}^{-1},\\ n_2 &= 2_0 2_2 \cdots 2_{2k-6}2_{2k+4}2_{2k+8}2_{2k+10}\cdots 2_{2k+2\ell+2}\\ &= m_+A_{2,2k-3}^{-1}A_{1,2k-2}^{-1}A_{3,2k-1}^{-1}A_{2,2k+1}^{-1}A_{2,2k-1}^{-1}A_{3,2k+1}^{-1}A_{1,2k+2}^{-1}A_{2,2k+3}^{-1},\\ n_3 &= 2_0 2_2 \cdots 2_{2k-10}1_{2k-7}3_{2k-5}2_{2k+8}2_{2k+10}\cdots 2_{2k+2\ell+2}\\ &= n_2A_{2,2k-5}^{-1}A_{2,2k-7}^{-1}A_{1,2k-4}^{-1}A_{3,2k-5}^{-1}A_{2,2k-3}^{-1}. \end{aligned}$$

Moreover, n_1, n_2, n_3 are not in $\chi_q(m_1'')\chi_q(m_2'')$. Therefore the only dominant monomial in $\chi_q(m_+)$ is m_+.

4.6. **The case of $\mathcal{R}_{k,2\ell,\ell}^{(s)}$.** Let $m_+ = R_{k,2\ell,\ell}^{(s)}$ with $k\in\mathbb{Z}_{\geq 1}, \ell\in\mathbb{Z}_{\geq 0}$. Without loss of generality, we may assume that $s=0$. Then

$$m_+ = (2_0 2_2\cdots 2_{2k-2})(1_{2k+1}1_{2k+3}\cdots 1_{2k+4\ell-1})(3_{2k+3}3_{2k+7}\cdots 3_{2k+4\ell-1}).$$

Let

$$U = I\times\{aq^s : s\in\mathbb{Z}, s\leq 2k+4\ell-1\}.$$

Let $\mathcal{M}$ be the finite set consisting of the following monomials

$$m_0 = m_+,\ m_1 = m_0A_{2,2k-1}^{-1},\ m_2 = m_1A_{2,2k-3}^{-1}, \ldots, m_k = m_{k-1}A_{2,1}^{-1}.$$

By Theorem 2.2,

$$\mathrm{trunc}_{m_+\mathcal{Q}_U^-}\,\chi_q(m_+) = \sum_{m\in\mathcal{M}} m$$

and hence $\mathrm{trunc}_{m_+\mathcal{Q}_U^-}\,\chi_q(m_+)$ is special. Therefore $\mathcal{R}_{k,2\ell,\ell}^{(s)}$ is special.

4.7. **The case of $\mathcal{R}^{(s)}_{k,2\ell+1,\ell}$.** Let $m_+ = R^{(s)}_{k,2\ell+1,\ell}$ with $k \in \mathbb{Z}_{\geq 1}, \ell \in \mathbb{Z}_{\geq 0}$. Without loss of generality, we may assume that $s = 0$. Then

$$m_+ = (2_0 2_2 \cdots 2_{2k-2})(1_{2k+1} 1_{2k+3} \cdots 1_{2k+4\ell-1} 1_{2k+4\ell+1})(3_{2k+3} 3_{2k+7} \cdots 3_{2k+4\ell-1})$$

Let

$$U = I \times \{aq^s : s \in \mathbb{Z}, s \leq 2k + 4\ell - 1\}.$$

Let $\mathcal{M}$ be the finite set consisting of the following monomials

$$m_0 = m_+,\ m_1 = m_0 A^{-1}_{2,2k-1},\ m_2 = m_1 A^{-1}_{2,2k-3}, \ldots, m_k = m_{k-1} A^{-1}_{2,1}.$$

By Theorem 2.2,

$$\text{trunc}_{m_+ \mathcal{Q}^-_U} \chi_q(m_+) = \sum_{m \in \mathcal{M}} m$$

and hence $\text{trunc}_{m_+ \mathcal{Q}^-_U} \chi_q(m_+)$ is special. Therefore $\mathcal{R}^{(s)}_{k,2\ell+1,\ell}$ is special.

4.8. **The case of $\mathcal{R}^{(s)}_{k,2\ell+2,\ell}$.** Let $m_+ = R^{(s)}_{k,2\ell+2,\ell}$ with $k \in \mathbb{Z}_{\geq 1}, \ell \in \mathbb{Z}_{\geq 0}$. Without loss of generality, we may assume that $s = 0$. Then

$$m_+ = (2_0 2_2 \cdots 2_{2k-2})(1_{2k+1} 1_{2k+3} \cdots 1_{2k+4\ell-1} 1_{2k+4\ell+1} 1_{2k+4\ell+3})(3_{2k+3} 3_{2k+7} \cdots 3_{2k+4\ell-1}).$$

Let

$$U = I \times \{aq^s : s \in \mathbb{Z}, s \leq 2k + 4\ell - 1\}.$$

Let $\mathcal{M}$ be the finite set consisting of the following monomials

$$\begin{aligned}
&m_+,\ m_1 = m_+ A^{-1}_{2,2k-1},\ m_2 = m_1 A^{-1}_{2,2k-3}, \ldots, m_k = m_{k-1} A^{-1}_{2,1},\\
&m_{11} = m_1 A^{-1}_{3,2k+4\ell+1},\ m_{21} = m_{11} A^{-1}_{3,2k+4\ell-3}, \ldots, m_{\ell,1} = m_{\ell-1,1} A^{-1}_{3,2k+5},\\
&m_{12} = m_2 A^{-1}_{3,2k+4\ell+1},\ m_{22} = m_{12} A^{-1}_{3,2k+4\ell-3}, \ldots, m_{\ell,2} = m_{\ell-1,2} A^{-1}_{3,2k+5},\\
&\vdots\\
&m_{1k} = m_k A^{-1}_{3,2k+4\ell+1},\ m_{2k} = m_{1k} A^{-1}_{3,2k+4\ell-3}, \ldots, m_{\ell,k} = m_{\ell-1,k} A^{-1}_{3,2k+5}.
\end{aligned}$$

By Theorem 2.2,

$$\text{trunc}_{m_+ \mathcal{Q}^-_U} \chi_q(m_+) = \sum_{m \in \mathcal{M}} m$$

and hence $\text{trunc}_{m_+ \mathcal{Q}^-_U} \chi_q(m_+)$ is special. Therefore $\mathcal{R}^{(s)}_{k,2\ell+2,\ell}$ is special.

4.9. **The cases of $\mathcal{R}^{(s)}_{0,2\ell+i,\ell}$, $i = 0, 1, 2$.** These cases can be proved by using similar arguments as the cases for $\mathcal{R}^{(s)}_{k,2\ell+i,\ell}$, $i = 0, 1, 2$, $k \in \mathbb{Z}_{\geq 1}, \ell \in \mathbb{Z}_{\geq 0}$.

4.10. **The case of $\mathcal{U}_{k,\ell}^{(s)}$.** Let $m_+ = U_{k,\ell}^{(s)}$ with $k, \ell \in \mathbb{Z}_{\geq 0}$. Without loss of generality, we may assume that $s = 0$. Then

$$m_+ = (2_0 2_2 \cdots 2_{2k-2})(3_{2k+1} 3_{2k+3} \cdots 3_{2k+2\ell-1}).$$

Let

$$U = I \times \{aq^s : s \in \mathbb{Z}, s \leq 2k + 2\ell - 1\}.$$

Let $\mathcal{M}$ be the finite set consisting of the following monomials

$$\begin{aligned}
&m_+, \ m_1 = m_+ A_{2,2k-1}^{-1}, \ m_2 = m_1 A_{2,2k-3}^{-1}, \ldots, m_k = m_{k-1} A_{2,1}^{-1}, \\
&m_{11} = m_1 A_{1,2k}^{-1}, \\
&m_{12} = m_2 A_{1,2k}^{-1}, \ m_{22} = m_{12} A_{1,2k-2}^{-1}, \\
&\vdots \\
&m_{1k} = m_k A_{1,2k}^{-1}, \ m_{2k} = m_{1k} A_{1,2k-2}^{-1}, \ldots, m_{kk} = m_{k-1,k} A_{1,2}^{-1}.
\end{aligned}$$

By Theorem 2.2,

$$\operatorname{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+) = \sum_{m \in \mathcal{M}} m$$

and hence $\operatorname{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+)$ is special. Therefore $\mathcal{U}_{k,\ell}^{(s)}$ is special.

4.11. **The case of $\mathcal{V}_{k,\ell}^{(s)}$.** Let $m_+ = V_{k,\ell}^{(s)}$ with $k, \ell \in \mathbb{Z}_{\geq 0}$. Without loss of generality, we may assume that $s = 0$. Then

$$m_+ = (3_0 3_4 \cdots 3_{4k-4})(3_{4k+2} 3_{4k+6} \cdots 3_{4k+4\ell-2}).$$

Case 1. $k = 1$. In this case,

$$m_+ = 3_0 (3_6 3_{10} \cdots 3_{4\ell+2}).$$

Let

$$U = I \times \{aq^s : s \in \mathbb{Z}, s \leq 4\ell + 2\}.$$

Let $\mathcal{M}$ be the finite set consisting of the following monomials

$$\begin{aligned}
&m_0 = m_+, \ m_1 = m_0 A_{3,2}^{-1}, \ m_2 = m_1 A_{2,4}^{-1}, \\
&m_3 = m_2 A_{2,2}^{-1}, \ m_4 = m_2 A_{1,5}^{-1}, \ m_5 = m_4 A_{2,2}^{-1}, \ m_6 = m_5 A_{1,3}^{-1}.
\end{aligned}$$

By Theorem 2.2,

$$\operatorname{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+) = \sum_{m \in \mathcal{M}} m$$

and hence $\operatorname{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+)$ is special. Therefore $\mathcal{V}_{1,\ell}^{(s)}$ is special.

Case 2. $k > 1$. Let

$$\begin{aligned}
&m_1' = 3_0 3_4 \cdots 3_{4k-8}, \ m_2' = 3_{4k-4} 3_{4k+2} 3_{4k+6} \cdots 3_{4k+4\ell-2}, \\
&m_1'' = 3_0 3_4 \cdots 3_{4k-4}, \ m_2'' = 3_{4k+2} 3_{4k+6} \cdots 3_{4k+4\ell-2}.
\end{aligned}$$

Then $\mathscr{M}(L(m_+)) \subset \mathscr{M}(\chi_q(m_1')\chi_q(m_2')) \cap \mathscr{M}(\chi_q(m_1'')\chi_q(m_2''))$.

By using similar arguments as the case of $\mathcal{T}_{k,0,m}^{(s)}$, we show that the only possible dominant monomials in $\chi_q(m_1')\chi_q(m_2')$ are m_+ and

$$n_1 = (3_0 3_4 \cdots 3_{4k-12}) 2_{4k-7} 2_{4k-5} (3_{4k+2} 3_{4k+6} \cdots 3_{4k+4\ell-2}) = m_+ A_{3,4k-6}^{-1}.$$

Moreover, n_1 is not in $\chi_q(m_1'')\chi_q(m_2'')$. Therefore the only dominant monomial in $\chi_q(m_+)$ is m_+.

