The Marginal Cost of Risk, Risk Measures, and Capital Allocation*

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Abstract

The Euler (or gradient) allocation technique defines a financial institution's marginal cost of a risk exposure via calculation of the gradient of a risk measure evaluated at the institution's current portfolio position. The technique, however, relies on an arbitrary selection of a risk measure. We reverse the sequence of this approach by calculating the marginal costs of risk exposures for a profit maximizing financial institution with risk-averse counterparties, and then identifying a closed-form solution for the risk measure whose gradient delivers the correct marginal costs. We compare the properties of allocations derived in this manner to those obtained through application of the Euler technique to Expected Shortfall (ES), showing that ES generally yields economically inefficient allocations.

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1 Introduction

Practitioners have long wrestled with the problem of allocating capital to the various risks within a financial institution. And, while no particular method will work for all types of decisions (see Merton and Perold (1993)), an attractive mathematical technology for allocation appeared in an academic literature starting with Schmock and Straumann (1999), Denault (2001), Tasche (2004), and Myers and Read (2001). Broadly speaking, these papers start with a differentiable risk measure and compute the marginal capital increase required to maintain the risk measure at a threshold value as a particular risk exposure within the portfolio is expanded, an approach referred to as "gradient" allocation or "Euler" allocation. Allocations of capital using variants of the Euler technology are now widely used in the course of pricing and performance measurement within the portfolios of financial institutions (see e.g. Society of Actuaries (2008) or McKinsey&Company (2011)).

The approach requires an arbitrarily chosen risk measure as a key input—and unfortunately, excepting highly specialized circumstances,[1] economic theory offers no guidance on the choice of the measure.[2] Yet the choice is not to be taken lightly: It has a profound influence on how the institution perceives risk. For example, Value-at-Risk (VaR)—perhaps the most widely used measure—is known to contain incentives for excessive risk-taking behavior (see Basak and Shapiro (2001)), and its use has been fingered by some as playing a key role in the recent financial crisis (see Nocera (2009)). Post-crisis soul-searching for alternatives to VaR has brought wider attention to the debate on the statistical properties of risk measures, with the likely outcome that measures such as Expected Shortfall (ES) will continue to gain traction among practitioners and regulators. Indeed, the ascent of ES seems well underway.[3] This ascent, however, has been predicated on the technical properties of ES (e.g., coherence) and has occurred independently of any sound economic reasoning.

In this paper, we reverse the usual approach: Rather than choosing a risk measure constraint to determine the marginal cost of risk and the allocation of capital within the firm, we calculate the latter quantities as by-products of the institution's optimization problem. We then derive the risk measure whose gradient yields the correct marginal costs and capital allocations.

The focus of our analysis is the optimal pricing behavior of a profit-maximizing financial institution with costly capital and risk-averse counterparties in the presence of a (possibly nonbinding) regulatory constraint tied to a risk measure. We start by considering a greatly simplified one-period model in an environment without securities markets but subsequently generalize the results to the case where both the firm and its consumers have access to securities markets and to multiple periods.

[1]Specifically, the issue is resolved trivially if consumer preferences are defined over a particular risk measure (Zanjani (2002)), or if a particular risk measure is assumed to constitute the payoff function in a cooperative game (see Denault (2001); Powers (2007)). Similarly, if the *institutional* preferences are defined by a particular risk-averse utility function of outcomes, a particular risk measure may be implied (see e.g. Föllmer and Schied (2010)). Alternatively, Adrian and Shin (2008) find a justification for using Value-at-Risk in a model with a limited commitment and a specialized risk structure.

[2]Early papers in the literature, such as Myers and Read (2001) and Tasche (2004), implicitly recognized this difficulty and ultimately alluded to regulation in justifying the choice of risk measure.

[3]Various papers make a case for ES over VaR (see e.g. Hull (2007), Jaschke (2001)), and both regulation and practice appear to be moving in this direction as well. For instance, the International Actuarial Association (2004) recommends using ES in a risk based regulatory framework, and ES was embedded in the Swiss Solvency Test and appears to be viewed favorably by US regulators (cf. NAIC (2009)).

In this environment, we identify three sources of "discipline" that feed into the marginal cost of risk faced by the firm (and, consequently, the resulting capital allocation). The first is the regulatory solvency constraint, which is a familiar feature of the existing literature: Risks are costly in that they force the firm to hold more capital due to regulation, and the firm must recognize this in its assessment of the cost of bearing a risk. The second derives from the firm's counterparties: When counterparties are not fully protected, the firm's marginal cost associated with the risk of a particular counterparty depends on how that risk affects the firm's other counterparties (and, thus, their willingness to pay for the firm's contracts)—so the firm must take these cross-effects into account in its assessment of the marginal cost of bearing a risk. The third and final source of discipline stems from the continuation value of the firm: Risks taken on in the current period may lead to bankruptcy of the firm and thus may destroy future profit flows, so the firm must also assess the cost of a risk with an eye to the threat it poses to future profit flows.

In general, the marginal cost of risk and, consequently, the optimal capital allocation rule, are determined by these three influences. Specifically, the optimal capital allocation rule is a weighted average of an "external" allocation rule implied by the regulatory constraint (if it binds), a "continuation" rule that derives from the firm's value as a going concern, and an "internal counterparty" allocation rule driven by the institution's uninsured counterparties. In the extreme case of no regulation and perfect competition (so that the firm is earning zero economic rents), the allocation rule simply boils down to the "internal counterparty" rule. Another extreme case is the scenario envisioned by Myers and Read or Tasche (a single period model with fully insured counterparties), in which case the economically optimal allocation follows from the Euler allocation principle as applied to the risk measure imposed by regulation. Intermediate cases, however, could feature marginal cost being driven mainly by the "internal counterparty" and/or the "continuation" allocation rule (if the regulatory constraint puts firm capitalization close to the level it would have chosen in the absence of regulation) or the "external" constraint (if regulation forces the firm to hold far more capital than is privately optimal).

We then investigate the connection between these economically derived capital allocations and the gradient method. Specifically, we "reverse-engineer" risk measures that—upon application of the gradient method—yield the economically correct capital allocations. Each of the three components of the optimal allocation rule discussed above are connected to a risk measure that yields the correct component capital allocation. The "external" allocation rule is of course connected to a risk measure by definition, as it arises from a risk measure imposed by a regulator. The more interesting finding is that the "internal counterparty" allocation rule can be implemented by applying the gradient allocation principle to a particular risk measure—the exponential of a weighted average of the logarithm of portfolio outcomes in states of default, with the weights being determined by the relative values placed on recoveries in the various states of default by the firm's counterparties. The weights are thus similar in concept to the "spectral" weights proposed by Acerbi (2002) as a means of capturing the "subjective risk aversion" of a financial institution, with the weights determined endogenously through the process of profit maximization. The risk measure itself is evidently a "tail" risk measure, although the functional transformations ultimately cause it to be non-convex. Finally, in a multi-period setting, the allocation stemming from the "continuation" value of the firm can be recovered by applying the gradient method to VaR—which thus arises endogenously in our model.

We then derive closed form allocation formulas where possible, and we use numerical techniques to compare the allocations resulting from the "internal counterparty" allocation rule to those arising from the application of the gradient allocation method to ES. We show that ES-based allocations generally fail to weight default outcomes properly. Specifically, in cases where counterparties are strongly risk-averse or where potential losses are large relative to counterparty wealth, ES-based allocations tend to underweight bad outcomes; in cases where counterparties are only weakly risk-averse or where potential losses are relatively small, ES-based allocations tend to overweight bad outcomes. These differences flow from a fundamental difference in the basis for allocation under the "internal counterparty" rule and under ES. The starting point for evaluation of a risk's impact under ES concerns its share of the institution's *losses* in default states, whereas the starting point under the "internal counterparty" rule is the risk's share of *recoveries*—as a risk's impact on recoveries in default states is ultimately what counterparties care about.

This distinction underscores the key point of the paper: The true marginal cost of risk and the associated allocation of capital should flow from the economic context of the problem. A risk measure chosen for its technical properties such as coherence, rather than for the specific economic circumstances, will generally fail to yield correct pricing and efficient allocation of capital from the perspective of its user. In the concluding section, we consider how changing the perspective of the user—from that of the firm owner/manager, as in this paper, to, say, a regulator—changes the economic context and, hence, the appropriate measure of risk.

2 Profit Maximization and Capital Allocation

We consider the optimization problem of a representative financial institution. We frame our model in terms of an insurance company, and our language will reflect this in that we refer to the financial contracts as providing "insurance coverage" and the counterparties of the institution as "consumers." The setup obviously fits other institutions providing similar contracts, such as reinsurance companies and private pension plan sponsors—and can be applied with little modification to institutions selling insurance-like contracts (such as credit default swaps) where the main risks emanate from risk in obligations to counterparties. The model can also be adapted to fit other institutions where capital allocation is relevant (such as commercial banks) but where the key risks emanate from the asset side of the balance sheet, by including an additional set of choice variables for investments. The key assumption of the model, however, is that the policyholders, counterparties, and/or debtholders of the institution are exposed to the failure of the institution—and their preferences for solvency drive the motivation for risk management.

To illustrate the main ideas, we will start by considering a greatly simplified one-period model in an environment without securities markets. Subsequently, we generalize the results to the case where both the firm and its consumers have access to securities markets and to multiple periods.

2.1 One-period Model without Security Markets

Formally, we consider an insurance company that has N consumers, with consumer i facing a loss L_i modeled as a continuous, non-negative, square-integrable random variable on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with (joint) density $f_{L_1,L_2,\ldots,L_N} : \mathbb{R}_+^N \to \mathbb{R}_+$. The firm determines the optimal level of assets a for the company, as well as levels of insurance coverage for the consumers, with the coverage indemnification level for consumer i denoted as a function of the loss experienced and a parameter $q_i \in \Phi$, where Φ is a compact choice set. For tractability, we focus on a quota share arrangement, i.e. a linear contract, where the insurer agrees to reimburse q_i per dollar of loss:

$$I_i = I_i(L_i, q_i) = q_i \times L_i. \tag{1}$$

However, generalizations are possible.[4]

If a consumer experiences a loss, she claims to the extent of the promised indemnification. If total claims are less than company assets, all are paid in full. If not, all claimants are paid at the same rate per dollar of coverage. The total claims submitted are:

$$I = I(L_1, L_2, \ldots, L_N, q_1, q_2, \ldots, q_N) = \sum_{j=1}^{N} I_j(L_j, q_j),$$

and we define the consumer's recovery as:

$$R_i = \min\left\{ I_i(L_i, q_i), \frac{a}{I}\, I_i(L_i, q_i) \right\}. \tag{2}$$

Accordingly, $\{I \geq a\} = \{\omega \in \Omega \,|\, I(\omega) \geq a\}$ denote the states in which the company defaults whereas $\{I < a\}$ are the solvent states. The expected value of recoveries for the i-th consumer is whence given by:

$$e_i = \mathbb{E}\left[R_i\right] = \underbrace{\mathbb{E}\left[R_i\,\mathbf{1}_{\{I<a\}}\right]}_{=e_i^Z} + \underbrace{\mathbb{E}\left[R_i\,\mathbf{1}_{\{I\geq a\}}\right]}_{=e_i^D}.$$

There is a frictional cost—including agency, taxes, and monitoring costs—associated with holding assets in the company. We represent the cost as a tax on assets:

$$\tau \times a, \tag{3}$$

although it is also possible to represent frictional costs as a tax on equity capital, as in:

$$\tau \times \left(a - \mathbb{E}\left[\sum_{i=1}^{N} \min\left\{ I_i(L_i, q_i), \frac{a}{I}\, I_i(L_i, q_i) \right\} \right] \right) \tag{4}$$

and this does not change the ensuing allocation result.

We denote the premium charged to the consumer i as p_i, and consumer utility may be expressed as:

$$v_i(a, \mathrm{w}_i - p_i, q_1, ..., q_N) = \mathbb{E}\left[U_i\left(\mathrm{w}_i - p_i - L_i + R_i\right)\right], \tag{5}$$

where w_i denotes consumer i's wealth, and we write $v_i'(\cdot) = \frac{\partial}{\partial \mathrm{w}_i} v_i(\cdot)$.

The firm then solves:

$$\max_{a, \{q_i\}, \{p_i\}} \sum p_i - \sum e_i - \tau a, \tag{6}$$

subject to participation constraints for each consumer:

$$v_i(a, \mathrm{w}_i - p_i, q_1, ..., q_N) \geq \gamma_i \quad \forall i \tag{7}$$

[4]For instance, a fixed policy limit as in $I_i = \min\{L_i, q_i\}$ in conjunction with binary loss distributions also fits our framework, although the lack of differentiability would formally require a separate treatment.

and subject to a differentiable solvency constraint imposed by the regulator:

$$s(q_1, ..., q_N) \leq a, \tag{8}$$

where s is imagined to arise from, for example, an externally supplied risk measure with a set threshold dictating the requisite capitalization for the firm. As is customary for risk measures (see e.g. the well-known *coherence* axioms by Artzner et al. (1999)), we assume that s is positive homogeneous of degree 1.

Let λ_k be the Lagrange multiplier associated with the participation constraint (7) for consumer k, and let ξ the multiplier associated with (8). The first order conditions for an interior solution are then:[5]

$$[q_i] \qquad -\sum_k \frac{\partial e_k}{\partial q_i} + \sum_k \lambda_k \frac{\partial v_k}{\partial q_i} - \frac{\partial s}{\partial q_i}\xi = 0, \tag{9}$$

$$[a] \qquad -\sum_k \frac{\partial e_k}{\partial a} - \tau + \sum_k \lambda_k \frac{\partial v_k}{\partial a} + \xi = 0, \tag{10}$$

$$[p_i] \qquad 1 - \lambda_i \frac{\partial v_i}{\partial \mathrm{w}} = 0. \tag{11}$$

We show in Appendix A that a profit-maximizing firm will be able to achieve the optimum by offering each consumer a smooth and monotonic premium schedule, where consumer i is free to choose any level of q_i desired. We denote the variable premium as $p_i^*(q_i)$ and consider its construction under the assumption that each consumer is a "price taker" and ignores the impact of her own purchase at the margin on the level of recoveries in states of default. This assumption is discussed in Zanjani (2010), who followed the transportation economics literature on congestion pricing (Keeler and Small (1977)) by using the assumption when calculating the optimal consumer pricing function.[6] With this assumption in place, the marginal price change at the optimal level of q_i must satisfy:

$$\left[\frac{\partial v_i}{\partial q_i} + \mathbb{E}\left[\mathbf{1}_{\{I \geq a\}}\, U_i' \,\frac{a}{I^2}\, I_i\, \frac{\partial I_i}{\partial q_i}\right]\right] - \frac{\partial v_i}{\partial \mathrm{w}} \frac{\partial p_i^*}{\partial q_i} = 0 \tag{12}$$

with $U_i' = U_i'\left(\mathrm{w}_i - p_i - L_i + R_i\right)$. The term in brackets represents how the consumer *perceives* the marginal benefit of additional coverage, which, due to the aforementioned assumption, differs from the true impact of coverage on the utility function by $\mathbb{E}\left[\mathbf{1}_{\{I \geq a\}}\, U_i' \,\frac{a}{I^2}\, I_i\, \frac{\partial I_i}{\partial q_i}\right]$.

Using (9) and (11), we see that (12) may be rewritten as:

$$\frac{\partial p_i^*}{\partial q_i} = \sum_k \frac{\partial e_k}{\partial q_i} + \frac{\partial s}{\partial q_i}\,\xi - \sum_{k \neq i} \frac{\frac{\partial v_k}{\partial q_i}}{v_k'} + \frac{\mathbb{E}\left[\mathbf{1}_{\{I \geq a\}}\, U_i' \,\frac{a}{I^2}\, I_i\, \frac{\partial I_i}{\partial q_i}\right]}{v_i'}$$

[5]Despite the apparent kinks, the objective function and the constraints are differentiable for continuous distributions as can be easily verified by an application of the Leibniz rule.

