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MASSEY PRODUCT AND TWISTED COHOMOLOGY OF A-INFINITY ALGEBRAS

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ABSTRACT. We study the twisted cohomology groups of A_∞-algebras defined by twisting elements and their behavior under morphisms and homotopies using the bar construction. We define higher Massey products on the cohomology groups of general A_∞-algebras and establish the naturality under morphisms and their dependency on defining systems. The above constructions are also considered for C_∞-algebras. We construct a spectral sequence converging to the twisted cohomology groups and show that the higher differentials are given by the A_∞-algebraic Massey products.

1. Introduction

The concept of A_∞-algebras was introduced by Stasheff [28, 29] for studying multiplication operations which satisfy associativity up to homotopy. Since then it has played a crucial role in homotopy theory. The A_∞-structure on the cohomology of a topological space determines the cohomology of its loop space and can be applied to the cohomology of fiber bundles [7]. Moreover, it determines the rational homotopy type of 1-connected spaces [11]. Recently, the subject finds applications in many areas of algebra, topology, geometry and mathematical physics, including homological mirror symmetry [13].

Motivated by the work on twisted cohomology of the de Rham complex [27, 1, 20, 16], we study the twisted cohomology of A_∞-algebras. In addition, we define higher Massey products on the cohomology of a general A_∞-algebra with a possibly non-associative multiplication. We then construct a spectral sequence converging to the twisted cohomology and relate the higher differentials to the higher Massey products.

The paper is organized as follows. In Section 2, we recall the notions of A_∞-algebras, morphisms of A_∞-algebras and homotopies of morphisms. We express these concepts using the bar construction. The A_∞-structure on the cohomology group is also described. The C_∞-algebras are discussed as a special case. In Section 3, we study the twisted cohomology group of a differential on the A_∞-algebra deformed by a twisting element. (These concepts simplify when the A_∞-algebra is C_∞.) We put special emphasis on the use of the bar

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construction, which clarifies many concepts and calculations. The twisted cohomology groups are naturally isomorphic if the twisting elements are homotopic equivalent. We show that a morphism of A_∞-algebras maps a twisting element to another and induces a homomorphism on the twisted cohomology groups. We also find the relation of the induced homomorphisms from two homotopic morphisms of A_∞-algebras. In Section 4, we introduce the triple and higher Massey products on the cohomology of a general A_∞-algebra. A crucial ingredient is the matric A_∞-algebra formed by matrices of elements in the original A_∞-algebra, together with its properties. We introduce an equivalence relation on the set of defining systems under which the (higher) Massey products take the same value; this is of interest even in the classical case. We establish some properties of the Massey product and the naturality under morphisms of A_∞-algebras. In particular, we clarify and generalize the folklore relation of the Massey product and the A_∞-structure on the cohomology (of differential graded algebras) to the context of A_∞-algebras. We also study the Massey products for C_∞-algebras. In Section 5, we construct a spectral sequence associated to a natural filtration on a $\mathbb{Z}$-graded A_∞-algebra (assuming the twisting element is of positive degree) that converges to the $\mathbb{Z}_2$-graded twisted cohomology group. We give a complete description of the higher differentials in terms of the higher Massey products of the A_∞-algebra. We show that a morphism of A_∞-algebras induces a morphism of spectral sequences. This result is applied to the quasi-isomorphism of the cohomology group and the original A_∞-algebra. In the Appendix, we present a construction of spectral sequence that is slightly different from what we can find in the literature but suits the purpose for the previous section.

2. A_∞-algebras and morphisms

In this section, we recall the definitions of A_∞- and C_∞-algebras, their morphisms, homotopy of morphisms, and the bar construction.

2.1. A_∞-algebras, morphisms and homotopy.

Definition 2.1. Let $\boldsymbol{k}$ be a field. An *A_∞-algebra* $(A,\{\mathsf{b}_n\})$ over $\boldsymbol{k}$ is a $\mathbb{Z}$- or $\mathbb{Z}_2$-graded vector space $A=\bigoplus_p A^p$ over $\boldsymbol{k}$ with graded homogeneous $\boldsymbol{k}$-multilinear maps $\mathsf{b}_n\colon A^{\otimes n}\to A$ $(n\geq 1)$ of degree 1 satisfying

$$\sum_{\substack{r,t\geq 0,\, s\geq 1\\ r+s+t=n}} \mathsf{b}_{r+1+t}\circ(1^{\otimes r}\otimes \mathsf{b}_s\otimes 1^{\otimes t})=0,$$

where $1=1_A$ is the identity map on A.

When maps are evaluated on elements, we follow Koszul's sign rule $(\mathsf{f}\otimes\mathsf{g})(x\otimes y)=(-1)^{|\mathsf{g}||x|}\mathsf{f}(x)\otimes\mathsf{g}(y)$, where f, g are graded homogeneous maps of degrees $|\mathsf{f}|$, $|\mathsf{g}|$ and x,

y are homogeneous elements of degrees $|x|$, $|y|$, respectively. So the above identity on $a_1, \ldots, a_n \in A$ is

$$\sum_{\substack{r,t\geq 0,\, s\geq 1\\ r+s+t=n}} \mathsf{b}_{r+1+t}(\bar{a}_1, \ldots, \bar{a}_r, \mathsf{b}_s(a_{r+1}, \ldots, a_{r+s}), a_{r+s+1}, \ldots, a_n) = 0,$$

where $\bar{a} = (-1)^{|a|} a$ for any homogeneous element $a \in A$. For $n = 1$, the identity is $\mathsf{b}_1 \circ \mathsf{b}_1 = 0$, i.e., $(A^\bullet, \partial)$, where $\partial = \mathsf{b}_1$, is a cochain complex. For $n \geq 2$, the meaning of the identities is best seen from the desuspended version. The A_∞-algebras defined here differ from the original ones (see [25, 12] for reviews) by a suspension. Let $\mathsf{s}\colon \mathsf{s}^{-1}A \to A$ be the suspension map of degree -1 given by the identification $(\mathsf{s}^{-1}A)^p = A^{p-1}$. Then the multilinear maps $\mathsf{m}_n = \pm\, \mathsf{s}^{-1} \circ \mathsf{b}_n \circ \mathsf{s}^{\otimes n}\colon (\mathsf{s}^{-1}A)^{\otimes n} \to \mathsf{s}^{-1}A$ (each with an appropriate sign) satisfy a similar set of identities. The identity for $n = 2$ is the Leibniz property for the product m_2 while that for $n = 3$ says that m_2 is associative up to a homotopy given by m_3. If $\mathsf{m}_n = 0$ (or equivalently, $\mathsf{b}_n = 0$) for all $n \geq 3$, then A is a differential graded algebra.

Definition 2.2. Let $(A, \{\mathsf{b}_n^A\})$, $(B, \{\mathsf{b}_n^B\})$ be A_∞-algebras. A *morphism* $\mathsf{f}\colon A \to B$ of A_∞-algebras is a family $\mathsf{f} = \{\mathsf{f}_n\}_{n\geq 1}$ of graded homogeneous $\boldsymbol{k}$-multilinear maps $\mathsf{f}_n\colon A^{\otimes n} \to B$ of degree 0 satisfying, for each $n \geq 1$, the identity

$$\sum_{\substack{r,t\geq 0,\, s\geq 1\\ r+s+t=n}} \mathsf{f}_{r+1+t} \circ (1^{\otimes r} \otimes \mathsf{b}_s^A \otimes 1^{\otimes t}) = \sum_{\substack{r\geq 0;\, i_1,\ldots,i_r>0\\ i_1+\cdots+i_r=n}} \mathsf{b}_r^B(\mathsf{f}_{i_1} \otimes \cdots \otimes \mathsf{f}_{i_r}).$$

For $n = 1$, the above identity means that $\mathsf{f}_1\colon (A, \partial^A) \to (B, \partial^B)$ is a morphism of cochain complexes. For $n = 2$, f_1 commutes with the operations $\mathsf{m}_2^A, \mathsf{m}_2^B$ (in the desuspended version) up to a homotopy given by f_2. A morphism f is called a *quasi-isomorphism* if f_1 is a quasi-isomorphism. It is *strict* if $\mathsf{f}_n = 0$ for all $n \geq 2$. The composition of two A_∞-algebra morphisms $\mathsf{f}\colon A \to B$ and $\mathsf{g}\colon B \to C$ is given by, for any $n \geq 1$,

$$(\mathsf{g} \circ \mathsf{f})_n = \sum_{\substack{r\geq 0;\, i_1,\ldots,i_r>0\\ i_1+\cdots+i_r=n}} \mathsf{g}_r \circ (\mathsf{f}_{i_1} \otimes \cdots \otimes \mathsf{f}_{i_r}).$$

In particular, $(\mathsf{g} \circ \mathsf{f})_1 = \mathsf{g}_1 \circ \mathsf{f}_1$.

Definition 2.3. Let $(A, \{\mathsf{b}_n^A\})$, $(B, \{\mathsf{b}_n^B\})$ be A_∞-algebras. Two morphisms $\{\mathsf{f}_n\}$, $\{\mathsf{g}_n\}$ from A to B are *homotopic* (through $\{\mathsf{h}_n\}$) if there exists a family of graded homogeneous $\boldsymbol{k}$-multilinear maps $\mathsf{h}_n\colon A^{\otimes n} \to B$ ($n \geq 1$) of degree -1 such that for any $n \geq 1$,

$$\mathsf{g}_n - \mathsf{f}_n = \sum \mathsf{b}_{r+1+t}^B \circ (\mathsf{g}_{i_1} \otimes \cdots \otimes \mathsf{g}_{i_r} \otimes \mathsf{h}_s \otimes \mathsf{f}_{j_1} \otimes \cdots \mathsf{f}_{j_t}) + \sum \mathsf{h}_{r+1+t} \circ (1^{\otimes r} \otimes \mathsf{b}_s^A \otimes 1^{\otimes t}),$$

where the first sum runs over $r, t \geq 0$, $s \geq 1$, $i_1+\cdots+i_r+j_1+\cdots+j_t = n$ ($i_1, \ldots, i_r, j_1, \ldots, j_t > 0$) and the second runs over $r, t \geq 0$, $s \geq 1$, $r + s + t = n$.

Homotopy is an equivalence relation on the set of morphisms.[1]

[1]See for example [25, 6, 15]. We thank B. Keller for providing the references.

2.2. **The bar construction.** A more conceptual way of describing A_∞-structures is through the use the bar construction [29]. Recall that the reduced tensor coalgebra on a vector space A over $\boldsymbol{k}$ is $\overline{T}A=\bigoplus_{n\ge1}A^{\otimes n}$. There is a comultiplication $\Delta\colon \overline{T}A\to\overline{T}A\otimes\overline{T}A$ given by

$$\Delta\,[a_1\otimes\cdots\otimes a_n]=\sum_{r=1}^{n-1}[a_1\otimes\cdots\otimes a_r]\otimes[a_{r+1}\otimes\cdots\otimes a_n],$$

where $0<r<n$ and $a_1,\dots,a_n\in A$. For an A_∞-algebra A, the map $\sum_{n\ge1}\mathsf{b}_n\colon \overline{T}A\to A$ lifts uniquely to a graded coderivation $\mathsf{b}\colon\overline{T}A\to\overline{T}A$ of degree 1 satisfying

$$\Delta\circ\mathsf{b}=(1\otimes\mathsf{b}+\mathsf{b}\otimes1)\circ\Delta.$$

The conditions that $(A,\{\mathsf{b}_n\})$ is an A_∞-algebra is equivalent to the identity $\mathsf{b}\circ\mathsf{b}=0$, i.e., b is a coalgebra differential on $\overline{T}A$ of degree 1.

Similarly, a collection of $\boldsymbol{k}$-multilinear maps $\mathsf{f}=\{\mathsf{f}_n\colon A^{\otimes n}\to B\}_{n\ge1}$ defines a map $\sum_{n\ge1}\mathsf{f}_n\colon\overline{T}A\to B$ which lifts uniquely to a coalgebra morphism $\mathsf{F}\colon\overline{T}A\to\overline{T}B$ of degree 0. If $(A,\{\mathsf{b}^A_n\})$, $(B,\{\mathsf{b}^B_n\})$ are A_∞-algebras, the condition that $\mathsf{f}=\{\mathsf{f}_n\}\colon A\to B$ is an A_∞-algebra morphism is equivalent to $\mathsf{F}\circ\mathsf{b}^A=\mathsf{b}^B\circ\mathsf{F}$, i.e., $\mathsf{F}\colon(\overline{T}A,\mathsf{b}^A)\to(\overline{T}B,\mathsf{b}^B)$ is a morphism of graded differential coalgebras. If $\mathsf{g}\colon B\to C$ is another morphism of A_∞-algebras, the composition of morphisms $\mathsf{g}\circ\mathsf{f}\colon A\to C$ corresponds to the usual composition $\mathsf{G}\circ\mathsf{F}\colon\overline{T}A\to\overline{T}C$.

Two A_∞-algebra morphisms $\mathsf{f},\mathsf{g}\colon A\to B$ are homotopic if and only if $\mathsf{F},\mathsf{G}\colon\overline{T}A\to\overline{T}B$ are homotopic as morphisms of graded differential coalgebras, i.e., there is a map $\mathsf{H}\colon\overline{T}A\to\overline{T}B$ of degree -1 such that

$$\mathsf{G}-\mathsf{F}=\mathsf{b}^B\circ\mathsf{H}+\mathsf{H}\circ\mathsf{b}^A,\quad \Delta^B\circ\mathsf{H}=(\mathsf{G}\otimes\mathsf{H}+\mathsf{H}\otimes\mathsf{F})\circ\Delta^A.$$

2.3. **C_∞-algebras and morphisms.** If A is a vector space over $\boldsymbol{k}$, the shuffle product [18] on $\overline{T}A$ is given by, for any $0<r<n$ and $a_1\dots,a_n\in A$,

$$[a_1\otimes\cdots\otimes a_r]\bowtie[a_{r+1}\otimes\cdots\otimes a_n]=\sum_{\sigma\in S_{r,n}}(-1)^{\varepsilon(\sigma)}\,a_{\sigma(1)}\otimes\cdots\otimes a_{\sigma(n)},$$

where $S_{r,n}$ is the set of permutations $\sigma\in S_n$ on the set $\{1,\dots,n\}$ such that $\sigma(i)<\sigma(j)$ if either $1\le i<j\le r$ or $r+1\le i<j\le n$ and $\varepsilon(\sigma)=\sum|a_i||a_j|$, summing over the pairs (i,j) with $1\le i\le r<j\le n$ and $\sigma(i)>\sigma(j)$. This product is graded commutative and associative on $\overline{T}A$, making it, together with the comultiplication Δ, a graded bi-algebra.

Definition 2.4. A C_∞-*algebra* $(A,\{\mathsf{b}_n\})$ is an A_∞-algebra such that for each $n\ge2$, $\mathsf{b}_n=0$ on $\overline{T}A\bowtie\overline{T}A$. A *morphism* $\mathsf{f}=\{\mathsf{f}_n\}_{n\ge1}\colon A\to B$ of C_∞-algebras is a morphism of A_∞-algebras such that for each $n\ge2$, $\mathsf{f}_n=0$ on $\overline{T}A\bowtie\overline{T}A$. Two C_∞-algebra morphisms $\mathsf{f},\mathsf{g}\colon A\to B$ are *homotopic* if they are homotopic as A_∞-algebra morphisms through $\{\mathsf{h}_n\}$ such that for each $n\ge2$, $\mathsf{h}_n=0$ on $\overline{T}A\bowtie\overline{T}A$.

For $n = 2$, the condition $\mathsf{b}_2(A \bowtie A) = 0$ is equivalent to the graded commutativity of the product m_2 on $\mathsf{s}^{-1}A$. Thus, C_∞-algebras are "graded commutative" A_∞-algebras (see [11] for a history of and references on C_∞-algebras). The conditions on C_∞-structures and morphisms can be stated concisely using the bar construction [11]. The A_∞-structure on A is a C_∞-structure if and only if b is a derivation on the graded bi-algebra $\overline{T}A$, i.e., for any $x, y \in \overline{T}A$,

$$\mathsf{b}(x \bowtie y) = \mathsf{b}(x) \bowtie y + \bar{x} \bowtie \mathsf{b}(y).$$

Similarly, an A_∞-morphism $\mathsf{f}\colon A \to B$ is C_∞ if and only if $\mathsf{F}\colon \overline{T}A \to \overline{T}B$ if a morphism of bi-algebras, i.e., for any $x, y \in \overline{T}A$,

$$\mathsf{F}(x \bowtie y) = \mathsf{F}(x) \bowtie \mathsf{F}(y).$$

Using the same method, one can further show that if $\mathsf{h} = \{\mathsf{h}_n\}$ is a homotopy of C_∞-algebra morphisms $\mathsf{f}, \mathsf{g}\colon A \to B$ if and only if for any $x, y \in \overline{T}A$,

$$\mathsf{H}(x \bowtie y) = \mathsf{G}(\bar{x}) \bowtie \mathsf{H}(y) + \mathsf{H}(x) \bowtie \mathsf{F}(y).$$

2.4. **A_∞-structure on the cohomology group.** Given an A_∞-algebra $(A, \{\mathsf{b}_n\})$, the cohomology $\mathrm{H}(A) = \mathrm{H}(A, \partial)$ is an associative graded algebra under the operation $\bar{\mathsf{b}}_2$ induced by b_2. In fact, the cohomology $\mathrm{H}(A)$ has an A_∞-algebra structure $\{\bar{\mathsf{b}}_n\}$ with $\bar{\mathsf{b}}_1 = 0$ [7, 8, 24, 13]. There is a quasi-isomorphism of A_∞-algebras $\mathrm{H}(A) \to A$ lifting the identity map of $\mathrm{H}(A)$. Such an A_∞-structure on $\mathrm{H}(A)$ is unique up to isomorphisms of A_∞-algebras.