4.12. **The case of $\mathcal{P}_{k,\ell}^{(s)}$.** Let $m_+ = P_{k,\ell}^{(s)}$ with $k, \ell \in \mathbb{Z}_{\geq 0}$. Without loss of generality, we may assume that $s = 0$. Then

$$m_+ = (3_0 3_4 \cdots 3_{4k-4}) 2_{4k+1} (3_{4k+6} 3_{4k+10} \cdots 3_{4k+4\ell+2}).$$

Case 1. $k = 0$. In this case,

$$m_+ = 2_1 (3_6 3_{10} \cdots 3_{4\ell+2}).$$

Let

$$U = I \times \{aq^s : s \in \mathbb{Z}, s \leq 4\ell + 2\}.$$

Let $\mathcal{M}$ be the finite set consisting of the following monomials

$$m_0 = m_+, \ m_1 = m_0 A_{2,2}^{-1}, \ m_2 = m_1 A_{1,3}^{-1}.$$

It is clear that $\mathcal{M}$ satisfies the conditions in Theorem 2.2. Therefore

$$\operatorname{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+) = \sum_{m \in \mathcal{M}} m$$

and hence $\operatorname{trunc}_{m_+ \mathcal{Q}_U^-} \chi_q(m_+)$ is special. Therefore $\mathcal{P}_{0,\ell}^{(s)}$ is special.

Case 2. $k > 0$. Let

$$\begin{aligned} m_1' &= 3_0 3_4 \cdots 3_{4k-4}, \ m_2' = 2_{4k+1} 3_{4k+6} 3_{4k+10} \cdots 3_{4k+4\ell+2}, \\ m_1'' &= 3_0 3_4 \cdots 3_{4k-4} 2_{4k+1}, \ m_2'' = 3_{4k+6} 3_{4k+10} \cdots 3_{4k+4\ell+2}. \end{aligned}$$

Then $\mathscr{M}(L(m_+)) \subset \mathscr{M}(\chi_q(m_1')\chi_q(m_2')) \cap \mathscr{M}(\chi_q(m_1'')\chi_q(m_2''))$. Note that we have shown that $m_2'' = T_{k,1,0}^{(0)}$ is special in the case of $T_{k,\ell,0}^{(s)}$. Therefore the FM algorithm applies to m_2''.

By using similar arguments as the case of $\mathcal{T}_{k,0,m}^{(s)}$, we show that the only possible dominant monomials in $\chi_q(m_1')\chi_q(m_2')$ are m_+ and

$$n_1 = (3_0 3_4 \cdots 3_{4k-8}) 2_{4k-3} 1_{4k} (3_{4k+6} 3_{4k+10} \cdots 3_{4k+4\ell+2}) = m_+ A_{3,4k-2}^{-1} A_{2,4k}^{-1}.$$

Moreover, n_1 is not in $\chi_q(m_1'')\chi_q(m_2'')$. Therefore the only dominant monomial in $\chi_q(m_+)$ is m_+.

4.13. **The case of $\mathcal{O}^{(s)}_{k,\ell}$.** Let $m_+ = O^{(s)}_{k,\ell}$ with $k, \ell \in \mathbb{Z}_{\geq 0}$. Without loss of generality, we may assume that $s = 0$. Then

$$m_+ = (2_0 2_2 \cdots 2_{2k-2}) 1_{2k+1} 1_{2k+3} (2_{2k+6} 2_{2k+8} \cdots 2_{2k+2\ell+4}).$$

Case 1. $k = 0$. In this case,

$$m_+ = 1_1 1_3 (2_6 2_8 \cdots 2_{2\ell+4}) = \tilde{T}^{(1)}_{2,\ell,0}.$$

By Theorem 3.10, $L(m_+) = \mathcal{O}^{(s)}_{0,\ell}$ is special.

Case 2. $k > 0$. Let

$$\begin{aligned} m'_1 &= 2_0 2_2 \cdots 2_{2k-2}, \ m'_2 = 1_{2k+1} 1_{2k+3} 2_{2k+6} 2_{2k+8} \cdots 2_{2k+2\ell+4}, \\ m''_1 &= 2_0 2_2 \cdots 2_{2k-2} 1_{2k+1} 1_{2k+3}, \ m''_2 = 2_{2k+6} 2_{2k+8} \cdots 2_{2k+2\ell+4}. \end{aligned}$$

Then $\mathscr{M}(L(m_+)) \subset \mathscr{M}(\chi_q(m'_1)\chi_q(m'_2)) \cap \mathscr{M}(\chi_q(m''_1)\chi_q(m''_2))$. Note that we have shown that $m''_2 = T^{(0)}_{0,k,2}$ is special in the case of $T^{(s)}_{0,k,r}$, $r \in \{1, 2\}$. Therefore the FM algorithm applies to m''_2.

By using similar arguments as the case of $\mathcal{T}^{(s)}_{k,0,m}$, we show that the only possible dominant monomials in $\chi_q(m'_1)\chi_q(m'_2)$ are m_+ and

$$\begin{aligned} n_1 &= (2_0 2_2 \cdots 2_{2k-4}) 3_{2k-1} 1_{2k+3} (2_{2k+6} 2_{2k+8} \cdots 2_{2k+2\ell+4}) \\ &= m_+ A^{-1}_{2,2k-1} A^{-1}_{1,2k}, \\ n_2 &= (2_0 2_2 \cdots 2_{2k-4}) 1_{2k+1} 1_{2k+3} (2_{2k+8} 2_{2k+10} \cdots 2_{2k+2\ell+4}) \\ &= n_1 A^{-1}_{3,2k+1} A^{-1}_{2,2k+3} A^{-1}_{2,2k+1} A^{-1}_{3,2k+3} A^{-1}_{1,2k+4} A^{-1}_{2,2k+5}, \\ n_3 &= (2_0 2_2 \cdots 2_{2k-6}) 1_{2k-3} 1_{2k+1} (2_{2k+8} 2_{2k+10} \cdots 2_{2k+2\ell+4}) \\ &= n_2 A^{-1}_{2,2k-3} A^{-1}_{3,2k-1} A^{-1}_{2,2k+1} A^{-1}_{1,2k+2}, \\ n_4 &= (2_0 2_2 \cdots 2_{2k-8}) 1_{2k-5} 1_{2k-3} (2_{2k+8} 2_{2k+10} \cdots 2_{2k+2\ell+4}) \\ &= n_3 A^{-1}_{2,2k-5} A^{-1}_{3,2k-3} A^{-1}_{2,2k-1} A^{-1}_{1,2k}. \end{aligned}$$

Moreover, n_1, n_2, n_3, n_4 are not in $\chi_q(m''_1)\chi_q(m''_2)$. Therefore the only dominant monomial in $\chi_q(m_+)$ is m_+.

5. Proof Theorem 3.3 and Theorem 3.12

In this section, we will prove Theorem 3.3 and Theorem 3.12. We use the FM algorithm to classify dominant monomials in $\chi_q(\mathcal{L})\chi_q(\mathcal{R})$, $\chi_q(\mathcal{T})\chi_q(\mathcal{B})$, and $\chi_q(\mathcal{S})$.

5.1. **Classification of dominant monomials in $\chi_q(\mathcal{L})\chi_q(\mathcal{R})$ and $\chi_q(\mathcal{T})\chi_q(\mathcal{B})$.**

Lemma 5.1. *We have the following cases.*

(1) Let

$$M = T^{(s)}_{k,\ell-1,0} T^{(s+4)}_{k-1,\ell}, k \geq 1, \ell \geq 1.$$

Then dominant monomials in $\chi_q(T^{(s)}_{k,\ell-1,0})\chi_q(T^{(s+4)}_{k-1,\ell,0})$ *are*

$$M_0 = M,\ M_1 = MA^{-1}_{2,s+4k+2\ell-2},\ M_2 = M_1A^{-1}_{2,s+4k+2\ell-4},\ \ldots,$$
$$M_{\ell-1} = M_{\ell-2}A^{-1}_{2,s+4k+2},\ M_\ell = M_{\ell-1}A^{-1}_{3,s+4k-2}A^{-1}_{2,s+4k},$$
$$M_{\ell+1} = M_\ell A^{-1}_{3,s+4k-6},\ M_{\ell+2} = M_{\ell+1}A^{-1}_{3,s+4k-10},\ \ldots,\ M_{k+\ell-1} = M_{k+\ell-2}A^{-1}_{3,s+2}.$$

The dominant monomials in $\chi_q(T^{(s)}_{k,\ell,0})\chi_q(T^{(s+4)}_{k-1,\ell-1,0})$ *are* $M_0, \ldots, M_{k+\ell-2}$.

(2) Let

$$M = T^{(s)}_{k,0,m-1}T^{(s+4)}_{k-1,0,m}, k \geq 1, m \geq 1.$$

Then dominant monomials in $\chi_q(T^{(s)}_{k,0,m-1})\chi_q(T^{(s+4)}_{k-1,0,m})$ *are*

$$M_0 = M,\ M_1 = MA^{-1}_{1,s+4k+2m-1},\ M_2 = M_1A^{-1}_{1,s+4k+2m-3},\ \ldots,$$
$$M_{m-1} = M_{m-2}A^{-1}_{1,s+4k+3},\ M_m = M_{m-1}A^{-1}_{3,s+4k-2}A^{-1}_{2,s+4k}A^{-1}_{1,s+4k+1},$$
$$M_{m+1} = M_mA^{-1}_{3,s+4k-6},\ M_{m+2} = M_{m+1}A^{-1}_{3,s+4k-10},\ \ldots,\ M_{k+m-1} = M_{k+m-2}A^{-1}_{3,s+2}.$$

The dominant monomials in $\chi_q(T^{(s)}_{k,0,m})\chi_q(T^{(s+4)}_{k-1,0,m-1})$ *are* $M_0, \ldots, M_{k+m-2}$.

(3) Let

$$M = S^{(s)}_{k,\ell-1}S^{(s+2)}_{k-1,\ell}, k, \ell \geq 1.$$

Then dominant monomials in $\chi_q(S^{(s)}_{k,\ell-1})\chi_q(S^{(s+2)}_{k-1,\ell})$ *are*

$$M_0 = M,\ M_1 = MA^{-1}_{2,s+2k+2\ell+1},\ M_2 = M_1A^{-1}_{2,s+2k+2\ell-1},\ \ldots,$$
$$M_{\ell-1} = M_{\ell-2}A^{-1}_{2,s+2k+5},\ M_\ell = M_{\ell-1}A^{-1}_{2,s+2k-1}A^{-1}_{3,s+2k+1}A^{-1}_{2,2k+3},$$
$$M_{\ell+1} = M_\ell A^{-1}_{2,s+2k-3},\ M_{\ell+2} = M_{\ell+1}A^{-1}_{2,s+2k-5},\ \ldots, M_{k+\ell-1} = M_{k+\ell-2}A^{-1}_{2,s+1}.$$

The dominant monomials in $\chi_q(S^{(s)}_{k,\ell})\chi_q(S^{(s+2)}_{k-1,\ell-1})$ *are* $M_0, \ldots, M_{k+\ell-2}$.