[6]The assumption is ubiquitous (although often implicit rather than stated) within many fields of economics. Its importance here is that, without it, the marginal cost associated with each consumer's risk will feature a cost related to an implicit allocation of capital that will not "add up" across consumers to the total capital of the firm. The real-world analogy may be a consumer consulting an insurance rating before choosing her carrier without contemplating the effects of her purchase on the firm's financial situation.

or, simplifying and using (10):

$$\frac{\partial p_i^*}{\partial q_i} = \frac{\partial e_i^Z}{\partial q_i} + \frac{\partial s}{\partial q_i}\left[\sum_k \frac{\partial e_k}{\partial a} + \tau - \sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'}\right] + \mathbb{E}\left[\mathbf{1}_{\{I \geq a\}} \sum_k \frac{U_k'}{v_k'} \frac{a}{I^2} I_k \frac{\partial I_i}{\partial q_i}\right].$$

Moving on:

$$\frac{\partial p_i^*}{\partial q_i} = \frac{\partial e_i^Z}{\partial q_i} + \frac{\partial s}{\partial q_i}\left[\mathbb{P}(I \geq a) + \tau - \sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'}\right] + \frac{\mathbb{E}\left[\mathbf{1}_{\{I \geq a\}} \sum_k \frac{U_k'}{v_k'} \frac{1}{I^2} I_k \frac{\partial I_i}{\partial q_i}\right]}{\sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'}} \times \sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'} \times a$$

or

$$\frac{\partial p_i^*}{\partial q_i} = \frac{\partial e_i^Z}{\partial q_i} + \frac{\partial s}{\partial q_i}\left[\mathbb{P}(I \geq a) + \tau - \sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'}\right] + \tilde{\phi}_i \times a \times \left[\sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'}\right], \tag{13}$$

where

$$\tilde{\phi}_i = \frac{\mathbb{E}\left[\mathbf{1}_{\{I \geq a\}} \sum_k \frac{U_k'}{v_k'} \frac{1}{I^2} I_k \frac{\partial I_i}{\partial q_i}\right]}{\mathbb{E}\left[\mathbf{1}_{\{I \geq a\}} \sum_k \frac{U_k'}{v_k'} \frac{I_k}{I}\right]}. \tag{14}$$

The last two terms of (13) imply an allocation of the marginal unit of capital to consumer that "adds up." More specifically, it can be verified that:

$$a \times \sum \tilde{\phi}_i \, q_i = a, \tag{15}$$

whereas the regulatory constraint "adds up" by the homogeneity assumption:

$$\sum \frac{\partial s}{\partial q_i} q_i = a. \tag{16}$$

Thus, the optimal marginal pricing condition (13) can be extended to fully allocate all of the firm's costs, including the cost of capital:

$$\sum \frac{\partial p_i^*}{\partial q_i} q_i = \sum \frac{\partial e_i^Z}{\partial q_i} q_i + \left[\mathbb{P}(I \geq a)\, a + \tau a\right].$$

Note that the cost of capital as captured in the bracketed term breaks down as:

$$\left[\sum_k \frac{\partial e_k}{\partial a} a + \tau a\right] = \sum_i \frac{\partial s}{\partial q_i} q_i \times \left[\mathbb{P}(I \geq a) + \tau - \sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'}\right] + \sum_i \tilde{\phi}_i \, q_i \times a \times \left[\sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'}\right].$$

So for any one individual consumer, their capital allocation has two components. The first derives from an "internal" marginal cost—driven by the cross-effects of consumers on each other:

$$\tilde{\phi}_i \, q_i \times a \times \left[\sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'}\right]$$

and the second originates from an "external" marginal cost imposed by regulators:

$$\frac{\partial s}{\partial q_i} q_i \times \left[\mathbb{P}(I \geq a) + \tau - \sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'}\right].$$

It is useful at this point to consider several different institutional scenarios.

Full Coverage by Deposit Insurance and Binding Regulation

If consumers are fully covered by deposit insurance, they are indifferent to the capitalization of their financial institution. Mathematically, this means that

$$\sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'} = 0,$$

so that (13) becomes:

$$\frac{\partial p_i^*}{\partial q_i} = \frac{\partial e_i^Z}{\partial q_i} + \frac{\partial s}{\partial q_i}\left[\mathbb{P}(I \geq a) + \tau\right]. \tag{17}$$

Thus, the marginal cost of risk, and the attendant allocation of capital, is completely determined by the gradient of the binding regulatory constraint. This is the world of Denault, Tasche, and others involved in the development of the gradient allocation principle. In this world, the marginal cost of risk is indeed completely determined by an arbitrarily chosen risk measure.

No Deposit Insurance and Non-Binding Regulation

At the opposite extreme is the case of an unregulated market with no deposit insurance. Here, $\xi = 0$, so (cf. Eq. (10)):

$$\sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'} = \left[\mathbb{P}(I \geq a) + \tau\right],$$

meaning that (13) becomes:

$$\frac{\partial p_i^*}{\partial q_i} = \frac{\partial e_i^Z}{\partial q_i} + \tilde{\phi}_i \times a \times \left[\mathbb{P}(I \geq a) + \tau\right]. \tag{18}$$

Thus, the marginal cost of risk and the attendant allocation of capital is driven completely by "internal" considerations. Specifically, (14) indicates that the allocation is driven by the time-zero value that affected consumers place on their contingent claims on recoveries in the various states of default.

General Case: Uninsured Consumers and Binding Regulation

In general, we may imagine the case where both of the considerations isolated above—an "external" regulatory constraint, and "internal" concerns driven by counterparty preferences—are influencing the marginal cost of risk. In this case, (13) remains in its original form:

$$\frac{\partial p_i^*}{\partial q_i} = \frac{\partial e_i^Z}{\partial q_i} + \frac{\partial s}{\partial q_i}\left[\mathbb{P}(I \geq a) + \tau - \sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'}\right] + \tilde{\phi}_i \times a \times \left[\sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'}\right], \tag{19}$$

but we are now able to see more clearly the two influences on capital allocation. When the regulatory constraint binds, we know that:

$$\mathbb{P}(I \geq a) + \tau > \sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'},$$

with the interpretation that regulation is forcing the institution to hold assets beyond the level that would be privately efficient from the perspective of serving its counterparties. However, the extent of this distortion is key to identifying whether internal counterparty concerns or external regulatory concerns guide capital allocation. If regulation comes close to replicating the private market outcome:

$$\mathbb{P}(I \geq a) + \tau \approx \sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'},$$

then the second term in (13) will be unimportant relative to the third term, and internal counterparty concerns will dominate. On the other hand, if regulation has the effect of pushing institutional capitalization well beyond the level that would prevail in the private market:

$$\mathbb{P}(I \geq a) + \tau \gg \sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'},$$

then the second term in (13) will overshadow the third term, and external regulatory concerns will dominate.

2.2 Allocation in a Security Market Equilibrium

To keep the setup simple, we limit our considerations to a one-period market with a finite number of securities (M), each security with potentially distinct payoffs in X states and assume that the risk-free rate is zero. More specifically, let $\Omega^{(S)} = \{\omega_1^{(S)}, \ldots, \omega_X^{(S)}\}$ be the set of these states with associated sigma-algebra $\mathcal{F}^{(S)}$ given by its power set and let $p_j^{(S)} = \mathbb{P}\left(\{\omega_j^{(S)}\}\right)$ denote the associated (physical) probabilities. Let then D be the $M \times X$ matrix with D_{ij} describing the payoff of the i^{th} security in state $\omega_j^{(S)}$, where we assume:

$$\text{span}(D) = \mathbb{R}^X.$$

This condition allows us to define unique state prices, consistent with the absence of arbitrage within the securities market, denoted by $\pi_j,\ j = 1, \ldots, X$. Thus, any arbitrary menu of securities-market-sub-state-contingent consumption can be purchased at time zero. However, it would be misleading to characterize markets as complete, since $\Omega^{(S)}$ does not provide a complete description of the "states of the world." Instead, we characterize the full probability space as $\left(\bar{\Omega}, \bar{\mathcal{F}}, \mathbb{P}\right)$, with:

$$\begin{aligned}
\bar{\Omega} &= \Omega^{(S)} \times \Omega = \left\{\bar{\omega} = (\omega^{(S)}, \omega) \middle| \omega^{(S)} \in \Omega^{(S)},\ \omega \in \Omega\right\}, \\
\bar{\mathcal{F}} &= \mathcal{F}^{(S)} \vee \mathcal{F}, \text{ and} \\
\mathbb{P}\left(\bar{A}\right) &= \sum_{j \in \Upsilon_A} p_j^{(S)} \times \mathbb{P}\left(A_j \middle| \{\omega_j^{(S)}\}\right)
\end{aligned}$$

for $\bar{A} = \bigcup_{j \in \Upsilon_A} \{\omega_j^{(S)}\} \times A_j \in \bar{F}$ with $A_j \in \mathcal{F}, j = 1, 2, \ldots, |\Upsilon_A|$.

Our problem now, however, is that the market is no longer complete so that we need a notion of what insurance liabilities are "worth" to the insurer when they cannot be hedged completely. We make the assumption that the insurance market is "small" relative to the securities market and, for purposes of valuing insurance liabilities, employ the so-called *minimal martingale measure* (cf. Föllmer and Schweizer (2010))

$$\mathbb{Q}\left(\bar{A}\right) = \sum_{j \in \Upsilon_A} \pi_j \times \mathbb{P}\left(A_j \,\middle|\, \{\omega_j^{(S)}\}\right), \; \bar{A} \subseteq \bar{\Omega},$$

i.e. $\mathbb{Q}$ is defined by the Radon-Nikodym derivative $\frac{\partial \mathbb{Q}}{\partial \mathbb{P}}((\omega_j^{(S)}, \omega)) = \frac{\pi_j}{p_j^{(S)}}$.

Consumer utility now depends on the individual's chosen security market allocation and may be expressed as:

$$v_i = \mathbb{E}^{\mathbb{P}}\left[U_i\left(W_i - p_i - L_i + R_i\right)\right] \text{ with } v_i' = \mathbb{E}^{\mathbb{P}}\left[U_i'\left(W_i - p_i - L_i + R_i\right)\right],$$

where W_i is $\mathcal{F}^{(S)}$-measurable with $\mathrm{w}_{ij} = W_i(\omega_j^{(S)})$ and $\sum_j \pi_j \, \mathrm{w}_{ij} = \mathrm{w}_i$ whereas L_i is $\bar{\mathcal{F}}$-measurable. The recovery R_i now depends both on insurance loss activity as well as portfolio decisions made within the insurance company. To elaborate on this, the budget constraint of the insurance company may be expressed as:

$$a = \sum_j \pi_j \, K_j \, a \Rightarrow 1 = \sum_j \pi_j \, K_j,$$

where $K_j \, a$ reflects consumption purchased in the securities market state $\omega_j^{(S)}$ or—more precisely—in the states of the world $\bar{\Omega}_j = \left\{\bar{\omega} = (\omega^{(S)}, \omega) \middle| \, \omega^{(S)} = \omega_j^{(S)}\right\}$. We write K to denote the corresponding $\mathcal{F}^{(S)}$-measurable random variable. Consumer i's recovery can then be expressed as:

$$R_i = \min\left\{I_i, \frac{K\,a}{I}\,I_i\right\},$$

and the fair valuation of claims is thus:

$$e_i = \mathbb{E}^{\mathbb{Q}}\left[R_i\right] = \underbrace{\mathbb{E}^{\mathbb{Q}}\left[R_i \, \mathbf{1}_{\{I < K\,a\}}\right]}_{=e_i^Z} + \underbrace{\mathbb{E}^{\mathbb{Q}}\left[R_i \, \mathbf{1}_{\{I \geq K\,a\}}\right]}_{=e_i^D}.$$

Analogously to before, we can now derive the capital allocation according to the company's marginal cost by working through its optimization problem (see Appendix B for details). We obtain a very similar allocation result as in the previous section: The cost of capital:

$$\left[\mathbb{E}^{\mathbb{Q}}\left[K\,a\,\mathbf{1}_{\{I \geq K\,a\}}\right] + \tau\,a\right],$$

which now naturally reflects state prices and the allocations to security market states K due to the company's asset allocation in the derivation of the default value, can be decomposed according to the marginal costs for each of the individual exposures as:

$$\left[\mathbb{E}^{\mathbb{Q}}\left[K\,a\,\mathbf{1}_{\{I \geq K\,a\}}\right] + \tau\,a\right] = \sum_i \frac{\partial s}{\partial q_i}\, q_i \left[\mathbb{E}^{\mathbb{Q}}\left[K\,\mathbf{1}_{\{I \geq K\,a\}}\right] + \tau - \sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'}\right] + \sum_i \tilde{\phi}_i\, q_i\, a \times \left[\sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'}\right], \tag{20}$$

where:

$$\tilde{\phi}_i = \frac{\mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{I \geq K\,a\}} \sum_k \frac{U'_k}{\mathbb{E}^{\mathbb{P}}[U'_k|\omega^{(S)}]} \frac{K}{I^2} I_k \frac{\partial I_i}{\partial q_i}\right]}{\mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{I \geq K\,a\}} \sum_k \frac{U'_k}{\mathbb{E}^{\mathbb{P}}[U'_k|\omega^{(S)}]} K \frac{I_k}{I}\right]} = \frac{\sum_j \pi_j\, K_j\, \mathbb{E}^{\mathbb{P}}\left[\mathbf{1}_{\{I \geq K_j\,a\}} \sum_k \frac{U'_k}{\mathbb{E}^{\mathbb{P}}[U'_k|\omega_j^{(S)}]} \frac{I_k}{I} \frac{\frac{\partial I_i}{\partial q_i}}{I} \middle| \omega_j^{(S)}\right]}{\sum_j \pi_j\, K_j\, \mathbb{E}^{\mathbb{P}}\left[\mathbf{1}_{\{I \geq K_j\,a\}} \sum_k \frac{U'_k}{\mathbb{E}^{\mathbb{P}}[U'_k|\omega_j^{(S)}]} \frac{I_k}{I} \middle| \omega_j^{(S)}\right]}. \tag{21}$$

Thus, we essentially have the same result as before, although the "internal" allocation rule now only applies in every "branch" of the security market where the incompleteness becomes material. In particular, after adjusting for state prices by conditioning on each "branch," capital allocation weights are still determined by consumer marginal utility.

However, this is no longer true in the limiting case of a complete market, i.e. the case when L_i and R_i are $\mathcal{F}^{(S)}$-measurable so that we can write $l_{ij} = L_i(\omega_j^{(S)})$ and $r_{ij} = R_i(\omega_j^{(S)})$, $l_{ij}, r_{ij} \in \mathbb{R}$. Here we obtain:

$$\tilde{\phi}_i = \frac{\mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{I \geq K\,a\}} \sum_k \frac{K}{I^2} I_k \frac{\partial I_i}{\partial q_i}\right]}{\mathbb{E}^{\mathbb{Q}}\left[K\,\mathbf{1}_{\{I \geq K\,a\}}\right]} = \frac{\mathbb{E}^{\mathbb{Q}}\left[K \frac{\frac{\partial I_i}{\partial q_i}}{I} \middle| I \geq K\,a\right]}{\mathbb{E}^{\mathbb{Q}}\left[K \middle| I \geq K\,a\right]}, \tag{22}$$

so that:

$$q_i \times \tilde{\phi}_i \times \mathbb{E}^{\mathbb{Q}}\left[K\,a\,\mathbf{1}_{\{I \geq K\,a\}}\right] = \mathbb{E}^{\mathbb{Q}}\left[\frac{K\,a}{I} I_i\,\mathbf{1}_{\{I \geq K\,a\}}\right]$$

is the fair price of the recovery. This is exactly the same allocation result as in Ibragimov et al. (2010), who extend the results from Myers and Read (2001) to the pro-rata sharing rule in the case of default we apply here (cf. Eq. (2)). It is important to note, however, that in the complete market case, purchasing protection from an insurance company with costly capital is unnecessary since consumers can hedge insurance risk themselves—so that here allocating capital may in fact be "inappropriate" (cf. Phillips et al. (1998)).[7]

2.3 A Multi-Period Version of the Model

In this section, we consider a generalization of the (one-period) setup to multiple periods. Let L_i^t denote the loss incurred by consumer i, $i \in \{1, 2, \ldots, N\}$, in period t, $t \in \{1, 2, \ldots\}$. We assume that L_i^t, $t > 0$—for fixed i—are independent and identically distributed, and we define the relevant filtration $\mathbf{F} = (\mathcal{F}_t)_{t \geq 0}$ via $\mathcal{F}_t = \sigma(L_i^s,\, i \in \{1, 2, \ldots, N\},\, s \leq t)$. The firm determines the optimal level of assets, a_t, in the beginning of each period (i.e. (a_t) is $\mathbf{F}$-predictable) for a period cost of $\tau \times a_t$. Similarly to before, the company chooses $\mathbf{F}$-predictable amounts q_t^i in $I_i^t = I(L_i^t, q_i^t) = q_i^t \times L_i^t$ and prices p_i^t at the beginning of the period, and we denote the total claims by $I^t = \sum_{j=1}^N I_j^t$.

Now the company defaults if $I^t > a_t$, so that the recovery paid to each consumer is $R_i^t = \min\{I_i^t, \frac{a_t}{I^t} I_i^t\}$ and the company shuts down in case of default, i.e. shareholder do not have access

[7] Ibragimov et al. (2010) deal with this by assuming "the insurees do not have direct access to the market for risk" whereas the insurer faces a "friction-free complete market for risk." In contrast, we explicitly allow for an incomplete market for risk where the insurer has an explicit risk-sharing function rather than solely intermediating between the insurees and the market for risk so that holding—and thus allocating—excess capital become material.

to future profits. The consumer's utility in period t is given by:

$$v_i^t(a_t, w_i^t - p_i^t, q_1^t, \ldots, q_N^t) = \mathbb{E}_{t-1}\left[U_i(w_i^t - p_i^t - L_i^t + R_i^t)\right],$$

where for simplicity we assume that wealth is homogeneous across periods, i.e. $w_i^t \equiv w_i$.[8]

The company solves:

$$\max_{\{a_t\},\{q_i^t\},\{p_i^t\}} V_0 = \sum_{t=1}^{\infty} \mathbb{E}\left[\mathbf{1}_{\{I^1 \le a_1, \ldots, I^{t-1} \le a_{t-1}\}} \times \left(\sum_i p_i^t - \sum_i \mathbb{E}_{t-1}\left[R_i^t\right] - \tau\, a_t\right)\right] \tag{23}$$

with constraints:

$$v_i^t(a_t, w_i^t - p_i^t, q_1^t, \ldots, q_N^t) \;\ge\; \gamma_i \;\forall i,\ \forall t, \tag{24}$$

$$s(q_1^t, \ldots, q_N^t) \;\le\; a_t \;\forall t. \tag{25}$$

Under the assumptions above, it is clear that there exists an optimal stationary policy:

$$(a_t, \{q_i^t\}, \{p_i^t\}) \equiv (a^*, \{q_i^*\}, \{p_i^*\})$$

that solves the Bellman equation:

$$V = \max_{a,\{q_i\},\{p_i\}} \sum_i p_i - \sum_i \underbrace{\mathbb{E}[R_i^t]}_{e_i} - \tau\, a + \mathbb{P}[I^t \le a] \times V \tag{26}$$

under conditions (24) and (25). Hence, we have a similar program as in the basic setup from Section 2.1 before, where the primary difference is the last term in (26) involving the value function.