We describe the A_∞-algebra $(\mathrm{H}(A), \{\bar{\mathsf{b}}_n\})$ and the quasi-isomorphism $\mathsf{q}\colon \mathrm{H}(A) \to A$. Let $\mathsf{p}_1\colon A \to \mathrm{H}(A)$ be a $\boldsymbol{k}$-linear map which sends any closed element to the cohomology class it represents; p_1 is determined by the choice of a subspace in A that is transverse to $\ker\partial$. Let $\mathsf{q}_1\colon \mathrm{H}(A) \to \ker\partial \subset A$ be a $\boldsymbol{k}$-linear map such that $\mathsf{p}_1 \circ \mathsf{q}_1 = 1_{\mathrm{H}(A)}$. Then there is a homogeneous $\boldsymbol{k}$-linear map $\mathsf{h}_1\colon A \to A$ of degree -1 such that $1_A - \mathsf{q}_1 \circ \mathsf{p}_1 = \partial \circ \mathsf{h}_1 + \mathsf{h}_1 \circ \partial$. The A_∞-structure $\{\bar{\mathsf{b}}_n\}$ on $\mathrm{H}(A)$ and the quasi-isomorphism $\mathsf{q} = \{\mathsf{q}_n\}\colon \mathrm{H}(A) \to A$ can be expressed explicitly as a sum over the oriented rooted planar trees [13]. Alternatively, they can be obtained inductively by [5]

$$\begin{aligned}
\bar{\mathsf{b}}_n &= \sum_{r=2}^{n} \sum_{\substack{i_1+\cdots+i_r=n \\ i_1,\ldots,i_r>0}} \mathsf{p}_1 \circ \mathsf{b}_r \circ (\mathsf{q}_{i_1} \otimes \cdots \otimes \mathsf{q}_{i_r}), \\
\mathsf{q}_n &= \sum_{r=2}^{n} \sum_{\substack{i_1+\cdots+i_r=n \\ i_1,\ldots,i_r>0}} \mathsf{h}_1 \circ \mathsf{b}_r \circ (\mathsf{q}_{i_1} \otimes \cdots \otimes \mathsf{q}_{i_r}),
\end{aligned}$$

respectively, for any $n \geq 2$. When A is a differential graded algebra, the inductive formulas simplify to [17]

$$\bar{\mathsf{b}}_n = \sum_{i=1}^{n-1} \mathsf{p}_1 \circ \mathsf{b}_2 \circ (\mathsf{q}_i \otimes \mathsf{q}_{n-i}), \quad \mathsf{q}_n = \sum_{i=1}^{n-1} \mathsf{h}_1 \circ \mathsf{b}_2 \circ (\mathsf{q}_i \otimes \mathsf{q}_{n-i}).$$

Finally, if $(A,\{\mathsf{b}_n\})$ is a C_∞-algebra, then so is $(\mathrm{H}(A),\{\bar{\mathsf{b}}_n\})$ [5].

3. Twisting elements and twisted cohomology

3.1. **Twisting elements and twisted cohomology of A_∞-algebras.** We construct a deformed differential on the A_∞-algebra and study its cohomology group. Let $(A,\{\mathsf{b}_n\})$ be an A_∞-algebra. We fix an element $h\in A$. Define a map $\tau_h\colon A\to\overline{T}A$ by

$$a\in A\mapsto \tau_h(a)=\sum_{n=0}^{\infty}\underbrace{h\otimes\cdots\otimes h}_{n\text{ times}}\otimes a=a+h\otimes a+h\otimes h\otimes a+\cdots.$$

τ_h is a $\boldsymbol{k}$-linear isomorphism from A onto its image, and the inverse map is given by $\tau_h^{-1}(x)=x-h\otimes x$ for $x\in\tau_h(A)$. We set $\tau_h(1)=1+\tau_h(h)\in\boldsymbol{k}\oplus\overline{T}A$. Then $\tau_h(a)=\tau_h(1)\otimes a$.

It is clear that $\Delta(\tau_h(a))=\tau_h(h)\otimes\tau_h(a)$ for any $h,a\in A$. In particular, $\Delta(\tau_h(h))=\tau_h(h)\otimes\tau_h(h)$, i.e., $\tau_h(h)$ can be regarded as a morphism of coalgebras from $\boldsymbol{k}$ (with the obvious comultiplication) to $\overline{T}A$.

We define a twisted differential $\partial_h\colon A\to A$ by[2]

$$a\in A\mapsto\partial_h a=\sum_{n=0}^{\infty}\mathsf{b}_{n+1}(\underbrace{h,\ldots,h}_{n\text{ times}},a)=\partial a+\mathsf{b}_2(h,a)+\mathsf{b}_3(h,h,a)+\cdots.$$

Lemma 3.1. *If $h\in A$, then*

$$\mathsf{b}\circ\tau_h=\tau_{\bar h}\circ\partial_h+\tau_{\bar h}(\partial_h h)\otimes\tau_h,$$

where $\bar h=(-1)^{|h|}h$.

Proof: By the definition of b, we have, for any $a\in A$,

$$\begin{aligned}
\mathsf{b}(\tau_h(a)) &= \sum_{n=0}^{\infty}\mathsf{b}(\underbrace{h\otimes\cdots\otimes h}_{n\text{ times}}\otimes a)\\
&= \sum_{n=0}^{\infty}\Bigg(\sum_{\substack{r,s\ge0\\ r+s=n}}\underbrace{\bar h\otimes\cdots\otimes\bar h}_{r\text{ times}}\otimes\mathsf{b}_{s+1}(\underbrace{h,\ldots,h}_{s\text{ times}},a)\;+\\
&\qquad+\sum_{\substack{r,t\ge0,s\ge1\\ r+s+t=n}}\underbrace{\bar h\otimes\cdots\otimes\bar h}_{r\text{ times}}\otimes\mathsf{b}_s(\underbrace{h,\ldots,h}_{s\text{ times}})\otimes\underbrace{h\otimes\cdots\otimes h}_{t\text{ times}}\otimes a\Bigg)\\
&= \tau_{\bar h}(1)\otimes\partial_h a+\tau_{\bar h}(1)\otimes\partial_h h\otimes\tau_h(a)\\
&= \tau_{\bar h}(\partial_h a)+\tau_{\bar h}(\partial_h h)\otimes\tau_h(a).
\end{aligned}$$

[2]During the preparation of the paper, we learned that the differential was previously introduced and studied in [25]. Our approach below relies instead on the bar construction. In the case when A is a C_∞-algebra, which is treated in §3.6, the same differential was also obtained recently by E. Getzler.

□

Corollary 3.2. *(i) For any $h \in A$,*

$$\sum_{r,t\geq 0} \mathsf{b}_{r+1+t}(\underbrace{\bar{h},\ldots,\bar{h}}_{r\ times},\partial_h h,\underbrace{h,\ldots,h}_{t\ times}) = 0;$$

(ii) $\partial_h h = 0$ if and only if $\mathsf{b}(\tau_h(h)) = 0$;

(ii) For any $h, a \in A$,

$$\partial_{\bar{h}}\partial_h a = -\sum_{r,t\geq 0} \mathsf{b}_{r+t+2}(\underbrace{\bar{h},\ldots,\bar{h}}_{r\ times},\partial_h h,\underbrace{h,\ldots,h}_{t\ times},a).$$

Proof: Taking $a = h$ in the proof of Lemma 3.1, we get

$$\mathsf{b}(\tau_h(h)) = \tau_{\bar{h}}(1) \otimes \partial_h h \otimes \tau_h(1),$$

from which (ii) follows. Applying b to both sides and projecting to A, we get (i). Applying b to the identity in Lemma 3.1, we get

$$\tau_h(\partial_{\bar{h}}\partial_h a) = -\tau_h(\partial_{\bar{h}}\bar{h}) \otimes \tau_{\bar{h}}(\partial_h a) - \mathsf{b}(\tau_{\bar{h}}(\partial_h h) \otimes \tau_h(a)).$$

(iii) follows by a projection onto A. □

Definition 3.3. If A is an A_∞-algebra, an element $h \in A$ is a *twisting element* if it is of even degree and $\partial_h h = 0$.

When A is a differential graded algebra, the condition $\partial_h h = 0$ reduces to the Maurer-Cartan equation $\partial h + \mathsf{b}_2(h,h) = 0$. In general, $\mathsf{b}(\tau_h(h)) = 0$ means that $\tau_h(h)$ is a morphism of differential graded coalgebras from $\boldsymbol{k}$ (with the trivial coderivation) to $\overline{T}A$. So $\tau_h(h) \in \mathrm{Hom}(\boldsymbol{k}, \overline{T}A)$ is a twisted cochain in the sense of Brown [3, 25]. We refer the readers to [31] for a history on twisting elements and twisting cochains, and to the references therein as well as [25, 9].

Theorem 3.4. *If h is a twisting element of an A_∞-algebra A, then*

(i) $\mathsf{b} \circ \tau_h = \tau_h \circ \partial_h$ on A;

(ii) b preserves the subspace $\tau_h(A) \subset \overline{T}A$;

(iii) $\partial_h^2 = 0$ on A.

Proof: (i) follows from Lemma 3.1 since $\bar{h} = h$ and $\partial_h h = 0$;
(ii) follows easily from (i);
(iii) follows from $\partial_h = \tau_h^{-1} \circ \mathsf{b} \circ \tau_h$ and $\mathsf{b}^2 = 0$. □

Definition 3.5. If h is a twisting element of the A_∞-algebra A, the *twisted cohomology* of A (twisted by h) is $\mathrm{H}_h(A) = \mathrm{H}(A, \partial_h)$.

We note that $\mathsf{s}^{-1}h$ is of odd degree if h is of even degree. Although A can be either $\mathbb{Z}$- or $\mathbb{Z}_2$-graded, the twisted cohomology $\mathrm{H}(A)$ is always $\mathbb{Z}_2$-graded. A special case is the cohomology of the de Rham complex twisted by a closed form of odd degree [27, 1, 20, 16].

3.2. **Induced morphisms on twisted cohomology groups.** Suppose $\mathsf{f}=\{\mathsf{f}_n\}\colon A\to B$ is a morphism of A_∞-algebras, inducing $\mathsf{F}\colon \overline{T}A\to\overline{T}B$ on the bar construction. For any $h\in A$, define $\mathsf{F}_h\colon A\to B$ by

$$a\in A\mapsto \mathsf{F}_h(a)=\sum_{n=0}^{\infty}\mathsf{f}_{n+1}(\underbrace{h,\dots,h}_{n\text{ times}},a).$$

Lemma 3.6. *For any $h\in A$, we have the identity*

$$\mathsf{F}\circ\tau^A_h=\tau^B_{\mathsf{F}_h(h)}\circ\mathsf{F}_h.$$

In particular, $\mathsf{F}\colon\tau^A_h(A)\subset\overline{T}A\to\tau^B_{\mathsf{F}_h(h)}(B)\subset\overline{T}B$ and $\mathsf{F}(\tau^A_h(h))=\tau^B_{\mathsf{F}_h(h)}(\mathsf{F}_h(h))$.

Proof: For any $a\in A$, by the definition of F, we have

$$\begin{aligned}
\mathsf{F}(\tau^A_h(a)) &= \sum_{n=0}^{\infty}\mathsf{F}(\underbrace{h\otimes\cdots\otimes h}_{n\text{ times}}\otimes a)\\
&= \sum_{n=0}^{\infty}\sum_{r=0}^{n}\sum_{\substack{i_0\ge0,\ i_1\dots,i_r\ge1\\ i_0+\cdots+i_r=n}}\mathsf{f}_{i_r}(\underbrace{h,\dots,h}_{i_r\text{ times}})\otimes\cdots\otimes\mathsf{f}_{i_1}(\underbrace{h,\dots,h}_{i_1\text{ times}})\otimes\mathsf{f}_{i_0+1}(\underbrace{h,\dots,h}_{i_0\text{ times}},a)\\
&= \sum_{r=0}^{\infty}\underbrace{\mathsf{F}_h(h)\otimes\cdots\otimes\mathsf{F}_h(h)}_{r\text{ times}}\otimes\mathsf{F}_h(a)\\
&= \tau^B_{\mathsf{F}_h(h)}(\mathsf{F}_h(a)).
\end{aligned}$$

The rest follows easily. $\square$

Theorem 3.7. *Suppose $\mathsf{f}\colon A\to B$ is a morphism of A_∞-algebras and h is a twisting element in A. Then*

(i) $\mathsf{F}_h(h)$ is a twisting element in B;

(ii) $\mathsf{F}_h\colon(A,\partial_h)\to(B,\partial_{\mathsf{F}_h(h)})$ is a cochain map, i.e., $\mathsf{F}_h\circ\partial^A_h=\partial^B_{\mathsf{F}_h(h)}\circ\mathsf{F}_h$;

(iii) if $\mathsf{g}\colon B\to C$ is another morphism of A_∞-algebras, then $(\mathsf{G}\circ\mathsf{F})_h=\mathsf{G}_{\mathsf{F}_h(h)}\circ\mathsf{F}_h$.

Proof: By Theorem 3.4(i), Lemma 3.6 and by $\mathsf{F}\circ\mathsf{b}^A=\mathsf{b}^B\circ\mathsf{F}$, we have

$$\begin{aligned}
\tau^B_{\mathsf{F}_h(h)}\circ\partial^B_{\mathsf{F}_h(h)}\circ\mathsf{F}_h &= \mathsf{b}^B\circ\tau^B_{\mathsf{F}_h(h)}\circ\mathsf{F}_h=F\circ\mathsf{b}^A\circ\tau^A_h=\mathsf{b}^B\circ\mathsf{F}\circ\tau^A_h\\
&= \mathsf{F}\circ\mathsf{b}^A\circ\tau^A_h=\mathsf{F}\circ\tau^A_h\circ\partial^A_h=\tau^B_{\mathsf{F}_h(h)}\circ\mathsf{F}_h\circ\partial^A_h.
\end{aligned}$$

(ii) follows from the injectivity of $\tau^B_{\mathsf{F}_h(h)}$ whereas (i) is from $\partial^B_{\mathsf{F}_h(h)}\mathsf{F}_h(h)=\mathsf{F}_h(\partial^A_h h)=0$. Next, we observe that, for any $a\in A$,

$$\mathsf{G}(\mathsf{F}(\tau^A_h(a)))=\mathsf{G}(\tau^B_{\mathsf{F}_h(h)}(\mathsf{F}_h(a)))=\tau^C_{\mathsf{G}_{\mathsf{F}_h(h)}(\mathsf{F}_h(h))}(\mathsf{G}_{\mathsf{F}_h(h)}(\mathsf{F}_h(a))).$$

On the other hand, we have

$$\mathsf{G}(\mathsf{F}(\tau_h^A(a))) = \tau^C_{(\mathsf{G}\circ\mathsf{F})_h(h)}((\mathsf{G}\circ\mathsf{F})_h(a)).$$

By setting $a = h$, we obtain $(\mathsf{G}\circ\mathsf{F})_h(h) = \mathsf{G}_{\mathsf{F}_h(h)}(\mathsf{F}_h(h))$ and (iii) follows. □

Corollary 3.8. *Under the above assumptions,*

(i) there is an induced homomorphism $(\mathsf{F}_h)_*\colon \mathrm{H}_h(A) \to \mathrm{H}_{\mathsf{F}_h(h)}(B)$*;*

(ii) there is a commutative diagram

$$\begin{tikzcd} & \mathrm{H}_{\mathsf{F}_h(h)}(B) \arrow[dr, "(\mathsf{G}_{\mathsf{F}_h(h)})_*"] & \\ \mathrm{H}_h(A) \arrow[ur, "(\mathsf{F}_h)_*"] \arrow[rr, "((\mathsf{G}\circ\mathsf{F})_h)_*"'] & & \mathrm{H}_{(\mathsf{G}\circ\mathsf{F})_h(h)}(C). \end{tikzcd}$$

3.3. **Homotopy equivalence of twisting elements.**

Definition 3.9. Two twisting elements h, h' in an A_∞-algebra $(A, \{\mathsf{b}_n\})$ are *homotopic* (through an element c) if there exists $c \in A$ (of odd degree) such that

$$h' - h = \sum_{r,t\geq 0} \mathsf{b}_{r+1+t}(\underbrace{h', \ldots, h'}_{r \text{ times}}, c, \underbrace{h, \ldots, h}_{t \text{ times}}).$$

Under this situation, define a map $\psi_c\colon A \to A$ by

$$a \in A \mapsto \psi_c(a) = a + \sum_{r,t\geq 0} \mathsf{b}_{r+t+2}(\underbrace{h', \ldots, h'}_{r \text{ times}}, c, \underbrace{h, \ldots, h}_{t \text{ times}}, a).$$

We note that $\psi_c(h) = h' - \partial_{h'} c$.

Lemma 3.10. *Suppose A is an A_∞-algebra and $h, h' \in A$ are two twisting elements. Then*

(i) h and h' are homotopic through $c \in A$ if and only if

$$\tau_{h'}(h') - \tau_h(h) = \mathsf{b}(\tau_{h'}(1) \otimes c \otimes \tau_h(1));$$

(ii) in this case, we have, for any $a \in A$,

$$\tau_{h'}(\psi_c(a)) - \tau_h(a) = \mathsf{b}(\tau_{h'}(c) \otimes \tau_h(a)) + \tau_{h'}(c) \otimes \tau_h(\partial_h a).$$

Moreover, $\psi_c(a)$ is the unique element satisfying this equality.

Proof: (i) Following the proof of Lemma 3.1 and using Definition 3.9, we get

$$\begin{aligned} & \mathsf{b}(\tau_{h'}(1) \otimes c \otimes \tau_h(1)) \\ = \ & \tau_{h'}(1) \otimes \partial_{h'} h' \otimes \tau_{h'}(1) \otimes c \otimes \tau_h(1) + \tau_{h'}(1) \otimes (h' - h) \otimes \tau_h(1) \\ & - \tau_{h'}(1) \otimes c \otimes \tau_{h'}(1) \otimes \partial_h h \otimes \tau_h(1) \\ = \ & \tau_{h'}(h') - \tau_h(h) \end{aligned}$$

since $\partial_{h'}h' = 0 = \partial_h h$.
(ii) follows from a similar calculation

$$\begin{aligned}
& \mathsf{b}(\tau_{h'}(c) \otimes \tau_h(a)) \\
= \; & \tau_{h'}(1) \otimes \partial_{h'}h' \otimes \tau_{h'}(1) \otimes c \otimes \tau_h(a) + \tau_{h'}(1) \otimes (\psi_c(a) - a) + \tau_{h'}(1) \otimes (h' - h) \otimes \tau_h(1) \otimes a \\
& - \tau_{h'}(1) \otimes c \otimes \tau_h(1) \otimes \partial_h h \otimes \tau_h(a) - \tau_{h'}(c) \otimes \tau_h(1) \otimes \partial_h a \\
= \; & \tau_{h'}(\psi_c(a)) - \tau_h(a) - \tau_{h'}(c) \otimes \tau_h(\partial_h a).
\end{aligned}$$

The uniqueness is clear. □

It can be shown that this homotopy is an equivalence relation [25, 9]. In fact, the equality in Lemma 3.10(i) means that $\tau_h(h)$ and $\tau_{h'}(h')$ are homotopic as coalgebra morphisms from $\boldsymbol{k}$ to $\overline{T}A$. More precisely, we have

Proposition 3.11. *Let A be an A_∞-algebra.*

(i) If $h, h' \in A$ are twisting elements that are homotopic through c, then ψ_c is a cochain map from (A, ∂_h) to $(A, \partial_{h'})$, i.e., $\psi_c \circ \partial_h = \partial_{h'} \circ \psi_c$.

(ii) If $h'' \in A$ is another twsiting element and h', h'' are homotopic through c', then h, h'' are homotopic through

$$c'' = c + c' + \sum_{r,s,t \geq 0} \mathsf{b}_{r+s+t+2}(\underbrace{h'', \ldots, h''}_{r \text{ times}}, c', \underbrace{h', \ldots, h'}_{s \text{ times}}, c, \underbrace{h, \ldots, h}_{t \text{ times}}).$$

(iii) Under the above condition, $\psi_{c'} \circ \psi_c$ and $\psi_{c''} \colon (A, \partial_h) \to (A, \partial_{h''})$ are homotopic cochain maps.