(4) Let

$$M = T^{(s)}_{0,\ell,r-1}T^{(s+2)}_{0,\ell-1,r}, \ell \geq 1, r \in \{1, 2\}.$$

Then dominant monomials in $\chi_q(T^{(s)}_{0,\ell,r-1})\chi_q(T^{(s+2)}_{0,\ell-1,r})$ *are*

$$M_0 = M,\ M_1 = MA^{-1}_{1,s+2\ell+2r-2},\ M_2 = M_1A^{-1}_{1,s+2\ell+2r-4},\ \ldots,$$
$$M_{r-1} = M_{r-2}A^{-1}_{1,s+2\ell+2},\ M_r = M_{r-1}A^{-1}_{2,s+2\ell}A^{-1}_{1,s+2\ell+1},$$
$$M_{r+1} = M_rA^{-1}_{2,s+2\ell-2},\ M_{r+2} = M_{r+1}A^{-1}_{2,s+2\ell-4},\ \ldots,\ M_{\ell+r-1} = M_{\ell+r-2}A^{-1}_{2,s+2}.$$

The dominant monomials in $\chi_q(T^{(s)}_{0,\ell,r})\chi_q(T^{(s+2)}_{0,\ell-1,r-1})$ *are* $M_0, \ldots, M_{\ell+r-2}$.

(5) Let

$$M = \tilde{T}^{(s)}_{k,0,m-1}\tilde{T}^{(s+2)}_{k-1,0,m}, k, m \geq 1.$$

Then dominant monomials in $\chi_q(\tilde{T}^{(s)}_{k,0,m-1})\chi_q(\tilde{T}^{(s+2)}_{k-1,0,m})$ *are*

$$M_0 = M,\ M_1 = MA^{-1}_{3,s+2k+4m-2},\ M_2 = M_1A^{-1}_{3,s+2k+4m-6},\ \ldots,$$
$$M_{m-1} = M_{m-2}A^{-1}_{3,s+2k+6},\ M_m = M_{m-1}A^{-1}_{1,s+2k-1}A^{-1}_{2,s+2k}A^{-1}_{3,2k+2},$$
$$M_{m+1} = M_mA^{-1}_{1,s+2k-3},\ M_{m+2} = M_{m+1}A^{-1}_{1,s+2k-5},\ \ldots,\ M_{k+m-1} = M_{k+m-2}A^{-1}_{1,s+1}.$$

The dominant monomials in $\chi_q(\tilde{T}^{(s)}_{k,0,m})\chi_q(\tilde{T}^{(s+2)}_{k-1,0,m-1})$ *are* $M_0, \ldots, M_{\ell+m-2}$.

(6) *Let*
$$M = R^{(s)}_{k,2\ell,\ell-1}R^{(s+2)}_{k-1,2\ell,\ell}, k \geq 1, \ell \geq 1.$$
Then dominant monomials in $\chi_q(R^{(s)}_{k,2\ell,\ell-1})\chi_q(R^{(s+2)}_{k-1,2\ell,\ell})$ *are*

$$M_0 = M,\ M_1 = MA^{-1}_{3,s+2k+4\ell-3},\ M_2 = M_1A^{-1}_{3,s+2k+4\ell-7},\ \ldots,$$
$$M_{\ell-1} = M_{\ell-2}A^{-1}_{3,s+2k+5},\ M_\ell = M_{\ell-1}A^{-1}_{2,s+2k-1}A^{-1}_{3,s+2k+1},$$
$$M_{\ell+1} = M_\ell A^{-1}_{2,s+2k-3},\ M_{\ell+2} = M_{\ell+1}A^{-1}_{2,s+2k-5},\ \ldots,\ M_{k+\ell-1} = M_{k+\ell-2}A^{-1}_{2,s+1}.$$

The dominant monomials in $\chi_q(R^{(s)}_{k,2\ell,\ell})\chi_q(R^{(s+2)}_{k-1,2\ell,\ell-1})$ *are* $M_0, \ldots, M_{k+\ell-2}$.

(7) *Let*
$$M = R^{(s)}_{k,2\ell,\ell}R^{(s+2)}_{k-1,2\ell+1,\ell}, k \geq 1, \ell \geq 1.$$
Then dominant monomials in $\chi_q(R^{(s)}_{k,2\ell,\ell})\chi_q(R^{(s+2)}_{k-1,2\ell+1,\ell})$ *are*

$$M_0 = M,\ M_1 = MA^{-1}_{1,s+2k+4\ell},\ M_2 = M_1A^{-1}_{1,s+2k+4\ell-2},\ \ldots,$$
$$M_{2\ell} = M_{2\ell-1}A^{-1}_{1,s+2k+2},\ M_{2\ell+1} = M_{2\ell}A^{-1}_{2,s+2k-1}A^{-1}_{1,s+2k},$$
$$M_{2\ell+2} = M_{2\ell+1}A^{-1}_{2,s+2k-3},\ M_{2\ell+3} = M_{2\ell+2}A^{-1}_{2,s+2k-5},\ \ldots,\ M_{k+2\ell} = M_{k+2\ell-1}A^{-1}_{2,s+1}.$$

The dominant monomials in $\chi_q(R^{(s)}_{k,2\ell+1,\ell})\chi_q(R^{(s+2)}_{k-1,2\ell,\ell})$ *are* $M_0, \ldots, M_{k+2\ell-1}$.

(8) *Let*
$$M = R^{(s)}_{k,2\ell+1,\ell}R^{(s+2)}_{k-1,2\ell+2,\ell}, k \geq 1, \ell \geq 1.$$
Then dominant monomials in $\chi_q(R^{(s)}_{k,2\ell+1,\ell})\chi_q(R^{(s+2)}_{k-1,2\ell+2,\ell})$ *are*

$$M_0 = M,\ M_1 = MA^{-1}_{1,s+2k+4\ell+2},\ M_2 = M_1A^{-1}_{1,s+2k+4\ell},\ \ldots,$$
$$M_{2\ell+1} = M_{2\ell}A^{-1}_{1,s+2k+2},\ M_{2\ell+2} = M_{2\ell+1}A^{-1}_{2,s+2k-1}A^{-1}_{1,s+2k},$$
$$M_{2\ell+3} = M_{2\ell+2}A^{-1}_{2,s+2k-3},\ M_{2\ell+4} = M_{2\ell+3}A^{-1}_{2,s+2k-5},\ \ldots,\ M_{k+2\ell+1} = M_{k+2\ell}A^{-1}_{2,s+1}.$$

The dominant monomials in $\chi_q(R^{(s)}_{k,2\ell+2,\ell})\chi_q(R^{(s+2)}_{k-1,2\ell+1,\ell})$ *are* $M_0, \ldots, M_{k+2\ell}$.

(9) *Let*
$$M = \tilde{T}^{(s)}_{k,\ell-1,0}\tilde{T}^{(s+2)}_{k-1,\ell}, k \geq 1, \ell \geq 1.$$

Then dominant monomials in $\chi_q(\tilde{T}^{(s)}_{k,\ell-1,0})\chi_q(\tilde{T}^{(s+2)}_{k-1,\ell,0})$ *are*

$$\begin{aligned}&M_0 = M,\ M_1 = MA^{-1}_{2,s+2k+2\ell-2},\ M_2 = M_1A^{-1}_{2,s+2k+2\ell-4},\ \ldots,\\&M_{\ell-1} = M_{\ell-2}A^{-1}_{2,s+2k+2},\ M_\ell = M_{\ell-1}A^{-1}_{1,s+2k-1}A^{-1}_{2,s+2k},\\&M_{\ell+1} = M_\ell A^{-1}_{1,s+2k-3},\ M_{\ell+2} = M_{\ell+1}A^{-1}_{1,s+2k-5},\ \ldots,\ M_{k+\ell-1} = M_{k+\ell-2}A^{-1}_{1,s+1}.\end{aligned}$$

The dominant monomials in $\chi_q(\tilde{T}^{(s)}_{k,\ell,0})\chi_q(\tilde{T}^{(s+2)}_{k-1,\ell-1,0})$ *are* $M_0,\ldots,M_{k+\ell-2}$.

Recall that $\sigma(x)=0$ *if* x *is even and* $\sigma(x)=1$ *if* x *is odd.*

(10) *Let*

$$M = U^{(s)}_{p,\ell-1}U^{(s+2)}_{p-1,\ell}, p \geq 2, \ell \geq 1.$$

Then dominant monomials in $\chi_q(U^{(s)}_{p,\ell-1})\chi_q(U^{(s+2)}_{p-1,\ell})$ *are*

$$\begin{aligned}&M_0 = M,\ M_1 = MA^{-1}_{3,s+2p+2\ell-3},\ M_2 = M_1A^{-1}_{2,s+2p+2\ell-7},\ \ldots,\\&M_{\lfloor\frac{\ell-1}{2}\rfloor} = M_{\lfloor\frac{\ell-1}{2}\rfloor-1}A^{-1}_{3,s+2p+2\sigma(\ell+1)+3},\ M_{\lfloor\frac{\ell-1}{2}\rfloor+1} = M_{\lfloor\frac{\ell-1}{2}\rfloor}A^{-1}_{2,s+2p-1}A^{-1}_{3,s+2p+1},\\&M_{\lfloor\frac{\ell-1}{2}\rfloor+2} = M_{\lfloor\frac{\ell-1}{2}\rfloor+1}A^{-1}_{2,s+2p-3},\ M_{\lfloor\frac{\ell-1}{2}\rfloor+3} = M_{\lfloor\frac{\ell-1}{2}\rfloor+2}A^{-1}_{2,s+2p-5},\ \ldots,\\&M_{p+\lfloor\frac{\ell-1}{2}\rfloor} = M_{p+\lfloor\frac{\ell-1}{2}\rfloor-1}A^{-1}_{2,s+1}.\end{aligned}$$

The dominant monomials in $\chi_q(U^{(s)}_{p,\ell})\chi_q(U^{(s+2)}_{p-1,\ell-1})$ *are* $M_0,\ldots,M_{p+\lfloor\frac{\ell-1}{2}\rfloor-1}$.

(11) *Let*

$$M = V^{(s)}_{k,\ell-1}V^{(s+4)}_{k-1,\ell}, k \geq 1, \ell \geq 1.$$

Then dominant monomials in $\chi_q(V^{(s)}_{k,\ell-1})\chi_q(V^{(s+4)}_{k-1,\ell})$ *are*

$$\begin{aligned}&M_0 = M,\ M_1 = MA^{-1}_{3,s+4k+4\ell-4},\ M_2 = M_1A^{-1}_{3,s+4k+4\ell-8},\ \ldots,\\&M_{\ell-1} = M_{\ell-2}A^{-1}_{3,s+4k+6},\ M_\ell = M_{\ell-1}A^{-1}_{3,s+4k-2}A^{-1}_{2,s+4k}A^{-1}_{2,s+4k-2}A^{-1}_{3,s+4k},\\&M_{\ell+1} = M_\ell A^{-1}_{3,s+4k-6},\ M_{\ell+2} = M_{\ell+1}A^{-1}_{3,s+4k-10},\ \ldots,\ M_{k+\ell-1} = M_{k+\ell-2}A^{-1}_{3,s+2}.\end{aligned}$$

The dominant monomials in $\chi_q(V^{(s)}_{k,\ell})\chi_q(V^{(s+4)}_{k-1,\ell-1})$ *are* $M_0,\ldots,M_{k+\ell-2}$.