Proceeding analogously to before (see Appendix C for details), we obtain the following marginal pricing condition:

$$\frac{\partial p_i^*}{\partial q_i} = \mathbb{E}\left[\frac{\partial I_i(L_i^t, q_i)}{\partial q_i} 1_{\{I^t \le a\}}\right] + V\, f_I(a)\, \underbrace{\mathbb{E}\left[\left.\frac{\partial I_i(L_i^t, q_i)}{\partial q_i}\right| I^t = a\right]}_{=\tilde{\theta}_i} \tag{27}$$

$$+ \left[\mathbb{P}\left(I^t > a\right) + \tau - \sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'} - V\, f_I(a)\right] \frac{\partial s}{\partial q_i}$$

$$+ \left[\sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'}\right] \times a \times \underbrace{\left[\frac{\mathbb{E}\left[\left(1_{\{I^t > a\}} \sum_k \frac{U_k'}{v_k'} \frac{I_k^t}{I^t}\right) \frac{\frac{\partial I_i^t}{\partial q_i}}{I^t}\right]}{\mathbb{E}\left[1_{\{I^t > a} \sum_k \frac{U_k'}{v_k'} \frac{I_k^t}{I^t}\right]}\right]}_{=\tilde{\phi}_i}. \tag{28}$$

[8]Formally, the consumers will form utilities over consumption in multiple periods. In particular, future (random) losses will also be material. Thus, here U should rather be interpreted as a value function (of end-of-period wealth) than as a utility function (of end-of-period consumption).

Thus, akin to the previous sections, again (28) implies an allocation of capital that "adds up" to the cost of capital:

$$\left[\mathbb{P}(I^t \geq a)\, a + \tau\, a\right] = \sum_i \tilde{\theta}_i\, q_i \times [V\, f_I(a)] + \sum_i \frac{\partial s}{\partial q_i}\, q_i \times \left[\mathbb{P}(I^t \geq a) + \tau - \sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'} - V\, f_I(a)\right]$$
$$+ \sum_i \tilde{\phi}\, q_i\, a \times \left[\sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'}\right].$$

In addition to the "external" ($\frac{\partial s}{\partial q_i}$) and "internal counter-party" ($\tilde{\phi}_i$) allocation rules from before, the allocation now features a third term—$\tilde{\theta}_i$—that is associated with the firm's value as a going concern. In order to obtain insights on the provenance of the corresponding weights, it again is helpful to consider a few specific situations.

The Limiting Case of Perfect Competition

In the limiting case of perfect (Bertrand) competition, the firm value V approaches zero, and, thus, so does the weight associated with the "going concern" allocation $\tilde{\theta}_i$. Hence, the limiting allocation is

$$\frac{\partial p_i^*}{\partial q_i} = \frac{\partial e_i^Z}{\partial q_i} + \frac{\partial s}{\partial q_i} \times \left[\mathbb{P}\left(I^t \geq a\right) + \tau - \sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'}\right] + \tilde{\phi}_i \times a \times \left[\sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'}\right], \tag{29}$$

i.e. we obtain the same allocation as in the single-period model (13). As before, we could now further break down this setting by distinguishing insured and uninsured consumers and binding and non-binding regulation to obtain allocations that are fully determined by "external" and "internal counterparty" considerations, respectively. In particular, in this case the remaining two weights in (29) adhere to the same interpretation as in the single period setting.

Monopolistic Competition, Full Coverage by Deposit Insurance, and Non-Binding Regulation

In this case, again consumers are indifferent to the capitalization of the firm and there is no external solvency constraint, so that the level—and the allocation—of firm capital is solely determined by the firm's value as a going concern:

$$\begin{aligned}\frac{\partial p_i^*}{\partial q_i} &= \mathbb{E}\left[\frac{\partial I_i(L_i^t, q_i)}{\partial q_i}\, 1_{\{I^t \leq a\}}\right] + V\, f_I(a)\, \mathbb{E}\left[\frac{\partial I_i(L_i^t, q_i)}{\partial q_i}\middle| I^t = a\right] \\ &= \mathbb{E}\left[\frac{\partial I_i(L_i^t, q_i)}{\partial q_i}\, 1_{\{I^t \leq a\}}\right] + \left[\mathbb{P}\left(I^t > a\right) + \tau\right] \times \mathbb{E}\left[\frac{\partial I_i(L_i^t, q_i)}{\partial q_i}\middle| I^t = a\right],\end{aligned} \tag{30}$$

where the latter equality follows from the first order condition for a in the absence of constraints (see Eq. (51) in Appendix C).

General Case: Monopolistic Competition, Uninsured Consumers, and Binding Regulation

Here we obtain (28), but we are able to see more clearly the—now—three influences on capital allocation. The latter two are exactly the same as before—with their relative importance determined by how close the regulatory requirement is to the capitalization level chosen by the consumers in an unregulated market. The relative importance of the "new" term $\tilde{\theta}_i$ that derives from the firm's value as a going concern depends on how different the capitalization would be if regulatory and consumer concerns were immaterial. In particular, if consumer and shareholder considerations in a private (unregulated) market would choose as similar capitalization as imposed by regulation:

$$\mathbb{P}(I^t \geq a) + \tau \approx \sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'} + V\, f_I(a),$$

then internal (counterparty and shareholder) concerns will dominate whereas regulatory concerns will be the key driver otherwise.

Inspecting the form of the regulatory and the shareholder-driven allocation, they are reminiscent of "conventional" allocation methods based on the gradients of risk measures. The next section elaborates on these relationships in more detail.

3 Capital Allocation and Risk Measures

Having solved the institutions optimization problem and—as a by-product—the ensuing allocation of costs to the risks within its portfolio, we could very well terminate the discussion. However, this "black board solution" offers little obvious consolation to a practitioner faced with the problem of allocating capital in a real-world setting, who may deem any attempt that relies on an explicit formalization of the firm's centralized optimization problem quixotic. This may be the reason why much of the applied risk and actuarial research has gravitated to allocation methods based on risk measures, particularly the so-called Euler or Gradient allocation. Yet—despite the large amount of papers written on the subject—there has been little scientific scrutiny on the practice of allocating capital based on risk measures and, particularly, little academic guidance on how to choose said risk measures.

This section aligns our results with those obtained from the Euler method. We first formally introduce the Euler allocation principle. We then derive the risk measures which—if the Euler method were applied to them—would yield our allocation results. Hence, we are providing theoretical guidance on the delicate issue of risk measure selection.

3.1 The Euler Allocation Principle

Although it is not always derived in this way, the Euler method is implied by the maximization of profits subject to a risk measure constraint. To illustrate, assume a company's profit function Π depends on the parameters (volumes) q_i, $1 \leq i \leq N$, and on capital a. Then maximizing profits subject to the *risk measure constraint*:

$$\rho\left(q_1, q_2, \ldots, q_N\right)\, \chi_\rho \leq a$$

yields:

$$\frac{\partial \Pi}{\partial q_i} = \left(-\frac{\partial \Pi}{\partial a}\right) \chi_\rho \frac{\partial \rho}{\partial q_i}$$

at the optimal values $a^*, q_i^*, 1 \leq i \leq N$. Here ρ is a suitable differentiable risk measure evaluated at the aggregate claims $\sum_{j=1}^N I_j(L_j, q_j)$ and χ_ρ is an exchange rate that converts risk to capital, which is often chosen to be unity if risk is measured in monetary units—i.e. in the case of a *monetary risk measure*. Hence, for the optimal portfolio, the risk-adjusted marginal return on marginal capital $\frac{\partial \Pi/\partial q_i}{\chi_\rho \times \partial \rho/\partial q_i}$ for each exposure is the same and equals the cost of a marginal unit of capital $-\frac{\partial \Pi}{\partial a}$. In other words, if the marginal performance of risk i as measured by its marginal return on marginal risk capital $\chi_\rho \times \frac{\partial \rho}{\partial q_i}$ exceeds (respectively, falls below) the cost of a marginal unit of capital, then increasing (respectively, decreasing) the weight q_i of that exposure by a small amount improves the overall performance of the portfolio.[9] Akin to the results of Tasche (2004) on the *suitability* of capital allocation principles, this motivates the interpretation of the marginal capital weighted by the corresponding volume $\chi_\rho \frac{\partial \rho}{\partial q_i} q_i^*$ as the amount of capital allocated to exposure i. In particular, for homogenous risks and risk measure, the allocations to the respective risks "add up" to the entire capital:

$$\sum_{j=1}^N \chi_\rho \frac{\partial \rho}{\partial q_j} q_j^* \quad = \quad a^*.$$

We refer to McNeil et al. (2005) and references therein for more details on the Euler principle, and to Denault (2001), Kalkbrener (2005), and Myers and Read (2001) for alternative derivations of the Euler principle based on cooperative game theory, formal axioms, or a contingent claim approach, respectively. Regardless of its provenance, the Euler methods strength is the feasibility of implementation as it is only necessary to calculate the partial derivatives of a *given* risk measure with respect to each exposure evaluated at the current portfolio.

3.2 Allocation Based on Conventional Risk Measures

As already indicated in the previous section, our allocation result collapses to this approach—based on the *given* risk measure s—in specific institutional circumstances when consumers are fully insured by deposit insurance (and close to perfect competition in the multi-period context):

$$\sum_{j=1}^N \frac{\partial s}{\partial q_j} q_j \left[\mathbb{P}(I \geq a) + \tau\right] = a \left[\mathbb{P}(I \geq a) + \tau\right]. \tag{31}$$

This is rather unsurprising as our model setting fits the framework above with our profit function from the previous subsection:

$$\begin{cases} \max_{a,\{q_i\}} \Pi\left(q_1, \ldots, q_N, a\right) = \sum_k p_k^*(q_k) - \sum_k e_k(q_1, \ldots, q_N, a) - \tau\, a, \\ s(q_1, \ldots, q_N) \leq a. \end{cases} \tag{32}$$

It is important to note, however, that this optimization problem *not equivalent* to the firm's formal optimization problem (6)-(8). In particular, the latter also entails the choice of the optimal premium

[9] Cf. Section 6.3.3 "Economic Justification of the Euler Principle" in McNeil et al. (2005).

functions p_i^*, which are embedded in (32) and require a solution of the original problem. Rather, the new problem (32) is to be understood as an auxiliary problem that—when fed with the optimal premium function from the original optimization problem—yields the same partial solution as the original problem and the same marginal costs at the optimum. The idea here is that if optimization within the firm is an inveterate process that is hard to trace, the simpler problem (32) can be used resulting in the easy-to-implement capital allocation (31). Yet, although this result reconfirms the Euler allocation as appropriate in specific institutional circumstances, it offers no guidance on how to choose s. Moreover, it is unclear if the Euler approach can be reconciled with the economic allocations under the other institutional arrangements discussed in Section 2.

One of the primary outcomes of the multi-period version of our model is that the allocation according to marginal costs in (28)—in contrast to the single period model—features terms:

$$\tilde{\psi}_i = \mathbb{E}\left[\left.\frac{\partial I_i(L_i, q_i)}{\partial q_i}\right| I = a\right],\ i = 1, \ldots, N.$$

Now $\tilde{\psi}_i$ is exactly the derivative of Value-at-Risk (VaR) with confidence level (cf. Property 1 on Page 229 in Gourieroux et al. (2000)):

$$\tilde{\alpha} = \mathbb{P}(I \leq a), \tag{33}$$

i.e. the allocation coincides with an Euler-allocation based on $\rho = \text{VaR}_{\tilde{\alpha}}$. Thus, in a regime with full deposit insurance and absent a binding regulatory requirements, a monopolist or a company operating in a monopolistic competition allocating capital according to marginal costs should choose Euler based on VaR:

$$\tilde{\psi}_i = \frac{\partial}{\partial q_i} \text{VaR}_{\tilde{\alpha}}(I).$$

Thats exactly the case (30) from Section 2.3.

Although this only holds in rather specific institutional circumstances, it is more than a curiosity since, on the one hand, deposit insurance is prevalent in most developed banking and (primary) insurance markets[10] and, on the other hand, regulatory capital requirements—though common in intermediary markets—frequently do not bind, i.e. solvency ratios exceed the required level (see e.g. Hanif et al. (2010)). Thus, the widespread use of VaR may not come as a surprise.

Furthermore, an Euler allocation based on VaR would of course also give the correct marginal costs under full deposit insurance if the regulatory requirement binds but is evaluated based on VaR, i.e. if VaR is externally supplied risk measure. Whether this is sufficient justification for imposing VaR when setting solvency requirements—as is the case for important examples such as Basel II for banks or Solvency II for insurance companies—is debatable. More precisely, while allocations based on VaR are “economically correct” from the firm’s perspective (i.e. they give the correct marginal cost), this is not necessarily true from the perspective of the society or the regulator. And as the regulator takes the responsibility in case of a shortfall due to deposit insurance, the private

[10]For example in the US, as of January 2012, the Federal Deposit Insurance Corporation (FDIC) insures bank deposits up to $250,000 per depositor per bank whereas for insurance the coverage of guarantee funds varies across lines of insurance and across the fifty states—although the coverage limits considerably exceeds average face values. For instance, according to the *Life & Health Insurance Guaranty Association System* “limits are established by state law and can vary from state to state, but most states provide at least $300,000 in life insurance death benefits” and “$100,000 in health insurance policy benefits.”

market outcome is already disturbed so that invoking basic equilibrium results in a free market environment is not possible. In this context, naturally the question arises how allocations would be set in such a free market and whether they also adhere to the Euler principle. The next section provides the answer.

3.3 A New Risk Measure for Allocating Capital

Consider now a company in a regime without (full) coverage by deposit insurance (but still assume close to perfect competition in the multi-period context). Then, capitalization will become material to consumers—which mathematically corresponds to the $\tilde{\phi}_i$'s emerging in the marginal cost allocations (13)/(28), where:

$$\tilde{\phi}_i = \frac{\mathbb{E}\left[\mathbf{1}_{\{I \geq a\}} \sum_k \frac{U_k}{v_k} \frac{I_k}{I} \frac{\frac{\partial I_i}{\partial q_i}}{I}\right]}{\mathbb{E}\left[\mathbf{1}_{\{I \geq a\}} \sum_k \frac{U_k}{v_k} \frac{I_k}{I}\right]}.$$

To align this allocation with the Euler method, introduce the probability measure $\tilde{\mathbb{P}}$ on $(\Omega, \mathcal{F})$ via its Radon-Nikodym derivative:

$$\frac{\partial \tilde{\mathbb{P}}}{\partial \mathbb{P}} = \frac{\sum_k \frac{U_k'}{v_k'} \frac{I_k}{I} \mathbf{1}_{\{I \geq a^*\}}}{\mathbb{E}\left[\sum_k \frac{U_k'}{v_k'} \frac{I_k}{I} \mathbf{1}_{\{I \geq a^*\}}\right]},$$

where I, U, etc. are evaluated at the concurrent (optimal) values a^*, $p_i^*(q_i^*)$, q_i^*, $1 \leq i \leq N$. Note that $\tilde{\mathbb{P}}$ is absolutely continuous with respect to the original probability measure $\mathbb{P}$ but the measures are not equivalent since under $\tilde{\mathbb{P}}$ all the probability mass is concentrated in default states. On the set of strictly positive $\tilde{\mathbb{P}}$-square integrable random variables:

$$L_+^2\left(\Omega, \mathcal{F}, \tilde{\mathbb{P}}\right) = \left\{X \in L^2\left(\Omega, \mathcal{F}, \tilde{\mathbb{P}}\right) | X > 0 \ \tilde{\mathbb{P}}\text{-a.s.}\right\},$$

we define the risk measure:[11]

$$\tilde{\rho}(X) = \exp\left\{\mathbb{E}^{\tilde{\mathbb{P}}}\left[\log\{X\}\right]\right\}.$$

Obviously $\tilde{\rho}$ is monotonic ($\tilde{\rho}(X) \leq \tilde{\rho}(Y)$ for $X \leq Y$ a.s.), positive homogenous ($\tilde{\rho}(a\,X) = a\,\tilde{\rho}(X)$, $a > 0$), and it satisfies the constancy condition $\tilde{\rho}(c) = c$ for $c > 0$ (see Frittelli and Gianin (2002) for a discussion of properties of risk measures). However, $\tilde{\rho}$ is neither translation-invariant nor sub-additive, and is therefore *not coherent* and *not convex*. In fact, $\tilde{\rho}$ is not even a *monetary risk measure* and thus may not qualify for the use as an external risk measure.[12] However, it is the correct risk measure for internal capital allocation based on the Euler principle.

