学位論文

Computational Algebraic Analyses for Unimodular or Lawrence-Type Integer Programs

単模および Lawrence 型整数計画問題に対する計算代数的解析

平成14年12月 博士(理学)申請

東京大学理学系研究科情報科学専攻
石関 隆幸

Computational Algebraic Analyses
for
Unimodular or Lawrence-Type Integer Programs

単模および Lawrence 型整数計画問題の計算代数的解析

by

Takayuki Ishizeki

石関 隆幸

A Dissertation

博士論文

Submitted to

the Department of Information Science

the Graduate School of Science

the University of Tokyo

in Partial Fulfillment of the Requirements

for the Degree of Doctor of Science

Thesis Supervisor: Hiroshi Imai　今井　浩
Professor of Information Science

December 2002

ABSTRACT

There have been a number of studies of the application of computational algebraic approaches to solve integer programming problems, using Gröbner bases or standard pairs. By connecting these methods with existing combinatorial methods for integer programming problems, we can understand both methods deeply, and they have allowed new analysis of their structures and construction of algorithms.

For an ideal over a polynomial ring, Gröbner bases and the set of standard pairs have duality as a Gröbner basis is a generator of the initial ideal, while the standard pair decomposition is a nice decomposition of the set of monomials which does not contained in the initial ideal. Such duality and consideration of a subclass of integer programming problems where the duality theorem holds might give deeper relation of two methods, and yield complexity bounds by making use of the characteristics of the subclass, which could not be derived for general integer programming problems. This thesis focuses on subclasses where the coefficient matrix is unimodular or of the Lawrence type.

Problems in which the coefficient matrix A is unimodular are a mathematically nice subclass in the sense that the system $yA \leq c$ becomes totally dual integral (TDI) and each standard pair corresponds to a dual feasible basis. In this dissertation, the method using standard pairs is shown to be equivalent to calculation of the reduced cost for each basis. Therefore, the number of standard pairs, which is equal to that of dual feasible bases, gives the complexity of this approach. The maximum number of dual feasible bases can be described by the normalized volume of the polytope, defined by homogenizing A. These results give a unified approach to analyze the number of vertices of dual polyhedron via Gröbner bases and volume computations.

We also focused on transportation problems and minimum cost flow problems. The Gröbner basis approach for minimum cost flow problems is a variant of the classical cycle-canceling algorithm: i.e., for any feasible flow the polynomial size of negative-cost cycles in the residual network is chosen based on various rules and the flow is augmented along these cycles as great a degree as possible. On the other hand, the standard pair approach first finds the set of standard pairs, and solves linear systems of equations for each standard pair until a non-negative integer solution is obtained.

For transportation problems, several results about the number of primal and dual feasible polyhedra have been reported. We give computational algebraic proof for these results using

above approach. Furthermore, we study minimum cost flow problems on acyclic tournament graphs, which are the most fundamental type of directed graphs. The size of dual feasible bases for minimum cost flow problems is shown to be less than the Catalan number, and the lower bound for the size of primal feasible bases is shown to be of exponential order. For a network optimization problem, the duality between the Gröbner basis and the set of standard pairs corresponds to the relation between circuits and dual feasible co-trees, dually, cutsets and primal feasible trees. As this relation has not been clarified previously, our results using the computational algebraic duality are of interest.

On the other hand, in combinatorial optimization, Lawrence-type matrices arise in many situations, e.g., in capacitated integer programming problems and some multidimensional transportation problems. Furthermore, Lawrence-type matrices are used in mathematical statistics in sampling or enumeration of multi-way contingency tables of type $2 \times M \times \cdots \times N$ with fixed marginal sums. The problem that counts the number of 2-dimensional contingency tables with fixed marginal sums has been shown to be #P-complete. On the other hand, Markov Chain Monte Carlo methods that have polynomial-time mixing times have been studied for tables of type $2 \times \cdots \times 2 \times N$.