(12) *Let*

$$M = P^{(s)}_{k,\ell-1}P^{(s+4)}_{k-1,\ell}, k \geq 1, \ell \geq 1.$$

Then dominant monomials in $\chi_q(P^{(s)}_{k,\ell-1})\chi_q(P^{(s+4)}_{k-1,\ell})$ *are*

$$\begin{aligned}&M_0 = M,\ M_1 = MA^{-1}_{3,s+4k+4\ell},\ M_2 = M_1A^{-1}_{3,s+4k+4\ell-4},\ \ldots,\\&M_{\ell-1} = M_{\ell-2}A^{-1}_{3,s+4k+8},\ M_\ell = M_{\ell-1}A^{-1}_{2,s+4k+2}A^{-1}_{3,s+4k+4},\\&M_{\ell+1} = M_\ell A^{-1}_{3,s+4k-2}A^{-1}_{2,s+4k},\ M_{\ell+2} = M_{\ell+1}A^{-1}_{3,s+4k-6},\\&M_{\ell+3} = M_{\ell+2}A^{-1}_{3,s+4k-10},\ \ldots,\ M_{k+\ell} = M_{k+\ell-1}A^{-1}_{3,s+2}.\end{aligned}$$

The dominant monomials in $\chi_q(P^{(s)}_{k,\ell})\chi_q(P^{(s+4)}_{k-1,\ell-1})$ *are* $M_0,\ldots,M_{k+\ell-1}$.

(13) Let

$$M = O^{(s)}_{k,\ell-1} O^{(s+2)}_{k-1,\ell}, k \geq 1, \ell \geq 1.$$

Then dominant monomials in $\chi_q(O^{(s)}_{k,\ell-1})\chi_q(O^{(s+2)}_{k-1,\ell})$ *are*

$$\begin{aligned}
&M_0 = M,\ M_1 = MA^{-1}_{2,s+2k+2\ell+4},\ M_2 = M_1 A^{-1}_{2,s+2k+2\ell+2},\ \ldots,\\
&M_{\ell-1} = M_{\ell-2}A^{-1}_{2,s+2k+7},\ M_\ell = M_{\ell-1}A^{-1}_{1,s+2k+4}A^{-1}_{2,s+2k+5},\\
&M_{\ell+1} = M_\ell A^{-1}_{1,s+2k+2},\ M_{\ell+2} = M_{\ell+1}A^{-1}_{2,s+2k-1}A^{-1}_{1,2k},\\
&M_{\ell+3} = M_{\ell+2}A^{-1}_{2,s+2k-3},\ M_{\ell+4} = M_{\ell+3}A^{-1}_{2,s+2k-5},\ \ldots,\ M_{k+\ell+1} = M_{k+\ell}A^{-1}_{2,s+1}.
\end{aligned}$$

The dominant monomials in $\chi_q(O^{(s)}_{k,\ell})\chi_q(O^{(s+2)}_{k-1,\ell-1})$ *are* $M_0, \ldots, M_{k+\ell}$.

In each case, for each i, the multiplicity of M_i in the corresponding product of q-characters is 1.

Proof. We prove the case of $\chi_q(T^{(s)}_{k,\ell-1,0})\chi_q(T^{(s+4)}_{k-1,\ell,0})$. The other cases are similar. Let $m'_1 = T^{(s)}_{k,\ell-1,0}$, $m'_2 = T^{(s+4)}_{k-1,\ell,0}$. Without loss of generality, we assume that $s = 0$. Then

$$\begin{aligned}
m'_1 &= (3_0 3_4 \cdots 3_{4k-4})(2_{4k+1}2_{4k+3}\cdots 2_{4k+2\ell-3}),\\
m'_2 &= (3_4 3_6 \cdots 3_{4k-4})(2_{4k+1}2_{4k+3}\cdots 2_{4k+2\ell-1}).
\end{aligned}$$

Let $m = m_1 m_2$ be a dominant monomial, where $m_i \in \chi_q(m'_i), i = 1, 2$. Let

$$m_3 = 3_4 3_6 \cdots 3_{4k-4}, \quad m_4 = 2_{4k+1}2_{4k+3}\cdots 2_{4k+2\ell-1}.$$

If $m_2 \in \chi_q(m_3)(\chi_q(m_4) - m_4)$, then $m = m_1m_2$ is right negative and hence m is not dominant. Therefore $m_2 \in \chi_q(m_3)m_4$.

Suppose that $m_2 \in \mathscr{M}(L(m'_2)) \cap \mathscr{M}((\chi_q(m_3) - m_3)m_4)$. By the FM algorithm for $L(m'_2)$, m_2 must have a factor 3^{-1}_{4k}. By Lemma 3.1, m_1 does not have the factor 3_{4k}. Therefore $m = m_1m_2$ is not dominant. This is a contradiction. Therefore $m_2 = m_3m_4 = m'_2$.

If

$$\begin{aligned}
m_1 \in \chi_q(&3_0 3_4 \cdots 3_{4k-4} 2_{4k+1}2_{4k+3}\cdots 2_{4k+2\ell-5})\times\\
&\times (\chi_q(2_{4k+2\ell-3}) - 2_{4k+2\ell-3} - 2^{-1}_{4k+2\ell-3}1_{4k+2\ell-4}3_{4k+2\ell-4}),
\end{aligned}$$

then $m = m_1m_2$ is right-negative and hence not dominant. Therefore m_1 is in one of the following polynomials

$$\chi_q(3_0 3_4 \cdots 3_{4k-4} 2_{4k+1}2_{4k+3}\cdots 2_{4k+2\ell-5})2_{4k+2\ell-3}, \tag{5.1}$$

$$\chi_q(3_0 3_4 \cdots 3_{4k-4} 2_{4k+1}2_{4k+3}\cdots 2_{4k+2\ell-5})2^{-1}_{4k+2\ell-1}1_{4k+2\ell-2}3_{4k+2\ell-2}. \tag{5.2}$$

If m_1 is in (5.1), then $m_1 = m'_1$. The dominant monomial we obtain is $M_0 = m'_1m'_2$. If m_1 is the highest monomial in (5.2), then we obtain the dominant monomial $M_1 = m_1m'_2$. Suppose that m_1 is in

$$\begin{aligned}
\mathscr{M}(L(m'_1)) \ \cap\ \mathscr{M}\big(&\chi_q(3_0 3_4 \cdots 3_{4k-4} 2_{4k+1}2_{4k+3}\cdots 2_{4k+2\ell-7})\times\\
&\times(\chi_q(2_{4k+2\ell-5}) - 2_{4k+2\ell-5})2^{-1}_{4k+2\ell-3}1_{4k+2\ell-4}3_{4k+2\ell-4}\big).
\end{aligned}$$

By the FM algorithm for $L(m_1')$, m_1 is in one of the following polynomials

$$\chi_q(3_0 3_4 \cdots 3_{4k-4} 2_{4k+1} 2_{4k+3} \cdots 2_{4k+2\ell-7}) 2^{-1}_{4k+2\ell-3} 1_{4k+2\ell-4} 3_{4k+2\ell-4} 2^{-1}_{4k+2\ell-1} 1_{4k+2\ell-2} 3_{4k+2\ell-2}, \tag{5.3}$$

$$\chi_q(3_0 3_4 \cdots 3_{4k-4} 2_{4k+1} 2_{4k+3} \cdots 2_{4k+2\ell-7}) 1_{4k+2\ell-4} 3^{-1}_{4k+2\ell} 1_{4k+2\ell-2} 3_{4k+2\ell-2}. \tag{5.4}$$

If m_1 is in (5.4), then $m = m_1 m_2$ is right-negative and hence m is not dominant. This is a contradiction. Therefore m_1 is in (5.3). If m_1 is the highest monomial in (5.3), then we obtain the dominant monomial $M_2 = m_1 m_2'$. Suppose that m_1 is not the highest monomial in (5.3). Then by the FM algorithm, m_1 is in the set

$$\begin{aligned} &\chi_q(3_0 3_4 \cdots 3_{4k-4} 2_{4k+1} 2_{4k+3} \cdots 2_{4k+2\ell-9}) \times \\ &\times 2^{-1}_{4k+2\ell-5} 1_{4k+2\ell-6} 3_{4k+2\ell-6} 2^{-1}_{4k+2\ell-3} 1_{4k+2\ell-4} 3_{4k+2\ell-4} 2^{-1}_{4k+2\ell-1} 1_{4k+2\ell-2} 3_{4k+2\ell-2}. \end{aligned}$$

If m_1 is the highest monomial in the above set, then we obtain the dominant monomial $M_3 = m_1 m_2'$. Continue this procedure, we obtain dominant monomials $M_4, \ldots, M_{\ell-1}$ and the remaining dominant monomials are of the form $m_1 m_2'$, where m_1 is a non-highest monomial in

$$\begin{aligned} &\mathscr{M}(L(m_1')) \cap \mathscr{M}\big(\chi_q(3_0 3_4 \cdots 3_{4k-4}) \times \\ &\qquad \times 2^{-1}_{4k+3} 2^{-1}_{4k+5} \cdots 2^{-1}_{4k+2\ell-1} 1_{4k+2} 1_{4k+4} \cdots 1_{4k+2\ell-2} 3_{4k+2} 3_{4k+4} \cdots 3_{4k+2\ell-2}\big). \end{aligned}$$

Suppose that m_1 is a non-highest monomial in the above set. Since the non-highest monomials in $\chi_q(3_0 3_4 \cdots 3_{4k-4})$ are right-negative, we need cancelations of factors with negative powers of some monomial in $\chi_q(3_0 3_4 \cdots 3_{4k-4})$ with

$$1_{4k+2} 1_{4k+4} \cdots 1_{4k+2\ell-2} 3_{4k+2} 3_{4k+4} \cdots 3_{4k+2\ell-2} 2_{4k+1}.$$

The only cancelation can happen is to cancel 1_{4k+2} or 3_{4k+2} or 3_{4k+4} or 2_{4k+1}. Since 1^2_{4k}, 3^2_{4k}, and 3^2_{4k-2} do not appear in

$$\begin{aligned} &\mathscr{M}(L(m_1')) \cap \mathscr{M}\big(\chi_q(3_0 3_4 \cdots 3_{4k-4}) \times \\ &\qquad \times 2^{-1}_{4k+3} 2^{-1}_{4k+5} \cdots 2^{-1}_{4k+2\ell-1} 1_{4k+2} 1_{4k+4} \cdots 1_{4k+2\ell-2} 3_{4k+2} 3_{4k+4} \cdots 3_{4k+2\ell-2}\big), \end{aligned}$$

1_{4k+2}, 3_{4k+2} and 3_{4k+4} cannot be canceled. Therefore we need a cancelation with 2_{4k+1}. The only monomials in $\chi_q(3_0 3_4 \cdots 3_{4k-4})$ which can cancel 2_{4k+1} is in one of the following polynomials

$$\begin{aligned} &\chi_q(3_0 3_4 \cdots 3_{4k-8}) 1_{4k} 2_{4k-3} 2^{-1}_{4k+1}, \\ &\chi_q(3_0 3_4 \cdots 3_{4k-8}) 1_{4k-2} 1_{4k} 2^{-1}_{4k-1} 2^{-1}_{4k+1} 3_{4k-2}, \\ &\chi_q(3_0 3_4 \cdots 3_{4k-8}) 2^{-1}_{4k+1} 2^{-1}_{4k+3} 3_{4k}. \end{aligned}$$