[11] While its functional form seems similar to that of the so-called *entropic risk measure* (which has recently gained popularity in the mathematical finance literature (see e.g. Föllmer and Schied (2002) or Detlefsen and Scandolo (2005))), note that the roles of the exponential function and the logarithm are interchanged.

[12] While sub-additivity is subject of an ongoing debate (see e.g. Dhaene et al. (2008) or Kou et al. (2012)), at least translation invariance is generally deemed adequate for an *external risk measure* and even "*necessary for the risk-capital interpretation* [...] *to make sense*" (see p. 239 in McNeil et al. (2005)).

More precisely, define $\tilde{\chi}_\rho = \frac{a^*}{\tilde{\rho}\left(\sum_{j=1}^N I_j(L_j, q_j^*)\right)}$ as the "exchange rate" between units of risk and capital. Then the application of the Euler principle with profit function $\Pi\left(q_1, \ldots, q_N, a\right)$ as in (32) and risk measure constraint:

$$\underbrace{\tilde{\rho}\left(\sum_{j=1}^N I_j(L_j, q_j)\right)}_{=\tilde{\rho}(q_1,\ldots,q_N)} \tilde{\chi}_\rho \leq a, \tag{34}$$

yields:

$$\begin{aligned} \sum_{j=1}^N \tilde{\chi}_\rho \frac{\partial \tilde{\rho}}{\partial q_j} q_j^* \left[\mathbb{P}(I \geq a^*) + \tau\right] &= \sum_{j=1}^N \tilde{\chi}_\rho \, \mathbb{E}^{\tilde{\mathbb{P}}}\left[\frac{\partial I_j / \partial q_j}{I}\right] \tilde{\rho} \, q_j^* \left[\mathbb{P}(I \geq a^*) + \tau\right] \\ &= \sum_{j=1}^N \tilde{\phi}_j \, q_j^* \, a^* \left[\mathbb{P}(I \geq a^*) + \tau\right] = a^* \left[\mathbb{P}(I \geq a^*) + \tau\right], \end{aligned} \tag{35}$$

which is exactly the allocation (18) derived in the previous section. While it may appear aberrant to make use of the optimal levels a^* and q_i^* in the definition of $\tilde{\chi}_\rho$ and the Radon-Nikodym derivative $\frac{\partial \tilde{\mathbb{P}}}{\partial \mathbb{P}}$, note that this is akin to employing the optimal premium functions p^* for deriving the Euler allocation based on the externally supplied risk measure in the previous Subsection 3.2. Just like there, optimizing Π subject to the risk measure constraint (34) is to be understood as simple auxiliary problem that circumvents the formalization of the firm's (innate) full centralized optimization problem but yields the same partial solution and—particularly—the same capital allocation according to marginal costs. Moreover, by inserting the definition of $\tilde{\chi}_\rho$, we obtain a potentially less peculiar and more familiar representation of the amount of capital allocated to consumer i (cf. Schmock and Straumann (1999)):

$$a^* \times q_i^* \times \frac{\frac{\partial \tilde{\rho}}{\partial q_i}(q_1^*, \ldots, q_N^*)}{\tilde{\rho}(q_1^*, \ldots, q_N^*)}.$$

Hence, the correct internal capital allocation according to marginal cost can be implemented by the Euler principle relying on the risk measure $\tilde{\rho}$—a risk measure that, surprisingly, is neither coherent nor convex.

To further illustrate the properties of this risk measure and particularly its relation to conventional risk measures, evaluate $\tilde{\rho}$ as a function of the aggregate loss $I = \sum_{j=1}^N I_j(L_j, q_j^*)$—which of course is its primary purpose in our context:

$$\begin{aligned} \tilde{\rho}(I) &= \exp\left\{\mathbb{E}^{\tilde{\mathbb{P}}}\left[\log\{I\}\right]\right\} = \exp\left\{\mathbb{E}\left[\frac{\partial \tilde{\mathbb{P}}}{\partial \mathbb{P}} \log\{I\}\right]\right\} = \exp\left\{\mathbb{E}\left[\mathbb{E}\left[\left.\frac{\partial \tilde{\mathbb{P}}}{\partial \mathbb{P}}\right| I\right] \log\{I\}\right]\right\} \\ &= \exp\left\{\mathbb{E}\left[\left.\mathbb{E}\left[\left.\frac{\sum_k \frac{U_k'}{v_k'} \frac{I_k}{I} \mathbf{1}_{\{I \geq a^*\}}}{\mathbb{E}\left[\left.\sum_k \frac{U_k'}{v_k'} \frac{I_k}{I}\right| I \geq a^*\right]}\right| I\right] \log\{I\}\right| I \geq a^*\right]\right\} \\ &= \exp\left\{\mathbb{E}\left[\left.\tilde{\psi}(I) \, \log\{I\}\right| I \geq a^*\right]\right\}. \end{aligned} \tag{36}$$

Thus, $\tilde{\rho}$ in this sense is in fact a tail risk measure, and hence is related to Expected Shortfall (ES). Here, the weights $\tilde{\psi}(\cdot)$ perform a role similar to the *risk spectrum* within the so-called *spectral risk* *measures* as introduced in Acerbi (2002). According to the author, the "*subjective risk aversion of an investor can be encoded*" in this function, which may justify overweighting bad outcomes, although he does not provide guidance on how to choose an explicit form. In contrast, in our setting the weights represent an adjustment to objective probabilities based on the value placed by claimants on recoveries in various states of default. Thus, the pivotal characteristics for our weights lie in the primitives of the firm's profit maximization problem (namely, the preferences of counterparties)—which ultimately determine the overall choice of capitalization as well as the values consumers place on state contingent recoveries—rather than in a subjectively specified concave preference function for the firm, which will generally fail to capture limited liability.

In the absence of weights, however, it is worth noting that the concavity of the logarithmic function will, in the course of the application of the Euler allocation methods, tend to penalize bad outcomes *less* heavily than ES. In fact, it is evident from (36) that the Euler method will effectively weight all aggregate loss outcomes in excess of the firm's capital equally, regardless of size, when $\tilde{\psi}(\cdot) \equiv 1$. The reason for this is that $\tilde{\psi}(\cdot) \equiv 1$ implies that the firm's counterparties are risk-neutral, and, thus, the value of the firm in all states of default, regardless of how extreme the default, is simply the firm's assets. At the margin, the counterparties evaluate changes in risk simply from the perspective of how the expected value of recoveries from the firm are affected, and recoveries in mild states of default are weighted no differently from severe ones. This is also the reason why $\tilde{\rho}$ is not sub-additive or translation-invariant: Adding a constant in high loss states is less precarious than in low loss states because of limited liability. Under risk aversion, on the other hand, $\tilde{\psi}(\cdot) \neq 1$, and counterparties may well weight recoveries under severe states of default more heavily than mild ones.

In other institutional circumstances, i.e. with binding regulation and/or monopoly profits, in addition to the constraint associated with the new risk measure $\tilde{\rho}$, we have constraints associated with the continuation value (VaR) and with the external solvency constraint (s), i.e. we have multiple risk measure constraints. For instance, with binding regulation but uninsured consumers in the single period model—the third and general case from Section 2.1—we have:

$$\begin{cases} s(q_1, \ldots, q_N) \leq a, \\ \tilde{\rho}(q_1, \ldots, q_N)\, \tilde{\chi}_\rho \leq a. \end{cases}$$

Thus, the allocation is determined by the gradients of both risk measures as well as the associated Lagrange multipliers for s and $\tilde{\rho}$, which at the optimum $(q_1^*, \ldots, q_N^*, a^*)$ equal (cf. Equation (19)) $\left[\mathbb{P}(I \geq a) + \tau - \sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'}\right]$ and $\left[\sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'}\right]$, respectively. More specifically, we have

$$\begin{aligned} \sum_{j=1}^{N} \frac{\partial s}{\partial q_j}\, q_j^* \left[\mathbb{P}(I \geq a^*) + \tau - \sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'}\right] + \sum_{j=1}^{N} \tilde{\chi}_\rho \frac{\partial \tilde{\rho}}{\partial q_j}\, q_j^* \left[\sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'}\right] &= a^* \left[\mathbb{P}(I \geq a^*) + \tau\right] \\ \Rightarrow \quad \sum_{j=1}^{N} \frac{\partial}{\partial q_j} \left((1-\zeta)\, s + \zeta\, \tilde{\chi}_\rho\, \tilde{\rho}\right) q_j^* &= a^*, \end{aligned}$$

where the weight ζ is defined as

$$\zeta = \frac{\sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'}}{\mathbb{P}(I \geq a) + \tau}.$$

Hence, the Euler principle still applies, but the supporting risk measure is a weighted average between the external risk measure s and the internal risk measure $\tilde{\rho}$. The weights are determined by the consumers' marginal preference for capitalization of the company relative to their marginal utility of own wealth.

To determine which considerations dominate, it is therefore necessary to assess the consumers' proclivity for capitalization—unless of course s and $\tilde{\chi}_\rho\,\tilde{\rho}$ coincide, i.e. if $\tilde{\chi}_\rho\,\tilde{\rho}$ is the externally supplied risk measure. Similarly, a regulator providing full deposit insurance to avoid bank runs caused by informational inefficiencies that is interested in replicating the the first-best outcome in an efficient competitive market should impose $\tilde{\chi}_\rho\,\tilde{\rho}$. There are, of course, potential contentions on the use of $\tilde{\chi}_\rho\,\tilde{\rho}$ as an external risk measure. For instance, one may retort that the choice of $\tilde{\chi}_\rho$ is baseless and that fixing the weighting function associated with the probability measure $\tilde{\mathbb{P}}$ requires fixing a slew of parameters rendering the approach arbitrary. However, fixing the parameters of risk measures—such as the confidence levels of VaR or ES—is no less arbitrary than choosing $\tilde{\chi}_\rho$, and having various parameters corresponding to different lines of business may actually result in a more refined and better adapted answer than one-size-fits-all solutions. In particular, this may improve risk taking behavior as it is important to realize that binding regulatory requirements not only affect the level at which capital is set, but also how the firm will economize on said capital.

In any case, in order to gain insights on how the counterparty-driven internal allocation effectively differs from the allocation based on the external risk measure, it is sufficient to study risk measure $\tilde{\rho}$ and to compare ensuing allocations to conventional approaches. In particular, we are interested how the economic weight assigned to various outcomes under $\tilde{\rho}$ differs from what would be obtained from the use of more popular risk measures. The next section sheds more light on these issues by discussing several examples.

4 Comparison of Capital Allocation Methods

In this section, we consider the practical implications of allocating capital based on the method discussed in the previous sections. In particular, we compare the resulting allocations to those obtained when applying the Euler technique to Expected Shortfall—perhaps the most widely endorsed measure within the academic and practitioner community. We then illustrate in the context of several examples.

4.1 The Case of Exponential Losses

Assume that there are N identical consumers with wealth level w in a regime with non-binding regulation that face independent, Exponentially distributed losses $L_i \sim \text{Exp}(\nu),\ 1 \leq i \leq N$. Assume further that all consumers exhibit a constant absolute risk aversion of $\alpha < \nu$, and that their participation constraint is given by the autarky level

$$\gamma = \gamma_i = \mathbb{E}\left[U(w - L_i)\right] = -e^{-\alpha\,\mathrm{w}}\frac{\nu}{\nu - \alpha}.$$

Then, the optimization problem (6)/(7)/(8) may be written as

$$\begin{cases} \max_{a,q,p} \Big\{ N \times p - N \times q \times \Big[\frac{1}{\nu} \Gamma_{N-1,\nu}\left(\frac{a}{q}\right) - \frac{\nu^{N-1}}{(N-1)!} e^{-\nu \frac{a}{q}} \left(\frac{a}{q}\right)^{N-1} \left(\frac{1}{\nu} + \frac{1}{N}\frac{a}{q}\right) \Big] \\ \qquad -a \times \bar{\Gamma}_{N,\nu}\left(\frac{a}{q}\right) - \tau \times a \Big\} \\ \text{subject to} \\ \gamma \leq e^{-\alpha(\mathrm{w}-p)} \Big\{ \frac{\nu}{\nu-(1-q)\alpha} \Big[\Gamma_{N-1,\nu}\left(\frac{a}{q}\right) - \frac{e^{-\frac{a}{q}(\nu-(1-q)\alpha)} \nu^{N-1}}{((1-q)\alpha)^{N-1}} \Gamma_{N-1,(1-q)\alpha}\left(\frac{a}{q}\right) \Big] \\ \qquad + \sum_{k=0}^{\infty} \left(\frac{\alpha}{\nu}\right)^k \frac{(N-1)!}{(N-1+k)!} e^{-\nu a} \sum_{j=0}^{N-1} \frac{(a\,\nu)^j}{j!} \frac{(N-j+k-1)!}{(N-j-1)!} \bar{\Gamma}_{N-j+k,\nu}\left(a\left(\frac{1-q}{q}\right)\right) \Big\}, \end{cases} \tag{37}$$

where $\bar{\Gamma}_{m,b}(x) = 1 - \Gamma_{m,b}(x)$ and $\Gamma_{m,b}(\cdot)$ denotes the cumulative distribution function of the Gamma distribution with parameters m and b (see the Appendix D for the derivation of (37)).

For the allocation of capital to the individual consumers, we trivially obtain

$$q\,\tilde{\phi}_i = N^{-1},\ i = 1, 2, \ldots, N,$$

which is the same when applying the Euler technique with *any* risk measure. More specifically,

$$q\,\tilde{\phi}_i \overset{\text{Eq. (14)}}{=} \frac{\mathbb{E}\left[\mathbf{1}_{\{qL \geq a\}} \mathbb{E}\left[\sum_{j=1}^{N} U'\left(\mathrm{w} - p - L_j + a\frac{L_j}{L}\right) \frac{L_j}{L}\frac{L_i}{L} \Big| L\right]\right]}{\mathbb{E}\left[\mathbf{1}_{\{qL \geq a\}} \sum_{j=1}^{N} U'\left(\mathrm{w} - p - L_j + a\frac{L_j}{L}\right) \frac{L_j}{L}\right]} = \frac{1}{N}\mathbb{E}\left[\hat{\psi}(L) \Big| qL \geq a\right],$$

where $L = \sum_j L_j$,

$$\hat{\psi}(l) = \mathbf{1}_{\{ql \geq a\}}\, \hat{c}_{N,\nu,\alpha,a,q} \sum_{k=0}^{\infty} \frac{(k+1)\,(\alpha(l-a))^k}{(N+k)!}, \tag{38}$$

and $\hat{c}_{N,\nu,\alpha,a,q}$ is a constant ensuring that $\mathbb{E}\left[\hat{\psi}(L)\Big| qL \geq a\right] = 1$.

For the risk measure $\tilde{\rho}$, we have

$$\tilde{\rho}(I) = \tilde{\rho}(q\,L) = \exp\left\{\mathbb{E}\left[\tilde{\psi}(I)\ \log\{I\}\Big|\ I \geq a\right]\right\} = \exp\left\{\mathbb{E}\left[\hat{\psi}(L)\ \log\{qL\}\Big|\ qL \geq a\right]\right\},$$

with $\hat{\psi}(\cdot)$ the corresponding weighting function.[13] Hence, the risk measure $\tilde{\rho}$ in this case naturally accounts for risk aversion (α) as well as for diversification effects (N).

For the allocation based on the Expected Shortfall (ES) according to the *Euler principle*, it is well known that (see e.g. Dhaene et al. (2009)):[14]

$$\frac{a_i}{a} = \frac{q\,\mathbb{E}\left[L_i|\,q\,L \geq a\right]}{\mathbb{E}\left[I|\,I \geq a\right]} = \frac{\mathbb{E}\left[\mathbb{E}[L_i|L]|\,q\,L \geq a\right]}{\mathbb{E}\left[L|\,q\,L \geq a\right]} = \frac{1}{N}\,\mathbb{E}\left[\text{const} \times L|\,q\,L \geq a\right],$$

i.e. the Expected Shortfall can be associated with a linear weighting function of the loss states. Since $\hat{\psi}(\cdot)$ is increasing and strictly convex for all risk aversion levels $\alpha > 0$, there always exists a

[13] The derivation of these equations, a closed form solution for $\hat{c}_{N,\nu,\alpha,a,q}$ as well as a representation of $\hat{\psi}(\cdot)$ not involving an infinite sum for implementation purposes all are provided in Appendix D.

[14] Here, we always assume that the confidence level is chosen corresponding to the counterparty-driven allocation, namely $\mathbb{P}(q\,L \geq a)$ in this case.

loss level l_0 such that the weighting function for the counterparty-driven allocation will be higher for all loss levels greater than l_0. In this sense, the allocation based on $\tilde{\rho}$ always appears more conservative in the current setting. However, we also see that for fixed parameters,

$$\hat{\psi}(l) \longrightarrow (\mathbb{P}(q\,L \geq a))^{-1} > \frac{a}{\mathbb{E}[qL|\,q\,L \geq a]},\ N \to \infty,$$

which is the left end-point for the Expected Shortfall weighting function. Similarly, for $\alpha = 0$, we obtain $\hat{\psi}(l) \equiv (\mathbb{P}(q\,L \geq a))^{-1}$, i.e. a flat weighting function. Thus, for large enough companies or risk-neutral consumers, the weight on relatively low loss levels will always be higher for the counterparty-driven allocation, rendering it to appear less conservative.