While the relationship between vector matroids defined by matrices of Lawrence type and Gröbner bases has been studied in detail, there have been few investigations of standard pairs for matrices of Lawrence type. This dissertation focuses on standard pairs that correspond to dual feasible bases. We present here a bijection between the set of such standard pairs and the set of bases of the vector matroid, and describe the matroidal structure of these standard pairs. In particular, in cases in which the matrix is unimodular, this relation indicates that the number of dual feasible bases is equal to the number of bases of the vector matroid. As a corollary, we analyze (i) the number of dual feasible bases for the capacitated minimum cost flow problem on an acyclic tournament graph, (ii) the number of dual feasible bases for multidimensional transportation problems of type $2 \times 2 \times \cdots \times 2 \times M \times N$, and (iii) the number of primal feasible bases for multidimensional transportation problems of type $2 \times \cdots \times 2 \times 2 \times N$.

論文要旨

整数計画問題に対して，近年 Gröbner 基底や standard pair を用いた計算代数的手法が研究されている．これらの手法と既存の組合せ的手法の橋渡しを行うことにより，双方の手法の理解が高まり，新たな構造解析手法やアルゴリズムの構築が期待される．

多項式環上のイデアルにおいて，Gröbner 基底は初項イデアルの生成系であり，standard pair の集合は初項イデアルに含まれない単項式の集合の分解であるため，双対の関係にあると言える．このような双対性に着目し，さらに双対定理の成り立つ整数計画問題のクラスを考えることにより，より豊かな橋渡しができ，一般的な整数計画問題からは得られないような計算量の上下限を得ることができると期待される．本論文では，そのような整数計画問題のクラスとして，係数行列が単模のとき，および Lawrence 型であるときに着目する．

係数行列 A が単模であるような問題は，不等式系 $yA \leq c$ が完全双対整数性 (TDI) を満たし，さらに各 standard pair が双対実行可能基底に対応する，という点で性質の良い問題のクラスである．本論文では，係数行列が単模であるような整数計画問題に対して standard pair を用いた方法が，その standard pair に対応する基底での被約コストを計算することと等価であることを示す．よって，standard pair の数 (つまり，双対実行可能基底の数) がこの方法の計算量を与える．さらに，双対実行可能基底の数の最大値が，行列 A を斉次化して定義される多面体の正規化体積で表せることを示す．これにより，Gröbner 基底から体積計算を通じて双対多面体の頂点数を解析するという統一的なアプローチを与えることができる．

さらに，本論文では特に輸送問題，および最小費用流問題に着目する．これらの問題に対する Gröbner 基底を用いたアプローチは，既存の負閉路消去アルゴリズム，つまり実行可能流に対して残余ネットワーク内の負閉路を見つけて流し変えていく方法の変形である．一方，standard pair を用いたアプローチでは，まず standard pair の集合を求め，非負整数解が得られるまで，各 pair に対応する線型連立方程式を解く方法である．

輸送問題の主問題および双対問題の実行可能領域の頂点数に関しては，これまでに既存の結果が知られているが，ここでは上のアプローチによる計算代数的な証明を与える．さらに，無閉路トーナメントグラフ上の最小費用流問題に着目し，最小費用流問題に対する双対実行可能基底の数が高々 Catalan 数となること，および主実行可能基底の数が少なくとも指数オーダになることを示す．ネットワーク最適化問題において，Gröbner 基底と双対実行可能基底の関係はサーキットと双対実行可能補木 (双対的には，カットセットと主実

行可能木) の関係に対応する．これらの関係はほとんど明らかにされていないので，この結果は計算代数的双対性を用いた解析の面白い結果である．

一方，組合せ最適化において，Lawrence 型の行列は容量付き整数計画問題やある多次元輸送問題など，多くの問題に現れる．また，数理統計学において，各行和を固定した $2 \times M \times \cdots \times N$ 型の多次元分割表のサンプリングや数え上げに Lawrence 型の行列が用いられることが知られている．行和の固定された 2 元分割表を数え上げる問題が #P-complete であることが知られている一方で， $2 \times 2 \times \cdots \times 2 \times N$ 型の分割表に対して，mixing time が多項式時間となるマルコフ連鎖モンテカルロ法が提案されている．