Proof: (i) We use Lemma 3.10(ii) in two ways. Applying b on the formula, we get

$$\mathsf{b}(\tau_{h'}(c) \otimes \tau_h(\partial_h a)) = \tau_{h'}(\partial_{h'}\psi_c(a)) - \tau_h(\partial_h a).$$

On the other hand, replacing a by $\partial_h a$ in the same formula, we get

$$\mathsf{b}(\tau_{h'}(c) \otimes \tau_h(\partial_h a)) = \tau_{h'}(\psi_c(\partial_h a)) - \tau_h(\partial_h a).$$

Therefore $\psi_c \circ \partial_h = \partial_{h'} \circ \psi_c$ on A.
(ii) First, we calculate

$$\begin{aligned}
& \mathsf{b}(\tau_{h''}(1) \otimes c' \otimes \tau_{h'}(1) \otimes c \otimes \tau_h(1)) \\
= \; & \tau_{h''}(1) \otimes (h'' - h') \otimes \tau_{h'}(1) \otimes c \otimes \tau_h(1) - \tau_{h''}(1) \otimes c' \otimes \tau_{h'}(1) \otimes (h' - h) \otimes \tau_h(1) \\
& + \tau_{h''}(1) \otimes (c'' - c - c') \otimes \tau_h(1) \\
= \; & \tau_{h''}(1) \otimes c'' \otimes \tau_h(1) - \tau_{h'}(1) \otimes c \otimes \tau_h(1) - \tau_{h''}(1) \otimes c' \otimes \tau_{h'}(1).
\end{aligned}$$

Applying b to both sides and using Lemma 3.10(i), we get

$$\mathsf{b}(\tau_{h''}(1) \otimes c'' \otimes \tau_h(1)) = (\tau_{h'}(h') - \tau_h(h)) + (\tau_{h''}(h'') - \tau_{h'}(h')) = \tau_{h''}(h'') - \tau_h(h).$$

The result follows from Lemma 3.10(i).
(iii) A similar calculation shows that, for any $a \in A$,

$$\begin{aligned} &\mathsf{b}(\tau_{h''}(c') \otimes \tau_{h'}(c) \otimes \tau_h(a)) + \tau_{h''}(c') \otimes \tau_{h'}(c) \otimes \tau_h(\partial_h a) \\ = \; &\tau_{h''}(c') \otimes \tau_h(a) - \tau_{h'}(c) \otimes \tau_h(a) - \tau_{h''}(c') \otimes \tau_{h'}(\psi_c(a)) + \tau_{h''}(\psi_{c',c}(a)), \end{aligned}$$

where the map $\psi_{c',c} \colon A \to A$ is defined by

$$a \in A \mapsto \psi_{c',c}(a) = \sum_{r,s,t \geq 0} \mathsf{b}_{r+s+t+3}(\underbrace{h'', \ldots, h''}_{r \text{ times}}, c', \underbrace{h', \ldots, h'}_{s \text{ times}}, c, \underbrace{h, \ldots, h}_{t \text{ times}}, a).$$

Applying b to the above, we get, after simplifying and using (i), the desired relation

$$\psi_{c'}(\psi_c(a)) - \psi_{c''}(a) = \partial_{h''}\psi_{c',c}(a) + \psi_{c',c}(\partial_h a).$$

□

Corollary 3.12. *Let A be an A_∞-algebra and $h, h' \in A$, two twisting elements that are homotopic through c. Then*

(i) $\psi_c \colon A \to A$ induces an isomorphism $(\psi_c)_* \colon \mathrm{H}_h(A) \to \mathrm{H}_{h'}(A)$;

(ii) if h'' is another twisting element and h', h'' are homotopic through c', then there is a commutative diagram

$$\begin{tikzcd} & \mathrm{H}_{h'}(A) \arrow[dr, "(\psi_{c'})_*"] & \\ \mathrm{H}_h(A) \arrow[ur, "(\psi_c)_*"] \arrow[rr, "(\psi_{c''})_*"'] & & \mathrm{H}_{h''}(A), \end{tikzcd}$$

where c'' is given by Proposition 3.11(i).

Proof: As in [9], c' can be chosen such that c'' in Proposition 3.11(ii) vanishes. By Proposition 3.11(iii), ψ_c induces an isomorphism $(\psi_c)_* \colon \mathrm{H}_h(A) \to \mathrm{H}_{h'}(A)$ with inverse $(\psi_{c'})_*$. The rest follows. □

3.4. **Morphisms on homotopic twisting elements.** Let A be an A_∞-algebra. Suppose $h', h \in A$ satisfy the relation

$$h' - h = \sum_{r,t \geq 0} \mathsf{b}_{r+1+t}(\underbrace{\overline{h'}, \ldots, \overline{h'}}_{r \text{ times}}, c, \underbrace{h, \ldots, h}_{t \text{ times}})$$

for some $c \in A$ (cf. Definition 3.9, but h, h' need not be twisting elements). Let $\mathsf{f} \colon A \to B$ be a morphism of A_∞-algebras. We set

$$\mathsf{F}_{h',h}(c) = \sum_{r,t \geq 0} \mathsf{f}_{r+1+t}(\underbrace{\overline{h'}, \ldots, \overline{h'}}_{r \text{ times}}, c, \underbrace{h, \ldots, h}_{t \text{ times}})$$

and define a map $\mathsf{F}_{h',h}(c, \cdot) \colon A \to A$ by

$$a \in A \mapsto \mathsf{F}_{h',h}(c, a) = \sum_{r,t \geq 0} \mathsf{f}_{r+t+2}(\underbrace{\overline{h'}, \ldots, \overline{h'}}_{r \text{ times}}, c, \underbrace{h, \ldots, h}_{t \text{ times}}, a).$$

Lemma 3.13. *(i) In the above notations, we have the identities*

$$\mathsf{F}(\tau^A_{\overline{h'}}(1)\otimes c\otimes\tau^A_h(1))=\tau^B_{\overline{\mathsf{F}_{h'}(h')}}(1)\otimes\mathsf{F}_{h',h}(c)\otimes\tau^B_{\mathsf{F}_h(h)}(1).$$

(ii) For any $a\in A$, we have

$$\mathsf{F}(\tau^A_{\overline{h'}}(c)\otimes\tau^A_h(a))=\tau^B_{\overline{\mathsf{F}_{h'}(h')}}(\mathsf{F}_{h',h}(c))\otimes\tau^B_{\mathsf{F}_h(h)}(\mathsf{F}_h(a))+\tau^B_{\overline{\mathsf{F}_{h'}(h')}}(\mathsf{F}_{h',h}(c,a)).$$

Proof: (i) Following the proof of Lemma 3.6, we get

$$\begin{aligned}
&\mathsf{F}(\tau^A_{\overline{h'}}(1)\otimes c\otimes\tau^A_h(1))\\
=&\sum_{\substack{r\geq 0;\,i_0,\dots,i_r>0\\ t\geq 0;\,j_0\dots,j_t>0}}\mathsf{f}_{i_r}(\underbrace{\overline{h'},\dots,\overline{h'}}_{i_r\text{ times}})\otimes\cdots\otimes\mathsf{f}_{i_1}(\underbrace{\overline{h'},\dots,\overline{h'}}_{i_1\text{ times}})\otimes\mathsf{f}_{i_0+j_0+1}(\underbrace{\overline{h'},\dots,\overline{h'}}_{i_0\text{ times}},c,\underbrace{h,\dots,h}_{j_0\text{ times}})\otimes\\
&\qquad\otimes\mathsf{f}_{j_1}(\underbrace{h,\dots,h}_{j_1\text{ times}})\otimes\cdots\otimes\mathsf{f}_{j_t}(\underbrace{h,\dots,h}_{j_t\text{ times}})\\
=&\sum_{r,t\geq 0}\underbrace{\overline{\mathsf{F}_{h'}(h')}\otimes\cdots\otimes\overline{\mathsf{F}_{h'}(h')}}_{r\text{ times}}\otimes\mathsf{F}_{h',h}(c)\otimes\underbrace{\mathsf{F}_h(h)\otimes\cdots\otimes\mathsf{F}_h(h)}_{t\text{ times}}\\
=&\ \tau^B_{\overline{\mathsf{F}_{h'}(h')}}(1)\otimes\mathsf{F}_{h',h}(c)\otimes\tau^B_{\mathsf{F}_h(h)}(1).
\end{aligned}$$

(ii) can be proved by a similar calculation. □

Theorem 3.14. *Let $\mathsf{f}\colon A\to B$ be a morphism of A_∞-algebras. Suppose $h,h'\in A$ are twisting elements that are homotopic through c. Then*

(i) $\mathsf{F}_h(h),\mathsf{F}_{h'}(h')\in B$ are twisting elements that are homotopic through $\mathsf{F}_{h',h}(c)$;

(ii) the diagram

$$\begin{tikzcd}
(A,\partial^A_h) \arrow[r, "\psi^A_c"] \arrow[d, "F_h"'] & (A,\partial^A_{h'}) \arrow[d, "F_{h'}"] \\
(B,\partial^B_{\mathsf{F}_h(h)}) \arrow[r, "\psi^B_{FF_{h',h}(c)}"'] & (B,\partial^B_{\mathsf{F}_{h'}(h')}).
\end{tikzcd}$$

commutes up to a cochain homotopy.

Proof: (i) Applying F to the formula in Lemma 3.10(i) and using Lemma 3.13(i), we get

$$\begin{aligned}
&\tau^B_{\mathsf{F}_{h'}(h')}(\mathsf{F}_{h'}(h'))-\tau^B_{\mathsf{F}_h(h)}(\mathsf{F}_h(h))=\mathsf{F}(\mathsf{b}^A(\tau^A_{h'}(1)\otimes c\otimes\tau^A_h(1)))\\
=&\ \mathsf{b}^B(\mathsf{F}(\tau^A_{h'}(1)\otimes c\otimes\tau^A_h(1)))=\mathsf{b}^B(\tau^B_{\mathsf{F}_{h'}(h')}(1)\otimes\mathsf{F}_{h',h}(c)\otimes\tau^B_{\mathsf{F}_h(h)}(1)).
\end{aligned}$$

The result is then proved by using Lemma 3.10(i) again.

(ii) Applying F to the formula in Lemma 3.10(ii) and using Lemma 3.13(ii), we get

$$\begin{aligned}
&\tau^B_{\mathsf{F}_{h'}(h')}(\mathsf{F}_{h'}(\psi^A_c(a))) - \tau^B_{\mathsf{F}_h(h)}(\mathsf{F}_h(a))\\
= \;& \mathsf{b}^B(\tau^B_{\mathsf{F}_{h'}(h')}(\mathsf{F}_{h',h}(c)) \otimes \tau^B_{\mathsf{F}_h(h)}(\mathsf{F}_h(a)) + \tau^B_{\mathsf{F}_{h'}(h')}(\mathsf{F}_{h',h}(c,a)))\\
&\quad + \tau^B_{\mathsf{F}_{h'}(h')}(\mathsf{F}_{h',h}(c)) \otimes \tau^B_{\mathsf{F}_h(h)}(\mathsf{F}_h(\partial_h a)) + \tau^B_{\mathsf{F}_{h'}(h')}(\mathsf{F}_{h',h}(c,\partial_h a))\\
= \;& \mathsf{b}^B(\tau^B_{\mathsf{F}_{h'}(h')}(\mathsf{F}_{h',h}(c)) \otimes \tau^B_{\mathsf{F}_h(h)}(\mathsf{F}_h(a))) + \tau^B_{\mathsf{F}_h(h)}(\partial_{\mathsf{F}_h(h)}\mathsf{F}_h(a))\\
&\quad + \tau^B_{\mathsf{F}_{h'}(h')}(\partial_{\mathsf{F}_{h'}(h')}\mathsf{F}_{h',h}(c,a) + \mathsf{F}_{h',h}(c,\partial_h a)).
\end{aligned}$$

On the other hand, applying Lemma 3.10(ii) to B, we get

$$\begin{aligned}
&\tau^B_{\mathsf{F}_{h'}(h')}(\psi^B_{\mathsf{F}_{h',h}(c)}(\mathsf{F}_h(a))) - \tau^B_{\mathsf{F}_h(h)}(\mathsf{F}_h(a))\\
= \;& \mathsf{b}^B(\tau^B_{\mathsf{F}_{h'}(h')}(\mathsf{F}_{h',h}(c)) \otimes \tau^B_{\mathsf{F}_h(h)}(\mathsf{F}_h(a))) + \tau^B_{\mathsf{F}_{h'}(h')}(\mathsf{F}_{h',h}(c)) \otimes \tau^B_{\mathsf{F}_h(h)}(\partial_{\mathsf{F}_h(h)}\mathsf{F}_h(a)).
\end{aligned}$$

Comparing the two calculations and by the injectivity of $\tau^B_{\mathsf{F}_{h'}(h')}$, we have

$$\mathsf{F}_{h'}(\psi^A_c(a)) - \psi^B_{\mathsf{F}_{h',h}(c)}(\mathsf{F}_h(a)) = \partial_{\mathsf{F}_{h'}(h')}\mathsf{F}_{h',h}(c,a) + \mathsf{F}_{h',h}(c,\partial_h a),$$

which establishes the desired homotopy through $\mathsf{F}_{h',h}(c,\cdot)$. □

3.5. **Homomorphisms on cohomology induced by homotopic morphisms.** Suppose $\{\mathsf{f}_n\}$ and $\{\mathsf{g}_n\}$ are two A_∞-morphisms from A to B that are homotopic through $\{\mathsf{h}_n\}$. In terms of graded differential coalgebras, $\mathsf{F}, \mathsf{G}\colon \overline{T}A \to \overline{T}B$ are homotopic through H. For $h \in A$, let $\mathsf{H}_h\colon A \to B$ be a map defined by

$$a \in A \mapsto \mathsf{H}_h(a) = \sum_{n=0}^{\infty} \mathsf{h}_{n+1}(\underbrace{h,\ldots,h}_{n\text{ times}},a).$$

Lemma 3.15. *(i) If $h \in A$, then*

$$\mathsf{H}(\tau^A_h(h)) = \tau^B_{\overline{\mathsf{G}_h(h)}}(1) \otimes \mathsf{H}_h(h) \otimes \tau^B_{\mathsf{F}_h(h)}(1).$$

(ii) In addition, for any $a \in A$,

$$\mathsf{H}(\tau^A_h(a)) = \tau^B_{\overline{\mathsf{G}_h(h)}}(\mathsf{H}_h(a)) + \tau^B_{\overline{\mathsf{G}_h(h)}}(\mathsf{H}_h(h)) \otimes \tau^B_{\mathsf{F}_h(h)}(\mathsf{F}_h(a)).$$

Proof: (i) follows from a direct calculation

$$\begin{aligned}
&\mathsf{H}(\tau^A_h(h))\\
= \;& \sum_{\substack{r,t\ge 0, s>0\\ i_1,\ldots,i_r>0\\ j_1,\ldots,j_t>0}} \mathsf{g}_{i_1}(\underbrace{\bar h,\ldots,\bar h}_{i_1\text{ times}}) \otimes \cdots \otimes \mathsf{g}_{i_r}(\underbrace{\bar h,\ldots,\bar h}_{i_r\text{ times}}) \otimes \mathsf{h}_s(\underbrace{h,\ldots,h}_{s\text{ times}}) \otimes \mathsf{f}_{j_1}(\underbrace{h,\ldots,h}_{j_1\text{ times}}) \otimes \cdots \otimes \mathsf{f}_{j_t}(\underbrace{h,\ldots,h}_{j_t\text{ times}})\\
= \;& \sum_{r,t\ge 0} \underbrace{\overline{\mathsf{G}_h(h)} \otimes \cdots \otimes \overline{\mathsf{G}_h(h)}}_{r\text{ times}} \otimes \mathsf{H}_h(h) \otimes \underbrace{\mathsf{F}_h(h) \otimes \cdots \otimes \mathsf{F}_h(h)}_{t\text{ times}}\\
= \;& \tau^B_{\overline{\mathsf{G}_h(h)}}(1) \otimes \mathsf{H}_h(h) \otimes \tau^B_{\mathsf{F}_h(h)}(1).
\end{aligned}$$

The proof of (ii) is similar. $\square$

Theorem 3.16. *Suppose $\{\mathsf{f}_n\}$, $\{\mathsf{g}_n\}$ are two A_∞-morphisms from A to B that are homotopic through $\{\mathsf{h}_n\}$. Let h be a twisting element in A. Then*

(i) $\mathsf{F}_h(h)$, $\mathsf{G}_h(h)$ are twisting elements in B that are homotopic through $\mathsf{H}_h(h) \in B$;

(ii) $\psi_{\mathsf{H}_h(h)} \circ \mathsf{F}_h$ and $\mathsf{G}_h\colon (A, \partial_h) \to (B, \partial_{\mathsf{G}_h(h)})$ are two maps of cochain complexes that are homotopic.

Proof. (i) Applying $\mathsf{G} - \mathsf{F} = \mathsf{b}^B \circ \mathsf{H} + \mathsf{H} \circ \mathsf{b}^A$ to $\tau_h^A(h) \in \overline{T}A$ and using Lemma 3.15(i), we get,

$$\begin{aligned}\tau^B_{\mathsf{G}_h(h)}(\mathsf{G}_h(h)) - \tau^B_{\mathsf{F}_h(h)}(\mathsf{F}_h(h)) &= \mathsf{b}^B(\mathsf{H}(\tau_h^A(h))) + \mathsf{H}(\mathsf{b}^A(\tau_h^A(h)))\\ &= \mathsf{b}^B(\tau^B_{\mathsf{G}_h(h)}(1) \otimes \mathsf{H}_h(h) \otimes \tau^B_{\mathsf{F}_h(h)}(1)).\end{aligned}$$

The result then follows from Lemma 3.10(i).
(ii) Applying the same formula to $\tau_h^A(a) \in \overline{T}A$ ($a \in A$) and using Lemma 3.15(ii), we get

$$\begin{aligned}&\tau^B_{\mathsf{G}_h(h)}(\mathsf{G}_h(a)) - \tau^B_{\mathsf{F}_h(h)}(\mathsf{F}_h(a))\\ =\ &\mathsf{b}^B(\tau^B_{\mathsf{G}_h(h)}(\mathsf{H}_h(a)) + \tau^B_{\mathsf{G}_h(h)}(\mathsf{H}_h(h)) \otimes \tau^B_{\mathsf{F}_h(h)}(\mathsf{F}_h(a))) + \mathsf{H}(\tau_h^A(\partial_h a)) + \mathsf{H}(\mathsf{b}(\tau_h^A(h)))\\ =\ &\mathsf{b}^B(\tau^B_{\mathsf{G}_h(h)}(\mathsf{H}_h(h)) \otimes \tau^B_{\mathsf{F}_h(h)}(\mathsf{F}_h(a))) + \tau^B_{\mathsf{G}_h(h)}(\mathsf{H}_h(h)) \otimes \mathsf{b}(\tau^B_{\mathsf{F}_h(h)}(\mathsf{F}_h(a)))\\ &+\tau^B_{\mathsf{G}_h(h)}(\partial_{\mathsf{G}_h(h)}\mathsf{H}_h(a) + \mathsf{H}_h(\partial_h a)).\end{aligned}$$

Therefore

$$\mathsf{G}_h - \psi_{\mathsf{H}_h(h)} \circ \mathsf{F}_h = \partial_{\mathsf{G}_h(h)} \circ \mathsf{H}_h + \mathsf{H}_h \circ \partial_h$$

by the uniqueness in Lemma 3.10(ii). $\square$

Corollary 3.17. *Under the above conditions, there is a commutative diagram*

$$\begin{tikzcd} & \mathrm{H}_h(A) \arrow[dl, "(\mathsf{F}_h)_*"'] \arrow[dr, "(\mathsf{G}_h)_*"] & \\ \mathrm{H}_{\mathsf{F}_h(h)}(B) \arrow[rr, "(\psi_{\mathsf{H}_h(h)})_*"'] & & \mathrm{H}_{\mathsf{G}_h(h)}(B). \end{tikzcd}$$

3.6. **Twisting elements and twisted cohomology of C_∞-algebras.** When the A_∞-structure is C_∞, there are a number of simplifications. We summarize the results in the following

Proposition 3.18. *Suppose $\boldsymbol{k}$ is a field of characteristic 0.*

(i) If A is a C_∞-algebra, then any closed element $h \in A$ of even degree is a twisting element.