Therefore m_1 is in one of the following sets

$$\mathscr{M}(L(m_1')) \cap \mathscr{M}\big(\chi_q(3_0 3_4 \cdots 3_{4k-8}) 1_{4k} 2_{4k-3} 2^{-1}_{4k+1} \times \\ \times 2^{-1}_{4k+3} 2^{-1}_{4k+5} \cdots 2^{-1}_{4k+2\ell-1} 1_{4k+2} 1_{4k+4} \cdots 1_{4k+2\ell-2} 3_{4k+2} 3_{4k+4} \cdots 3_{4k+2\ell-2}\big), \tag{5.5}$$

$$\mathscr{M}(L(m_1')) \cap \mathscr{M}\big(\chi_q(3_0 3_4 \cdots 3_{4k-8}) 1_{4k-2} 1_{4k} 2^{-1}_{4k-1} 2^{-1}_{4k+1} 3_{4k-2} \times \\ \times 2^{-1}_{4k+3} 2^{-1}_{4k+5} \cdots 2^{-1}_{4k+2\ell-1} 1_{4k+2} 1_{4k+4} \cdots 1_{4k+2\ell-2} 3_{4k+2} 3_{4k+4} \cdots 3_{4k+2\ell-2}\big), \tag{5.6}$$

$$\mathscr{M}(L(m_1')) \cap \mathscr{M}\big(\chi_q(3_0 3_4 \cdots 3_{4k-8}) 2^{-1}_{4k+1} 2^{-1}_{4k+3} 3_{4k} \times \\ \times 2^{-1}_{4k+3} 2^{-1}_{4k+5} \cdots 2^{-1}_{4k+2\ell-1} 1_{4k+2} 1_{4k+4} \cdots 1_{4k+2\ell-2} 3_{4k+2} 3_{4k+4} \cdots 3_{4k+2\ell-2}\big). \tag{5.7}$$

If m_1 is in (5.6), then we need to cancel 2^{-1}_{4k-1}. But 2^{-1}_{4k-1} cannot be canceled by any monomials in $\chi_q(3_0 3_4 \cdots 3_{4k-8})$ or by m_2'. Hence m_1 is not in (5.6).

If m_1 is in (5.7), then we need to cancel 2^{-1}_{4k+3}. But 2^{-1}_{4k+3} cannot be canceled by any monomials in $\chi_q(3_0 3_4 \cdots 3_{4k-8})$ or by m_2'. Therefore m_1 is not in (5.7). Hence m_1 is in (5.5).

If m_1 is the highest monomial in (5.5) with respect to $\leq$ defined in (2.4), then we obtain the dominant monomial $M_\ell = m_1 m_2'$. Suppose that m_1 a non-highest monomial in (5.5). By the FM algorithm, m_1 must in

$$\mathscr{M}(L(m_1')) \cap \mathscr{M}\big(\chi_q(3_0 3_4 \cdots 3_{4k-12}) 3^{-1}_{4k-4} 2_{4k-7} 2_{4k-5} 1_{4k} 2_{4k-3} 2^{-1}_{4k+1} \times \\ \times 2^{-1}_{4k+3} 2^{-1}_{4k+5} \cdots 2^{-1}_{4k+2\ell-1} 1_{4k+2} 1_{4k+4} \cdots 1_{4k+2\ell-2} 3_{4k+2} 3_{4k+4} \cdots 3_{4k+2\ell-2}\big).$$

If m_1 is the highest monomial in the above set, then we obtain the dominant monomial $M_{\ell+1} = m_1 m_2'$. Continue this procedure, we can show that the only remaining dominant monomials are $M_{\ell+2}, \ldots, M_{k+\ell-1}$.

It is clear that the multiplicity of $M_i, i = 1, \ldots, k+\ell-1$, in $\chi_q(m_1)\chi_q(m_2)$ is 1. □

5.2. **Products of sources are special.**

Lemma 5.2. *Let $[\mathcal{S}]$ be the last summand in one of the relations in Theorem 3.3. Then $\mathcal{S}$ is special.*

Proof. We prove the case of $\mathcal{S}$ in the first relation in Theorem 3.3. The other cases are similar.

Let $S = \chi_q(R^{(s+1)}_{2k-1,\ell,\lfloor \frac{\ell}{2} \rfloor})\chi_q(T^{(s+4k+4)}_{\lfloor \frac{\ell-1}{2} \rfloor,0,0})$, $\ell = 2m$, $m \in \mathbb{Z}_{\geq 0}$. Let

$$n_1 = 2_{s+1} 2_{s+3} \cdots 2_{s+4k-3}, \ n_1' = 1_{s+4k} 1_{s+4k+2} \cdots 1_{s+4k+4m-2},$$
$$n_1'' = 3_{s+4k+2} 3_{s+4k+6} \cdots 3_{s+4k+4m-2}, \ n_2 = 3_{s+4k+4} 3_{s+4k+8} \cdots 3_{s+4k+4m-4}.$$

Then $R^{(s+1)}_{2k-1,\ell,\lfloor\frac{\ell}{2}\rfloor} = n_1n_1'n_1'', T^{(s+4k+4)}_{\lfloor\frac{\ell-1}{2}\rfloor,0,0} = n_2$. Let $m = m_1m_2$ be a dominant monomial, where

$$m_1 \in \chi_q(R^{(s+1)}_{2k-1,\ell,\lfloor\frac{\ell}{2}\rfloor}),\ m_2 \in \chi_q(T^{(s+4k+4)}_{\lfloor\frac{\ell-1}{2}\rfloor,0,0}).$$

If $m_2 \neq n_2$ or $m_1 \in \chi_q(n_1n_1'')(\chi_q(n_1') - n_1')$ or $m_1 \in \chi_q(n_1n_1')(\chi_q(n_1'') - n_1'')$, then m is right-negative which contradicts the fact that m is dominant. Therefore $m_2 = n_2$, $m_1 \in \chi_q(n_1)n_1'n_1''$.

If m_1 is the highest monomial in $\chi_q(n_1)n_1'n_1''$, then we obtain a dominant monomial $n_1n_1'n_1''n_2$. Suppose that m_1 is a non-highest monomial in $\chi_q(n_1)n_1'n_1''$. Then m_1 is in $(\chi_q(n_1) - n_1)n_1'n_1''$. By the FM algorithm, m_1 is one of the following monomials

$$\bar{m}_1 = n_1n_1'n_1''A^{-1}_{2,s+4k-2},\ \bar{m}_2 = \bar{m}_1A^{-1}_{2,s+4k-4},\ \ldots,\ \bar{m}_{2k-1} = \bar{m}_{2k-2}A^{-1}_{2,s+2}.$$

Therefore $m = m_1m_2 = m_1n_2$ is one of the following monomials

$$\bar{m}_1n_2 = n_1n_1'n_1''A^{-1}_{2,s+4k-2}n_2,\ \bar{m}_2n_2 = \bar{m}_1A^{-1}_{2,s+4k-4}n_2,\ \ldots,\ \bar{m}_{2k-1}n_2 = \bar{m}_{2k-2}A^{-1}_{2,s+2}n_2.$$

Hence m is not a dominant monomial. This is a contradiction. Therefore the only dominant monomial in S is $R^{(s+1)}_{2k-1,\ell,\lfloor\frac{\ell}{2}\rfloor}T^{(s+4k+4)}_{\lfloor\frac{\ell-1}{2}\rfloor,0,0}$. □

5.3. **Proof of Theorem 3.3.** In Section 4, we have shown that $\mathcal{R}^{(s)}_{0,2\ell+i,\ell}$, $i = 0, 1, 2$, and $\mathcal{U}^{(s)}_{k,\ell}$ are special. Therefore by Lemma 5.2, (3.13) and (3.26) are true. By Lemmas 5.1 and 5.2, the dominant monomials in the q-characters of the left hand side and of the right hand side of every relation in Theorem 3.3 are the same. The theorem follows.

6. Proof of Theorem 3.4 and Theorem 3.13

By Lemma 5.2, $\mathcal{S}$ is special and hence irreducible. Therefore we only have to show that each $\mathcal{T} \otimes \mathcal{B}$ in Theorem 3.4 and Theorem 3.13 is irreducible. It suffices to prove that for each non-highest dominant monomial M in $\mathcal{T}\otimes\mathcal{B}$, we have $\mathscr{M}(L(M)) \not\subset \mathscr{M}(\mathcal{T}\otimes\mathcal{B})$. The idea is similar as in [Her06], [MY12b], [LM12].

Lemma 6.1. *We consider the same cases as in Lemma 5.1. In each case M_i are the dominant monomials described by that lemma.*

(1) For $k \geq 1, \ell \geq 1$, let

$$\begin{aligned}
&n_1 = M_1A^{-1}_{2,s+4k+2\ell-2},\ n_2 = M_2A^{-1}_{2,s+4k+2\ell-4},\ \ldots,\\
&n_{\ell-1} = M_{\ell-1}A^{-1}_{2,s+4k+2},\ n_\ell = M_\ell A^{-1}_{3,s+4k-2}A^{-1}_{2,s+4k},\\
&n_{\ell+1} = M_{\ell+1}A^{-1}_{3,s+4k-6},\ n_{\ell+2} = M_{\ell+2}A^{-1}_{3,s+4k-10},\ \ldots,\ n_{k+\ell-2} = M_{k+\ell-2}A^{-1}_{3,s+6}.
\end{aligned}$$

Then for $i = 1, \ldots, k + \ell - 2$, $n_i \in \chi_q(M_i)$ and $n_i \notin \chi_q(T^{(s)}_{k,\ell,0})\chi_q(T^{(s+4)}_{k-1,\ell-1,0})$.

(2) *For* $k \geq 1, m \geq 1$, *let*

$$n_1 = M_1 A^{-1}_{1,s+4k+2m-1},\ n_2 = M_2 A^{-1}_{1,s+4k+2m-3},\ \ldots,$$
$$n_{m-1} = M_{m-1} A^{-1}_{1,s+4k+3},\ n_m = M_m A^{-1}_{3,s+4k-2} A^{-1}_{2,s+4k} A^{-1}_{1,s+4k+1},$$
$$n_{m+1} = M_{m+1} A^{-1}_{3,s+4k-6},\ n_{m+2} = M_{m+2} A^{-1}_{3,s+4k-10},\ \ldots,\ n_{k+m-2} = M_{k+m-2} A^{-1}_{3,s+6}.$$

Then for $i = 1, \ldots, k+m-2$, $n_i \in \chi_q(M_i)$ *and* $n_i \notin \chi_q(T^{(s)}_{k,0,m})\chi_q(T^{(s+4)}_{k-1,0,m-1})$.