Lawrence 型の行列に対して，行列の定めるベクトルマトロイドと Gröbner 基底との関係は良く研究されているが，standard pair については良く分かっていない．そこで本論文では，特に双対実行可能基底に対応する standard pair に着目する．まず，双対実行可能基底に対応する standard pair の集合とベクトルマトロイドの基集合の間の全単射を与え，さらにそのような standard pair たちのマトロイド的な構造を示す．特に，この関係は双対実行可能基底の数とベクトルマトロイドの基の数が等しいことを表している．さらにその系として，無閉路トーナメントグラフ上での容量付き最小費用流問題に対する双対実行可能基底の数，$2 \times 2 \times \cdots \times 2 \times M \times N$ 型の多次元輸送問題に対する双対実行可能基底の数，$2 \times 2 \times \cdots \times 2 \times 2 \times N$ 型の多次元輸送問題の主実行可能基底の数を解析する．

Acknowledgements

First of all I would like to thank Prof. Hiroshi Imai, my supervisor. He gave me lots of advice on my research and encouraged me. His advice about my research and presentations in conferences were very precise and useful. I am also indebted to Prof. Hiroshi Imai and Mr. Hiroki Nakayama, the co-authors of the preliminary papers that make up this dissertation.

I am grateful to Prof. Ryuichi Hirabayashi who chaired seminars at the Tokyo University of Science. It was a wonderful experience for me to meet him at the RAMP Symposium in Kyoto four years ago. I would also like to thank Prof. Ryuichi Hirabayashi, Prof. Yoshiko T. Ikebe, Dr. Kenji Kashiwabara, Dr. Masahiro Hachimori, Dr. Fumihiko Takeuchi, Mr. Masafumi Ito, Ms. Miho Ito, Mr. Kazuto Araki, and Mr. Hiroki Nakayama for many fruitful and encouraging discussions in the seminars.

In addition, I would like to acknowledge Prof. Takayuki Hibi and Dr. Hidefumi Ohsugi for organizing meetings at Osaka University and at Kyoto University. The presentations and discussions at these meetings were very useful, and I made a number of friends there, notably, Mr. Daisuke Ikegami, Mr. Shigeru Kuroda, and Mr. Tomonori Kitamura.

I would like to thank to Prof. Nobuki Takayama, from whose lectures on computational algebra I learned a great deal about Gröbner bases and standard pair decompositions. His lectures stimulated this dissertation.

I am also grateful to Prof. Yasuko Matsui, Dr. Kenji Kashiwabara and Dr. Masahiro Hachimori who chaired the COMA SEMI (COmbinatorial MAthematics SEMInar). I learned a great deal about combinatorics and operations research at this seminar, and participated in many fruitful discussions. Especially I thank Prof. Yasuko Matsui. The study in Chapter 4 was motivated by her talks about contingency tables.

I appreciate the people who are willing to be in my thesis committee: Prof. Yoshio Oyanagi, Prof. Masami Hagiya, Prof. Yasumasa Kanada, Prof. Kazuo Murota, and Prof. Reiji Suda.

Thanks are also due to Prof. Ken'ichi Asai, Dr. Jumpei Niwa, Dr. Hirotada Kobayashi, Mr. Yasuhito Asano, Ms. Sonoko Moriyama, and the other members and alumni of the Imai Laboratory and Ms. Akiko Shintani. A special debt of thanks is due to Mr. Yasuhito Asano, my colleague of five years, for his encouragement.

Finally, I would like to thank my parents, my sister and friends for their supports during the years of my graduate study.