(ii) If $\mathsf{f}\colon A \to B$ is a morphism of C_∞-algebras, then $\mathsf{F}_h(h) = \mathsf{f}_1(h)$.

(iii) If two C_∞-algebra morphisms $\mathsf{f}, \mathsf{g}\colon A \to B$ are homotopic through $\{\mathsf{h}_n\}$, then $\mathsf{H}_h(h) = \mathsf{h}_1(h)$.

Proof: This is because $\mathsf{b}_1(h) = 0$ and

$$\mathsf{b}_n(\underbrace{h, \ldots, h}_{n \text{ times}}) = \frac{1}{n}\, \mathsf{b}_n([\underbrace{h \otimes \cdots \otimes h}_{n-1 \text{ times}}] \bowtie [h]) = 0$$

for all $n \geq 2$. Similarly, for all $n \geq 2$, $\mathsf{f}_n(\underbrace{h, \ldots, h}_{n \text{ times}}) = 0$ and $\mathsf{h}_n(\underbrace{h, \ldots, h}_{n \text{ times}}) = 0$. Therefore $\mathsf{F}_h(h) = \mathsf{f}_1(h)$ and $\mathsf{H}_h(h) = \mathsf{h}_1(h)$. □

4. Higher Massey products on the cohomology of A_∞-algebras

The triple Massey product [19], was generalized to the context of A_∞-algebras [30]. We give an explicit construction of the higher Massey products, usually defined for differential graded algebras [14, 21], for A_∞-algebras. Furthermore, we introduce an equivalence relation on the set of defining systems under which the Massey product takes the same value in the cohomology; this clarifies the dependency of the Massey product on the defining systems. We establish some properties of the Massey products and the naturality under morphisms of A_∞-algebras. Finally, we study the triple and higher Massey products of C_∞-algebras.

4.1. **Triple Massey product.** We first review the definition of the triple Massey product on the cohomology of an A_∞-algebra A. Given classes $\alpha_1, \alpha_2, \alpha_3 \in \mathrm{H}(A)$, let $a_{01}, a_{12}, a_{23} \in A$ be their representatives, respectively. Suppose $\bar{\mathsf{b}}_2(\alpha_1, \alpha_2) = 0 = \bar{\mathsf{b}}_2(\alpha_2, \alpha_3)$. Then there are $a_{02}, a_{13} \in A$ such that $\mathsf{b}_2(a_{01}, a_{12}) = -\mathsf{b}_1(a_{02})$ and $\mathsf{b}_2(a_{12}, a_{23}) = -\mathsf{b}_1(a_{13})$. If A is a differential graded algebra, then $\mathsf{b}_2(a_{01}, a_{13}) + \mathsf{b}_2(a_{02}, a_{23})$ would be a cocycle representing the usual triple Massey product. However, when A is a general A_∞-algebra, this expression is no longer closed. Instead, with a correction term from b_3, we define [30]

$$\mu(a_{01}, a_{12}, a_{23}; a_{02}, a_{13}) = \mathsf{b}_2(a_{01}, a_{13}) + \mathsf{b}_2(a_{02}, a_{23}) + \mathsf{b}_3(a_{01}, a_{12}, a_{23}).$$

It is closed since

$$\begin{aligned} & \mathsf{b}_1(\mu(a_{01}, a_{12}, a_{23}; a_{02}, a_{13})) \\ = \; & \mathsf{b}_2(\mathsf{b}_2(a_{01}, a_{12}), a_{23}) + \mathsf{b}_2(\overline{a_{01}}, \mathsf{b}_2(a_{12}, a_{23})) + \mathsf{b}_1\mathsf{b}_3(a_{01}, a_{12}, a_{23}) \\ = \; & 0. \end{aligned}$$

Therefore the generalization of the triple Massey product for A_∞-algebras should be defined by the cocycle $\mu(a_{01}, a_{12}, a_{23}; a_{02}, a_{13})$.

We now study how it depends on the various choices made. First, a_{01}, a_{12}, a_{23} can each differ by a coboundary. Suppose, for example, $a'_{12} = a_{12} + \mathsf{b}_1(c_{12})$ is used instead, then we can choose $a'_{02} = a_{02} + \mathsf{b}_2(a_{01}, c_{12})$, $a'_{13} = a_{13} + \mathsf{b}_2(c_{12}, a_{23})$ and, accordingly, the difference in the cocycles is

$$\mu(a_{01}, a'_{12}, a_{23}; a'_{02}, a'_{13}) - \mu(a_{01}, a_{12}, a_{23}; a_{02}, a_{13}) = \mathsf{b}_1\mathsf{b}_3(a_{01}, c_{12}, a_{23}).$$

So the two cocycles represent the same class in $\mathrm{H}(A)$. In addition, each of a_{02}, a_{13} can differ by a cocycle; this results a difference in $\bar{\mathsf{b}}_2(\alpha_1, \mathrm{H}(A)) + \bar{\mathsf{b}}_2(\mathrm{H}(A), \alpha_3)$ of the class represented by $\mu(a_{01}, a_{12}, a_{23}; a_{02}, a_{13})$. We define the triple Massey product $\langle\alpha_1, \alpha_2, \alpha_3\rangle$ as the set of the cohomology classes which arises from all such choices of a_{02}, a_{13}. Thus $\langle\alpha_1, \alpha_2, \alpha_3\rangle$ is an element of $\mathrm{H}(A)/(\bar{\mathsf{b}}_2(\alpha_1, \mathrm{H}(A)) + \bar{\mathsf{b}}_2(\mathrm{H}(A), \alpha_3))$.

Finally, we establish the naturality of the triple Massey product. Let $\mathsf{f} = \{\mathsf{f}_n\} \colon A \to B$ be a morphism of A_∞-algebras. If $\alpha_1, \alpha_2, \alpha_3 \in \mathrm{H}(A)$ satisfy $\bar{\mathsf{b}}_2^A(\alpha_1, \alpha_2) = 0 = \bar{\mathsf{b}}_2^A(\alpha_2, \alpha_3)$, then $\beta_1 = (\mathsf{f}_1)_*\alpha_1, \beta_2 = (\mathsf{f}_1)_*\alpha_2, \beta_3 = (\mathsf{f}_1)_*\alpha_3 \in \mathrm{H}(B)$ satisfy the corresponding relations $\bar{\mathsf{b}}_2^B(\beta_1, \beta_2) = 0 = \bar{\mathsf{b}}_2^B(\beta_2, \beta_3)$. The latters are represented by $b_{01} = \mathsf{f}_1(a_{01}), b_{12} = \mathsf{f}_1(a_{12}), b_{23} = \mathsf{f}_1(a_{23}) \in B$, respectively. We can choose $b_{02} = \mathsf{f}_1(a_{02}) + \mathsf{f}_2(a_{01}, a_{12})$ and $b_{13} = \mathsf{f}_1(a_{13}) + \mathsf{f}_2(a_{12}, a_{23})$. A straightforward calculation shows that

$$\mu^B(b_{01}, b_{12}, b_{23}; b_{02}, b_{13}) = \mathsf{f}_1(\mu^A(a_{01}, a_{12}, a_{23}; a_{02}, a_{13})).$$

Thus $\langle\beta_1, \beta_2, \beta_3\rangle \supset (\mathsf{f}_1)_*\langle\alpha_1, \alpha_2, \alpha_3\rangle$.

4.2. **Matric A_∞-algebras.** If A is any vector space over $\boldsymbol{k}$, we denote by $\mathrm{Mat}^{(m)}(A)$ the set of $m \times m$ matrices with components in A. We write $\boldsymbol{a} = (a_{ij})_{1 \le i,j \le m}$, where $a_{ij} \in A$ is the (i,j)-component of $\boldsymbol{a} \in \mathrm{Mat}^{(m)}(A)$. Given two vector spaces A and B, we define a product $\odot \colon \mathrm{Mat}^{(m)}(A) \otimes \mathrm{Mat}^{(m)}(B) \to \mathrm{Mat}^{(m)}(A \otimes B)$ by

$$(\boldsymbol{a} \odot \boldsymbol{b})_{ij} = \sum_{k=1}^{m} a_{ik} \otimes b_{kj}, \quad 1 \le i, j \le m,$$

where $\boldsymbol{a} \in \mathrm{Mat}^{(m)}(A)$ and $\boldsymbol{b} \in \mathrm{Mat}^{(m)}(B)$. This product is associative under the natural identification of tensor products of vector spaces.

If A is an A_∞-algebra, then $\mathrm{Mat}^{(m)}(A)$ has a grading given by $\mathrm{Mat}^{(m)}(A)^p = \mathrm{Mat}^{(m)}(A^p)$ and there is a collection of $\boldsymbol{k}$-multilinear maps $\mathsf{b}_n^{(m)} \colon \mathrm{Mat}^{(m)}(A)^{\otimes n} \to \mathrm{Mat}^{(m)}(A)$ $(n \ge 1)$ defined by

$$\mathsf{b}_n^{(m)}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n) = \mathsf{b}_n(\boldsymbol{a}_1 \odot \cdots \odot \boldsymbol{a}_n),$$

where $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in \mathrm{Mat}^{(m)}(A)$ and b_n acts on $\mathrm{Mat}^{(m)}(A^{\otimes n})$ component-wise.

Proposition 4.1. *If $(A, \{\mathsf{b}_n\})$ is an A_∞-algebra, then so is $(\mathrm{Mat}^{(m)}(A), \{\mathsf{b}_n^{(m)}\})$.*

Proof: For any $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in \mathrm{Mat}^{(m)}(A)$, we have

$$\begin{aligned}
&\sum_{\substack{r,t \ge 0,\, s \ge 1 \\ r+s+t=n}} \mathsf{b}_{r+1+t}^{(m)}(\overline{\boldsymbol{a}_1}, \ldots, \overline{\boldsymbol{a}_r}, \mathsf{b}_s^{(m)}(\boldsymbol{a}_{r+1}, \ldots, \boldsymbol{a}_{r+s}), \boldsymbol{a}_{r+s+1}, \ldots, \boldsymbol{a}_n) \\
= &\sum_{\substack{r,t \ge 0,\, s \ge 1 \\ r+s+t=n}} \mathsf{b}_{r+1+t}(\overline{\boldsymbol{a}_1} \odot \cdots \odot \overline{\boldsymbol{a}_r} \odot \mathsf{b}_s(\boldsymbol{a}_{r+1} \odot \cdots \odot \boldsymbol{a}_{r+s}) \odot \boldsymbol{a}_{r+s+1} \odot \cdots \odot \boldsymbol{a}_n),
\end{aligned}$$

which is zero by Definition 2.1 since b_n acts on $\mathrm{Mat}^{(m)}(A)$ component-wise. $\square$

Definition 4.2. If $(A, \{\mathsf{b}_n\})$ is an A_∞-algebra, the *$m \times m$ matric A_∞-algebra* on A is $(\mathrm{Mat}^{(m)}(A), \{\mathsf{b}_n^{(m)}\})$.

More generally, if R is a ring that contains the field $\boldsymbol{k}$, then we have an A_∞-algebra $R \otimes_{\boldsymbol{k}} A$ that also has an R-module structure. The maps $\mathsf{b}_n^R \colon (R \otimes_{\boldsymbol{k}} A)^{\otimes n} \to R \otimes_{\boldsymbol{k}} A$ is given by

$$\mathsf{b}_n^R(r_1 \otimes a_1, \dots, r_n \otimes a_n) = (r_1 \cdots r_n) \otimes \mathsf{b}_n(a_1, \dots, a_n),$$

where $r_1, \dots, r_n \in R$ and $a_1, \dots, a_n \in A$. The matric A_∞-algebra is the special case when $R = \mathrm{Mat}^{(m)}(\boldsymbol{k})$.

If $\mathsf{f} = \{\mathsf{f}_n\} \colon A \to B$ is a morphism of A_∞-algebras, then we define $\boldsymbol{k}$-multilinear maps $\mathsf{f}_n^{(m)} \colon (\mathrm{Mat}^{(m)}(A))^{\otimes n} \to \mathrm{Mat}^{(m)}(B)$ for all $n \geq 1$ by

$$\mathsf{f}_n^{(m)}(\boldsymbol{a}_1, \dots, \boldsymbol{a}_n) = \mathsf{f}_n(\boldsymbol{a}_1 \odot \cdots \odot \boldsymbol{a}_n),$$

where $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n \in \mathrm{Mat}^{(m)}(A)$ and f_n acts on $\mathrm{Mat}^{(m)}(A^{\otimes n})$ component-wise.

Proposition 4.3. *If $\mathsf{f} = \{\mathsf{f}_n\} \colon A \to B$ is a morphism of A_∞-algebras, then so is $\mathsf{f}^{(m)} = \{\mathsf{f}_n^{(m)}\} \colon \mathrm{Mat}^{(m)}(A) \to \mathrm{Mat}^{(m)}(B)$.*

Proof: The proof is similar to that of Proposition 4.1, using Definition 2.2. □

If $\mathsf{f}, \mathsf{g} \colon A \to B$ are two morphisms of A_∞-algebras that are homotopic through $\mathsf{h} = \{\mathsf{h}_n\}$, then we can define $\boldsymbol{k}$-multilinear maps $\mathsf{h}_n^{(m)} \colon (\mathrm{Mat}^{(m)}(A))^{\otimes n} \to \mathrm{Mat}^{(m)}(B)$ for all $n \geq 1$ by

$$\mathsf{h}_n^{(m)}(\boldsymbol{a}_1, \dots, \boldsymbol{a}_n) = \mathsf{h}_n(\boldsymbol{a}_1 \odot \cdots \odot \boldsymbol{a}_n),$$

where $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n \in \mathrm{Mat}^{(m)}(A)$ and h_n acts on $\mathrm{Mat}^{(m)}(A^{\otimes n})$ component-wise.

Proposition 4.4. *If $\mathsf{f}, \mathsf{g} \colon A \to B$ are two morphisms of A_∞-algebras that are homotopic through $\mathsf{h} = \{\mathsf{h}_n\}$, then $\mathsf{f}^{(m)}, \mathsf{g}^{(m)} \colon \mathrm{Mat}^{(m)}(A) \to \mathrm{Mat}^{(m)}(B)$ are homotopic through $\mathsf{h}^{(m)} = \{\mathsf{h}_n^{(m)}\}$.*

Proof: The proof is similar to that of Proposition 4.1, using Definition 2.3. □

Let $\mathrm{Mat}_+^{(m)}(A) \subset \mathrm{Mat}^{(m)}(A)$ be the subspace of strictly upper-triangular matrices in A, that is, $\boldsymbol{a} = (a_{ij}) \in \mathrm{Mat}_+^{(m)}(A)$ if $a_{ij} = 0$ for all $i \geq j$. Since $\mathrm{Mat}_+^{(m)}(A)$ is preserved by the product $\odot$, we have

Corollary 4.5. *(i) If $(A, \{\mathsf{b}_n\})$ is an A_∞-algebra, then so is $(\mathrm{Mat}_+^{(m)}(A), \{\mathsf{b}_n^{(m)}\})$.*

(ii) If $\mathsf{f} = \{\mathsf{f}_n\} \colon A \to B$ is a morphism of A_∞-algebras, then so is

$$\mathsf{f}^{(m)} = \{\mathsf{f}_n^{(m)}\} \colon \mathrm{Mat}_+^{(m)}(A) \to \mathrm{Mat}_+^{(m)}(B).$$

(iii) If $\mathsf{f}, \mathsf{g} \colon A \to B$ are two morphisms of A_∞-algebras that are homotopic through $\mathsf{h} = \{\mathsf{h}_n\}$, then $\mathsf{f}^{(m)}, \mathsf{g}^{(m)} \colon \mathrm{Mat}_+^{(m)}(A) \to \mathrm{Mat}_+^{(m)}(B)$ are homotopic through $\mathsf{h}^{(m)} = \{\mathsf{h}_n^{(m)}\}$.

If A is a vector space over $\boldsymbol{k}$, then we regard elements of $A^{\oplus m} = \underbrace{A \oplus \ldots \oplus A}_{m \text{ times}}$ as column vectors and we write $\boldsymbol{x} = (x_i)_{1 \le i \le m}$, where $x_i \in A$. If A, B are two vector spaces, there is a multiplication $\odot$: $\mathrm{Mat}^{(m)}(A) \otimes B^{\oplus m} \to (A \otimes B)^{\oplus m}$ from the usual matrix multiplication on vectors. If $\boldsymbol{a} = (a_{ij}) \in \mathrm{Mat}^{(m)}(A)$ and $\boldsymbol{x} = (x_i)_{1 \le i \le m} \in B^{\oplus m}$, then $\boldsymbol{a} \odot \boldsymbol{x} = (\sum_{j=1}^m a_{ij} \otimes x_j)_{1 \le i \le m}$. This multiplication satisfies the standard associativity upon natural identification of tensor products of vector spaces. If A is an A_∞-algebra, then $A^{\oplus m}$ is an A_∞-module over $\mathrm{Mat}^{(m)}(A)$ or $\mathrm{Mat}_+^{(m)}(A)$. If $\boldsymbol{a} \in \mathrm{Mat}^{(m)}(A)$ and $\boldsymbol{x} \in A^{\oplus m}$, we write

$$\partial_{\boldsymbol{a}} \boldsymbol{x} = \sum_{n=0}^{\infty} \mathsf{b}_n(\underbrace{\boldsymbol{a} \odot \cdots \odot \boldsymbol{a}}_{n \text{ times}} \odot \boldsymbol{x}),$$

where b_n acts on the column vectors component-wise.