Exponential Losses: Parametrizations

Nr.	N	ν	τ	α	w	a	p	q
1	5	2.0	0.050	0.25	3.0	1.4663	0.2598	0.5713
2	5	2.0	0.050	1.25	3.0	4.0036	0.7401	0.9494

Table 1: Parametrizations of the Exponential Losses model.

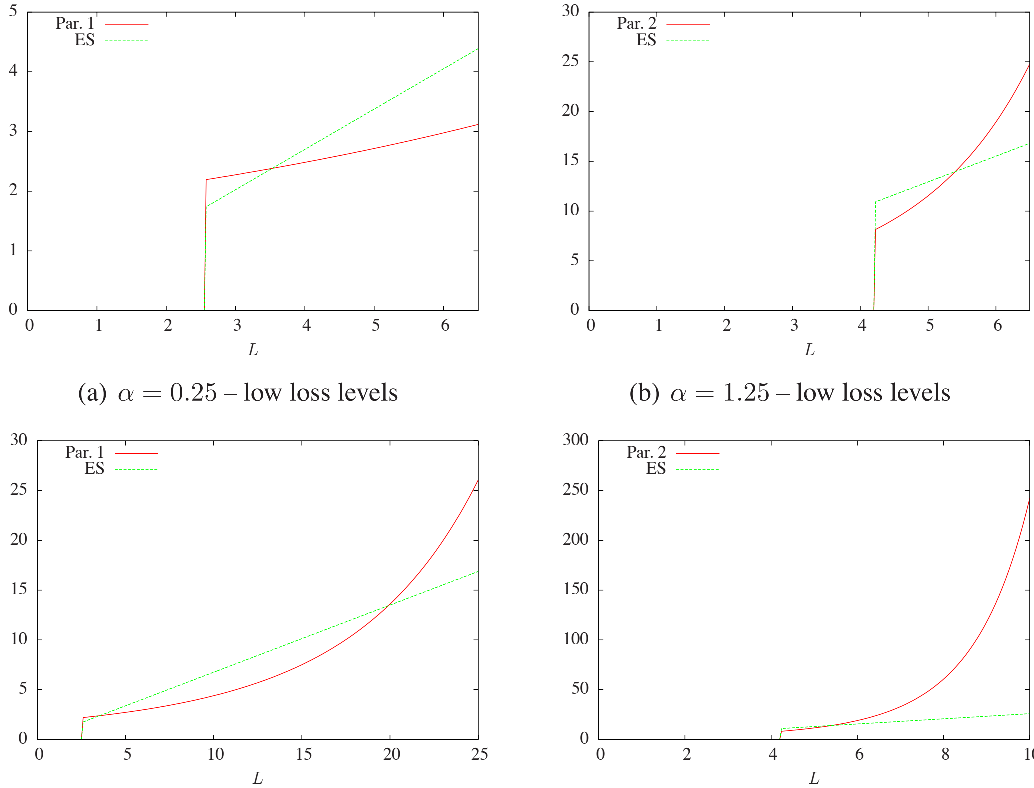
To further analyze this relationship, in Table 1 we present two parametrizations of the setup and the corresponding optimal parameters a, p, and q as solutions of the program (37). The properties are as expected: a, p, and q all are increasing in risk aversion. Figure 1 plots the weighting function $\hat{\psi}$ against the linear weighting function associated with the Expected Shortfall for varying risk aversion levels. We find two qualitatively different shapes:[15] For the high risk aversion level, $\hat{\psi}$ crosses the linear weighting function once from below; thus, in this case, relatively lower loss states are weighted more heavily for the allocation based on the Expected Shortfall, whereas the weighting is higher for the counter-party driven allocation in high loss states. For the low risk aversion level, $\hat{\psi}$ crosses the linear weighting function twice; in this case, the weighting function within the new risk measure $\tilde{\rho}$ puts more mass on low and extremely high loss states, while the weights are smaller for intermediate loss states.

Hence, when relying on the Expected Shortfall for the purpose of internal capital allocation, the loss-specific weights may be too conservative or not conservative enough, depending on, among other factors, company size or the risk aversion level. These considerations are naturally taken into account by the risk measure $\tilde{\rho}$.

4.2 The Case of Homogenous Bernoulli Losses

Again, we consider N identical consumers with wealth level w in a regime with non-binding regulation whose preferences are given by the (same) smooth utility function $U(\cdot)$. However, in

[15] Analyses with respect to other parameters such as company size N or the expected loss $1/\nu$ show similar results.



(a) $\alpha = 0.25$ – low loss levels (b) $\alpha = 1.25$ – low loss levels

(c) $\alpha = 0.25$ – high loss levels (d) $\alpha = 1.25$ – high loss levels

Figure 1: Weighting function $\hat{\psi}$ for varying risk aversion parameter α.

contrast to the previous section, we now assume that the consumers face Bernoulli distributed losses $L_i,\ 1 \le i \le N$, with loss level l and loss probability π. Their participation constraint once again is given by the autarky level

$$\gamma = \gamma_i = \mathbb{E}\left[U(\mathrm{w} - L_i)\right].$$

In this case, the optimization problem (6)/(7)/(8) takes the form

$$\begin{cases} \max_{a,q,p} \Big\{ N \times \Big(p - \pi \times \sum_{k=0}^{N-1} \binom{N-1}{k} \pi^k (1-\pi)^{N-1-k} \times \\ \qquad \Big[q\, l\, \mathbf{1}_{\left\{k < \frac{a}{ql} - 1\right\}} + \frac{a}{k+1} \mathbf{1}_{\left\{k \ge \frac{a}{ql} - 1\right\}} \Big] \Big) - \tau \times a \Big\} \\ \text{subject to} \\ \gamma \le (1-\pi)\, U(\mathrm{w} - p) + \pi \times \sum_{k=0}^{N-1} \binom{N-1}{k} \pi^k (1-\pi)^{N-1-k} \times \\ \qquad \Big[U\left(\mathrm{w} - p - (1-q)l\right) \mathbf{1}_{\left\{k < \frac{a}{ql} - 1\right\}} + U\left(\mathrm{w} - p - l + \frac{a}{k+1}\right) \mathbf{1}_{\left\{k \ge \frac{a}{ql} - 1\right\}} \Big]. \end{cases} \tag{39}$$

For the allocation to the individual consumer, similarly to the previous section, we obtain

$$q\,\tilde{\phi}_i = \frac{1}{N} \mathbb{E}\underbrace{\left[\frac{1}{\mathbb{E}\left[\mathbf{1}_{\left\{|\Gamma| \ge \frac{a}{q\,l}\right\}} U'\left(\mathrm{w} - p - l + \frac{a}{|\Gamma|}\right)\right]} \mathbf{1}_{\left\{|\Gamma| \ge \frac{a}{q\,l}\right\}} U'\left(\mathrm{w} - p - l + \frac{a}{|\Gamma|}\right)\right]}_{=\bar{\psi}(|\Gamma|)} = \frac{1}{N},$$

where $|\Gamma|$ denotes the number of total losses. And the risk measure now takes the form

$$\tilde{\rho}(I) = \tilde{\rho}(q\,l\,|\Gamma|) = \exp\left\{\mathbb{E}\left[\tilde{\psi}(I)\,\log\{I\}\middle|\,I \geq a\right]\right\} = \exp\left\{\mathbb{E}\left[\bar{\psi}(|\Gamma|)\,\log\{q\,l\,|\Gamma|\}\middle|\,|\Gamma \geq \frac{a}{q\,l}\right]\right\},$$

whereas for the Expected Shortfall based allocation, again similarly to Section 4.1,

$$\frac{a_i}{a} = \frac{q\,\mathbb{E}\,[L_i|q\,L \geq a]}{\mathbb{E}\,[I|I \geq a]} = \frac{1}{N}\,\mathbb{E}\left[\text{const}\,|\Gamma||\,|\Gamma| \geq \frac{a}{ql}\right],$$

so again it can be associated with a linear weighting function.

Hence, in this case, the weighting function for $\tilde{\rho}$ is a composition of the marginal utility and a reciprocal function. As such, it will always be increasing if consumers are risk-averse, and again we obtain a flat allocation for the risk neutral case. For $\mathrm{w} \geq p + l$, a sufficient condition for the concavity of $\bar{\psi}$ is a level of relative prudence smaller than two (see e.g. Kimball (1990) for the concept of relative prudence). However, for high levels of relative prudence, a convex shape is possible. Thus, again, it is not immediately clear how the counterparty-driven allocation compares to the linear weighting implied by the Expected Shortfall measure.

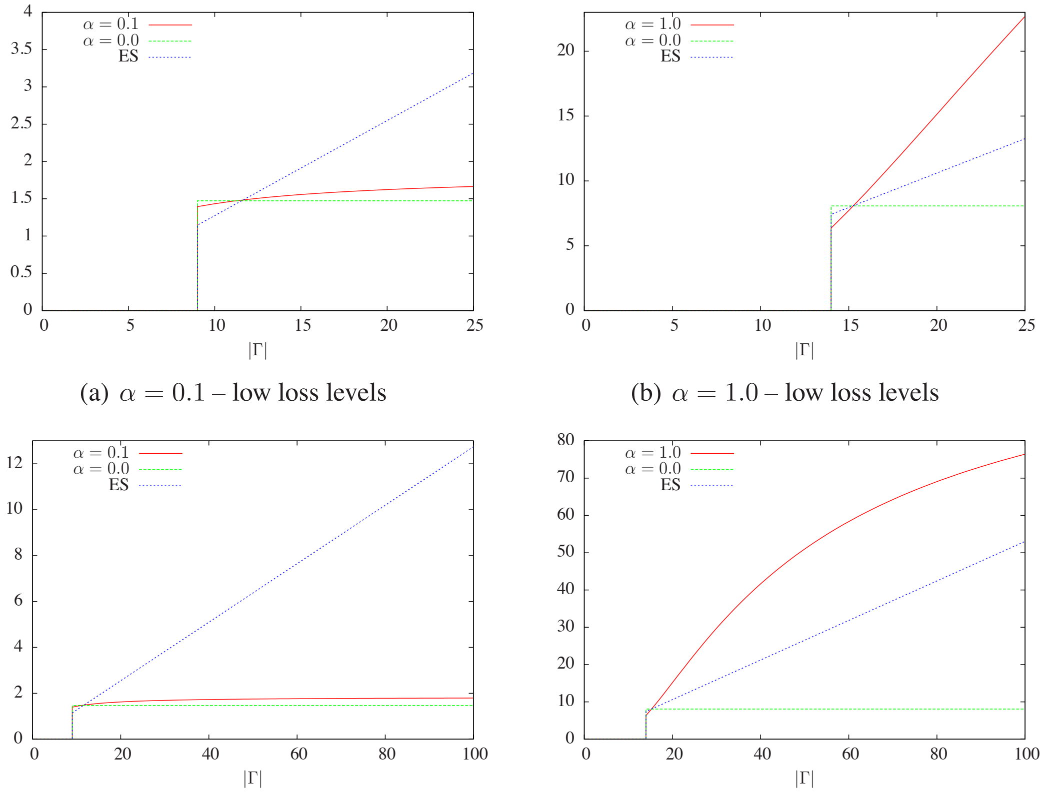
Homogeneous Bernoulli Losses: Parametrizations

Nr.	Type	N	l	π	τ	α/γ	w	a	p	q
1	CARA	100	3.00	0.10	0.05	0.1	5.0	24.8970	0.2727	0.9762
2	CARA	100	3.00	0.10	0.05	1.0	5.0	40.4654	1.0585	0.9969
3	CRRA	10	3.25	0.45	0.05	0.5	5.0	12.9492	1.2893	0.8704
4	CRRA	10	3.25	0.45	0.05	1.0	5.0	16.5461	1.6882	0.9350
5	CRRA	10	3.25	0.45	0.05	2.0	5.0	19.5488	2.1840	0.9693

Table 2: Parametrizations of the Bernoulli model.

Table 2 now displays several parametrizations in this setup for different Constant Absolute Risk Aversion (CARA) and Constant Relative Risk Aversion (CRRA) utility functions, where the optimal parameters a, p, and q are determined as solutions of program (39); again, they exhibit the expected relationships. Figure 2 now shows the weighting function $\bar{\psi}$ for parametrizations 1 and 2 (CARA utility) as well as the corresponding weighting function for the Expected Shortfall (ES) and risk-neutral consumers ($\alpha = 0$). In contrast to the previous section, $\bar{\psi}$ is concave in both cases. We again we obtain two qualitatively different relationships: For the low risk aversion, $\bar{\psi}$ crosses the linear weighting function from above, implying a less conservative state-specific allocation; for the strong risk averter, on the other side, the crossing is from below, ensuing a more conservative configuration. So yet again, risk aversion appears to play an important role.

For the CRRA case, the contract and firm parameters were chosen so that wealth is close to premium level plus losses, since for $\mathrm{w} = p + l$, $\bar{\psi}$ will be concave (convex) if and only if relative



(a) $\alpha = 0.1$ – low loss levels

(b) $\alpha = 1.0$ – low loss levels

(c) $\alpha = 0.1$ – high loss levels

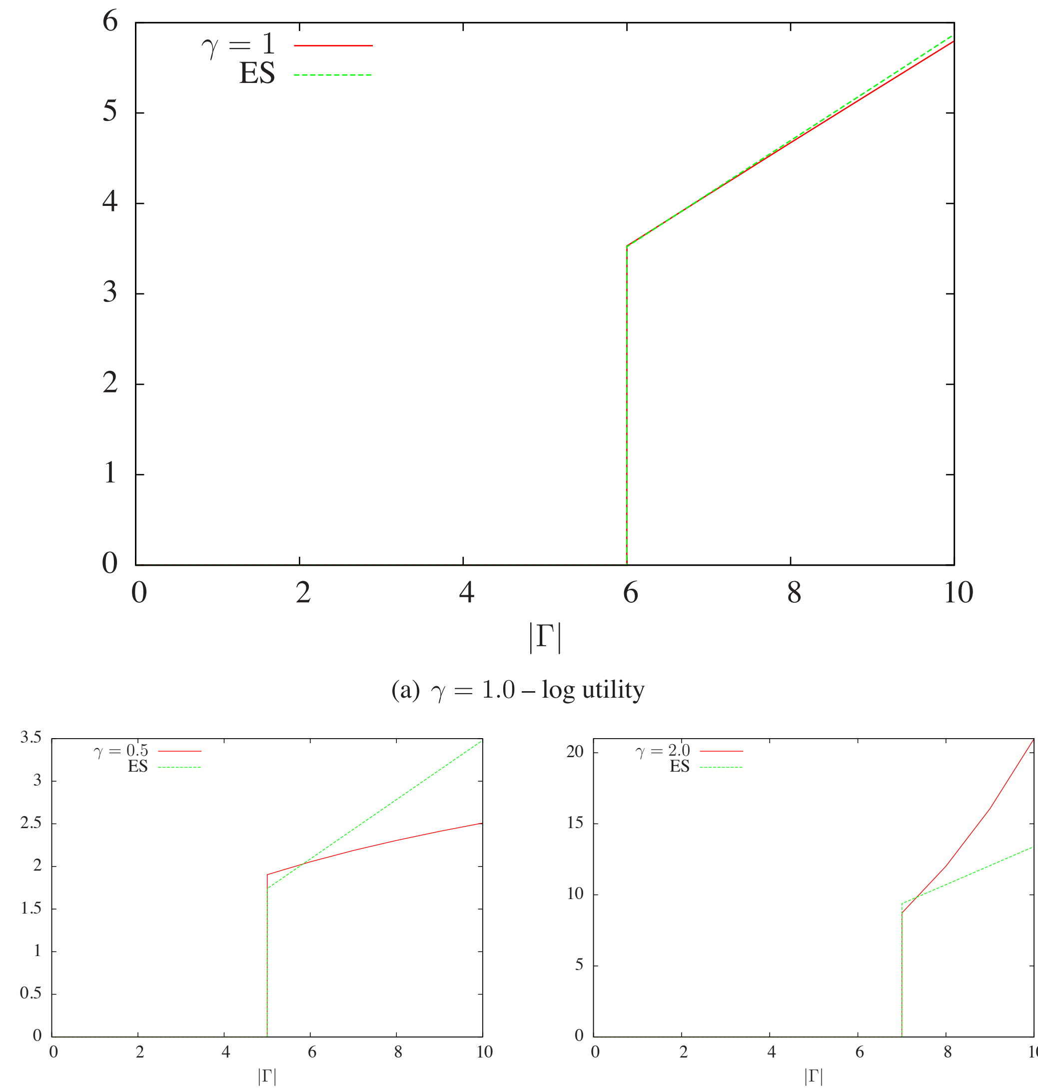
(d) $\alpha = 1.0$ – high loss levels

Figure 2: Weighting function $\bar{\psi}$ for varying absolute risk aversion parameter α; parametrizations 1 and 2 (CARA utility).

prudence is smaller (greater) than two. In particular, for log-utility, where relative prudence is constant and equals two, the weighting will be linear and, therefore, identical to the Expected Shortfall weights. This can also be seen from Figure 3, where the weighting functions for different relative risk aversion levels γ are plotted: For log-utility and w $\approx p + l$, the ES weighting and $\bar{\psi}$ are roughly identical. In contrast, for the lower constant relative risk aversion—and consequently a lower level of relative prudence—the crossing is from above and the shape is concave (panel (b)). For the higher level of constant relative risk aversion and whence more prudent consumers, the crossing is from above and the shaper of $\bar{\psi}$ is convex (panel (c)).[16]

Thus, we find that for the shape of the weighting function $\bar{\psi}$ and accordingly for the comparison of the resulting statewise allocation with that implied by the Expected Shortfall, other characteristics of the consumers' utility functions (in addition to risk aversion) are relevant. In particular, we can give a positive answer to the existence question raised towards the end of Section 3: Indeed, there exist special cases where the weighting is linear so that the counterparty-driven allocation and the Expected Shortfall based allocation are identical, although in general—of course—they will differ.