Takayuki Ishizeki

December 2002, Tokyo

Contents

1 Introduction **1**

1.1 Background 1

1.2 Unimodular Integer Programs 4

1.3 Lawrence-Type Integer Programs 5

1.4 Our Contributions 7

1.4.1 Unimodular Integer Programs 7

1.4.2 Lawrence-Type Integer Programs 8

1.5 Organization of this Dissertation 10

2 Algebraic Preliminaries: Computational Algebra and Integer Programs **12**

2.1 Gröbner Bases 12

2.1.1 Definitions 12

2.1.2 Gröbner Bases in Integer Programs 15

2.2 Standard Pair Decompositions 19

2.2.1 Definitions 19

2.2.2 Standard Pair Decompositions in Integer Programs 21

3 Dualistic Computational Algebraic Analyses of Unimodular Integer Programs **25**

3.1 Introduction 25

3.2 Unimodular Integer Programs: Definitions and Properties 27

3.3 Standard Pairs for Unimodular Integer Programs 31

3.4 Analyses of Primal Minimum Cost Flow Problems 36

3.4.1 Gröbner Bases and Their Size 36
3.4.2 Standard Pair Decompositions and Their Size 42
3.5 Analyses of Dual Minimum Cost Flow Problems 43
3.5.1 Gröbner Bases and Their Size 44
3.5.2 Standard Pair Decompositions and Their Size 46
3.6 Analyses of Transportation Problems 53
3.6.1 Number of Vertices of Dual Transportation Polyhedra 54
3.6.2 Number of Vertices of Transportation Polytopes on $K_{2,n}$ 55
3.7 Summary 58

4 Standard Pair Decompositions of Lawrence-Type Integer Programming Problems 60

4.1 Introduction 60
4.2 Lawrence-type Integer Programs: Definitions and Properties 63
4.3 Characterizations of Standard Pair Decompositions of Lawrence-Type Matrices 65
4.3.1 Vector Matroids and Standard Pair Decompositions of Lawrence-Type Matrices 65
4.3.2 Application to Minimum Cost Flow Problems with Upper Bound Constraints 69
4.3.3 Application to Multidimensional Transportation Problems of Type $2 \times \cdots \times 2 \times M \times N$ 70
4.4 Standard Pair Decompositions of Dual Problems of Lawrence-Type Integer Programs 76
4.4.1 Vector Matroids and Dual Problems of Lawrence-Type Integer Programs 77
4.4.2 Application to Dual Problems of Multidimensional Transportation Problems of Type $2 \times \cdots \times 2 \times 2 \times N$ 79
4.5 Summary 79

5 Concluding Remarks **81**

A Matroids and Oriented Matroids **83**

References **87**

List of Figures

1.1 Classes of matrices and cost vectors. 5

1.2 Gomory integer programs, unimodular integer programs and Lawrence-type integer programs. 11

2.1 Acyclic tournament graph with 3 vertices. 16

2.2 Two types of initial ideals for Example 2.12. A point (p, q, r) in the figure corresponds to the monomial $x_{1,2}^p x_{1,3}^q x_{2,3}^r$. 18

2.3 Two types of standard pair decompositions for Example 2.12. 20

2.4 Two regular triangulations for Example 2.12. 22

2.5 Regular triangulations of $\mathrm{cone}(A)$ and $\mathrm{conv}(A)$. 23

2.6 Dual polyhedron $P_{(3,1,2)}$ and regular triangulation $\Delta_{(3,1,2)}$. 24

3.1 Definition of σ_1 and σ_2. 35

3.2 Circuit corresponding to g_{ijk} (left) and circuit corresponding to g_{ijkl} (right). 38

3.3 The condition (A) of the definition of T_σ. 48

3.4 Two trees T^1 (middle) and T^2 (right) for the current spanning tree (left). 50

3.5 Cutsets corresponding to binomials $g_{(p)}^{(1)}$ (left) and $g_{(p,t)}^{(2)}$ (right). 50

3.6 Cutsets corresponding to binomials (i), (ii) and (iii). 52

3.7 Polytope P_3 and P_4. 56

3.8 Supporting hyperplane and visible facets. 57

3.9 $\mathrm{conv}(P \cup \{\boldsymbol{p}\})$ is equal to the union of P and the set of simplices $\mathrm{conv}(F \cup \{\boldsymbol{p}\})$ that F is visible from $\boldsymbol{p}$. 57