Lemma 4.6. *If $\boldsymbol{a} \in \mathrm{Mat}^{(m)}(A)$ and $\boldsymbol{x} \in A^{\oplus m}$, let $\tilde{\boldsymbol{a}} = \left(\begin{smallmatrix} \boldsymbol{a} & \boldsymbol{x} \\ 0 & 0 \end{smallmatrix}\right) \in \mathrm{Mat}^{(m+1)}(A)$. Then $\partial_{\tilde{\boldsymbol{a}}} \tilde{\boldsymbol{a}} = \left(\begin{smallmatrix} \partial_{\boldsymbol{a}} \boldsymbol{a} & \partial_{\boldsymbol{a}} \boldsymbol{x} \\ 0 & 0 \end{smallmatrix}\right)$.*

Proof: This follows from $\overbrace{\tilde{\boldsymbol{a}} \odot \cdots \odot \tilde{\boldsymbol{a}}}^{n \text{ times}} = \left(\begin{smallmatrix} \overbrace{\boldsymbol{a} \odot \cdots \odot \boldsymbol{a}}^{n \text{ times}} & \overbrace{\boldsymbol{a} \odot \cdots \odot \boldsymbol{a}}^{n-1 \text{ times}} \odot \boldsymbol{x} \\ 0 & 0 \end{smallmatrix}\right)$ for any $n \ge 1$. □

Finally, using the inclusions

$$\cdots \subset \mathrm{Mat}^{(m)}(A) \subset \mathrm{Mat}^{(m+1)}(A) \subset \cdots \quad \text{and} \quad \cdots \subset \mathrm{Mat}_+^{(m)}(A) \subset \mathrm{Mat}_+^{(m+1)}(A) \subset \cdots,$$

we get the A_∞-algebras $(\mathrm{Mat}^{(\infty)}(A), \{\mathsf{b}_n^{(\infty)}\})$ and $(\mathrm{Mat}_+^{(\infty)}(A), \{\mathsf{b}_n^{(\infty)}\})$ as direct limits. The results in the section hold also for $m = \infty$.

4.3. **Higher Massey products.** We generalize the triple Massey product for A_∞-algebras [30] discussed in §4.1 to higher Massey products of m cohomology classes. We now use the labeling $0 \le i, j \le m$ for the components of $\boldsymbol{a} = (a_{ij}) \in \mathrm{Mat}^{(m+1)}(A)$. If $\boldsymbol{a}, \boldsymbol{a}' \in \mathrm{Mat}_+^{(m+1)}(A)$, we write $\boldsymbol{a} \approx \boldsymbol{a}'$ if $a_{ij} = a'_{ij}$ for all i, j except $i = 0, j = m$. We have a simple

Lemma 4.7. *If $\boldsymbol{a}, \boldsymbol{a}' \in \mathrm{Mat}_+^{(m+1)}(A)$ and $\boldsymbol{b}, \boldsymbol{b}' \in \mathrm{Mat}_+^{(m+1)}(B)$ satisfy $\boldsymbol{a} \approx \boldsymbol{a}'$ and $\boldsymbol{b} \approx \boldsymbol{b}'$, then $\boldsymbol{a} \odot \boldsymbol{b} = \boldsymbol{a}' \odot \boldsymbol{b}'$.*

Definition 4.8. Let $(A, \{\mathsf{b}_n\})$ be an A_∞-algebra and $m \ge 2$. The matrix $\boldsymbol{a} = (a_{ij})_{0 \le i,j \le m} \in \mathrm{Mat}_+^{(m+1)}(A)$ is a *defining system* for $\alpha_1, \ldots, \alpha_m \in \mathrm{H}(A)$ if

(i) $\partial_{\boldsymbol{a}} \boldsymbol{a} \approx 0$; let $\mu(\boldsymbol{a}) = (\partial_{\boldsymbol{a}} \boldsymbol{a})_{0m} \in A$;

(ii) each α_i $(i = 1, \ldots, m)$ is represented by $a_{i-1,i} \in A$, which is closed by (i).

When $m = 2$, a defining system of $\alpha_1, \alpha_2 \in \mathrm{H}(A)$ is simply a choice of representatives $a_{01}, a_{12} \in A$ of α_1, α_2, respectively, and $\mu(a_{01}, a_{12}) = \mathsf{b}_2(a_{01}, a_{12})$ is a cocycle representing the class $\bar{\mathsf{b}}_2(a_{01}, a_{12}) \in \mathrm{H}(A)$. The case $m = 3$ is about the triple Massey product discussed in §4.1. For a general m, the formula is

$$\mu(\boldsymbol{a}) = \sum_{r=1}^{m} \sum_{0=i_0<i_1<\cdots<i_r=m} \mathsf{b}_r(a_{i_0i_1}, a_{i_1i_2}, \cdots, a_{i_{r-1}i_r}).$$

We remark that if $\boldsymbol{a} \in \mathrm{Mat}_+^{(m+1)}(A)$, then

$$\partial_{\boldsymbol{a}}\boldsymbol{a} = \sum_{n=1}^{\infty} \mathsf{b}_n^{(m+1)}(\underbrace{\boldsymbol{a}, \dots, \boldsymbol{a}}_{n \text{ times}}) \in \mathrm{Mat}_+^{(m+1)}(A)$$

can be regarded as the "curvature" of $\boldsymbol{a}$ in the context of (non-associative) A_∞-algebras. Furthermore, $\mu(\boldsymbol{a}) = 0$ if and only if $\boldsymbol{a}$ is a twisting element of $\mathrm{Mat}_+^{(m+1)}(A)$. In this case, the equations $\partial_{\boldsymbol{a}}\boldsymbol{a} = 0$ generalizes the Maurer-Cartan equation for flat connections. In general, applying Corollary 3.2(i) to $\mathrm{Mat}_+^{(m+1)}(A)$, we get

$$\sum_{r,t\geq 0} \mathsf{b}_{r+1+t}(\underbrace{\bar{\boldsymbol{a}} \odot \cdots \odot \bar{\boldsymbol{a}}}_{r \text{ times}} \odot \partial_{\boldsymbol{a}}\boldsymbol{a} \odot \underbrace{\boldsymbol{a} \odot \cdots \odot \boldsymbol{a}}_{t \text{ times}}) = 0.$$

This is the non-associative, A_∞-algebraic version of the Bianchi identity. (See [14, 22, 2] for the case of differential graded algebras). Furthermore, by Corollary 3.2(iii), we get

$$\partial_{\bar{\boldsymbol{a}}}\partial_{\boldsymbol{a}}\boldsymbol{c} = -\sum_{r,t\geq 0} \mathsf{b}_{r+t+2}(\underbrace{\bar{\boldsymbol{a}} \odot \cdots \odot \bar{\boldsymbol{a}}}_{r \text{ times}} \odot \partial_{\boldsymbol{a}}\boldsymbol{a} \odot \underbrace{\boldsymbol{a} \odot \cdots \odot \boldsymbol{a}}_{t \text{ times}} \odot \boldsymbol{c}),$$

where $\boldsymbol{c}$ can be either in $\mathrm{Mat}^{(m+1)}(A)_+$ or in $A^{\oplus(m+1)}$. This reflects the familiar relation in geometry between the square of the connection and the curvature.

Proposition 4.9. *(i) If $\boldsymbol{a}$ is a defining system of $\alpha_1, \dots, \alpha_m \in \mathrm{H}(A)$, then $\mathsf{b}_1(\mu(\boldsymbol{a})) = 0$.*

(ii) If $\boldsymbol{a}'$ is another defining system and $\boldsymbol{a}' \approx \boldsymbol{a}$, then $\mu(\boldsymbol{a}') - \mu(\boldsymbol{a}) = \mathsf{b}_1(a'_{0m} - a_{0m})$ and hence $[\mu(\boldsymbol{a}')] = [\mu(\boldsymbol{a})] \in \mathrm{H}(A)$.

Proof: (i) Since $\partial_{\boldsymbol{a}}\boldsymbol{a} \approx 0$, by Lemma 4.7, the Bianchi identity reduces to $\mathsf{b}_1(\partial_{\boldsymbol{a}}\boldsymbol{a}) = 0$ and hence $\mathsf{b}_1(\mu(\boldsymbol{a})) = 0$.
(ii) is obvious.

Definition 4.10. The (higher) Massey product of the cohomology classes $\alpha_1, \dots, \alpha_m \in \mathrm{H}(A)$ is *defined* if there is a defining system $\boldsymbol{a}$ of them. The class $[\mu(\boldsymbol{a})] \in H(A)$ is the *(higher) Massey product* of $\alpha_1, \dots, \alpha_m$ *through the defining system* $\boldsymbol{a}$. The *(higher) Massey product* of $\alpha_1, \dots, \alpha_m \in \mathrm{H}(A)$ is the set $\langle \alpha_1, \dots, \alpha_m \rangle \subset \mathrm{H}(A)$ of elements $[\mu(\boldsymbol{a})]$, where $\boldsymbol{a}$ runs over all the defining systems of $\alpha_1, \dots, \alpha_m$.

We establish a property of the higher Massey product, generalizing the case when A is a differential graded algebra [14, 22].

Theorem 4.11. *Let A be an A_∞-algebra. Suppose the Massey product of $\alpha_1, \dots, \alpha_m \in \mathrm{H}(A)$ is defined. Then for any $\gamma \in \mathrm{H}(A)$, the Massey product of $\overline{\alpha_1}, \cdots, \overline{\alpha_{m-1}}, \bar{\mathsf{b}}_2(\alpha_m, \gamma)$ is also defined, and $\langle \overline{\alpha_1}, \cdots, \overline{\alpha_{m-1}}, \bar{\mathsf{b}}_2(\alpha_m, \gamma) \rangle \supset -\mathsf{b}_2(\langle \alpha_1, \cdots, \alpha_m \rangle, \gamma)$.*

Proof: Let $\boldsymbol{a} \in \mathrm{Mat}_+^{(m+1)}(A)$ be a defining system of $\alpha_1, \ldots, \alpha_m$. We write $\boldsymbol{a} = \left(\begin{smallmatrix} \boldsymbol{a}_0 & \boldsymbol{a}' \\ 0 & 0 \end{smallmatrix}\right)$, where $\boldsymbol{a}_0 \in \mathrm{Mat}_+^{(m)}(A)$ and $\boldsymbol{a}' \in A^{\oplus m}$. By Lemma 4.6, $\partial_{\boldsymbol{a}_0} \boldsymbol{a}_0 = 0$ in $\mathrm{Mat}_+^{(m)}(A)$ and $\partial_{\boldsymbol{a}_0} \boldsymbol{a}' = \left(\begin{smallmatrix} \mu(\boldsymbol{a}) \\ \boldsymbol{0} \end{smallmatrix}\right)$ in $A^{\oplus m}$. Let $c \in \ker \mathsf{b}_1 \subset A$ be a representative of γ and set $\boldsymbol{c} = \left(\begin{smallmatrix} \boldsymbol{0} \\ c \end{smallmatrix}\right) \in A^{\oplus(m+1)}$. Then $\partial_{\boldsymbol{a}} \boldsymbol{c} = \left(\begin{smallmatrix} \boldsymbol{b}' \\ 0 \end{smallmatrix}\right)$ for some $\boldsymbol{b}' \in A^{\oplus m}$. Let $\boldsymbol{b} = \left(\begin{smallmatrix} \overline{\boldsymbol{a}_0} & \boldsymbol{b}' \\ 0 & 0 \end{smallmatrix}\right) \in \mathrm{Mat}_+^{(m+1)}(A)$. Then $b_{i-1,i} = \overline{a_{i-1,i}}$ for $i = 1, \ldots, m-1$ and $b_{m-1,m} = \mathsf{b}_2(a_{m-1,m}, c)$, which are representatives of $\overline{\alpha_1}, \ldots, \overline{\alpha_{m-1}}, \bar{\mathsf{b}}_2(\alpha_m, \gamma) \in \mathrm{H}(A)$, respectively. Using Lemma 4.6 again, we get

$$\partial_{\boldsymbol{b}} \boldsymbol{b} = \begin{pmatrix} \partial_{\overline{\boldsymbol{a}_0}} \overline{\boldsymbol{a}_0} & \partial_{\overline{\boldsymbol{a}_0}} \boldsymbol{b}' \\ 0 & 0 \end{pmatrix} = (\boldsymbol{0} \quad \partial_{\bar{\boldsymbol{a}}} \partial_{\boldsymbol{a}} \boldsymbol{c})$$

since $0 = \overline{\partial_{\boldsymbol{a}_0} \boldsymbol{a}_0} = -\partial_{\overline{\boldsymbol{a}_0}} \overline{\boldsymbol{a}_0}$. By Corollary 3.2(iii) and Lemma 4.7, we have

$$\partial_{\bar{\boldsymbol{a}}} \partial_{\boldsymbol{a}} \boldsymbol{c} = -\bar{\mathsf{b}}_2(\partial_{\boldsymbol{a}} \boldsymbol{a} \odot \boldsymbol{c}) = -\begin{pmatrix} \mathsf{b}_2(\mu(\boldsymbol{a}), c) \\ \boldsymbol{0} \end{pmatrix} \in A^{\oplus(m+1)}.$$

So $\boldsymbol{b}$ is a defining system of $\overline{\alpha_1}, \ldots, \overline{\alpha_{m-1}}, \bar{\mathsf{b}}_2(\alpha_m, \gamma)$ and $\mu(\boldsymbol{b}) = -\mathsf{b}_2(\mu(\boldsymbol{a}), c)$. $\square$

A similar statement can be made for the Massey product of $\bar{\mathsf{b}}_2(\bar{\gamma}, \alpha_1), \alpha_2, \ldots, \alpha_m$.

4.4. **Homotopy equivalence of defining systems.** In Proposition 4.9, we saw that the cohomology class of $\mu(\boldsymbol{a})$ does not depend on the $(0,m)$-component of $\boldsymbol{a}$. In fact, it is invariant under a wider equivalence of the defining systems.

Definition 4.12. Let A be an A_∞-algebra. Two defining systems $\boldsymbol{a}, \boldsymbol{a}'$ of $\alpha_1, \ldots, \alpha_m \in \mathrm{H}(A)$ are *homotopic* (through $\boldsymbol{c}$) if there is $\boldsymbol{c} \in \mathrm{Mat}_+^{(m+1)}(A)$ such that

$$\boldsymbol{a}' - \boldsymbol{a} = \sum_{r,t \geq 0} \mathsf{b}_{r+1+t}(\underbrace{\overline{\boldsymbol{a}'} \odot \cdots \odot \overline{\boldsymbol{a}'}}_{r \text{ times}} \odot \boldsymbol{c} \odot \underbrace{\boldsymbol{a} \odot \cdots \odot \boldsymbol{a}}_{t \text{ times}}).$$

$\boldsymbol{a}$ and $\boldsymbol{a}'$ are *equivalent* if there is $\boldsymbol{c} \in \mathrm{Mat}_+^{(m+1)}(A)$ such that

$$\boldsymbol{a}' - \boldsymbol{a} \approx \sum_{r,t \geq 0} \mathsf{b}_{r+1+t}(\underbrace{\overline{\boldsymbol{a}'} \odot \cdots \odot \overline{\boldsymbol{a}'}}_{r \text{ times}} \odot \boldsymbol{c} \odot \underbrace{\boldsymbol{a} \odot \cdots \odot \boldsymbol{a}}_{t \text{ times}}).$$

Clearly, if $\boldsymbol{a} \approx \boldsymbol{a}'$ or if $\boldsymbol{a}$ and $\boldsymbol{a}'$ are homotopic, then they are equivalent. In fact, this equivalence is the weakest relation with this property. When $m = 3$, the equivalence of the defining systems reflects the ambiguity in defining the triple Massey product in §4.1.

If $\boldsymbol{a} \in \mathrm{Mat}_+^{(m+1)}(A)$, then the map $\tau_{\boldsymbol{a}}^{(m+1)} \colon \mathrm{Mat}_+^{(m+1)}(A) \to \overline{T}\mathrm{Mat}_+^{(m+1)}(A)$ was given in §3.1. We define another map $\tau_{\boldsymbol{a}} \colon \mathrm{Mat}_+^{(m+1)}(A) \to \mathrm{Mat}_+^{(m+1)}(\overline{T}A)$ by the composition of $\tau_{\boldsymbol{a}}^{(m+1)}$ with the product $\odot$. That is, we have

$$\tau_{\boldsymbol{a}} \colon \boldsymbol{b} \in \mathrm{Mat}_+^{(m+1)}(A) \mapsto \sum_{n=0}^{\infty} \underbrace{\boldsymbol{a} \odot \cdots \odot \boldsymbol{a}}_{n \text{ times}} \odot \boldsymbol{b}.$$

Let $\tau_{\boldsymbol{a}}(1) = 1 + \tau_{\boldsymbol{a}}(\boldsymbol{a}) \in \mathrm{Mat}_+^{(m+1)}(\boldsymbol{k} \oplus \overline{T}A)$. We recall that b acts on $\mathrm{Mat}_+^{(m+1)}(\overline{T}A)$ component-wise.

Lemma 4.13. *Let $(A,\{\mathsf{b}_n\})$ be an A_∞-algebra.*

(i) If $\boldsymbol{a}\in \mathrm{Mat}_+^{(m+1)}(A)$ is a defining system, then $\mathsf{b}\circ\tau_{\boldsymbol{a}}=\tau_{\bar{\boldsymbol{a}}}\circ\partial_{\boldsymbol{a}}$ on $\mathrm{Mat}_+^{(m+1)}(A)$.

(ii) Two defining systems $\boldsymbol{a},\boldsymbol{a}'\in \mathrm{Mat}_+^{(m+1)}(A)$ are homotopic through $\boldsymbol{c}$ if and only if

$$\tau_{\boldsymbol{a}'}(\boldsymbol{a}')-\tau_{\boldsymbol{a}}(\boldsymbol{a})=\mathsf{b}(\tau_{\overline{\boldsymbol{a}'}}(1)\odot\boldsymbol{c}\odot\tau_{\boldsymbol{a}}(1)).$$

Proof: (i) By Lemma 3.1(i), we have

$$\mathsf{b}^{(m+1)}\circ\tau_{\boldsymbol{a}}^{(m+1)}=\tau_{\bar{\boldsymbol{a}}}^{(m+1)}\circ\partial_{\boldsymbol{a}}+\tau_{\bar{\boldsymbol{a}}}^{(m+1)}(\partial_{\boldsymbol{a}}\boldsymbol{a})\circ\tau_{\boldsymbol{a}}^{(m+1)}.$$

Taking the product $\odot$, the second term on the right hand side vanishes by Lemma 4.7 and we get the result.