[16]The observation that for $\gamma = 1$ we obtain the Expected Shortfall is a curious analogy to the so-called *power spectral risk measures* from Dowd et al. (2008), who show that these measures degenerate into the Expected Loss for $\gamma = 1$, and that there is a qualitative difference between the case $\gamma < 1$ and $\gamma > 1$.



(a) $\gamma = 1.0$ – log utility

(b) $\gamma = 0.5$

(c) $\gamma = 2.0$

Figure 3: Weighting function $\bar{\psi}$ for varying relative risk aversion parameter γ; parametrizations 3, 4, and 5 (CRRA utility).

4.3 The Case of Heterogenous Bernoulli Losses

Similarly to the last section, we consider consumers that face Bernoulli distributed losses. However, in contrast to the previous setup, we now allow for heterogeneity in consumer preferences as well as in the losses. More specifically, we assume that there are m groups of consumers, where group i contains N_i identical consumers with wealth level w_i and utility function $U_i(\cdot)$ that face independent losses l_i occurring with a probability π_i, $i = 1, \ldots, m$. The participation constraint again is given by their autarky levels:

$$\gamma_i = \mathbb{E}\left[U_i(\mathrm{w}_i - L_i)\right] = \pi_i\, U_i(\mathrm{w}_i - l_i) + (1 - \pi_i)\, U_i(\mathrm{w}_i).$$

The optimization problem (6)/(7)/(8) can then be easily set up by noticing that the number of losses in the different groups follow independent Binomial(N_i, π_i) distributions.

For the counterparty-based allocation, we obtain for each group i

$$\begin{aligned} q_i\,\tilde{\phi}_i \;=\; & \tilde{c} \sum_{k_1}^{N_1} \cdots \sum_{k_i}^{N_i} \cdots \sum_{k_m}^{N_m} \binom{N_1}{k_1} \cdots \binom{N_i}{k_i} \cdots \binom{N_m}{k_m} \\ & \times \pi_1^{k_1} \ldots \pi_i^{k_i} \ldots \pi_m^{k_m} (1-\pi_1)^{N_1-k_1} \ldots (1-\pi_i)^{N_i-k_i} \ldots (1-\pi_m)^{N_m-k_m} \\ & \times \underbrace{\left[\mathbf{1}_{\{k_1 q_1 l_1 + \ldots + k_m q_m l_m \geq a\}} \left\{ \sum_{j=1}^{m} \frac{k_j\, U'\left(\mathrm{w}_j - p_j - l_j + q_j\, l_j\, \frac{a}{k_1 q_1 l_1 + \ldots + k_m q_m l_m}\right) l_j}{v_j'\,(k_1 q_1 l_1 + \ldots + k_m q_m l_m)} \right\} \right]}_{=\text{const}\times\frac{\partial\tilde{\mathbb{P}}}{\partial\mathbb{P}}} \\ & \times \frac{k_i\, q_i\, l_i}{k_1 q_1 l_1 + \ldots + k_m q_m l_m}, \end{aligned} \tag{40}$$

where $\tilde{c}$ is a constant such that $\sum_i q_i\,\tilde{\phi}_i = 1$. Thus, while the analytical form of the weights $\tilde{\psi}(I) = \mathbb{E}\left[\frac{\partial\tilde{\mathbb{P}}}{\partial\mathbb{P}}\middle| I\right]$ is less transparent in this case, again we notice that they immediately depend on the marginal utilities of recoveries in various states of default. For the allocation based on the Expected Shortfall, on the other hand, we obtain

$$\begin{aligned} \frac{a_i}{a} \;=\; & \text{const} \sum_{k_1}^{N_1} \cdots \sum_{k_i}^{N_i} \cdots \sum_{k_m}^{N_m} \binom{N_1}{k_1} \cdots \binom{N_i}{k_i} \cdots \binom{N_m}{k_m} \\ & \times \pi_1^{k_1} \ldots \pi_i^{k_i} \ldots \pi_m^{k_m} (1-\pi_1)^{N_1-k_1} \ldots (1-\pi_i)^{N_i-k_i} \ldots (1-\pi_m)^{N_m-k_m} \\ & \mathbf{1}_{\{k_1 q_1 l_1 + \ldots + k_m q_m l_m \geq a\}} (k_i\, q_i\, l_i), \end{aligned} \tag{41}$$

i.e. it is of a similar form as (40) but now 1) does not contain the adjustment based on the marginal utilities $\frac{\partial\tilde{\mathbb{P}}}{\partial\mathbb{P}}$, and 2) the state-specific loss for consumer i, $(k_i\, q_i\, l_i)$, is not scaled by the aggregate loss—which, in the counterparty-driven allocation, is a consequence of the proportional partitioning of the recoveries in states of default.

To assess the consequences of these adjustments, we consider the case of identical group sizes $N_i = N$, identical CARA preferences and wealth levels throughout the population, identical loss probabilities $\pi_i = \pi$, but differing loss levels. More specifically, in Table 3, we present model parametrizations for a setup with $m = 3$ groups, whose members face loss levels $l_1 = 1$, $l_2 = 2$, and $l_3 = 3$, respectively. For the first five parametrizations, the only difference in the assumed parameters is the group size but remarkably the influence on premiums $\{p_i\}$ and the choice parameters $\{q_i\}$ is marginal, and may be completely attributable to numerical inaccuracies. Since the participation constraint binds, it appears that the solution features an adjustment of the asset level such that the utility level is the same in all cases. In particular, since the resulting asset level is concave in the group size due to obvious diversification effects, this implies that the companies' monopoly rents increase disproportionately in relation to the firm size.

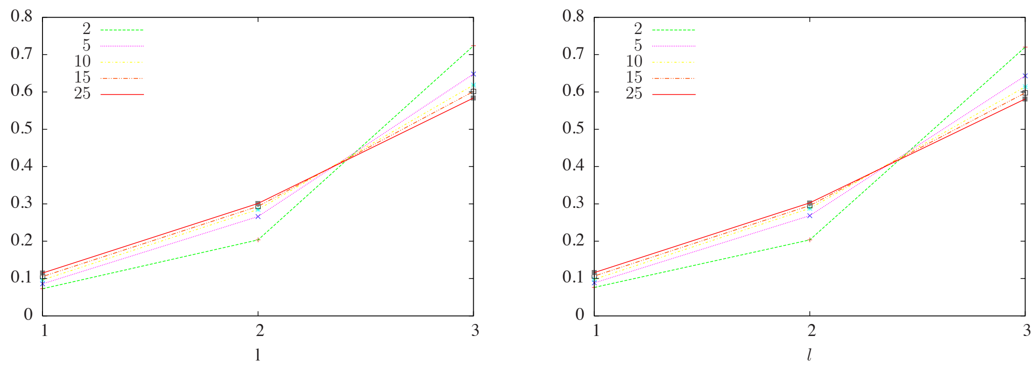
Heterogenous Bernoulli Losses: Parametrizations

Nr.	α	N	π	τ	w	a	p_1	p_2	p_3	q_1	q_2	q_3
1	1.00	2	0.1	0.01	5	5.37	0.16	0.49	1.06	0.99	0.99	1.00
2	1.00	5	0.1	0.01	5	8.84	0.16	0.49	1.06	0.99	1.00	1.00
3	1.00	10	0.1	0.01	5	13.63	0.16	0.49	1.06	0.99	1.00	1.00
4	1.00	15	0.1	0.01	5	17.86	0.16	0.49	1.06	0.99	1.00	1.00
5	1.00	25	0.1	0.01	5	25.76	0.16	0.49	1.06	0.99	1.00	1.00
6	1.25	10	0.1	0.01	5	14.07	0.18	0.60	1.31	0.99	1.00	1.00

Table 3: Parametrizations of the heterogeneous Bernoulli model with CARA preferences and loss levels $(l_1, l_2, l_3) = (1, 2, 3)$.

Figure 4 shows the counterparty-based allocation of capital to the three groups for parametrizations 1 through 5 as well as the corresponding allocations based on the Expected Shortfall (ES). At first glance, the two graphs look roughly identical. More specifically, for small group sizes, both of them entail a disproportionate allocation of capital to the high loss group ($l = 3$), whereas for an increasing group size, the allocations become more and more linear. This clearly is a consequence of diversification: For small group sizes, and individual loss in the high loss group is closely linked to a company default, whereas for extremely large group sizes, individual losses in either group become essentially independent of company default. Taking into account that positive contributions for either allocation (40) or (41) are only accrued in states where 1) the individual in view incurs a loss and 2) the company defaults, the similarity of the graphs becomes understandable. In particular, this effect appears to dominate other influences stemming from, for instance, risk aversion.

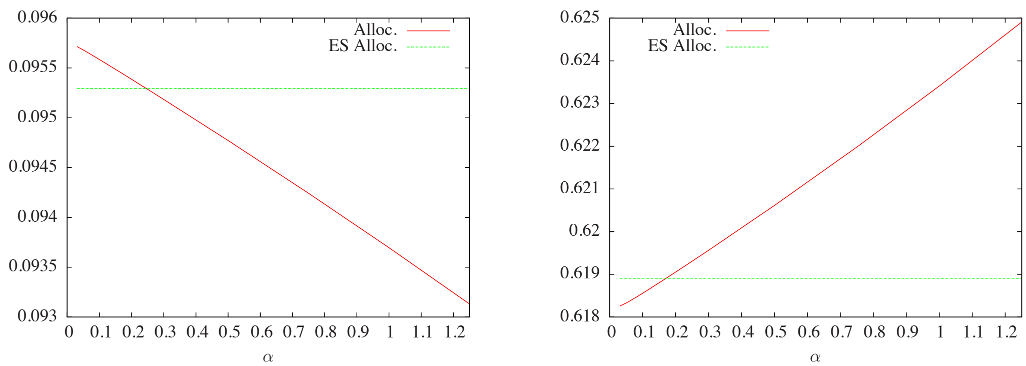
To demonstrate that such effects are present nonetheless when adjusting the focus, Figure 5 illustrates the allocation of capital to the low ($l = 1$) and high ($l = 3$) loss group for parametrization 6 and varying levels of absolute risk aversion. Obviously, the allocation based on the expected shortfall (ES) is independent of consumer preferences. For the counterparty-driven allocation, on the other hand, we find a decreasing and slightly concave dependence for the low loss group (panel (a)), and an increasing and slightly convex relationship for the high loss group (panel (b)). In particular, there exist risk aversion levels for which either of the allocations yield higher/lower charges for the high/low loss group, although in this case moderate levels of risk aversion cause the counterparty-driven allocation to penalize the high loss group more heavily.



(a) CARA Utilty, $\alpha = 1$

(b) ES

Figure 4: Allocation for varying company sizes N; parametrizations 1-5.



(a) Small loss ($l = 1$)

(b) Large loss ($l = 3$)

Figure 5: Allocation for varying absolute risk aversion parameter α; parametrization 6.

5 Conclusion

The early literature on capital allocation recognized that risk measure selection was a thorny issue that could be resolved only through careful consideration of institutional context. The subsequent literature on capital allocation celebrated, refined, and justified (in mathematical terms) the technique of Euler allocation technique while ignoring the institutional context, with the consequence that the demon of arbitrary risk measure selection has never been exorcised. The demon, however, cannot and should not be ignored: Risk measure selection has a profound influence on an organization's perception of the cost of risk.

Instead of starting with a risk measure, this paper starts with primitives and calculates the marginal cost of risk from the perspective of a profit-maximizing firm with risk-averse counterparties. But, recognizing ease of implementation is paramount to various users of capital allocation methods—who will not generally have access to the blackboard parameters of a profit maximization problem—we take the additional step of identifying the risk measure whose gradient yields allocations consistent with marginal cost. To get economically correct allocations, one of course needs considerable information to choose the appropriate weighting function, yet the problem of weighting functions is generally true of spectral risk measures as well—so, from the standpoint of practice, this measure starts from an economically correct foundation for the context of a profit-maximizing firm and offers no greater level of complication than is already present.

Surprisingly, this risk measure is neither coherent nor convex; nevertheless, it is the only one that yields the appropriate allocation of capital for the profit-maximizing firm. We have shown that ES-based allocation could either be underweighting *or* overweighting severe states of default, depending on the nature of customer risk aversion, and this raises the interesting possibility that a transition away from a system of regulation based on risk measure-based solvency assessment to one based on market (counterparty) discipline will not necessarily mitigate the oft-lamented failure of financial institutions to penalize "tail" risk.

The reason for the difference in allocations derives from a fundamental difference in the source of marginal cost in an economic model based on counterparty risk aversion and one based on the imposition of ES or spectral ES. In the economic model, marginal cost derives from the impact of the expansion of a particular risk exposure on the *recoveries* of the counterparties to the firm, which is determined by the assets of the firm and the value that counterparties place on those assets in various states of default—while other risk measures such as ES tend to focus on loss outcomes themselves rather than actual recoveries. In general, we have shown that the marginal cost of risk to the firm depends on the risk preferences of its counterparties, even in cases where it faces a binding regulatory constraint—so much that the influence of the regulatory constraint on capital allocation could be dominated by the influence of counterparty risk preferences.

The calculations in this paper are done from the perspective of a profit-maximizing firm, but one could also contemplate the calculus of a regulator or social planner. In some cases, the calculus will be similar. For example, a regulator without responsibility for unpaid losses (i.e., if no deposit insurance scheme exists) but in a context where counterparties are uninformed will view risk in manner similar to the profit-maximizing firm. However, a regulator responsible for unpaid losses would have to consider the extent of that responsibility in selecting a risk measure, as well as other issues—such as bankruptcy costs not internalized by private firms and the production cost technology associated with deposit insurance—that would determine the optimal level of capitalization for financial institutions as well as the social cost of risk. These issues are of course complex and well beyond the scope of this paper, but they are intriguing areas for future research.

Appendix

A Implementation of the Firm's Optimization Problem via a Premium Schedule

In the text we consider the solution of maximizing (6) subject to (7) and (8). We claim further that—if the consumer acts as a "price taker" with respect to the recovery rates offered by the company within the various states of default—that the company can implement the optimum by offering a smooth and monotonically increasing premium schedule that allows each consumer to freely choose the level of coverage desired for the premium indicated by the schedule. It is subsequently shown that the marginal price increase associated with coverage must satisfy (13) when evaluated at the optimum. It remains to be shown that this premium schedule exists and can be used to implement the optimum.

A complication arises in modeling the consumer as a price-taker with free choice of coverage level. To introduce the consumer's ignorance of his own influence on recoveries, we define price schedule described above as $p_i^*(\cdot)$ and modify the original utility function to

$$\tilde{v}_i\left(\mathrm{w}_i - p_i^*(q_i), q_i; \tilde{a}, \tilde{q}_1, \ldots, \tilde{q}_N\right) = \mathbb{E}\left[U_i\left(\mathrm{w}_i - p_i^*(q_i) - L_i + \tilde{R}_i\right)\right], \tag{42}$$

where:

$$\tilde{R}_i = \tilde{R}_i\left(q_i; \tilde{a}, \tilde{q}_1, \ldots, \tilde{q}_N\right) = \min\left\{I_i(L_i, q_i), \frac{\tilde{a}}{\sum_{j=1}^N I_j\left(L_j, \tilde{q}_j\right)} I_i(L_i, q_i)\right\}.$$

The idea here is to fix recovery rates by fixing the quantities $\tilde{a}$ and $\{\tilde{q}_i\}$, leaving the consumer with the free choice of q_i—but with the caveat that this choice does not influence recovery rates.[17]

The firm's objective function is identical to the previous one, except that 1) the firm now specifies a price function rather than a single price point, and 2) the firm fixes the recovery rates for purposes of consumer incentive compatibility by choosing $\tilde{a}$ and $\{\tilde{q}_i\}$ instead of the "true" levels of a and $\{q_i\}$:

$$\max_{\tilde{a}, \{p_i^*(\cdot)\}, \{\tilde{q}_i\}} \left\{\sum p_i^*(\tilde{q}_i) - \sum e_i - \tau\tilde{a}\right\} \tag{43}$$

The firm still faces the previous constraints (7) and (8),

$$\begin{aligned} v_i\left(\tilde{a}, \mathrm{w}_i - p_i^*(\tilde{q}_i), \tilde{q}_1, \ldots, \tilde{q}_N\right) &\geq \gamma_i, \\ s\left(\tilde{q}_1, \ldots, \tilde{q}_N\right) &\leq a, \end{aligned}$$

[17] Alternatively, we could also specify

$$\hat{\tilde{v}}_i\left(\mathrm{w}_i - p_i^*(q_i), q_i; \tilde{a}, \tilde{q}_1, \ldots, \tilde{q}_N\right) = \tilde{v}_i\left(\mathrm{w}_i - p_i^*(q_i), q_i; \frac{q_i}{\tilde{q}_i}\left(\tilde{a}, \tilde{q}_1, \ldots, \tilde{q}_N\right)\right),$$

where the consumer is cognizant of her own coverage but "scales" the company according to her own coverage level and this would not change the presentation in the main text. Again, the important point is that the consumer expects the *same recovery per dollar of coverage* in default states independent of her choice of q_i.

and in addition the new constraint:

$$\tilde{q}_i \in \arg\max_{q_i} \tilde{v}_i(\mathrm{w}_i - p_i^*(q_i), q_i\,; \tilde{a}, \tilde{q}_1, ..., \tilde{q}_N), \ \ \forall i. \tag{44}$$

Equation (44) is an incentive compatibility constraint requiring the choice of coverage level to be consistent with the consumer optimizing, given her perception of own utility (which ignores her own impact on recovery rates) and the selected pricing function.