4.1 Matroid polyhedron (left) and Base polyhedron (right) for Example 4.3. 68

4.2 Multidimensional transportation problem of the type $2 \times 2 \times 3$. 71

A.1 The graphic matroid of this graph is same to $M[A]$ in Example A.5. . . 85
A.2 An acyclic orientation of K_4. 86

List of Tables

3.1 Dual algebraic approaches for primal and dual minimum cost flow problems 28

3.2 Number of reduced Gröbner bases, maximum and minimum of cardinality. 41

3.3 Number of reduced Gröbner bases of dual problems, maximum and minimum of cardinality. 47

3.4 Divided and multiplied variables while reducing by $g^{(2)}_{(1,1)}$ and $g^{(1)}_{(1)}$. . . . 51

3.5 Divided and multiplied variables while reducing by $g^{(2)}_{(i_r,1)}$ and f_D. . . . 53

4.1 Dual algebraic approaches for primal and dual multidimensional transportation problems of several types. 63

4.2 Vector matroids $M[A]$ of A and computational algebraic methods for $I_{\Lambda(A)}$. Each inclusion in the second column becomes equal if A is unimodular. 63

4.3 The number of dual feasible bases of MTPs and that of bases of vector matroids. $A_{M,N}$ is the incidence matrix of the bipartite graph $K_{M,N}$. $M := M_{s-1}$, $N := M_s$ and $\Lambda^k(A) := \Lambda(\Lambda^{k-1}(A))$. 73

4.4 The number of dual feasible bases of MTPs and that of bases of vector matroids. $A_{M,N}$ is the incidence matrix of the bipartite graph $K_{M,N}$. $M := M_{s-1}$, $N := M_s$ and $\Lambda^k(A) := \Lambda(\Lambda^{k-1}(A))$. 76

Chapter 1

Introduction

1.1 Background

Integer programs form one of the most important classes in combinatorial optimization, and many researches on both theoretical and algorithmic aspects have been done. On algorithms side, useful algorithms such as the branch-and-bound method, the cutting-plane method or relaxation methods have been developed and implemented in the last fifty years (e.g., see [63]). On the other hand, theoretical studies seem to be less than algorithmic ones.

Network optimization problems form a nice subclass of integer programs in the sense that both theoretical structure and algorithms have been studied in detail. While general integer programs have been shown to be NP-complete, many network optimization problems have shown to be solved in polynomial time. One of the most important causes of the polynomial-time solvability is *unimodularity* of coefficient matrices.

In recent years, different theoretical approaches via computational algebra theory have been studied. Main methods used are *Gröbner bases* and *standard pair decompositions* of *toric ideals*. These methods connect integer programs with combinatorial or algebraic results about toric ideals such as triangulations, normalized volumes or hypergeometric systems.

Gröbner bases for polynomial ideals was introduced by Buchberger in 1965 [12]. Many algorithms and applications using Gröbner bases have been reported: algebraic algorithms in mathematical software systems [16, 17, 54, 83], algebraic geometry [17, 50], invariant theory [70], geometric designs [64], dynamical systems [25], systems of linear differential equations [55, 60], coding theory [17, 49, 36], etc.

This thesis concerns one of such applications of Gröbner bases to *integer programs* such as

$$IP_{A,\boldsymbol{c}}(\boldsymbol{b}) := minimize\{\boldsymbol{c} \cdot \boldsymbol{x} \mid A\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \in \mathbb{N}^n\},$$