(ii) Following the proof of Lemma 3.10(i), we get

$$\begin{aligned}
&\mathsf{b}^{(m+1)}(\overline{\tau_{\boldsymbol{a}'}^{(m+1)}(1)}\otimes\boldsymbol{c}\otimes\tau_{\boldsymbol{a}}^{(m+1)}(1))\\
=\ &\tau_{\boldsymbol{a}'}^{(m+1)}(1)\otimes\overline{\partial_{\boldsymbol{a}'}\boldsymbol{a}'}\otimes\overline{\tau_{\boldsymbol{a}'}^{(m+1)}(1)}\otimes\boldsymbol{c}\otimes\tau_{\boldsymbol{a}}^{(m+1)}(1)+\tau_{\boldsymbol{a}'}^{(m+1)}(1)\otimes(\boldsymbol{a}'-\boldsymbol{a})\otimes\tau_{\boldsymbol{a}}^{(m+1)}(1)\\
&-\tau_{\boldsymbol{a}'}^{(m+1)}(1)\otimes\bar{\boldsymbol{c}}\otimes\overline{\tau_{\boldsymbol{a}}^{(m+1)}(1)}\otimes\partial_{\boldsymbol{a}}\boldsymbol{a}\otimes\tau_{\boldsymbol{a}}^{(m+1)}(1).
\end{aligned}$$

Taking the product $\odot$, the first and third terms on the right hand side vanish by Lemma 4.7 since $\partial_{\boldsymbol{a}}\boldsymbol{a}\approx\partial_{\boldsymbol{a}'}\boldsymbol{a}'\approx 0$. The result follows. $\square$

Theorem 4.14. *If $\boldsymbol{a}$ and $\boldsymbol{a}'\in\mathrm{Mat}_+^{(m+1)}(A)$ are two equivalent defining systems, then $[\mu(\boldsymbol{a}')]=[\mu(\boldsymbol{a})]\in \mathrm{H}(A)$.*

Proof: By Proposition 4.9(ii), it suffices to show the result when $\boldsymbol{a}$ and $\boldsymbol{a}'$ are homotopic. By Lemma 4.13(i) and Lemma 4.7, we have

$$\mathsf{b}(\tau_{\boldsymbol{a}}(\boldsymbol{a}))=\tau_{\boldsymbol{a}}(\partial_{\boldsymbol{a}}\boldsymbol{a})=\partial_{\boldsymbol{a}}\boldsymbol{a}.$$

So, applying b to the formula in Lemma 4.13(ii), we get $\partial_{\boldsymbol{a}'}\boldsymbol{a}'=\partial_{\boldsymbol{a}}\boldsymbol{a}$ and $\mu(\boldsymbol{a}')=\mu(\boldsymbol{a})$. $\square$

4.5. **Naturality of the higher Massey products.** Given a morphism $\mathsf{f}=\{\mathsf{f}_n\}\colon A\to B$ of A_∞-algebras, the induced morphism $\mathsf{f}^{(m+1)}=\{\mathsf{f}_n^{(m+1)}\}\colon \mathrm{Mat}^{(m+1)}(A)\to\mathrm{Mat}^{(m+1)}(B)$ of A_∞-algebras determines to a morphism $\mathsf{F}^{(m+1)}\colon \overline{T}\mathrm{Mat}^{(m+1)}(A)\to\overline{T}\mathrm{Mat}^{(m+1)}(B)$ of coalgebras. For any $\boldsymbol{a},\boldsymbol{b}\in\mathrm{Mat}^{(m+1)}(A)$, we have

$$\mathsf{F}_{\boldsymbol{a}}^{(m+1)}(\boldsymbol{b})=\sum_{n=0}^{\infty}\mathsf{f}_n(\underbrace{\boldsymbol{a}\odot\cdots\odot\boldsymbol{a}}_{n\text{ times}}\odot\boldsymbol{b}),$$

which we denote by $\mathsf{F}_{\boldsymbol{a}}(\boldsymbol{b})$ for short.

Theorem 4.15. *Suppose $\mathsf{f}\colon A\to B$ of A_∞-algebras and $m\geq 2$.*

(i) If $\boldsymbol{a}=(a_{ij})_{0\leq i,j\leq m}\in\mathrm{Mat}_+^{(m+1)}(A)$ is a defining system of $\alpha_1,\dots,\alpha_m\in\mathrm{H}(A)$, then $\mathsf{F}_{\boldsymbol{a}}(\boldsymbol{a})\in\mathrm{Mat}_+^{(m+1)}(B)$ is a defining system of $(\mathsf{f}_1)_*\alpha_1,\dots,(\mathsf{f}_1)_*\alpha_m\in\mathrm{H}(B)$, and $\mu^B(\mathsf{F}_{\boldsymbol{a}}(\boldsymbol{a}))=\mathsf{f}_1(\mu^A(\boldsymbol{a}))$.

(ii) If $\boldsymbol{a}$ and $\boldsymbol{a}'$ are two equivalent defining systems, then so are $\mathsf{F}_{\boldsymbol{a}}(\boldsymbol{a})$ and $\mathsf{F}_{\boldsymbol{a}'}(\boldsymbol{a}')$.

Proof: (i) Using Lemma 4.13(i) and following the proof of Theorem 3.7(ii), we have

$$\partial_{\mathsf{F}_{\boldsymbol{a}}(\boldsymbol{a})}(\mathsf{F}_{\boldsymbol{a}}(\boldsymbol{a})) = \mathsf{F}_{\boldsymbol{a}}(\partial_{\boldsymbol{a}}\boldsymbol{a}).$$

Since $\partial_{\boldsymbol{a}}\boldsymbol{a} \approx 0$, the right hand side is equal to $\mathsf{f}_1(\partial_{\boldsymbol{a}}\boldsymbol{a}) \approx 0$. Therefore $\mathsf{F}_{\boldsymbol{a}}(\boldsymbol{a})$ is a defining system and $\mu^B(\mathsf{F}_{\boldsymbol{a}}(\boldsymbol{a})) = \mathsf{f}_1(\mu^A(\boldsymbol{a}))$. Next, we note that $(\mathsf{F}_{\boldsymbol{a}}(\boldsymbol{a}))_{i-1,i} = \mathsf{f}_1(a_{i-1,i})$. Since $a_{i-1,i}$ is closed and represents α_i (for each $1 \le i \le m$), we get $\mathsf{b}_1^B(\mathsf{f}_1(a_{i-1,i})) = \mathsf{f}_1(\mathsf{b}_1^A(a_{i-1,i})) = 0$ and that $\mathsf{f}_1(a_{i-1,i})$ represents $(\mathsf{f}_1)_*\alpha_i$.
(ii) First, if $\boldsymbol{a} \approx \boldsymbol{a}'$, then $\mathsf{F}_{\boldsymbol{a}}(\boldsymbol{a}) \approx \mathsf{F}_{\boldsymbol{a}'}(\boldsymbol{a}')$. If $\boldsymbol{a}$ and $\boldsymbol{a}'$ are homotopic through $\boldsymbol{c}$, then by the proofs of Lemma 3.13(i) and Theorem 3.14(i) (which can be adjusted without assuming twisting elements), we get

$$\begin{aligned}&\mathsf{F}_{\boldsymbol{a}'}(\boldsymbol{a}') - \mathsf{F}_{\boldsymbol{a}}(\boldsymbol{a})\\ = &\sum_{r,t\ge 0} \mathsf{b}^B_{r+1+t}(\underbrace{\overline{\mathsf{F}_{\boldsymbol{a}'}(\boldsymbol{a}')} \odot \cdots \odot \overline{\mathsf{F}_{\boldsymbol{a}'}(\boldsymbol{a}')}}_{r \text{ times}} \odot \mathsf{F}_{\boldsymbol{a}',\boldsymbol{a}}(\boldsymbol{c}) \odot \underbrace{\mathsf{F}_{\boldsymbol{a}}(\boldsymbol{a}) \odot \cdots \odot \mathsf{F}_{\boldsymbol{a}}(\boldsymbol{a})}_{t \text{ times}}),\end{aligned}$$

where

$$\mathsf{F}_{\boldsymbol{a}',\boldsymbol{a}}(\boldsymbol{c}) = \sum_{r,t\ge 0} \mathsf{f}_{r+1+t}(\underbrace{\overline{\boldsymbol{a}'} \odot \cdots \odot \overline{\boldsymbol{a}'}}_{r \text{ times}} \odot \boldsymbol{c} \odot \underbrace{\boldsymbol{a} \odot \cdots \odot \boldsymbol{a}}_{t \text{ times}}).$$

By Lemma 4.13(ii), $\mathsf{F}_{\boldsymbol{a}}(\boldsymbol{a})$ and $\mathsf{F}_{\boldsymbol{a}'}(\boldsymbol{a}')$ are homotopic through $\mathsf{F}_{\boldsymbol{a}',\boldsymbol{a}}(\boldsymbol{c})$. □

Corollary 4.16. *Let $\mathsf{f}\colon A \to B$ be a morphism of A_∞-algebras. If the Massey product of $\alpha_1 \dots, \alpha_m \in \mathrm{H}(A)$ is defined, then so is that of $(\mathsf{f}_1)_*(\alpha_1) \dots, (\mathsf{f}_1)_*(\alpha_m) \in \mathrm{H}(B)$, and $\langle (\mathsf{f}_1)_*(\alpha_1) \dots, (\mathsf{f}_1)_*(\alpha_m)\rangle \supset (\mathsf{f}_1)_*\langle \alpha_1 \dots, \alpha_m\rangle$.*

This generalizes the naturality of the triple Massey product in §4.1.

Recall that there is an A_∞-structure $\{\bar{\mathsf{b}}_n\}$ on $\mathrm{H}(A)$ after choosing the maps p_1 and q_1 (§2.4). When A is a differential graded algebra, it was a folklore that the A_∞-structure on $\mathrm{H}(A)$ gives the Massey products [29, 7, 26, 17]. The precise statement seems to be a long standing puzzle.[3] We now establish the exact relationship in the more general context of A_∞-algebras.

Proposition 4.17. *Let A be an A_∞-algebra and $\alpha_1, \dots, \alpha_m \in \mathrm{H}(A)$. If $\bar{\mathsf{b}}_{j-i}(\alpha_{i+1}, \dots, \alpha_j) = 0$ for any i, j satisfying $0 \le i < j \le m$, $j - i < m$, then the Massey product of $\alpha_1, \dots, \alpha_m$ is defined and $\bar{\mathsf{b}}_m(\alpha_1, \dots, \alpha_m) \in \langle \alpha_1, \dots, \alpha_m\rangle$.*

Proof: Notice that $\mathrm{H}(\mathrm{H}(A)) = \mathrm{H}(A)$ as $\bar{\mathsf{b}}_1 = 0$. Let $\boldsymbol{\alpha} \in \mathrm{Mat}_+^{(m+1)}(\mathrm{H}(A))$ be given by $\alpha_{i-1,i} = \alpha_i$ $(1 \le i \le m)$ and $\alpha_{ij} = 0$ if $j \ne i+1$. By the assumption, $\boldsymbol{\alpha}$ is a defining system of

[3] It was claimed in Theorem 3.1 of [17] that when A is a differential graded algebra, the conclusion of Proposition 4.17 holds under a weaker assumption that for any $j-i < m$, the Massey product of $\alpha_{i+1}, \dots, \alpha_j$ is defined and contains 0. We think that the stronger condition $\bar{\mathsf{b}}_{j-i}(\alpha_{i+1}, \dots, \alpha_j) = 0$ as in Propistion 4.17 is necessary even when A is a differential graded algebra.

$\alpha_1,\dots,\alpha_m \in \mathrm{H}(\mathrm{H}(A)) = \mathrm{H}(A)$ and $\mu^{\mathrm{H}(A)}(\boldsymbol{\alpha}) = \bar{\mathsf{b}}_m(\alpha_1,\dots,\alpha_m)$. It suffices to show that the Massey product of $\alpha_1,\dots,\alpha_m$ is defined and contains $\mu^{\mathrm{H}(A)}(\boldsymbol{\alpha})$. Applying Theorem 4.15(i) to the quasi-isomorphism $\mathsf{q}\colon \mathrm{H}(A) \to A$ in §2.4, we get a defining system $\boldsymbol{a} = \mathsf{Q}_{\boldsymbol{\alpha}}(\boldsymbol{\alpha}) \in \mathrm{Mat}_+^{(m+1)}(A)$ of $[\mathsf{q}_1(\alpha_i)] = \alpha_i$ $(1 \le i \le m)$. More explicitly, $a_{ij} = \mathsf{q}_{j-i}(\alpha_{i+1},\dots,\alpha_j)$ for $0 \le i < j \le m$. Therefore the Massey product of $\alpha_1,\dots,\alpha_m$ is defined. Furthermore, $\mathsf{q}_1(\mu^{\mathrm{H}(A)}(\boldsymbol{\alpha})) = \mu^A(\boldsymbol{a})$ and hence $\mu^{\mathrm{H}(A)}(\boldsymbol{\alpha}) = [\mu^A(\boldsymbol{a})] \in \langle \alpha_1,\dots,\alpha_m\rangle$. □

It is clear from the proof that the Massey product of $\alpha_1,\dots,\alpha_m$ as elements of $\mathrm{H}(\mathrm{H}(A))$ through the defining system $\boldsymbol{\alpha}$ is identical to that as elements of $\mathrm{H}(A)$ through the defining system $\boldsymbol{a} = \mathsf{Q}_{\boldsymbol{\alpha}}(\boldsymbol{\alpha})$.

Finally, we consider two morphisms $\mathsf{f},\mathsf{g}\colon A \to B$ of A_∞-algebras that are homotopic through h. Recall that $\mathsf{f}^{(m+1)}, \mathsf{g}^{(m+1)}\colon \mathrm{Mat}_+^{(m+1)}A \to \mathrm{Mat}_+^{(m+1)}(B)$ are morphisms of A_∞-algebras that are homotopic through $\mathsf{h}^{(m+1)}$. As before, we use $\mathsf{H}_{\boldsymbol{a}}(\boldsymbol{a})$ to denote

$$\mathsf{H}_{\boldsymbol{a}}^{(m+1)}(\boldsymbol{a}) = \sum_{n=1}^{\infty} \mathsf{h}_n(\underbrace{\boldsymbol{a}\odot\cdots\odot\boldsymbol{a}}_{n \text{ times}}).$$

Theorem 4.18. *Suppose $\mathsf{f},\mathsf{g}\colon A \to B$ are two morphisms of A_∞-algebras that are homotopic through h. Let $\boldsymbol{a} \in \mathrm{Mat}_+^{(m+1)}(A)$ be a defining system of $\alpha_1 \dots,\alpha_m \in \mathrm{H}(A)$, where $m \ge 2$. Then $\mathsf{F}_{\boldsymbol{a}}(\boldsymbol{a})$ and $\mathsf{G}_{\boldsymbol{a}}(\boldsymbol{a})$ are two equivalent defining systems of $(\mathsf{f}_1)_*\alpha_i = (\mathsf{g}_1)_*\alpha_i \in \mathrm{H}(B)$ $(1 \le i \le m)$ and hence define the same Massey product.*

Proof: Using Lemma 3.15(i) and following the proof of Theorem 3.16(i), we get

$$\begin{aligned} &\tau_{\mathsf{G}_{\boldsymbol{a}}(\boldsymbol{a})}(\mathsf{G}_{\boldsymbol{a}}(\boldsymbol{a})) - \tau_{\mathsf{F}_{\boldsymbol{a}}(\boldsymbol{a})}(\mathsf{F}_{\boldsymbol{a}}(\boldsymbol{a})) \\ =\ &\mathsf{b}^B(\tau^B_{\overline{\mathsf{G}_{\boldsymbol{a}}(\boldsymbol{a})}} \odot \mathsf{H}_{\boldsymbol{a}}(\boldsymbol{a}) \odot \tau^B_{\mathsf{F}_{\boldsymbol{a}}(\boldsymbol{a})}) + \mathsf{H}(\tau^A_{\boldsymbol{a}}(\partial_{\boldsymbol{a}}\boldsymbol{a})). \end{aligned}$$

Since $\mathsf{H}(\tau^A_{\boldsymbol{a}}(\partial_{\boldsymbol{a}}\boldsymbol{a})) = \mathsf{H}(\partial_{\boldsymbol{a}}\boldsymbol{a}) \approx 0$, the result follows from Lemma 4.13(ii) and Theorem 4.14. □

4.6. **Massey products on the cohomology of C_∞-algebras.** Let $(A,\{\mathsf{b}_n\})$ be a C_∞-algebra. We consider first the triple Massey product (§4.1). Let $\alpha_1,\alpha_2,\alpha_3 \in \mathrm{H}(A)$ and let $a_{01},a_{12},a_{23} \in A$ be their representatives, respectively. Suppose $\bar{\mathsf{b}}_2(\alpha_1,\alpha_2) = \bar{\mathsf{b}}_2(\alpha_2,\alpha_3) = \bar{\mathsf{b}}_2(\alpha_3,\alpha_1) = 0$. Then there exist $a_{02},a_{13},a'_{21} \in A$ such that $\mathsf{b}_2(a_{01},a_{12}) = -\mathsf{b}_1(a_{02})$, $\mathsf{b}_2(a_{12},a_{23}) = -\mathsf{b}_1(a_{13})$ and $\mathsf{b}_2(a_{23},a_{01}) = -\mathsf{b}_1(a'_{21})$. Recall that

$$[\alpha_1\otimes\alpha_2] \bowtie \alpha_3 = \alpha_1\otimes\alpha_2\otimes\alpha_3 + (-1)^{|\alpha_2||\alpha_3|}\alpha_1\otimes\alpha_3\otimes\alpha_2 + (-1)^{(|\alpha_1|+|\alpha_2|)|\alpha_3|}\alpha_3\otimes\alpha_1\otimes\alpha_2;$$

the same formula can be written for $[a_{01}\otimes a_{12}] \bowtie a_{23}$. We can check, using $\mathsf{b}_2(A \bowtie A) = 0$ and $\mathsf{b}_3([A\otimes A]\bowtie A) = 0$, that

$$\begin{aligned} \mu(a_{01},a_{12},a_{23};a_{02},a_{13}) &+ (-1)^{|\alpha_2||\alpha_3|}\mu(a_{01},a_{23},a_{12};(-1)^{|\alpha_1||\alpha_3|}a'_{21},(-1)^{|\alpha_2||\alpha_3|}a_{13}) \\ &+ (-1)^{(|\alpha_1|+|\alpha_2|)|\alpha_3|}\mu(a_{23},a_{01},a_{12};a'_{21},a_{02}) = 0. \end{aligned}$$

Consequently, we have

$$0 \in \langle \alpha_1, \alpha_2, \alpha_3 \rangle + (-1)^{|\alpha_2||\alpha_3|} \langle \alpha_1, \alpha_3, \alpha_2 \rangle + (-1)^{(|\alpha_1|+|\alpha_2|)|\alpha_3|} \langle \alpha_3, \alpha_1, \alpha_2 \rangle.$$

Similarly, corresponding to $\alpha_1 \bowtie [\alpha_2 \otimes \alpha_3]$, we have

$$0 \in \langle \alpha_1, \alpha_2, \alpha_3 \rangle + (-1)^{|\alpha_1||\alpha_2|} \langle \alpha_2, \alpha_1, \alpha_3 \rangle + (-1)^{(|\alpha_2|+|\alpha_3|)|\alpha_1|} \langle \alpha_2, \alpha_3, \alpha_1 \rangle.$$

We now consider higher Massey products. Recall the notations $S_{r,n}$ and $\varepsilon(\sigma)$ in §2.3.