It is evident that the firm's profits under this maximization can be no better than those achieved under the original program (maximizing (6) subject to (7) and (8)), since we have simply added another constraint and choosing the premium schedule at different points than $\tilde{q}_i$ is immaterial to the company's profits. It is therefore clear that, given optimal choices $\hat{a}$, $\{\hat{q}_i\}$, and $\{\hat{p}_i\}$ to the original program, the firm would maximize profits under the new setup if it could choose those same asset and coverage levels *and* find a pricing function $p_i^*(\cdot)$ that both satisfies $p_i^*(\hat{q}_i) = \hat{p}_i$ and induces consumers to choose the original solution:

$$\hat{q}_i \in \arg\max_{q_i} \tilde{v}_i(\mathrm{w}_i - p_i^*(q_i), q_i\,; \hat{a}, \hat{q}_1, ..., \hat{q}_N), \ \ \forall i.$$

The following lemma shows that this function exists.

Lemma A.1. *Suppose $\hat{a}$, $\{\hat{q}_i\}$, and $\{\hat{p}_i\}$ are the optimal choices maximizing (6) subject to (7) and (8). Then, for each i, there exists a smooth, monotonically increasing function $p_i^*(\cdot)$ satisfying:*

1. $p_i^*(\hat{q}_i) = \hat{p}_i$.
2. $\hat{q}_i \in \arg\max_{q_i} \tilde{v}_i(w_i - p_i^*(q_i), q_i\,; \hat{a}, \hat{q}_1, ..., \hat{q}_N)$.

Proof. Start by noting that it is evident that the constraints (7) all bind. Note further that the function of x:

$$g(x) = \tilde{v}_i(\mathrm{w}_i - x, 0\,; \hat{a}, \hat{q}_1, ..., \hat{q}_N)$$

is monotonically decreasing and, hence, invertible, so that we may uniquely define:

$$p_i^*(0) = g^{-1}(\gamma_i), \tag{45}$$

which obviously satisfies:

$$\tilde{v}_i(\mathrm{w}_i - p_i^*(0), 0\,; \hat{a}, \hat{q}_1, ..., \hat{q}_N) = \gamma_i.$$

Furthermore, let $p_i^*(\cdot)$ be a solution to the differential equation (initial value problem):[18]

$$\frac{\partial p_i^*(x)}{\partial x} = \frac{\frac{\partial}{\partial x}\tilde{v}_i(\mathrm{w}_i - p_i^*(x), x\,; \hat{a}, \hat{q}_1, ..., \hat{q}_N)}{\frac{\partial}{\partial \mathrm{w}}\tilde{v}_i(\mathrm{w}_i - p_i^*(x), x\,; \hat{a}, \hat{q}_1, ..., \hat{q}_N)}, \ p_i^*(0) = g^{-1}(\gamma_i), \tag{46}$$

on the compact choice set for q_i. Due to Peano's Theorem, we are guaranteed existence of such a function and that it is smooth. Moreover, since $\frac{\partial \tilde{v}_i}{\partial \mathrm{w}}$, $\frac{\partial \tilde{v}_i}{\partial q_i} > 0$, we know that the function is monotonically increasing.

[18] Here, $\frac{\partial \tilde{v}_i}{\partial \mathrm{w}}$ and $\frac{\partial \tilde{v}_i}{\partial q_i}$ denote the derivatives with respect to the first and the second argument of $\tilde{v}_i$, respectively.

Moving on, by construction we know that:

$$\begin{aligned}
& \tilde{v}_i(\mathrm{w}_i - p_i^*(q_i), q_i \,; \hat{a}, \hat{q}_1, ..., \hat{q}_N) \\
= \; & \gamma_i + \int_0^{q_i} \left[\frac{\partial}{\partial q_i} \tilde{v}_i(\mathrm{w}_i - p_i^*(x), x \,; \hat{a}, \hat{q}_1, ..., \hat{q}_N) - \frac{\partial}{\partial \mathrm{w}} \tilde{v}_i(\mathrm{w}_i - p_i^*(x), x \,; \hat{a}, \hat{q}_1, ..., \hat{q}_N) \times \frac{\partial p_i^*(x)}{\partial x} \right] dx \\
= \; & \gamma_i + 0, \; q_i > 0. \qquad (47)
\end{aligned}$$

In particular,

$$\tilde{v}_i(\mathrm{w}_i - p_i^*(\hat{q}_i), \hat{q}_i \,; \hat{a}, \hat{q}_1, ..., \hat{q}_N) = \gamma_i,$$

which, since it is evident that the constraints (7) all bind in the original optimization, can be true if and only if:

$$p_i^*(\hat{q}_i) = \hat{p}_i,$$

proving the first part of the lemma. Moreover, (47) directly implies that:

$$\hat{q}_i \in \arg\max_{q_i} \tilde{v}_i(\mathrm{w}_i - p_i^*(q_i), q_i \,; \hat{a}, \hat{q}_1, ..., \hat{q}_N).$$

proving the second part. □

B Identities in Section 2.2

In the setting of Section 2.2, the firm's problem becomes:

$$\max_{a, \{q_i\}, \{p_i\}, \{K_j\}, \{\mathrm{w}_{ij}\}} \sum p_i - \sum e_i - \tau\, a,$$

subject to:

$$\begin{aligned}
v_i & \geq \gamma_i, \\
s(q_1, \ldots, q_N) & \leq a, \\
\sum_j \pi_j\, K_j & = 1, \\
\sum_j \pi_j\, \mathrm{w}_{ij} & = \mathrm{w}_i.
\end{aligned}$$

In addition to a new set of optimality conditions connected with $\{K_j\}$ and $\{\mathrm{w}_{ij}\}$, we have the same set of first order conditions:

$$\begin{aligned}
[q_i] \qquad & -\sum_k \frac{\partial e_k}{\partial q_i} + \sum_k \lambda_k \frac{\partial v_k}{\partial q_i} - \frac{\partial s}{\partial q_i} \xi = 0, \\
[a] \qquad & -\sum_k \frac{\partial e_k}{\partial a} - \tau + \sum_k \lambda_k \frac{\partial v_k}{\partial a} + \xi = 0, \\
[p_i] \qquad & 1 - \lambda_i \frac{\partial v_i}{\partial \mathrm{w}} = 0.
\end{aligned}$$

The first order condition for $\{\mathrm{w}_{ij}\}$ is:

$$
\begin{aligned}
[\mathrm{w}_{ij}] \qquad & \lambda_i \frac{\partial v_i}{\partial \mathrm{w}_{ij}} - \eta_i \, \pi_j = 0 \qquad (48)\\
& \Leftrightarrow \lambda_i \, p_j^{(S)} \, \mathbb{E}^{\mathbb{P}}\left[U_i'(W_i - p_i - L_i + R_i)|\omega_j^{(S)}\right] - \eta_i \, \pi_j = 0,
\end{aligned}
$$

where $\{\eta_i\}$ are the Lagrange multipliers for the individual wealth constraints. Since

$$
\begin{aligned}
0 &= \sum_j \left(\lambda_i \, p_j^{(S)} \, \mathbb{E}^{\mathbb{P}}\left[U_i'(W_i - p_i - L_i + R_i)|\omega_j^{(S)}\right] - \eta_i \, \pi_j\right)\\
&= \lambda_i \frac{\partial v_i}{\partial \mathrm{w}} - \eta_i,
\end{aligned}
$$

with $[p_i]$ we obtain $\eta_i \equiv 1$. Thus, we also have:

$$
\frac{\pi_j}{p_j^{(S)}} = \frac{\mathbb{E}^{\mathbb{P}}\left[U_i'(W_i - p_i - L_i + R_i)|\omega_j^{(S)}\right]}{\frac{\partial v_i}{\partial \mathrm{w}}}. \qquad (49)
$$

As before, we seek a pricing function satisfying:

$$
\left[\frac{\partial v_i}{\partial q_i} + \mathbb{E}^{\mathbb{P}}\left[\mathbf{1}_{\{I \geq K\,a\}} \, U_i' \, \frac{K\,a}{I^2} \, I_i \, \frac{\partial I_i}{\partial q_i}\right]\right] - \frac{\partial v_i}{\partial \mathrm{w}} \frac{\partial p_i^*}{\partial q_i} = 0.
$$

Proceeding analogously to Section 2.1, we arrive at the marginal pricing condition associated with a decentralized implementation:

$$
\frac{\partial p_i^*}{\partial q_i} = \sum_k \frac{\partial e_k}{\partial q_i} + \frac{\partial s}{\partial q_i}\,\xi - \sum_{k \neq i} \frac{\frac{\partial v_k}{\partial q_i}}{v_k'} + \frac{\mathbb{E}^{\mathbb{P}}\left[\mathbf{1}_{\{I \geq K\,a\}} \, U_i' \, \frac{K\,a}{I^2} \, I_i \, \frac{\partial I_i}{\partial q_i}\right]}{v_i'}.
$$

Simplifying, we obtain:

$$
\begin{aligned}
\frac{\partial p_i^*}{\partial q_i} &= \frac{\partial e_i^Z}{\partial q_i} + \frac{\partial s}{\partial q_i}\left[\sum_k \frac{\partial e_k}{\partial a} + \tau - \sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'}\right] + \mathbb{E}^{\mathbb{P}}\left[\mathbf{1}_{\{I \geq K\,a\}} \sum_k \frac{U_k'}{v_k'} \, \frac{K\,a}{I^2} \, I_k \, \frac{\partial I_i}{\partial q_i}\right]\\
&= \frac{\partial e_i^Z}{\partial q_i} + \frac{\partial s}{\partial q_i}\left[\mathbb{E}^{\mathbb{Q}}\left[K\,\mathbf{1}_{\{I \geq K\,a\}}\right] + \tau - \sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'}\right] + \frac{\mathbb{E}^{\mathbb{P}}\left[\mathbf{1}_{\{I \geq K\,a\}} \sum_k \frac{U_k'}{v_k'} \, \frac{K}{I^2} \, I_k \, \frac{\partial I_i}{\partial q_i}\right]}{\sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'}} \times \sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'} \times a\\
&= \frac{\partial e_i^Z}{\partial q_i} + \frac{\partial s}{\partial q_i}\left[\mathbb{E}^{\mathbb{Q}}\left[K\,\mathbf{1}_{\{I \geq K\,a\}}\right] + \tau - \sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'}\right] + \tilde{\phi}_i \times a \times \left[\sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'}\right],
\end{aligned}
$$

i.e. (20), where:

$$
\tilde{\phi}_i = \frac{\mathbb{E}^{\mathbb{P}}\left[\mathbf{1}_{\{I \geq K\,a\}} \sum_k \frac{U_k'}{v_k'} \, \frac{K}{I^2} \, I_k \, \frac{\partial I_i}{\partial q_i}\right]}{\mathbb{E}^{\mathbb{P}}\left[\mathbf{1}_{\{I \geq K\,a\}} \sum_k \frac{U_k'}{v_k'} \, K \, \frac{I_k}{I}\right]} \overset{(49)}{=} \frac{\mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{I \geq K\,a\}} \sum_k \frac{U_k'}{\mathbb{E}^{\mathbb{P}}[U_k'|\omega^{(S)}]} \, \frac{K}{I^2} \, I_k \, \frac{\partial I_i}{\partial q_i}\right]}{\mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{I \geq K\,a\}} \sum_k \frac{U_k'}{\mathbb{E}^{\mathbb{P}}[U_k'|\omega^{(S)}]} \, K \, \frac{I_k}{I}\right]},
$$

i.e. (21). In the limiting case of a complete market, clearly $U_i'(\mathrm{w}_{ij} - p_i - l_{ij} + r_{ij})$ is $\mathcal{F}^{(S)}$-measurable, so that we immediately obtain (22).

C Identities in Section 2.3

The first order conditions of program (26) are:[19]

$$[q_i] \qquad -\sum_k \frac{\partial e_k}{\partial q_i} - V\mathbb{E}\left[\frac{\partial I_i(L_i,q_i)}{\partial q_i}\,\frac{\mathbf{1}_{\{I=a\}}}{dt}\right] + \sum_k \lambda_k \frac{\partial v_k}{\partial q_i} - \xi\frac{\partial s}{\partial q_i} = 0, \tag{50}$$

$$[a] \qquad -\sum_k \frac{\partial e_k}{\partial a} - \tau + \sum_k \lambda_k \frac{\partial v_k}{\partial a} + \xi + V\, f_I(a) = 0, \tag{51}$$

$$[p_i] \qquad 1 - \lambda_i \frac{\partial v_i}{\partial \mathrm{w}} = 0, \tag{52}$$

where—as before—λ_i and ξ denote the Lagrange multipliers for the first set of conditions and the second condition, respectively, and f_I denotes the cumulative density function of I.

Again, we assume each consumer is a "price taker" and ignores the impact of her own purchase at the margin on the level of recoveries in states of default, so that the marginal price change at the optimal level of q_i must satisfy:

$$\frac{\partial v_i}{\partial \mathrm{w}}\,\frac{\partial p_i^*}{\partial q_i} = \frac{\partial v_i}{\partial q_i} + \mathbb{E}\left[1_{\{I>a\}}\,\frac{a}{(I)^2}\,I_i\,\frac{\partial I_i}{\partial q_i}\,U_i'\right].$$

Therefore:

$$\begin{aligned}
\frac{\partial p_i^*}{\partial q_i} &= \frac{1}{\frac{\partial v_i}{\partial \mathrm{w}}}\left[\frac{\partial v_i}{\partial q_i} + \mathbb{E}\left[1_{\{I>a\}}\,\frac{a}{(I)^2}\,I_i\,\frac{\partial I_i}{\partial q_i}\,U_i'\right]\right] \\
&= \sum_k \frac{\partial e_k}{\partial q_i} + \xi\,\frac{\partial s}{\partial q_i} + V\,\mathbb{E}\left[\frac{\partial I_i(L_i,q_i)}{\partial q_i}\,\frac{\mathbf{1}_{\{I=a\}}}{dt}\right] - \sum_{k\neq i}\frac{\frac{\partial v_k}{\partial q_i}}{v_k'} + \frac{\mathbb{E}\left[1_{\{I>a\}}\,\frac{a}{(I)^2}\,I_i\,\frac{\partial I_i}{\partial q_i}\,U_i'\right]}{v_i'} \\
&= \frac{\partial e_k^Z}{\partial q_i} + \left[\mathbb{P}(I\geq a) + \tau - \sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'} - V\,f_I(a)\right]\frac{\partial s}{\partial q_i} + V\,f_I(a)\,\mathbb{E}\left[\frac{\partial I_i}{\partial q_i}\middle|\,I=a\right] \\
&\quad + \mathbb{E}\left[1_{\{I>a\}}\,\sum_k \frac{U_k'}{v_k'}\,\frac{a}{(I)^2}\,I_k\,\frac{\partial I_i}{\partial q_i}\right]
\end{aligned}$$

and since:

$$\sum_k \frac{\frac{\partial v_k}{\partial a}}{v_k'} = \mathbb{E}\left[1_{\{I>a\}}\,\sum_k \frac{U_k'}{v_k'}\,\frac{I_k}{I}\right],$$

we obtain (28).

[19]For simplicity, we omit the "t" super- and subscripts in case no ambiguity arises.