($A \in \mathbb{Z}^{d \times n}$, $\boldsymbol{b} \in \mathbb{Z}^d$, $\boldsymbol{c} \in \mathbb{R}^n$ and $\mathbb{N}$ is the set of non-negative integers), which was first reported in 1991 by Conti and Traverso [13]. This approach utilized the discreteness of *toric ideals*, which have been studied in detail in algebraic geometry. Their algorithm first finds a feasible solution and a Gröbner basis for the toric ideal, and if a monomial corresponding to the feasible solution can be reduced by the Gröbner basis, then it reduces as many times as possible, and the optimal solution is automatically obtained by the resulting monomial. As this reduction corresponds to a move to another feasible solution with less cost, this approach is an algebraic interpretation of the algorithm using *test sets* [14]. This approach is, in itself, very interesting, as it is an application of computational algebraic methods to difficult problems like these, and provides a method for algebraic analyses of structure and algebraic algorithms for integer programs [13, 72, 76, 77, 79, 86], though this approach does not yield improved complexity bounds, as compared with existing methods, and has not been demonstrated as solving difficult practical cases that cannot be handled by existing methods.

Standard pair decompositions of monomial ideals was introduced by Sturmfels et al. [73] to give a characteristic invariant (called the *arithmetic degree*) in algebraic geometry. Many applications of standard pair decompositions were reported to the theory of algebraic geometry, especially to algebraic combinatorics of monomial ideals (for more details of recent results, see [75]).

Hoçten and Thomas reported another computational algebraic approach to integer programs [34] (also appeared in Hoçten's Ph.D thesis [31]) that uses *standard pair decompositions* of monomial ideals defined from toric ideals. Their algorithm first finds the set of standard pairs, and successively solves linear systems of equations, each of which is defined from one standard pair, until a non-negative integer solution is obtained. In this algorithm, the number of standard pairs, i.e., the arithmetic degree, is an important invariant that indicates the complexity of this approach. Although, like the approach using Gröbner bases, this approach does not yield improved complexity bounds, it does provide a new subclass of integer programs called the *Gomory family* [33], which generalizes the *total dual integrality* [23, 53, 63].

Standard pairs are related with Gomory's relaxation [28], which is one of the classical relaxation approaches to integer programs. Gomory's relaxation considers a family of group relaxations, each of which is indexed by a face of a simplicial complex called a *regular triangulation* [33] (different relaxations correspond to different faces). Each standard pair also has an index of a face of regular triangulation (different pairs may have same index), and a group relaxation indexed by a face τ solves the given integer program if and only if there exists some standard pair whose index is τ' with $\tau \subseteq \tau'$ [33, 78]. The family $IP_{A,\boldsymbol{c}}$ of all integer programs $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$ as $\boldsymbol{b}$ varies in $\mathbb{N}A := \{A\boldsymbol{u} \mid \boldsymbol{u} \in \mathbb{N}^n\} \subseteq \mathbb{Z}^d$ is called the *Gomory family* if each program in $IP_{A,\boldsymbol{c}}$ is solved by a group relaxation which is indexed by a *maximal* face.

For an ideal over a polynomial ring, the reduced Gröbner basis and the set of standard pairs are dual in the sense that the complement of the monomials in the initial ideal, which is generated by the initial terms of polynomials in the reduced Gröbner basis, is the set of standard monomials, a nice decomposition of which is the standard pair decomposition. This kind of duality may shed new light on duality in combinatorial optimization, and by considering a nice subclass of integer programming problems where the duality theorem holds, we might be able to obtain some complexity bounds by making use of the characteristics of the subclass, which could not be derived for general integer programming problems.

For these reasons, this dissertation focuses on Gröbner bases and standard pairs for several well-studied subclasses of integer programs: unimodular integer programs and Lawrence-type integer programs. In particular, we focus on the arithmetic degree, one important invariant for computational algebraic approaches to integer programs, and the number of dual feasible bases, which, in the case of non-degenerate programs, is the number of vertices of the feasible region for its dual problem. Generally, the number of dual feasible bases is equal to or less than the arithmetic degree [33]. As almost all of the problems we consider are non-degenerate, the number of dual feasible bases characterizes the feasible region of the dual problem and is also an important invariant of the program. This dissertation describes analysis of the computational algebraic complexity of unimodular and Lawrence-type integer programs using the duality of Gröbner bases and standard pairs, which will be important to provide new duality in combinatorial optimization and complexity bounds, which could not be derived using general integer programs.