Proposition 4.19. *Let A be a C_∞-algebra and let $\alpha_1, \dots, \alpha_m \in \mathrm{H}(A)$. Fix $r < m$. Suppose for any $\sigma \in S_{r,m}$ and any $i, j = 0, \dots, m$ with $0 < j-i < m$, we have $\bar{\mathsf{b}}_{j-i}(\alpha_{\sigma(i+1)}, \dots, \alpha_{\sigma(j)}) = 0$. Then the Massey product of $\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(m)}$ is defined for any $\sigma \in S_{r,m}$ and we have*

$$0 \in \sum_{\sigma \in S_{r,m}} (-1)^{\varepsilon(\sigma)} \langle \alpha_{\sigma(1)}, \dots, \alpha_{\sigma(m)} \rangle.$$

Proof: By Proposition 4.17, the assumption implies that for any $\sigma \in S_{r,m}$, the Massey product of $\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(m)}$ is defined and $\bar{\mathsf{b}}_m(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(m)}) \in \langle \alpha_{\sigma(1)}, \dots, \alpha_{\sigma(m)} \rangle$. Since $(\mathrm{H}(A), \{\bar{\mathsf{b}}_n\})$ is a C_∞-algebra [5], we have

$$0 = \sum_{\sigma \in S_{r,m}} (-1)^{\varepsilon(\sigma)} \bar{\mathsf{b}}_m(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(m)})$$

and the result follows. □

5. Spectral sequence and higher Massey products

5.1. **A spectral sequence for the twisted cohomology.** So far, the A_∞-algebra A is either $\mathbb{Z}$- or $\mathbb{Z}_2$-graded. In both cases, we write $A = A^{\bar{0}} \oplus A^{\bar{1}}$, where $A^{\bar{0}}, A^{\bar{1}}$ are the even, odd parts of A, respectively. Here $\bar{k}$ means the integer k modulo 2. If $h \in A^{\bar{0}}$ is a twisting element, then the twisted differential $\partial_h \colon A^{\bar{k}} \to A^{\overline{k+1}}$ defines a $\mathbb{Z}_2$-graded cochain complex $(A^\bullet, \partial_h)$ and the twisted cohomology $\mathrm{H}_h(A)$ is always $\mathbb{Z}_2$-graded.

We now assume that A is $\mathbb{Z}$-graded, i.e., $A = \bigoplus_{k \in \mathbb{Z}} A^k$. If the degree of the twisting element h is non-negative, then both the component h_0 in A^0 and the positive-degree part $h - h_0$ are twisting elements. The twisted cohomology groups defined by h_0 and by $h - h_0$ behave very differently (see [20] for the case of twisted de Rham complex[4]). We now assume $h_0 = 0$, i.e., the twisting element h has positive degree. Denote the twisted differential ∂_h on A by d. Then there is a natural filtration F of the $\mathbb{Z}_2$-graded cochain complex $(A^\bullet, \mathsf{d})$ given by $\mathrm{F}^p A = \bigoplus_{n=p}^\infty A^n$, or

$$\mathrm{F}^p A^{\bar{k}} = \bigoplus_{\substack{n \ge p \\ n = k \bmod 2}} A^n$$

[4] In geometry, ∂_h is a superconnection while ∂_{h_0} is a usual connection, as $\mathsf{s}^{-1} h_0$ is of degree 1.

(see [27, 1, 20, 16] when A is the de Rham complex). The graded components are $\mathrm{Gr}^p A = A^p$, or

$$\mathrm{Gr}^p A^{\bar{k}} = \begin{cases} A^p, & \text{if } p = k \bmod 2, \\ 0, & \text{if } p \neq k \bmod 2. \end{cases}$$

Lemma 5.1. *There is a spectral sequence $\{E_r^{p\bar{q}}, \mathsf{d}_r\}$ converging to twisted cohomology $\mathrm{H}_h(A)$. Moreover,*

(i) $E_r^{p\bar{1}} = 0$ for any $p \in \mathbb{Z}$ and $r \geq 0$;

(ii) $\mathsf{d}_r = 0$ if r is even;

(iii) $E_{2m}^{p\bar{0}} = E_{2m+1}^{p\bar{0}}$ for any $p \in \mathbb{Z}$ and $m \geq 1$.

(iv) $E_2^{p\bar{0}} = E_3^{p\bar{0}} = \mathrm{H}^p(A)$.

Proof: The filtration is clearly exhaustive and weakly convergent. By Theorem 3.2 of [23], the corresponding spectral sequence converges to the twisted cohomology group.
(i) $E_r^{p\bar{1}} = 0$ for all $r \geq 0$ since $E_0^{p\bar{1}} = \mathrm{Gr}^p A^{\overline{p+1}} = 0$.
(ii) If r is even, then either q or $q - r + 1$ is odd. So $\mathsf{d}_r \colon E_r^{p\bar{q}} \to E_r^{p+r,\overline{q-r+1}}$ is zero.
(iii) follows from (ii).
(iv) We have $E_1^{p\bar{0}} = E_0^{p\bar{0}} = A^p$ and $\mathsf{d}_1 \colon A^p \to A^{p+1}$ is the (untwisted) differential $\partial = \mathsf{b}_1$. So $E_2^{p\bar{0}} = \mathrm{H}^p(A)$ and the rest follows from (iii). □

We postpone the discussion on the general d_r to §5.3. Instead, generalizing the work on the twisted de Rham complex [27, 1], we describe the differentials d_3 and d_5 when A is an A_∞-algebra. For simplicity, we assume that the twisting element h is of homogeneous degree 2 (corresponding a closed 3-form for the de Rham complex). Then $\mathsf{b}_n(\underbrace{h, \ldots, h}_{n \text{ times}}) = 0$ for any $n \geq 1$. Any element of $E_3^{p\bar{0}}$ is a class $[x] \in \mathrm{H}^p(A)$ represented by a closed element $x \in A^p$, i.e., $\mathsf{b}_1(x) = 0$. The map d_3 is given by $\mathsf{d}_3[x] = [\mathsf{b}_2(h, x)]$. It is easy to check that $\mathsf{b}_2(h, x)$ is closed (since h and x both are) and the class $[\mathsf{b}_2(h, x)] \in \mathrm{H}^{p+3}(A)$ does not depend on the choice of the representative of $[x]$.

If $\mathsf{d}_3[x] = [0]$, i.e., $\mathsf{b}_2(h, x) = -\mathsf{b}_1(x')$ for some $x' \in A^{p+2}$, then $[x]$ represents a class $[x]_5 \in E_4^{p\bar{0}} = E_5^{p\bar{0}}$. When A is a differential graded algebra, then $\mathsf{d}_5[x]_5 = [\mathsf{b}_2(h, x')]_5$ (see [27, 1] for the case of de Rham complex). But for a general A_∞-algebra, we claim that $\mathsf{d}_5[x]_5 = [\mathsf{b}_2(h, x') + \mathsf{b}_3(h, h, x)]_5$. Note that $\mathsf{b}_2(h, x') + \mathsf{b}_3(h, h, x) = \mu(h, h, x; 0, x')$ is closed and represents a class in the triple Massey product $\langle [h], [h], [x] \rangle$. Furthermore,

$$\mathsf{b}_2(h, \mathsf{b}_2(h, x') + \mathsf{b}_3(h, h, x)) = -\mathsf{b}_1(\mathsf{b}_3(h, h, x') + \mathsf{b}_4(h, h, h, x))$$

is zero in $E_3^{p+7,\bar{0}}$ and hence $[\mathsf{b}_2(h, x') + \mathsf{b}_3(h, h, x)] \in \mathrm{H}^{p+5}(A) = E_3^{p+5,\bar{0}}$ indeed descends to a class in $E_4^{p+5,\bar{0}} = E_5^{p+5,\bar{0}}$. We note that x' is not unique; we can add an arbitrary closed element to x'. Yet the freedom in x' does not exhaust all the defining systems of $[h], [h], [x]$. This results in a smaller ambiguity of the triple Massey product in $\bar{\mathsf{b}}_2([h], \mathrm{H}(A))$ (rather than $\bar{\mathsf{b}}_2([h],\mathrm{H}(A))+\mathsf{b}_2(\mathrm{H}(A),[x])$ as in §4.1). Happily, it is the subspace $\bar{\mathsf{b}}_2([h],\mathrm{H}(A))$ that descends to zero in $E_4=E_5$.

5.2. **Twisting elements and matric A_∞-algebras.** We continue to assume that A is a $\mathbb{Z}$-graded A_∞-algebra and has the filtration given in §5.1. If $a\in A$ is an element of positive, even degree, we write $a=a_2+a_4+a_6+\cdots$, where $a_{2i}\in A^{2i}$. We define $\boldsymbol{a}^{(m)}=(a_{ij})_{0\le i,j\le m-1}\in\mathrm{Mat}_+^{(m)}(A)$ as the strictly upper-triangular matrix given by $a_{ij}=a_{2(j-i)}$ if $i<j$. Likewise, if $x\in\mathrm{F}^pA^{\bar p}$, we write $x=x_p+x_{p+2}+\cdots$, where $x_{p+2i}\in A^{p+2i}$ $(i\ge 0)$. We define a column vector $\boldsymbol{x}_p^{(m)}=(x_{p-2(m-i-1)})_{0\le i\le m-1}\in A^{\oplus m}$. In components, we have

$$\boldsymbol{a}^{(m)}=\begin{pmatrix}0 & a_2 & a_4 & \cdots & a_{2m-4} & a_{2m-2}\\ & 0 & a_2 & \cdots & a_{2m-6} & a_{2m-4}\\ & & \ddots & \ddots & \vdots & \vdots\\ & & & & a_2 & a_4\\ & & & & 0 & a_2\\ & & & & & 0\end{pmatrix},\qquad \boldsymbol{x}_p^{(m)}=\begin{pmatrix}x_{p+2m-2}\\ x_{p+2m-4}\\ \vdots\\ x_{p+4}\\ x_{p+2}\\ x_p\end{pmatrix}.$$

We denote the zero matrix and the zero vector by $\mathbf{0}^{(m)}$ and $\mathbf{0}_p^{(m)}$, respectively.

Given two graded $\mathbb{Z}$-graded vector spaces A,B and two elements $a\in A^{\bar 0}$, $b\in B^{\bar 0}$ of positive degrees, then for each $m\ge 2$, the matrix that corresponds to $a\otimes b\in(A\otimes B)^{\bar 0}$ is $\boldsymbol{a}^{(m)}\odot\boldsymbol{b}^{(m)}$. Similarly, if $x\in\mathrm{F}^pB^{\bar p}$, then for each $m\ge 2$, $a\otimes x$ determines the column vectors $\boldsymbol{a}^{(m)}\odot\boldsymbol{x}_p^{(m)}\in(A\otimes B)^{\oplus m}$. Finally, by taking a direct limit, we can define $\boldsymbol{a}^{(\infty)}$ and $\boldsymbol{x}_p^{(\infty)}$ with similar properties.

Lemma 5.2. *Let A be an A_∞-algebra. If $h\in A^{\bar 0}$ is of positive degree and $x\in\mathrm{F}^pA^{\bar p}$, $y\in\mathrm{F}^{p+1}A^{\overline{p+1}}$, then*

(i) $\partial_hx-y\in\mathrm{F}^{p+2m+1}A$ if and only if $\partial_{\boldsymbol{h}^{(m)}}\boldsymbol{x}_p^{(m)}=\boldsymbol{y}_{p+1}^{(m)}$. In particular, $\partial_hx=y$ if and only if $\partial_{\boldsymbol{h}^{(m)}}\boldsymbol{x}_p^{(m)}=\boldsymbol{y}_{p+1}^{(m)}$ for all $m\ge 2$.

(ii) $\partial_hh\in\mathrm{F}^{p+2m+1}A$ if and only if $\partial_{\boldsymbol{h}^{(m)}}\boldsymbol{h}^{(m)}=\mathbf{0}^{(m)}$. In particular, h is a twisting element in A if and only if $\boldsymbol{h}^{(m)}$ is one in $\mathrm{Mat}_+^{(m)}(A)$ for any $m\ge 2$.

Proof: (i) If $\partial_hx-y\in\mathrm{F}^{p+2m+1}A$, by comparing the homogeneous components of ∂_hx and y in $A^{p+1}\oplus A^{p+3}\oplus\cdots\oplus A^{p+2m-1}$, we get, for each $i=1,\dots,m$,

$$\sum_{r=1}^{i}\sum_{\substack{i_1,\dots,i_r\ge 1\\ i_1+\cdots+i_r=i}}\mathsf{b}_r(h_{2i_1},\dots,h_{2i_{r-1}},x_{p+2i_r})=0.$$

These equalities are equivalent to $\partial_{\boldsymbol{h}^{(m)}}\boldsymbol{x}_p^{(m)}=\boldsymbol{y}_{p+1}^{(m)}$. The rest is straightforward.
(ii) is proved similarly. □

Combining Lemmas 4.6 and 5.2, we get

Corollary 5.3. *Suppose $h\in A^{\bar 0}$ is of positive degree and $x\in\mathrm{F}^pA^{\bar p}$. Then $\left(\begin{smallmatrix}\boldsymbol{h}^{(m)} & \boldsymbol{x}_p^{(m)}\\ 0 & 0\end{smallmatrix}\right)\in\mathrm{Mat}_+^{(m+1)}(A)$ is a defining system if and only if $\partial_hh\in\mathrm{F}^{2m+1}A$ and $\partial_hx\in\mathrm{F}^{p+2m-1}A$. In* *this case, the component of $\partial_h x$ in A^{p+2m-1} is*

$$(\partial_h x)_{p+2m-1} = \mu\begin{pmatrix} \boldsymbol{h}^{(m)} & \boldsymbol{x}_p^{(m)} \\ 0 & 0 \end{pmatrix}.$$

In particular, $\left(\begin{smallmatrix} \boldsymbol{h}^{(m)} & \boldsymbol{x}_p^{(m)} \\ 0 & 0 \end{smallmatrix}\right)$ *is a twisting element if and only if $\partial_h h \in \mathrm{F}^{2m+1}A$ and $\partial_h x \in \mathrm{F}^{p+2m+1}A$.*

5.3. **Higher differentials and higher Massey products.** Let A be an A_∞-algebra and $h \in A$, a twisting element of positive degree. With the filtration F^pA defined in §5.1, we recall that $\mathrm{Gr}^pA = A^p$ and the results in Lemma 5.1. We apply the Appendix on spectral sequences to the $\mathbb{Z}_2$-graded setting of twisted differential. We want to describe the spaces $B_{2m+1}^{p\bar{0}} \subset Z_{2m+1}^{p\bar{0}} \subset A^p$, $E_{2m+1}^{p\bar{0}} = Z_{2m+1}^{p\bar{0}}/B_{2m+1}^{p\bar{0}}$ and the maps $\mathsf{d}_{2m+1}\colon E_{2m+1}^{p\bar{0}} \to E_{2m+1}^{p+2m+1,\bar{0}}$ for all $m \geq 0$.

Theorem 5.4. *Let $x \in A^p$. Then*

(i) $x_p \in Z_{2m+1}^{p\bar{0}}$ if and only if there exists $x = x_p + x_{p+2} + \cdots + x_{p+2m-2}$, where $x_{p+2i} \in A^{p+2i}$ $(0 \leq i < m)$, such that $\partial_{\boldsymbol{h}^{(m)}} \boldsymbol{x}_p^{(m)} = \boldsymbol{0}_{p+1}^{(m)}$, or equivalently, $\left(\begin{smallmatrix} \boldsymbol{h}^{(m)} & \boldsymbol{x}_p^{(m)} \\ 0 & 0 \end{smallmatrix}\right) \in \mathrm{Mat}_+^{(m+1)}(A)$ *is a twisting element. In this case,* $0 \in \langle \underbrace{[h_2], \ldots, [h_2]}_{m-1 \text{ times}}, [x_p] \rangle$.

(ii) $x_p \in B_{2m+1}^{p\bar{0}}$ if and only if there exists $y = y_{p-2m+1} + y_{p-2m+3} + \cdots + y_{p-1}$, where $y_{p-2i+1} \in A^{p-2i+1}$ $(0 \leq i < m)$, such that $\partial_{\boldsymbol{h}^{(m)}} \boldsymbol{y}_{p-2m+1}^{(m)} = \begin{pmatrix} x_p \\ \boldsymbol{0}_{p-2m+2}^{(m-1)} \end{pmatrix}$*, or equivalently,* $\left(\begin{smallmatrix} \boldsymbol{h}^{(m)} & \boldsymbol{y}_p^{(m)} \\ 0 & 0 \end{smallmatrix}\right) \in \mathrm{Mat}_+^{(m+1)}(A)$ *is a defining system (of* $\underbrace{[h_2], \ldots, [h_2]}_{m-1 \text{ times}}, [y_{p-2m+1}]$*) and* $x_p = \mu\left(\begin{smallmatrix} \boldsymbol{h}^{(m)} & \boldsymbol{y}_p^{(m)} \\ 0 & 0 \end{smallmatrix}\right)$*. In this case,* $x_p \in \langle \underbrace{h_2, \ldots, h_2}_{m-1 \text{ times}}, y_{p-2m+1} \rangle$.

(iii) Suppose $x_p \in Z_{2m+1}^{p\bar{0}}$ represents a class $[x_p]_{2m+1} \in E_{2m+1}^{p\bar{0}}$. Let x be given by (i) and let $\tilde{x} = x + x_{p+2m}$ with any choice of $x_{p+2m} \in A^{p+2m}$. Then $\partial_{\boldsymbol{h}^{(m+1)}} \tilde{\boldsymbol{x}}_p^{(m+1)} = \begin{pmatrix} z_{p+2m+1} \\ \boldsymbol{0}_{p+1}^{(m)} \end{pmatrix}$ *for some $z_{p+2m+1} \in A^{p+2m+1}$, or equivalently,* $\left(\begin{smallmatrix} \boldsymbol{h}^{(m+1)} & \tilde{\boldsymbol{x}}_p^{(m+1)} \\ 0 & 0 \end{smallmatrix}\right) \in \mathrm{Mat}_+^{(m+2)}(A)$ *is a defining system (of* $\underbrace{[h_2], \ldots, [h_2]}_{m \text{ times}}, [x_p]$*). Furthermore,* $z_{p+2m+1} = \mu\left(\begin{smallmatrix} \boldsymbol{h}^{(m+1)} & \tilde{\boldsymbol{x}}_p^{(m+1)} \\ 0 & 0 \end{smallmatrix}\right) \in A^{p+2m+1}$ *descends to a class in $E_{2m+1}^{p+2m+1,\bar{0}}$ which is equal to $\mathsf{d}_{2m+1}[x_p]_{2m+1}$.*

Proof: (i) By the description of $Z_r^{p\bar{q}}$ in the Appendix, $x_p \in Z_{2m+1}^{p\bar{0}}$ if and only if there is $x = x_p + x_{p-2} + \cdots + x_{p+2m-2} \in A^p \oplus A^{p+2} \oplus \cdots \oplus A^{p+2m-2}$ such that $\mathsf{d}x \in \mathrm{F}^{p+2m+1}A$. The rest follows from Lemma 5.2(i) and Corollary 5.3.
(ii) By the description of $B_r^{p\bar{q}}$ in the Appendix, $x_p \in B_{2m+1}^{p\bar{0}}$ if and only if there is $y = y_{p-2m+1} + y_{p-2m+3} + \cdots + y_{p-1} \in A^{p-2m+1} \oplus A^{p-2m+3} \oplus \cdots \oplus A^{p-1}$ such that $\mathsf{d}y - x_p \in \mathrm{F}^{p+3}A$. The rest follows from Lemma 5.2(i) and Corollary 5.3.