D Identities in Section 4.1

Derivation of Equation (37)

For consumer N, $L_N \sim Exp(\nu)$ and the loss incurred by "the other" consumers is $L_{-N} = \sum_{i=1}^{N-1} L_i \sim \text{Gamma}(N-1,\nu)$. Then

$$e = e_N = \underbrace{\mathbb{E}\left[q\, L_N\, \mathbf{1}_{\{q(L_{-N}+L_N)<a\}}\right]}_{\text{part } i.} + \underbrace{a\,\mathbb{E}\left[\frac{q\, L_N}{q\,(L_{-N}+L_N)}\,\mathbf{1}_{\{q(L_{-N}+L_N)\geq a\}}\right]}_{\text{part } ii.}.$$

For part $ii.$, note that $\frac{L_N}{L_{-N}+L_N}$ is Beta$(1, N-1)$ distributed independent of $L_{-N}+L_N \sim$ Gamma(N,ν). Hence, part $ii.$ can be written as

$$a\,\mathbb{P}\left(L_{-N}+L_N \geq \frac{a}{q}\right)\,\mathbb{E}\left[\frac{L_N}{L_{-N}+L_N}\right] = a\,\bar{\Gamma}_{N,\nu}\left(\frac{a}{q}\right)\,N^{-1}.$$

For part $i.$, we have

$$\begin{aligned}
&q\,\mathbb{E}\left[L_N\,\mathbf{1}_{\{q(L_{-N}+L_N)<a\}}\right]\\
=\;& q\int_0^\infty\int_0^\infty \mathbf{1}_{\{i+l<a/q\}}\,l\,\nu\,e^{-\nu\, l}\,\frac{\nu^{N-1}}{(N-2)!}\,i^{N-2}\,e^{-\nu\, i}\,dl\,di\\
=\;& q\frac{\nu^N}{(N-2)!}\int_0^{a/q}\int_0^{a/q-i} l e^{-\nu\, l}\,dl\;i^{N-2}\,e^{-\nu\, i}\,di\\
=\;& q\frac{\nu^N}{(N-2)!}\int_0^{a/q}\left[\frac{1}{\nu^2}-\frac{1}{\nu}\left(\frac{a}{q}+\frac{1}{\nu}\right)e^{-\nu\, a/q}\,e^{\nu\, i}+i\frac{1}{\nu}\,e^{-\nu\, a/q}\,e^{\nu\, i}\right]\,i^{N-2}\,e^{-\nu\, i}\,di\\
=\;& q\frac{\nu^{N-2}}{(N-2)!}\int_0^{a/q} i^{N-2}\,e^{-\nu\, i}\,di - q\frac{\nu^{N-1}}{(N-2)!}\,e^{-\nu\, a/q}\left[\left(\frac{a}{q}+\frac{1}{\nu}\right)\int_0^{a/q} i^{N-2}\,di-\int_0^{a/q} i^{N-1}\,di\right]\\
=\;& \frac{q}{\nu}\Gamma_{N-1,\nu}(a/q) - q\frac{\nu^{N-1}}{(N-1)!}e^{-\nu\, a/q}\left(\frac{a}{q}\right)^{N-1}\left[\frac{1}{\nu}+\frac{1}{N}\frac{a}{q}\right].
\end{aligned}$$

Therefore, since all consumers are identical, the objective function (6) takes the form displayed in (37). For condition (7), on the other hand, we have

$$\begin{aligned}
V \;=\;& V_N = \mathbb{E}\left[U\left(\mathrm{w}-p-L_N+R_N\right)\right]\\
=\;& \underbrace{\mathbb{E}\left[U\left(\mathrm{w}-p-(1-q)\,L_N\right)\,\mathbf{1}_{\{q(L_{-N}+L_N)<a\}}\right]}_{\text{part } i.}\\
&+\underbrace{\mathbb{E}\left[U\left(\mathrm{w}-p-L_N+a\frac{L_N}{L_{-N}+L_N}\right)\,\mathbf{1}_{\{q(L_{-N}+L_N)\geq a\}}\right]}_{\text{part } ii.}
\end{aligned}$$

For part i, we obtain

$$\begin{aligned}
&\mathbb{E}\left[U\left(w-p-(1-q)\,L_N\right)\,\mathbf{1}_{\{q(L_{-N}+L_N)<a\}}\right]\\
=\;& -\int_0^\infty\int_0^\infty \mathbf{1}_{\{i+l<a/q\}} e^{-\alpha(\mathrm{w}-p-(1-q)l)}\,\nu\, e^{-\nu\, l}\frac{\nu^{N-1}}{(N-2)!}\, i^{N-2}\, e^{-\nu\, i}\, dl\, di\\
=\;& -e^{-\alpha(\mathrm{w}-p)}\int_0^{a/q}\frac{1}{\nu-\alpha(1-q)}\left[1-\,e^{-a/q(\nu-(1-q)\alpha)-i(1-q)\nu+i\nu}\right]\frac{\nu^{N-1}}{(N-2)!}\, i^{N-2}\, e^{-\nu\, i}\, di\\
=\;& e^{-\alpha(\mathrm{w}-p)}\left[\frac{1}{\nu-\alpha(1-q)}\,\Gamma_{N-1,\nu}(a/q)-\frac{e^{-\frac{a}{q}(\nu-(1-q)\alpha)}\nu^{N-1}}{((1-q)\alpha)^{N-1}}\,\Gamma_{N-1,(1-q)\alpha}\left(\frac{a}{q}\right)\right].
\end{aligned}$$

For part $ii.$, note that

$$\begin{aligned}
&\mathbb{E}\left[U\left(\mathrm{w}-p-((L_{-N}+L_N)-a)\,\frac{L_N}{L_{-N}+L_N}\right)\,\mathbf{1}_{\{q(L_{-N}+L_N)\geq a\}}\right]\\
=\;& \int_0^1\int_0^\infty e^{-\alpha(\mathrm{w}-p-(l-a)y)}\mathbf{1}_{\{l\geq a/q\}}\,\frac{\nu^N}{(N-1)!}\, l^{N-1}\, e^{-\nu\, l}\,(N-1)\,(1-y)^{N-2}\, dl\, dy\\
=\;& -e^{-\alpha(\mathrm{w}-p)}\int_{a/q}^\infty\frac{\nu^N}{(N-1)!}e^{-\nu\, l}\, l^{N-1}\underbrace{\int_0^1 e^{(\alpha(l-a))y}(N-1)\,(1-y)^{N-2}\, dy}_{\mathrm{mgf}_{\mathrm{Beta}(1,N-1)}(\alpha(l-a))}\, dl\\
=\;& -e^{-\alpha(\mathrm{w}-p)}\int_{a/q}^\infty\frac{\nu^N}{(N-1)!}e^{-\nu\, l}\, l^{N-1}\sum_{k=0}^\infty\frac{(\alpha(l-a))^k}{(N-1+k)!}\, dl\\
=\;& -e^{-\alpha(\mathrm{w}-p)}\sum_{k=0}^\infty\sum_{j=0}^{N-1}\binom{N-1}{j}\, a^j\, e^{-\nu\, a}\frac{\alpha^k}{(N-1+k)!}\,\frac{(N+k-j-1)!}{\nu^{-j+k}}\\
&\qquad\times\int_{a/q}^\infty\frac{\nu^{N+k-j}}{(N+k-j-1)!}e^{-\nu\,(l-a)}\,(l-a)^{N+k-j-1}\, dl\\
=\;& -e^{-\alpha(\mathrm{w}-p)}\sum_{k=0}^\infty\sum_{j=0}^{N-1}\frac{(N-1)!}{(N-1+k)!}\frac{(a\nu)^j}{j!}\, e^{-\nu\, a}\left(\frac{\alpha}{\nu}\right)^k\frac{(N+k-j-1)!}{(N-1-j)!}\,\bar{\Gamma}_{N-j+k,\nu}\left(a\left(\frac{1-q}{q}\right)\right).
\end{aligned}$$

Derivation of Equation (38)

Similar to the previous part, for consumer N with $L=\sum_{i=1}^N L_i$:

$$\begin{aligned}
&\mathbb{E}\left[\sum_{j=1}^N U'\left(\mathrm{w}-p-L_j+a\,\frac{L_j}{L}\right)\frac{L_j}{L}\frac{L_N}{L}\middle|\, L\right]\\
=\;& \sum_{j=1}^{N-1}\underbrace{\mathbb{E}\left[U'\left(\mathrm{w}-p-(L-a)\,\frac{L_j}{L}\right)\frac{L_j}{L}\frac{L_N}{L}\middle|\, L\right]}_{\text{part } i.}+\underbrace{\mathbb{E}\left[U'\left(\mathrm{w}-p-(L-a)\,\frac{L_N}{L}\right)\left(\frac{L_N}{L}\right)^2\middle|\, L\right]}_{\text{part } ii.}.
\end{aligned}$$

Note that $\frac{L_j}{L},\frac{L_N}{L}\sim\mathrm{Beta}(1,N-1)$ and for the joint distribution

$$f_{\frac{L_j}{L},\frac{L_N}{L}}(x,y)=(1-x-y)^{N-3}\,(N-2)\,(N-1)\,\mathbf{1}_{\{x,y\geq 0,x+y\leq 1\}},\; j\neq N.$$

Whence, for part $i.$,

$$\begin{aligned}
&\mathbb{E}\left[U'\left(\mathrm{w}-p-(L-a)\,\frac{L_{N-1}}{L}\right)\frac{L_{N-1}}{L}\frac{L_N}{L}\,\middle|\,L\right]\\
&= \alpha\, e^{-\alpha(\mathrm{w}-p)}\int_0^1\int_0^{1-x} e^{\alpha(L-a)x}\,x\,y\,(N-1)\,(N-2)\,(1-x-y)^{N-3}\,dy\,dx\\
&= \alpha\, e^{-\alpha(\mathrm{w}-p)}\int_0^1 e^{\alpha(L-a)x}x\underbrace{\int_0^{1-x} y\,(N-1)\,(N-2)\,(1-x-y)^{N-3}\,dy}_{=(1-x)^{N-1}}\,dx\\
&= \alpha\, e^{-\alpha(\mathrm{w}-p)}\,\beta(2,N)\underbrace{\int_0^1 e^{\alpha(L-a)x}\,\frac{1}{\beta(2,N)}\,x\,(1-x)^{N-1}\,dx}_{=\mathrm{mgf}_{\mathrm{Beta}(2,N)}(\alpha(L-a))}\\
&= \alpha\, e^{-\alpha(\mathrm{w}-p)}\frac{1}{N(N+1)}(N+1)!\sum_{k=0}^{\infty}\frac{(k+1)\,\left(\alpha(L-a)\right)^k}{(N+k+1)!},
\end{aligned}$$

whereas for part $ii.$,

$$\begin{aligned}
&\mathbb{E}\left[U'\left(\mathrm{w}-p-(L-a)\,\frac{L_N}{L}\right)\left(\frac{L_N}{L}\right)^2\middle|\,L\right]\\
&= \alpha e^{-\alpha(\mathrm{w}-p)}\underbrace{\mathbb{E}\left[\exp\left\{\alpha(L-a)\frac{L_N}{L}\right\}\left(\frac{L_N}{L}\right)^2\middle|\,L\right]}_{=\frac{\partial^2}{\partial t^2}\mathrm{mgf}_{\mathrm{Beta}(1,N-1)}(t)\big|_{t=\alpha(L-a)}}\\
&= \alpha\, e^{-\alpha(\mathrm{w}-p)}(N-1)!\sum_{k=0}^{\infty}\frac{(k+1)\,(k+2)\,\left(\alpha(L-a)\right)^k}{(N+k+1)!},
\end{aligned}$$

so that

$$\begin{aligned}
&\mathbb{E}\left[\sum_{j=1}^{N}u'\left(w-p-L_j+a\,\frac{L_j}{L}\right)\frac{L_j}{L}\frac{L_N}{L}\,\middle|\,L\right]\\
&= \alpha\, e^{-\alpha(w-p)}(N-1)!\sum_{k=0}^{\infty}\frac{(N-1)\,(k+1)\,\left(\alpha(L-a)\right)^k+(k+1)\,(k+2)\,\left(\alpha(L-a)\right)^k}{(N+k+1)!}\\
&= \alpha\, e^{-\alpha(w-p)}(N-1)!\sum_{k=0}^{\infty}\frac{(k+1)\,(N+1+k)\,\left(\alpha(L-a)\right)^k}{(N+k+1)!}\\
&= \alpha\, e^{-\alpha(w-p)}(N-1)!\sum_{k=0}^{\infty}\frac{(k+1)\,\left(\alpha(L-a)\right)^k}{(N+k)!}.
\end{aligned}$$

For the denominator,

$$
\begin{aligned}
&\mathbb{E}\left[\mathbf{1}_{\{q\,L\geq a\}}\sum_{j=1}^{N}U'\left(\mathrm{w}-p-L_j-a\frac{L_j}{L}\right)\frac{L_j}{L}\right]\\
=\;&\mathbb{E}\left[\mathbf{1}_{\{q\,L\geq a\}}\,N\,\alpha\,e^{-\alpha(\mathrm{w}-p)}\,(N-1)!\sum_{k=0}^{\infty}\frac{(k+1)\;(\alpha(L-a))^k}{(N+k)!}\right]\\
=\;&N!\,\alpha\,e^{-\alpha(\mathrm{w}-p)}\sum_{k=0}^{\infty}\frac{k+1}{(N+k)!}\,\alpha^k\int_{a/q}^{\infty}\frac{\nu^N}{(N-1)!}(l-a)^k\,l^{N-1}\,e^{-\nu\,l}\,dl\\
=\;&N!\,\alpha\,e^{-\alpha(\mathrm{w}-p)}\sum_{k=0}^{\infty}\left(\frac{\alpha}{\nu}\right)^k\frac{k+1}{(N+k)!}\,e^{-\nu\,a}\sum_{j=0}^{N-1}\binom{N-1}{j}(a\nu)^j\,\frac{(N+k-j-1)!}{(N-1)!}\\
&\qquad\times\int_{a/q}^{\infty}\frac{\nu^{N+k-j}}{(N+k-j-1)!}e^{-\nu(l-a)}\,(l-a)^{N+k-j-1}\,dl\\
=\;&N!\,\alpha\,e^{-\alpha(\mathrm{w}-p)}\sum_{k=0}^{\infty}\left(\frac{\alpha}{\nu}\right)^k\frac{k+1}{(N+k)!}\,e^{-\nu\,a}\sum_{j=0}^{N-1}\frac{(a\nu)^j}{j!}\frac{(N+k-j-1)!}{(N-j-1)!}\,\bar{\Gamma}_{N+k-j,\nu}\left(a\left(\frac{1-q}{q}\right)\right).
\end{aligned}
$$

Hence,

$$
q\,\tilde{\phi}_i=\frac{1}{N}\mathbb{E}\left[\mathbf{1}_{\{q\,L\geq a\}}\underbrace{\frac{\sum_{k=0}^{\infty}\frac{(k+1)\,(\alpha(L-a))^k}{(N+k)!}}{\sum_{k=0}^{\infty}\left(\frac{\alpha}{\nu}\right)^k\frac{k+1}{(N+k)!}\,e^{-\nu\,a}\sum_{j=0}^{N-1}\frac{(a\nu)^j}{j!}\frac{(N+k-j-1)!}{(N-j-1)!}\,\bar{\Gamma}_{N+k-j,\nu}\left(a\left(\frac{1-q}{q}\right)\right)}}_{=\hat{\psi}(L)}\right].
$$

For implementation purposes, the numerator can be expressed as

$$
\begin{aligned}
\sum_{k=0}^{\infty}\frac{(k+1)\,t^k}{(N+k)!}\bigg|_{t=\alpha(L-a)}&=\frac{\partial}{\partial t}\left[\sum_{k=0}^{\infty}\frac{t^{k+1}}{(N+k)!}\right]\bigg|_{t=\alpha(L-a)}=\frac{\partial}{\partial t}\left[t^{-(N-1)}\sum_{k=N}^{\infty}\frac{t^k}{k!}\right]\bigg|_{t=\alpha(L-a)}\\
&=\frac{\partial}{\partial t}\left[t^{-(N-1)}\left(e^t-\sum_{k=0}^{N-1}\frac{t^k}{k!}\right)\right]\bigg|_{t=\alpha(L-a)}\\
&=-(N-1)t^{-N}\left(e^t-\sum_{k=0}^{N-1}\frac{t^k}{k!}\right)+t^{-(N-1)}\left(e^t-\sum_{k=0}^{N-1}\frac{k\,t^{k-1}}{k!}\right)\bigg|_{t=\alpha(L-a)}\\
&=\left(e^t-\sum_{k=0}^{N-2}\frac{t^k}{k!}\right)\times\left(t^{-N+1}-t^{-N}\,(N-1)\right)+\frac{(N-1)}{(N-1)!}\,t^{-1}\bigg|_{t=\alpha(L-a)}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\tilde{\psi}(I) &= \tilde{\psi}(qL) = \mathbb{E}\left[\frac{\partial\tilde{\mathbb{P}}}{\mathbb{P}}\middle| L\right] \\
&= \text{const}\ \mathbb{E}\left[\sum_{j=1}^{N} \alpha \exp\left\{-\alpha\left(\mathrm{w} - p - L_j + a\frac{L_j}{L}\right)\right\}\ \frac{L_j}{L}\middle| L\right] \\
&= \text{const}\ N\ \mathbb{E}\left[e^{\alpha(L-a)L_j/L}\frac{L_j}{L}\middle| L\right] \\
&= \text{const}\ \frac{\partial}{\partial x}\mathrm{mgf}_{\mathrm{Beta}(1,N-1)}(x)\bigg|_{x=\alpha(L-a)} = \hat{\psi}(L),
\end{aligned}
$$

since $\mathbb{E}\left[\tilde{\psi}(I)\right] = 1$.

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