1.2 Unimodular Integer Programs

A row-full rank matrix $A \in \mathbb{Z}^{d\times n}$ is called *unimodular* if each nonsingular submatrix of order d has determinant ± 1.

Integer programs whose coefficient matrices are unimodular are the most fundamental subclass of integer programs in the sense that, for any integral matrix A and integral vector $\boldsymbol{b}$ for which the linear relaxation of $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$ is feasible, the feasible region of the linear relaxation is integral (i.e., any vertex of the region is integral) [80]. In particular, if A is unimodular, the optimal solution for the linear relaxation of $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$ is also the optimal solution to $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$. In algebraic approaches for integer programs, unimodularity guarantees that the set of circuits of a vector matroid, the union of all reduced Gröbner bases, and the Graver basis [29] coincide. The incidence matrix of a directed graph is unimodular (e.g., see [63]).

There are additional related subclasses of integer programs, which are involved with the integrality of feasible regions. A family of unimodular integer programs can be considered as a special subset of *total dual integral (TDI)* systems [23], in the sense that the system $\boldsymbol{y}A \leq \boldsymbol{c}$ is TDI for *any* $\boldsymbol{c}$ if and only if A is unimodular. If the system $\boldsymbol{y}A \leq \boldsymbol{c}$ is TDI, then the set of standard pair decompositions corresponds to the set of vertices of the dual polyhedron of $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$, or equivalently, the set of dual feasible bases of $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$ [33]. In addition, if $\boldsymbol{y}A \leq \boldsymbol{c}$ is TDI, then the family $IP_{A,\boldsymbol{c}}$ becomes a Gomory family [33].

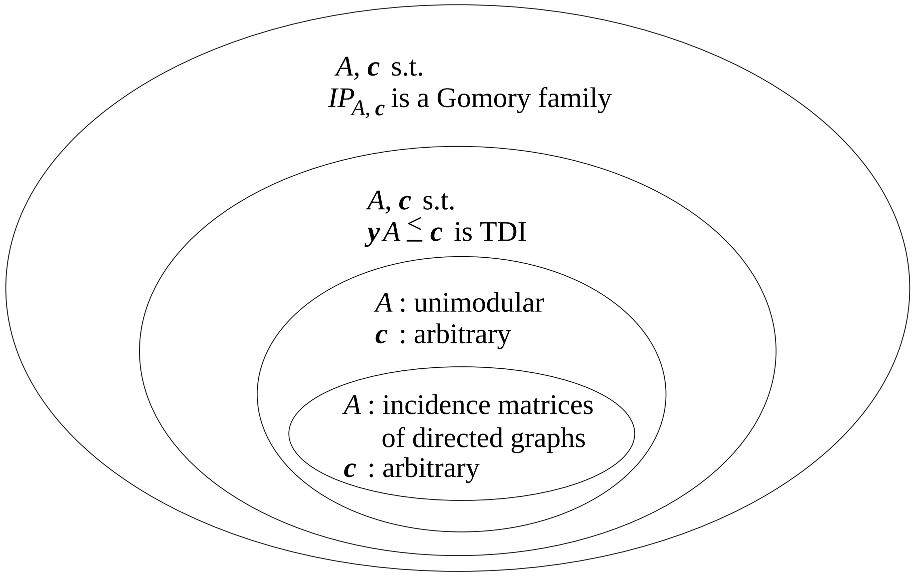

Figure 1.1: Classes of matrices and cost vectors.