(iii) By the description of d_r in the Appendix, $\mathsf{d}_{2m+1}[x_p]_{2m+1}$ is represented by $z_{p+2m+1} \in Z_{2m+1}^{p+2m+1,\bar{0}} \subset A^{p+2m+1}$ such that $d\tilde{x} - z_{p+2m+1} \in \mathrm{F}^{p+2m+3}A$. The rest follows from Lemma 5.2(i) and Corollary 5.3. $\square$

Given x_p in Theorem 5.4(i), the element x satisfying the condition is not unique. As in §5.1, this freedom does not exhaust all the defining systems of $\underbrace{[h_2], \ldots, [h_2]}_{m-1 \text{ times}}, [x_p]$. Let $x_p + x'$, where $x' = x'_{p+2} + \cdots + x'_{p+2(m-1)}$, be another choice and let $z'_{p+2m+1} = \mu\begin{pmatrix} \boldsymbol{h}^{(m+1)} & \tilde{\boldsymbol{x}}_p'^{(m+1)} \\ 0 & 0 \end{pmatrix}$, where $\tilde{x} = x_p + x' + x'_{p+2m}$ for some $x'_{p+2m} \in A^{p+2m}$. Then the difference

$$\begin{aligned} z'_{p+2m+1} - z_{p+2m+1} &= \mu\begin{pmatrix} \boldsymbol{h}^{(m+1)} & \tilde{\boldsymbol{x}}_p'^{(m+1)} - \tilde{\boldsymbol{x}}_p^{(m+1)} \\ 0 & 0 \end{pmatrix} \\ &= \mu\begin{pmatrix} \boldsymbol{h}^{(m)} & \boldsymbol{x}_{p+2}'^{(m)} \\ 0 & 0 \end{pmatrix} + \mathsf{b}_1(x'_{p+2m} - x_{p+2m}) \end{aligned}$$

descends to an element in the image of d_{2m-1} which is a Massey product of $\underbrace{[h_2], \ldots, [h_2]}_{m-2 \text{ times}}, [x'_{p+2} - x_{p+2}]$. Therefore the class in $E_{2m} = E_{2m+1}$ remains the same. This generalizes the discussion on the special case ($m = 2$) on d_5 and the triple Massey product in §5.1.

We also remark that the elements $x_{p+2}, \ldots, x_{p+2(m-1)}$ chosen in Theorem 5.4 for a given $m \geq 1$ can not be used recursively without correction for higher values of m. This phenomenon already appeared in the special case when A is the de Rham complex [16]. We now illustrate this fact for general A_∞-algebras in a more concise way. Suppose the element x_p in Theorem 5.4(i) is actually in $Z_{2m+3}^{p\bar{0}}$, then $\mathsf{d}_{2m+1}[x_p]_{2m+1}$ given by Theorem 5.4(iii) is zero in $E_{2m+1}^{p+2m+1,\bar{0}}$, i.e., $z_{p+2m+1} \in B_{2m+1}^{p+2m+1,\bar{0}}$. By Theorem 5.4(ii), there exists $y' = y'_{p+2} + y'_{p+4} + \cdots + y'_{p+2m}$, where $y'_{p+2i} \in A^{p+2i}$ ($1 \leq i \leq m$) such that $\partial_{\boldsymbol{h}^{(m)}} \boldsymbol{y}_{p+2}'^{(m)} = \begin{pmatrix} z_{p+2m+1} \\ \boldsymbol{0}_{p+3}^{(m-1)} \end{pmatrix}$. Therefore

$$\partial_{\boldsymbol{h}^{(m+1)}} \left(\tilde{\boldsymbol{x}}_p^{(m+1)} - \begin{pmatrix} \boldsymbol{y}_{p+2}'^{(m)} \\ 0_p \end{pmatrix} \right) = \boldsymbol{0}_{p+1}^{(m+1)}.$$

Thus when m increases by 1, the element x in Theorem 5.4(i) should be replaced by

$$x + x_{p+2m} - y' = x_p + (x_{p+2} - y'_{p+2}) + \cdots + (x_{p+2m} - y'_{p+2m}).$$

5.4. **Naturality of the spectral sequence.** If $\mathsf{f}\colon A \to B$ is a morphism of A_∞-algebras and $h \in A$ is a twisting element, then so is $\mathsf{F}_h(h) \in B$ and there is an induced homomorphism $(\mathsf{F}_h)_*\colon \mathrm{H}_h(A) \to \mathrm{H}_{\mathsf{F}_h(h)}(B)$. We want to describe explicitly the induced morphism of the spectral sequences. If h is of homogeneous degree 2 as in the example in §5.1 and if f is strict, then $\mathsf{F}_h(h) = \mathsf{f}_1(h) \in B$ is of homogeneous degree 2 as well. In this case, the morphism of spectral sequences is induced by f_1. It is easy to see, using the explicit formulas of d_3 and

d_5 in §5.1, that

$$(\mathsf{f}_1)_* \circ \mathsf{d}_3^A = \mathsf{d}_3^B \circ (\mathsf{f}_1)_*, \quad (\mathsf{f}_1)_* \circ \mathsf{d}_5^A = \mathsf{d}_5^B \circ (\mathsf{f}_1)_*.$$

In the general situation, we have

Proposition 5.5. *If $\mathsf{f}\colon A \to B$ is a morphism of A_∞-algebras and $h \in A$ is a twisting element, then the cochain map $\mathsf{F}_h\colon (A, \mathsf{d}^A) \to (B, \mathsf{d}^B)$, where $\mathsf{d}^A = \partial_h$ and $\mathsf{d}^B = \partial_{\mathsf{F}_h(h)}$, is a morphism of filtered cochain complexes and hence there is a morphism of the spectral sequences $({}^A E_r^{p\bar{q}}, \mathsf{d}_r^A) \to ({}^B E_r^{p\bar{q}}, \mathsf{d}_r^B)$. For all $m \geq 0$, the homomorphisms $(\mathsf{f}_1)_*\colon {}^A E_{2m+1}^{p\bar{0}} \to {}^B E_{2m+1}^{p\bar{0}}$ are induced by $\mathsf{f}_1\colon A \to B$ and $(\mathsf{f}_1)_* \circ \mathsf{d}_{2m+1}^A = \mathsf{d}_{2m+1}^B \circ (\mathsf{f}_1)_*$. In particular, we have $(\mathsf{f}_1)_*\colon {}^A E_\infty^{p\bar{0}} \to {}^B E_\infty^{p\bar{0}}$.*

Proof: The cochain map F_h clearly preserve the filtrations. On the graded components $\mathrm{Gr}^p A = A^p$, it is simply $\mathsf{f}_1\colon A^p \to B^p$. The rest is self-evident by the discussion of naturality in the Appendix. □

The result that the morphism of the spectral sequence is induced by f_1 alone is consistent with the naturality of the higher Massey products in §4.5. We note that ${}^A E_\infty^{p\bar{0}}$ are the graded components of $\mathrm{H}_h^{\bar{p}}(A)$. Although the homomorphisms on E_∞ depend solely on f_1, the induced map $(\mathsf{F}_h)_*\colon \mathrm{H}_h(A) \to \mathrm{H}_{\mathsf{F}_h(h)}(B)$ does depend on all f_n for $n \geq 1$.

We apply Proposition 5.5 to a number of cases.

First, suppose $\mathsf{q}\colon \mathrm{H}(A) \to A$ is the quasi-isomorphism in §2.4. Denote by $(\bar{E}_r^{p\bar{q}}, \bar{\mathsf{d}}_r)$ the spectral sequence associated to the $\mathbb{Z}$-graded A_∞-algebra $\mathrm{H}(A)$. Since $\bar{\mathsf{b}}_1 = 0$, we have $\bar{E}_3^{p\bar{0}} = \bar{E}_2^{p\bar{0}} = \bar{E}_1^{p\bar{0}} = \bar{E}_0^{p\bar{0}} = \mathrm{H}^p(A)$. The map $\mathsf{q}_1\colon \bar{E}_0^{p\bar{0}} = \mathrm{H}^p(A) \to E_0^{p\bar{0}} = A^p$ is a quasi-isomorphism of cochain complexes and hence the induced homomorphisms $(\mathsf{q}_1)_*\colon \bar{E}_r^{p\bar{0}} \to E_r^{p\bar{0}}$ are the identity maps for all $r \geq 2$. With the relation between the higher differentials and the higher Massey products (Theorem 5.4(iii)), this is consistent with the relation of (higher) Massey products with the A_∞-structure on $\mathrm{H}(A)$ (Proposition 4.17).

Next, if as in §3.3, $h, h' \in A$ are two twisting elements that are homotopic through c, then by Proposition 3.11(i) and Corollary 3.12(i), there is a cochain map $\psi_c\colon (A, \partial_h) \to (A, \partial_{h'})$, which preserves the filtration on A. Moreover, ψ_c is the identity map on the graded components. So the induced isomorphism identifies the two spectral sequences $(E_r^{p\bar{q}}, \mathsf{d}_r) = (E_r'^{p\bar{q}}, \mathsf{d}'_r)$, which converge to $\mathrm{H}_h(A)$ and $\mathrm{H}_{h'}(A)$, respectively. We note that although $\mathrm{H}_h(A)$ and $\mathrm{H}_{h'}(A)$ have identical graded components, the isomorphism between the total spaces is induced by ψ_c, not the identity map.

Finally, if $\mathsf{f}, \mathsf{g}\colon A \to B$ are two morphisms of A_∞-algebras that are homotopic through h, then $\mathsf{f}_1, \mathsf{g}_1\colon A \to B$ are cochain maps that are homotopic through h_1. Consequently, the morphisms on the spectral sequence ${}^A E_r^{p\bar{q}}$ induced by f and g are identical when $r \geq 2$. This is consistent with Theorem 3.16 and Corollary 3.17 since $\psi_{\mathsf{H}_h(h)}$ induces the identity isomorphism on spectral sequences.

Appendix. Spectral sequences

We summarize some aspects of spectral sequences used in Section 5. We refer the readers to numerous discussions in the literature (for example [4, 23]) although the version we present here is somewhat different. We will treat the cohomological spectral sequences only as the homological ones are parallel.

Let $(C^\bullet, \mathsf{d})$ be a ($\mathbb{Z}$-graded) cochain complex with a filtration F, that is, we have a descending d-invariant sequence

$$\cdots \supset \mathrm{F}^p C^\bullet \supset \mathrm{F}^{p+1} C^\bullet \supset \cdots$$

and thus each $(\mathrm{F}^p C^\bullet, \mathsf{d})$ is a cochain complex. Let $\mathrm{Gr}^p C^\bullet = \mathrm{F}^p C^\bullet / \mathrm{F}^{p+1} C^\bullet$ be the graded components; each $\mathrm{Gr}^p C^\bullet = \mathrm{F}^p C^\bullet / \mathrm{F}^{p+1} C^\bullet$ is also a cochain complex whose coboundary operator will also be denoted by d. Let

$$\tilde{Z}_r^{pq} = \mathrm{F}^p C^{p+q} \cap \mathsf{d}^{-1}(\mathrm{F}^{p+r} C^{p+q+1}), \quad \tilde{B}_r^{pq} = \mathrm{F}^p C^{p+q} \cap \mathsf{d}(\mathrm{F}^{p-r+1} C^{p+q-1})$$

(which are usually denoted by Z_r^{pq}, B_r^{pq}, respectively) and

$$E_r^{pq} = \tilde{Z}_r^{pq} / (\tilde{B}_r^{pq} + \tilde{Z}_{r-1}^{p+1,q-1}).$$

Then the coboundary operator d induces the higher differentials $\mathsf{d}_r \colon E_r^{pq} \to E_r^{p+r,q-r+1}$ with $\mathsf{d}_r^2 = 0$, and E_{r+1}^{pq} is the cohomology of (E_r^{pq}, d_r). The filtration is exhaustive and weakly convergent, then the spectral sequence converges to the cohomology groups (cf. Theorem 3.2 of [23]).

We reserve the notations Z_r^{pq}, B_r^{pq} for the new spaces

$$Z_r^{pq} = \tilde{Z}_r^{pq} / \tilde{Z}_{r-1}^{p+1,q-1}, \quad B_r^{pq} = \tilde{B}_r^{pq} / (\tilde{B}_r^{pq} \cap \tilde{Z}_{r-1}^{p+1,q-1}).$$

Then $E_r^{pq} = Z_r^{pq} / B_r^{pq}$. There are natural isomorphisms (through which we identify)

$$Z_r^{pq} = \{x \in \mathrm{Gr}^p C^{p+q} \,|\, \exists \tilde{x} \in \mathrm{F}^p C^{p+q} \text{ such that } x = \tilde{x} + \mathrm{F}^{p+1} C^{p+q},\ \mathsf{d}\tilde{x} \in \mathrm{F}^{p+r} C^{p+q+1}\}$$

and

$$B_r^{pq} = \{x \in \mathrm{Gr}^p C^{p+q} \,|\, \exists \tilde{y} \in \mathrm{F}^{p-r+1} C^{p+q-1} \text{ such that } \mathsf{d}\tilde{y} \in \mathrm{F}^p C^{p+q},\ x = \mathsf{d}\tilde{y} + \mathrm{F}^{p+1} C^{p+q}\}.$$

For example, $Z_0^{pq} = \mathrm{Gr}^p C^{p+q}$, $B_0^{pq} = 0$ and $E_0^{pq} = \mathrm{Gr}^p C^{p+q}$ whereas

$$Z_1^{pq} = \ker(\mathsf{d} \colon \mathrm{Gr}^p C^{p+q} \to \mathrm{Gr}^p C^{p+q+1}), \quad B_1^{pq} = \mathrm{im}(\mathsf{d} \colon \mathrm{Gr}^p C^{p+q-1} \to \mathrm{Gr}^p C^{p+q})$$

and $E_1^{pq} = H^{p+q}(\mathrm{Gr}^p C^\bullet, \mathsf{d})$ as expected. We have a sequence of inclusions

$$0 = B_0^{pq} \subset \cdots \subset B_r^{pq} \subset B_{r+1}^{pq} \subset \cdots \subset Z_{r+1}^{pq} \subset Z_r^{pq} \subset \cdots \subset Z_0^{pq} = \mathrm{Gr}^p C^{p+q}.$$

Finally, if the filtration is weakly convergent, then

$$Z_\infty^{pq} = \{x \in \mathrm{Gr}^p C^{p+q} \,|\, \exists \tilde{x} \in \mathrm{F}^p C^{p+q} \text{ such that } x = \tilde{x} + \mathrm{F}^{p+1} C^{p+q},\ \mathsf{d}\tilde{x} = 0\},$$

$$B_\infty^{pq} = \{x \in \mathrm{Gr}^p C^{p+q} \,|\, \exists \tilde{y} \in C^{p+q-1} \text{ such that } \mathsf{d}\tilde{y} \in \mathrm{F}^p C^{p+q},\ x = \mathsf{d}\tilde{y} + \mathrm{F}^{p+1} C^{p+q}\},$$

and $E_\infty^{pq} = Z_\infty^{pq} / B_\infty^{pq}$ are the graded components of the cohomology groups $\mathrm{H}(C^\bullet, \mathsf{d})$.

We now describe the higher differentials $\mathsf{d}_r\colon E_r^{pq} \to E_r^{p+r,q-r+1}$. If a class $[x] \in E_r^{pq} = Z_r^{pq}/B_r^{pq}$ is represented by $x \in Z_r^{pq} \subset \mathrm{Gr}^p C^{p+q}$, choose a lifting $\tilde{x} \in \mathrm{F}^p C^{p+q}$ of x. Then $\mathsf{d}\tilde{x} \in \mathrm{F}^{p+r} C^{p+q+1}$ and we denote by $z \in \mathrm{Gr}^{p+r} C^{p+q+1}$ its graded component. In fact, $z \in Z_r^{p+r,q-r+1}$ (since if we choose $\tilde{z} = \mathsf{d}\tilde{x}$, then $\mathsf{d}\tilde{z} = 0 \in \mathrm{F}^{p+2r} C^{p+q+2}$) and d_r is given by

$$\mathsf{d}_r[x] = [z] \in E_r^{p+r,q-r+1}.$$

We check that the result is independent of the choices made. If $\tilde{x}' \in \mathrm{Gr}^p C^{p+q}$ is another lifting of x, then $\tilde{x}' - \tilde{x} \in \mathrm{F}^{p+1} C^{p+q}$ and $\mathsf{d}(\tilde{x}' - \tilde{x}) \in \mathrm{F}^{p+r} C^{p+q+1}$. Since the graded component $z' \in \mathrm{Gr}^{p+r} C^{p+q}$ of $\mathsf{d}\tilde{x}'$ differs from z by an element in $B_r^{p+r,q-r+1}$, they descend to the same class in $E_r^{p+r,q-r+1}$. On the other hand, if $[x] = 0$ or $x \in B_r^{pq}$, then we can choose $\tilde{x} = \mathsf{d}\tilde{y}$, where $\tilde{y} \in \mathrm{F}^{p-r+1} C^{p+q-1}$. Thus $z = \mathsf{d}\tilde{x} = 0$ and $\mathsf{d}_r[x] = 0$, which is required by consistency.

Let $(C'^{\bullet}, \mathsf{d}')$ be another ($\mathbb{Z}$-graded) cochain complex with a filtration, also denoted by F. Suppose $\mathsf{f}\colon (C^{\bullet}, \mathsf{d}) \to (C'^{\bullet}, \mathsf{d}')$ is a morphism of filtered cochains, i.e., $\mathsf{f} \circ \mathsf{d} = \mathsf{d}' \circ \mathsf{f}$ and $\mathsf{f}\colon \mathrm{F}^p C^{\bullet} \to \mathrm{F}^p C'^{\bullet}$. Then there is an induced morphism of the associated spectral sequences $(E_r^{pq}, \mathsf{d}_r) \to (E_r'^{pq}, \mathsf{d}_r')$ (see, for example, Theorem 3.5 of [23]). More concretely, f induces cochain maps (using the same notation) $\mathsf{f}\colon \mathrm{Gr}^p C^{\bullet} \to \mathrm{Gr}^p C'^{\bullet}$. It can be shown that $\mathsf{f}(Z_r^{pq}) \subset Z_r'^{pq}$, $\mathsf{f}(B_r^{pq}) \subset B_r'^{pq}$ and hences f induces a map $\mathsf{f}_*\colon E_r^{pq} \to E_r'^{pq}$. Furthermore, we have $\mathsf{f}_* \circ \mathsf{d}_r = \mathsf{d}_r' \circ \mathsf{f}_*$, which can also be seen from the above explicit description of d_r. This gives a morphism between the two spectral sequences.

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