(3) *For* $k \geq 1, \ell \geq 1$, *let*

$$n_1 = M_1 A^{-1}_{2,s+2k+2\ell+1},\ n_2 = M_2 A^{-1}_{2,s+2k+2\ell-1},\ \ldots,$$
$$n_{\ell-1} = M_{\ell-1} A^{-1}_{2,s+2k+5},\ n_\ell = M_\ell A^{-1}_{2,s+2k-1} A^{-1}_{3,s+2k+1} A^{-1}_{2,2k+3},$$
$$n_{\ell+1} = M_{\ell+1} A^{-1}_{2,s+2k-3},\ n_{\ell+2} = M_{\ell+2} A^{-1}_{2,s+2k-5},\ \ldots, n_{k+\ell-2} = M_{k+\ell-2} A^{-1}_{2,s+3}.$$

Then for $i = 1, \ldots, k+\ell-2$, $n_i \in \chi_q(M_i)$ *and* $n_i \notin \chi_q(S^{(s)}_{k,\ell})\chi_q(S^{(s+2)}_{k-1,\ell-1})$.

(4) *For* $\ell \geq 1, r \in \{1,2\}$, *let*

$$n_1 = M_1 A^{-1}_{1,s+2\ell+2r-2},\ n_2 = M_2 A^{-1}_{1,s+2\ell+2r-4},\ \ldots,$$
$$n_{r-1} = M_{r-1} A^{-1}_{1,s+2\ell+2},\ n_r = M_r A^{-1}_{2,s+2\ell} A^{-1}_{1,s+2\ell+1},$$
$$n_{r+1} = M_{r+1} A^{-1}_{2,s+2\ell-2},\ n_{r+2} = M_{r+2} A^{-1}_{2,s+2\ell-4},\ \ldots,\ n_{\ell+r-2} = M_{\ell+r-2} A^{-1}_{2,s+4}.$$

Then for $i = 1, \ldots, \ell+r-2$, $n_i \in \chi_q(M_i)$ *and* $n_i \notin \chi_q(T^{(s)}_{0,\ell,r})\chi_q(T^{(s+2)}_{0,\ell-1,r-1})$.

(5) *For* $k, m \geq 1$, *let*

$$n_1 = M_1 A^{-1}_{3,s+2k+4m-2},\ n_2 = M_2 A^{-1}_{3,s+2k+4m-6},\ \ldots,$$
$$n_{m-1} = M_{m-1} A^{-1}_{3,s+2k+6},\ n_m = M_m A^{-1}_{1,s+2k-1} A^{-1}_{2,s+2k} A^{-1}_{3,2k+2},$$
$$n_{m+1} = M_{m+1} A^{-1}_{1,s+2k-3},\ n_{m+2} = M_{m+2} A^{-1}_{1,s+2k-5},\ \ldots,\ n_{k+m-2} = M_{k+m-2} A^{-1}_{1,s+3}.$$

Then for $i = 1, \ldots, k+m-2$, $n_i \in \chi_q(M_i)$ *and* $n_i \notin \chi_q(\tilde{T}^{(s)}_{k,0,m})\chi_q(\tilde{T}^{(s+2)}_{k-1,0,m-1})$.

(6) *For* $k \geq 1, \ell \geq 1$, *let*

$$n_1 = M_1 A^{-1}_{3,s+2k+4\ell-3},\ n_2 = M_2 A^{-1}_{3,s+2k+4\ell-7},\ \ldots,$$
$$n_{\ell-1} = M_{\ell-1} A^{-1}_{3,s+2k+5},\ n_\ell = M_\ell A^{-1}_{2,s+2k-1} A^{-1}_{3,s+2k+1},$$
$$n_{\ell+1} = M_{\ell+1} A^{-1}_{2,s+2k-3},\ n_{\ell+2} = M_{\ell+2} A^{-1}_{2,s+2k-5},\ \ldots,\ n_{k+\ell-2} = M_{k+\ell-2} A^{-1}_{2,s+3}.$$

Then for $i = 1, \ldots, k+\ell-2$, $n_i \in \chi_q(M_i)$ *and* $n_i \notin \chi_q(R^{(s)}_{k,2\ell,\ell})\chi_q(R^{(s+2)}_{k-1,2\ell,\ell-1})$.

(7) *For* $k \geq 1, \ell \geq 1$, *let*

$$n_1 = M_1 A^{-1}_{1,s+2k+4\ell},\ n_2 = M_2 A^{-1}_{1,s+2k+4\ell-2},\ \ldots,$$
$$n_{2\ell} = M_{2\ell} A^{-1}_{1,s+2k+2},\ n_{2\ell+1} = M_{2\ell+1} A^{-1}_{2,s+2k-1} A^{-1}_{1,s+2k},$$
$$n_{2\ell+2} = M_{2\ell+2} A^{-1}_{2,s+2k-3},\ n_{2\ell+3} = M_{2\ell+3} A^{-1}_{2,s+2k-5},\ \ldots,\ n_{k+2\ell-1} = M_{k+2\ell-1} A^{-1}_{2,s+3}.$$

Then for $i = 1, \ldots, k+2\ell-1$, $n_i \in \chi_q(M_i)$ *and* $n_i \notin \chi_q(R^{(s)}_{k,2\ell+1,\ell})\chi_q(R^{(s+2)}_{k-1,2\ell,\ell})$.

(8) *For* $k \geq 1, \ell \geq 1$, *let*

$$n_1 = M_1 A^{-1}_{1,s+2k+4\ell+2},\ n_2 = M_2 A^{-1}_{1,s+2k+4\ell},\ \ldots,$$
$$n_{2\ell+1} = M_{2\ell+1} A^{-1}_{1,s+2k+2},\ n_{2\ell+2} = M_{2\ell+2} A^{-1}_{2,s+2k-1} A^{-1}_{1,s+2k},$$
$$n_{2\ell+3} = M_{2\ell+3} A^{-1}_{2,s+2k-3},\ n_{2\ell+4} = M_{2\ell+4} A^{-1}_{2,s+2k-5},\ \ldots,\ n_{k+2\ell} = M_{k+2\ell} A^{-1}_{2,s+3}.$$

Then for $i = 1, \ldots, k+2\ell$, $n_i \in \chi_q(M_i)$ *and* $n_i \notin \chi_q(R^{(s)}_{k,2\ell+2,\ell})\chi_q(R^{(s+2)}_{k-1,2\ell+1,\ell})$.

(9) *For* $k \geq 1, \ell \geq 1$, *let*

$$n_1 = M_1 A^{-1}_{2,s+2k+2\ell-2},\ n_2 = M_2 A^{-1}_{2,s+2k+2\ell-4},\ \ldots,$$
$$n_{\ell-1} = M_{\ell-1} A^{-1}_{2,s+2k+2},\ n_\ell = M_\ell A^{-1}_{1,s+2k-1} A^{-1}_{2,s+2k},$$
$$n_{\ell+1} = M_{\ell+1} A^{-1}_{1,s+2k-3},\ n_{\ell+2} = M_{\ell+2} A^{-1}_{1,s+2k-5},\ \ldots,\ n_{k+\ell-2} = M_{k+\ell-2} A^{-1}_{1,s+3}.$$

Then for $i = 1, \ldots, k+\ell-2$, $n_i \in \chi_q(M_i)$ *and* $n_i \notin \chi_q(\tilde{T}^{(s)}_{k,\ell,0})\chi_q(\tilde{T}^{(s+2)}_{k-1,\ell-1,0})$.

(10) *For* $p \geq 2, \ell \geq 1$, *let*

$$n_1 = M_1 A^{-1}_{3,s+2p+2\ell-3},\ n_2 = M_2 A^{-1}_{2,s+2p+2\ell-7},\ \ldots,$$
$$n_{\lfloor \frac{\ell-1}{2} \rfloor} = M_{\lfloor \frac{\ell-1}{2} \rfloor} A^{-1}_{3,s+2p+2\sigma(\ell+1)+3},\ n_{\lfloor \frac{\ell-1}{2} \rfloor+1} = M_{\lfloor \frac{\ell-1}{2} \rfloor+1} A^{-1}_{2,s+2p-1} A^{-1}_{3,s+2p+1},$$
$$n_{\lfloor \frac{\ell-1}{2} \rfloor+2} = M_{\lfloor \frac{\ell-1}{2} \rfloor+2} A^{-1}_{2,s+2p-3},\ n_{\lfloor \frac{\ell-1}{2} \rfloor+3} = M_{\lfloor \frac{\ell-1}{2} \rfloor+3} A^{-1}_{2,s+2p-5},\ \ldots,$$
$$n_{p+\lfloor \frac{\ell-1}{2} \rfloor-1} = M_{p+\lfloor \frac{\ell-1}{2} \rfloor-1} A^{-1}_{2,s+3}.$$

Then for $i = 1, \ldots, p+\lfloor \frac{\ell-1}{2} \rfloor - 1$, $n_i \in \chi_q(M_i)$ *and* $n_i \notin \chi_q(U^{(s)}_{p,\ell})\chi_q(U^{(s+2)}_{p-1,\ell-1})$.

(11) *For* $k \geq 1, \ell \geq 1$, *let*

$$n_1 = M_1 A^{-1}_{3,s+4k+4\ell-4},\ n_2 = M_2 A^{-1}_{3,s+4k+4\ell-8},\ \ldots,$$
$$n_{\ell-1} = M_{\ell-1} A^{-1}_{3,s+4k+6},\ n_\ell = M_\ell A^{-1}_{3,s+4k-2} A^{-1}_{2,s+4k} A^{-1}_{2,s+4k-2} A^{-1}_{3,s+4k},$$
$$n_{\ell+1} = M_{\ell+1} A^{-1}_{3,s+4k-6},\ n_{\ell+2} = M_{\ell+2} A^{-1}_{3,s+4k-10},\ \ldots,\ n_{k+\ell-2} = M_{k+\ell-2} A^{-1}_{3,s+6}.$$

Then for $i = 1, \ldots, k+\ell-2$, $n_i \in \chi_q(M_i)$ *and* $n_i \notin \chi_q(V^{(s)}_{k,\ell})\chi_q(V^{(s+4)}_{k-1,\ell-1})$.

(12) *For* $k \geq 1, \ell \geq 1$, *let*

$$n_1 = M_1 A^{-1}_{3,s+4k+4\ell},\ n_2 = M_2 A^{-1}_{3,s+4k+4\ell-4},\ \ldots,$$
$$n_{\ell-1} = M_{\ell-1} A^{-1}_{3,s+4k+8},\ n_\ell = M_\ell A^{-1}_{2,s+4k+2} A^{-1}_{3,s+4k+4},$$
$$n_{\ell+1} = M_{\ell+1} A^{-1}_{3,s+4k-2} A^{-1}_{2,s+4k},\ n_{\ell+2} = M_{\ell+2} A^{-1}_{3,s+4k-6},$$
$$n_{\ell+3} = M_{\ell+3} A^{-1}_{3,s+4k-10},\ \ldots,\ n_{k+\ell-1} = M_{k+\ell-1} A^{-1}_{3,s+6}.$$

Then for $i = 1, \ldots, k+\ell-1$, $n_i \in \chi_q(M_i)$ *and* $n_i \notin \chi_q(P^{(s)}_{k,\ell})\chi_q(P^{(s+4)}_{k-1,\ell-1})$.

(13) For $k \geq 1, \ell \geq 1$, let

$$\begin{aligned}
&n_1 = M_1 A^{-1}_{2,s+2k+2\ell+4},\ n_2 = M_2 A^{-1}_{2,s+2k+2\ell+2},\ \ldots,\\
&n_{\ell-1} = M_{\ell-1} A^{-1}_{2,s+2k+7},\ n_\ell = M_\ell A^{-1}_{1,s+2k+4} A^{-1}_{2,s+2k+5},\\
&n_{\ell+1} = M_{\ell+1} A^{-1}_{1,s+2k+2},\ n_{\ell+2} = M_{\ell+2} A^{-1}_{2,s+2k-1} A^{-1}_{1,2k},\\
&n_{\ell+3} = M_{\ell+3} A^{-1}_{2,s+2k-3},\ n_{\ell+4} = M_{\ell+4} A^{-1}_{2,s+2k-5},\ \ldots,\ n_{k+\ell} = M_{k+\ell} A^{-1}_{2,s+3}.
\end{aligned}$$

Then for $i = 1, \ldots, k+\ell$, $n_i \in \chi_q(M_i)$ and $n_i \notin \chi_q(O^{(s)}_{k,\ell})\chi_q(O^{(s+2)}_{k-1,\ell-1})$.

Proof. We prove the case of $\chi_q(T^{(s)}_{k,\ell,0})\chi_q(T^{(s+4)}_{k-1,\ell-1,0})$. The other cases are similar. By definition, we have

$$\begin{aligned}
T^{(s)}_{k,\ell,0} &= (3_s 3_{s+4} \cdots 3_{s+4k-4})(2_{s+4k+1} 2_{s+4k+3} \cdots 2_{s+4k+2\ell-3} 2_{s+4k+2\ell-1}),\\
T^{(s+4)}_{k-1,\ell-1,0} &= (3_{s+4} \cdots 3_{s+4k-4})(2_{s+4k+1} 2_{s+4k+3} \cdots 2_{s+4k+2\ell-3}),\\
M_1 &= T^{(s)}_{k,\ell,0} T^{(s+4)}_{k-1,\ell-1,0} A^{-1}_{2,s+4k+2\ell-2}\\
&= T^{(s)}_{k,\ell,0} T^{(s+4)}_{k-1,\ell-1,0} 2^{-1}_{s+4k+2\ell-1} 1_{s+4k+2\ell-2} 3_{s+4k+2\ell-2}.
\end{aligned}$$

By $U_{q^2}(\hat{\mathfrak{sl}}_2)$ argument, it is clear that $n_1 = M_1 A^{-1}_{2,s+4k+2\ell-2}$ is in $\chi_q(M_1)$.

If n_1 is in $\chi_q(T^{(s)}_{k,\ell,0})\chi_q(T^{(s+4)}_{k-1,\ell-1,0})$, then $T^{(s)}_{k,\ell,0} A^{-1}_{2,s+4k+2\ell-2}$ is in $\chi_q(T^{(s)}_{k,\ell,0})$ which is impossible by the FM algorithm for $\mathcal{T}^{(s)}_{k,\ell,0}$. Similarly, $n_i \in \chi_q(M_i)$, $i = 2, \ldots, \ell-1$, but $n_2, \ldots, n_{\ell-1}$ are not in $\chi_q(T^{(s)}_{k,\ell,0})\chi_q(T^{(s+4)}_{k-1,\ell-1,0})$.

By definition,

$$\begin{aligned}
M_\ell \;=\; &(3_s 3_{s+4} \cdots 3_{s+4k-4})(3_s 3_{s+4} \cdots 3_{s+4k-8}) 2_{s+4k-3} \times\\
&\times (1_{s+4k} 1_{s+4k+2} 1_{s+4k+4} \cdots 1_{s+4k-2\ell-2})(3_{s+4k+2} 3_{s+4k+4} \cdots 3_{s+4k-2\ell-2}).
\end{aligned}$$

Let $U = \{(2, aq^{s+4k}), (3, aq^{s+4k-2})\} \subset I \times \mathbb{C}^\times$. Let $\mathcal{M}$ be the finite set consisting of the following monomials

$$m_0 = M_\ell,\ m_1 = m_0 A^{-1}_{3,s+4k-2},\ m_2 = m_1 A^{-1}_{2,s+4k}.$$

It is clear that $\mathcal{M}$ satisfies the conditions in Theorem 2.2. Therefore

$$\mathrm{trunc}_{M_\ell \mathcal{Q}_U^-}(\chi_q(M_\ell)) = \sum_{m \in \mathcal{M}} m$$

and hence $n_\ell = M_\ell A^{-1}_{3,s+4k-2} A^{-1}_{2,s+4k}$ is in $\chi_q(M_\ell)$.

If n_ℓ is in $\chi_q(T^{(s)}_{k,\ell,0})\chi_q(T^{(s+4)}_{k-1,\ell-1,0})$, then $T^{(s)}_{k,\ell,0} A^{-1}_{3,s+4k-2} A^{-1}_{2,s+4k}$ is in $\chi_q(T^{(s)}_{k,\ell,0})$ which is impossible by the FM algorithm for $\mathcal{T}^{(s)}_{k,\ell,0}$.

Similarly, for $i = \ell+1, \ldots, k+\ell-2$, we have $n_i \in \chi_q(M_i)$ and $n_i \notin \chi_q(T^{(s)}_{k,\ell,0})\chi_q(T^{(s+4)}_{k-1,\ell-1,0})$. □

7. Proof Proposition 3.5

Let $\mathcal{A}^{(s)}$ be a module in the system I. By (2.3), $\chi_q(\mathcal{A}^{(s)})$ is obtained from $\chi_q(\mathcal{A}^{(0)})$ by a shift of indices. For simplicity, we do not write the upper-subscripts "(s)" in the proof. The q-characters of Kirillov-Reshetikhin modules can be computed from $\chi_q(1_0), \chi_q(2_0), \chi_q(3_0)$, see [Her06].

It suffices to prove the following results.

(1) By using (3.9), we can compute $\chi_q(T_{0,\ell,r})$, $r \in \{0,1,2\}$, $\ell \in \mathbb{Z}_{\geq 0}$.
(2) By using (3.6), (3.7), and (3.8), we can compute $\chi_q(T_{k,0,m})$, $\chi_q(\tilde{T}_{k,0,m})$, $\chi_q(S_{k,m})$, $k, m \in \mathbb{Z}_{\geq 0}$.
(3) By using (3.5), (3.10)–(3.13) and the q-characters computed in (1), (2), we can compute $\chi_q(R_{k,2\ell+i,\ell})$, $\chi_q(T_{k,\ell,0})$, $i = 0, 1, 2$, $k, \ell \in \mathbb{Z}_{\geq 0}$.

Proof of (1). We use induction on ℓ, r. If $\ell \leq 1, r = 0$, then (1) is clearly true. Suppose that (1) is true for $\ell \leq \ell_1, r \leq r_1$, $\ell_1 \geq 1$, $r_1 \in \{0,1,2\}$. We will show that (1) is true for $\ell = \ell_1 + 1, r = r_1$ and $\ell = \ell_1, r = r_1 + 1$ respectively.

Suppose that $\ell = \ell_1 + 1, r = r_1$. If $r = 0$, then $T_{0,\ell,r}$ is a Kirillov-Reshetikhin modules and hence (1) is true. If $r = 1$, then by using (3.9), $T_{0,\ell,r}$ can be computed by using the q-characters for Kirillov-Reshetikhin modules and induction hypothesis. If $r = 2$, then by using (3.9), $T_{0,\ell,r}$ can be computed by using the q-characters for Kirillov-Reshetikhin modules, $\chi_q(0,k,1)$, $k \in \mathbb{Z}_{\geq 0}$, and induction hypothesis. Similarly, we can show that (1) is true for $\ell = \ell_1, r = r_1 + 1$.

Proof of (2). It suffices to show the following result.

(2') By using (3.6), (3.7), and (3.8), we can compute

$$\begin{aligned} &\chi_q(T_{k,0,m}),\ k \leq k_1, m \leq m_1, \quad \chi_q(\tilde{T}_{m,0,k}),\ k \leq k_1, m \leq m_1, \\ &\chi_q(S_{k,m}),\ k \leq 2k_1, m \leq m_1 - 1, \quad \chi_q(S_{m,k}),\ m \leq m_1 - 1, k \leq 2k_1. \end{aligned}$$

If $t_1 < 0$ or $t_2 < 0$, then we do not need to compute S_{t_1,t_2}.

We will prove (2') by using induction on k_1, m_1. If $k_1 = 1, m_1 = 1$, then (2') is clearly true. Suppose that (2') is true for $k_1 = k_2, m_1 = m_2$, $k_2, m_2 \geq 1$. We will show that (2') is true for $k_1 = k_2 + 1, m_1 = m_2$ and $k_1 = k_2, m_1 = m_2 + 1$ respectively.

Suppose that $k_1 = k_2 + 1, m_1 = m_2$. We only need to show that the following q-characters

$$\chi_q(S_{2k_2+1,m}),\ m \leq m_2 - 1, \quad \chi_q(S_{2k_2+2,m}),\ m \leq m_2 - 1, \quad \chi_q(T_{k_2+1,0,m}),\ m \leq m_2, \tag{7.1}$$

$$\chi_q(\tilde{T}_{m,0,k_2+1}),\ m \leq m_2, \quad \chi_q(S_{m,2k_2+1}),\ m \leq m_2 - 1, \quad \chi_q(S_{m,2k_2+2}),\ m \leq m_2 - 1, \tag{7.2}$$

can be computed. We compute the following q-characters

$$\begin{aligned}
&\chi_q(S_{2k_2,2m-1}),\ m\leq \lfloor\frac{m_2-1}{2}\rfloor, \quad \chi_q(\tilde{T}_{2k_2+1,0,m}),\ m\leq \lfloor\frac{m_2-1}{2}\rfloor,\\
&\chi_q(T_{k_2,0,m}),\ m\leq m_2-1, \quad \chi_q(S_{2k_2+1,m}),\ m\leq m_2-1, \quad \chi_q(T_{k_2+1,0,m}),\ m\leq m_2,\\
&\chi_q(S_{2k_2+1,m}),\ m\leq \lfloor\frac{m_2-1}{2}\rfloor, \quad \chi_q(\tilde{T}_{2k_2+2,0,m}),\ m\leq \lfloor\frac{m_2-1}{2}\rfloor,\\
&\chi_q(S_{2k_2+2,m}),\ m\leq m_2-1,
\end{aligned}$$

in the order as shown. At each step, we consider the module that we want to compute as a top module and use the corresponding relation in Theorem 3.3 and known q-characters. Therefore $\chi_q(S_{2k_2+1,m})$, $m\leq m_2-1$, $\chi_q(S_{2k_2+2,m})$, $m\leq m_2-1$, $\chi_q(T_{k_1+1,0,m})$, $m\leq m_2$, can be computed. The fact that the q-characters in (7.2) can be computed can be proved similarly. Therefore (2') is true for $k_1=k_2+1, m_1=m_2$.

Similarly, we can show that (2') is true for $k_1=k_2, m_1=m_2+1$.

Proof of (3). It suffices to prove the following result.

(3') By using (3.5), (3.10)–(3.13) and the q-characters computed in (1), (2), we can compute the following q-characters:

$$\begin{aligned}
&\chi_q(R_{k,2\ell,\ell}),\ k\leq 2k_1-3, \ell\leq \ell_1, \quad \chi_q(R_{k,2\ell,\ell-1}),\ k\leq 2k_1-1, \ell\leq \ell_1,\\
&\chi_q(R_{k,2\ell-1,\ell-1}),\ k\leq 2k_1-1, \ell\leq \ell_1, \quad \chi_q(T_{k,\ell,0}),\ k\leq k_1, \ell\leq 2\ell_1.
\end{aligned}$$

Let $k_2=1$, $\ell_2=1$. The q-characters of $R_{k,i,0}=T_{k,i,0}$, $i=0,1,2$, are computed in (1). Therefore (3') is true in this case by using (3.10) and the q-characters computed in (1).

Suppose that (3') is true for $k_1=k_2$, $\ell_1=\ell_2$, $k_2\geq 1$, $\ell_2\geq 0$. We will show that (3') is true for $k_1=k_2$, $\ell_1=\ell_2+1$ and $k_1=k_2+1$, $\ell_1=\ell_2$ respectively. Let $k_1=k_2$, $\ell_1=\ell_2+1$. We need to show that the following q-characters

$$\begin{aligned}
&\chi_q(R_{k,2\ell_2+2,\ell_2+1}),\ k\leq 2k_2-1, \quad \chi_q(R_{k,2\ell_2+2,\ell_2}),\ k\leq 2k_2-1,\\
&\chi_q(R_{k,2\ell_2+1,\ell_2}),\ k\leq 2k_2-1, \quad \chi_q(T_{k,2\ell_2+1,0}),\ k\leq k_2, \quad \chi_q(T_{k,2\ell_2+2,0}),\ k\leq k_2,
\end{aligned}$$

can be computed. We compute the following q-characters

$$\begin{aligned}
&\chi_q(T_{k-2,2\ell_2,0}),\ k\leq k_2, \quad \chi_q(R_{2k-3,2\ell_2+1,\ell_2}),\ k\leq k_2,\\
&\chi_q(T_{k-1,2\ell_2+1,0}),\ k\leq k_2, \quad \chi_q(R_{2k-1,2\ell_2+2,\ell_2}),\ k\leq k_2,\\
&\chi_q(R_{2k-1,2\ell_2+2,\ell_2+1}),\ k\leq k_2, \quad \chi_q(T_{k,2\ell_2+2,0}),\ k\leq k_2,
\end{aligned}$$

in the order as shown. At each step, we consider the module that we want to compute as a top module and use the corresponding relation in Theorem 3.3 and known q-characters. Therefore (3') is true for $k_1=k_2, \ell_1=\ell_2+1$.

Similarly, we can show that (3') is true for $k_1=k_2+1, \ell_1=\ell_2$.

8. Conjectural character formulas

Every $U_q(\hat{\mathfrak{g}})$-module V is also a $U_q(\mathfrak{g})$-module. We use $\mathrm{Res}(V)$ to denote the $U_q(\mathfrak{g})$-module obtained by restricting V to $U_q(\mathfrak{g})$. We use the system I to compute the characters of the restrictions of modules in the system I. According to these computations, we obtain the following conjecture.

Conjecture 8.1. *For $s \in \mathbb{Z}$, $k, \ell, m \in \mathbb{Z}_{\geq 0}$, $\mathrm{Res}(\mathcal{T}^{(s)}_{k,\ell,0})$, $\mathrm{Res}(\mathcal{T}^{(s)}_{k,0,m})$, $\mathrm{Res}(\tilde{\mathcal{T}}^{(s)}_{k,0,m})$ are decomposed to direct sums of irreducible $U_q(\mathfrak{g})$-modules as follows.*

$$\mathrm{Res}(\mathcal{T}^{(s)}_{k,\ell,0}) = \bigoplus_{j=0}^{i} \bigoplus_{i=0}^{\lfloor \frac{\ell}{2} \rfloor} V((2i-2j)\omega_1 + (\ell-2i)\omega_2 + k\omega_3),$$

$$\mathrm{Res}(\mathcal{T}^{(s)}_{k,0,m}) = \bigoplus_{i=0}^{\lfloor \frac{m}{2} \rfloor} V((m-2i)\omega_1 + k\omega_3),$$

$$\mathrm{Res}(\tilde{\mathcal{T}}^{(s)}_{k,0,m}) = \bigoplus_{i=0}^{\lfloor \frac{k}{2} \rfloor} V((k-2i)\omega_1 + m\omega_3).$$

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References

[CMY12] V. Chari, A. Moura, C. A. S. Young, *Prime representations from a homological perspective*, Mathematische Zeitschrift (2012), DOI: 10.1007/s00209-012-1088-7.

[CG11] V. Chari , J. Greenstein, *Minimal affinizations as projective objects*, J. Geom. Phys. **61** (2011), no. 3, 594–609.

[CP91] V. Chari, A. Pressley, *Quantum affine algebras.* Comm. Math. Phys. **142** (1991), no. 2, 261–283.

[CP94] V. Chari, A. Pressley, A guide to quantum groups, Cambridge University Press, Cambridge, 1994, xvi+651 pp.

[C95] V. Chari, *Minimal affnizations of representations of quantum groups: the rank* 2 *case*, Publ. Res. Inst. Math. Sci. **31** (1995), no. 5, 873–911.

[CP95a] V. Chari, A. Pressley, *Quantum affine algebras and their representations*, Representations of groups (Banff, AB, 1994), 59–78, CMS Conf. Proc., **16**, Amer. Math. Soc., Providence, RI, 1995.

[CP95b] V. Chari, A. Pressley, *Minimal affinizations of representations of quantum groups: the nonsimple-laced case*, Lett. Math. Phys. **35** (1995), no. 2, 99–114.

[CP96a] V. Chari, A. Pressley, *Minimal affinizations of representations of quantum groups: the irregular case*, Lett. Math. Phys. **36** (1996), no. 3, 247–266.

[CP96b] V. Chari, A. Pressley, *Minimal affinizations of representations of quantum groups: the simply laced case*, J. Algebra **184** (1996), no. 1, 1–30.

[Dri88] V. G. Drinfeld, *A new realization of Yangians and of quantum affine algebras.* (Russian) Dokl. Akad. Nauk SSSR **296** (1987), no. 1, 13–17; translation in Soviet Math. Dokl. **36** (1988), no. 2, 212–216.

[FM01] E. Frenkel, E. Mukhin, *Combinatorics of q-characters of finite-dimensional representations of quantum affine algebras*, Comm. Math. Phys. **216** (2001), no. 1, 23–57.

[FR92] I. Frenkel, N. Reshetikhin, *Quantum a?ne algebras and holonomic difference equations*, Comm. Math. Phys. **146** (1992), no. 1, 1–60.

[FR98] E. Frenkel, N. Yu. Reshetikin, *The q-characters of representations of quantum affine algebras and deformations of W-algebras*, Recent developments in quantum affine algebras and related topics (Raleigh, NC, 1998), 163–205, Contemp. Math., **248**, Amer. Math. Soc., Providence, RI, 1999.

[Her06] D. Hernandez, *The Kirillov-Reshetikhin conjecture and solutions of T-systems*, J. Reine Angew. Math. **596** (2006), 63–87.

[Her07] D. Hernandez, *On minimal affinizations of representations of quantum groups*, Comm. Math. Phys. **276** (2007), no. 1, 221–259.

[HL10] D. Hernandez, B. Leclerc, *Cluster algebras and quantum affine algebras*, Duke Math. J. **154** (2010), no. 2, 265–341.

[HL13] D. Hernandez, B. Leclerc, *A cluster algebra approach to q-characters of Kirillov-Reshetikhin modules*, arXiv:1303.0744, 1–45.

[IIKKN13a] R. Inoue, O. Iyama, B. Keller, A. Kuniba, T. Nakanishi, *Periodicities of T and Y-systems, dilogarithm identities, and cluster algebras I: Type B_r*, Publ. Res. Inst. Math. Sci. **49** (2013), no. 1, 1–42.

[IIKKN13b] R. Inoue, O. Iyama, B. Keller, A. Kuniba, T. Nakanishi, *Periodicities of T and Y-systems, dilogarithm identities, and cluster algebras II: Types C_r, F_4, and G_2*, Publ. Res. Inst. Math. Sci. **49** (2013), no. 1, 43–85.

[K83] A. N. Kirillov, *Combinatorial identities and completeness of states of the Heisenberg magnet Questions in quantum field theory and statistical physics*, 4. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **131** (1983), 88–105.

[K84] A. N. Kirillov, *Completeness of states of the generalized Heisenberg magnet*, (Russian) Automorphic functions and number theory, II. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **134** (1984), 169–189.

[K87] A. N. Kirillov, *Identities for the Rogers dilogarithmic function connected with simple Lie algebras*, J. Soviet Math. **47** (1989), no. 2, 2450–2459; translated from Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 164, Differentsialnaya Geom. Gruppy Li i Mekh. IX, **198** (1987), 121–133.

[KNS94] A. Kuniba, T. Nakanishi, J. Suzuki, *Functional relations in solvable lattice models. I. Functional relations and representation theory*, Internat. J. Modern Phys. A **9** (1994), no. 30, 5215–5266.

[KNS11] A. Kuniba, T. Nakanishi, J. Suzuki, *T-systems and Y-systems in integrable systems*, J. Phys. A **44** (2011), no. 10, 103001, 146 pp.

[KR90] A. N. Kirillov, N. Yu. Reshetikhin, *Representations of Yangians and multiplicities of the inclusion of the irreducible components of the tensor product of representations of simple Lie algebras*, (Russian) Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **160** (1987), Anal. Teor. Chisel i Teor. Funktsii. 8, 211–221, 301; translation in J. Soviet Math. **52** (1990), no. 3, 3156–3164.

[LM12] J. R. Li, E. Mukhin, *Extended T-system of type G_2*, arXiv:1208.4821, 1–29.

[Mou10] A. Moura, *Restricted limits of minimal affinizations*, Pacific J. Math **244** (2010), no. 2, 359–397.

[MF11] A. Moura, F. Pereira, *Graded limits of minimal affinizations and beyond: the multiplicity free case for type E6*, Algebra and Discrete Mathematics **12** (2011), no. 1, 69–115.

[MY12a] E. Mukhin, C. A. S. Young, *Path description of type B q-characters*, Advances in Mathematics, **231** (2012), 1119–1150.

[MY12b] E. Mukhin, C. A. S. Young, *Extended T-systems*, Selecta Mathematica, New Series, **18** (2012), 591–631.

[MY12c] E. Mukhin, C. A. S. Young, *Affinization of category O for quantum groups*, arXiv:1204.2769, 1–32.

[Nak03] H. Nakajima, *t-analogs of q-characters of Kirillov-Reshetikhin modules of quantum affine algebras*, Represent. Theory **7** (2003), 259–274.

[Nak11] H. Nakajima, *Quiver varieties and cluster algebras*, Kyoto J. Math. **51** (2011), 71–126.

[Nao12] K. Naoi, *Demazure modules and graded limits of minimal affinizations*, arXiv:1210.0175, 1–29.

[NT98] M. Nazarov, V. Tarasov, *Representations of Yangians with Gelfand-Zetlin bases*, J. Reine Angew. Math. **496** (1998), 181–212.

[Y12] C. A. S. Young, *Quantum loop algebras and l-root operators*, arXiv:1206.6657, 1–27.

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