1.3 Lawrence-Type Integer Programs

A matrix of the form $\Lambda(A) := \begin{pmatrix} A & O \\ I & I \end{pmatrix}$ ($A \in \mathbb{Z}^{d \times n}$, I is the $n \times n$ identity matrix and O is the $d \times n$ zero matrix) is said to be of *Lawrence type*, or to be the *Lawrence lifting* of A [8]. This type of matrix was first suggested by Lawrence (1980, unpublished) in oriented matroid theory, such that, for a given oriented matroid M, there exists a matroid $\Lambda(M)$ such that the face lattice of $\Lambda(M)$ is polytopal if and only if M is realizable [6]. If M is realizable, then $\Lambda(M)$ is realized by $\Lambda(B)$ for some matrix B. The convex hull of column vectors of $\Lambda(B)$ is called a *Lawrence polytope*, and its *Gale transform* (a specific duality in oriented matroids and polytopes) is centrally symmetrical [6]. The above Lawrence construction of oriented matroids, or Lawrence polytope, has been applied to the theory of convex polytopes [6, 8, 61, 68, 85].

In combinatorial optimization problems, Lawrence-type integer programming problems whose coefficient matrices are of Lawrence type arise in many situations. For example, the capacitated integer program $minimize\ \{\boldsymbol{c}\cdot\boldsymbol{x} \mid A\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{0} \leq \boldsymbol{x} \leq \boldsymbol{u}\}$ is equivalent to the problem $minimize\ \left\{\boldsymbol{c}\cdot\boldsymbol{x} \;\middle|\; \Lambda(A)\binom{\boldsymbol{x}}{\boldsymbol{s}} = \binom{\boldsymbol{b}}{\boldsymbol{u}},\ \ \binom{\boldsymbol{x}}{\boldsymbol{s}} \geq \boldsymbol{0}\right\}$ with the introduction of slack variables $\boldsymbol{s}$. In multidimensional transportation problems [84], coefficient matrices of several problems can be replaced by a Lawrence-type matrix.

In statistical analyses, Lawrence-type matrices, or *r-th Lawrence lifting* [62], appear in multi-way contingency tables with fixed marginal sums [3, 20, 62] since the conditions that a table has a fixed marginal sum can be expressed using Lawrence-type matrices or r-th Lawrence lifting. For enumeration of multi-way contingency tables with fixed marginal sums, it is necessary to enumerate feasible solutions for multidimensional transportation problems. However, the problem of counting the number of contingency tables with fixed marginal sums has been shown to be #P-complete [22] even the case of $2 \times N$ tables. Furthermore, for given marginal sums of a three dimensional contingency table, the problem to show the existence of a table satisfying marginal sums have shown to be NP-complete [37]. But, for sampling contingency tables, Markov Chain Monte Carlo methods whose mixing time (roughly, the number of iterations until converging to the stationary distribution) become polynomial of the size of input data have been studied for tables of type $2 \times N$ [21] and type $2 \times \cdots \times 2 \times N$ [51]. It is an open problem whether there exists a Markov Chain Monte Carlo method whose mixing time is polynomial for a $M \times N$ ($M, N \geq 3$) contingency table [81].

Algebraically, reduced Gröbner bases for toric ideals of Lawrence-type matrices and their relations with matroids have been studied in detail, and are applied to algorithms that compute universal Gröbner bases and Graver bases [71, 72]. On the other hand, standard pairs for Lawrence-type matrices and their matroidal structure have not been characterized in detail.

1.4 Our Contributions

1.4.1 Unimodular Integer Programs

The Gomory family described by Hoçten and Thomas [33] includes the family of unimodular integer programs. In their paper, they provided the lower bound for an arithmetic degree of any integer program, and introduced the results reported by Kannan [47] who studied the upper bound of an arithmetic degree.

For unimodular integer programs, this dissertation describes the approach for integer programs reported by Hoçten and Thomas [34] in terms of reduced costs, and gives the maximum number of arithmetic degrees by the normalized volume of a polytope defined by the coefficient matrix. This result gives a framework for computing the number of feasible bases via Gröbner bases and volume computations.

This dissertation also considers minimum cost flow problems, which form a well-known subclass of unimodular integer programming problems that can be solved in polynomial time. A variant of the cycle-canceling algorithm represents a Gröbner basis approach for minimum cost flow problems. For any feasible flow, strongly polynomial time algorithms [27, 65, 66] use selection rules to choose the polynomial size of negative-cost cycles from the set of negative-cost cycles in the residual network, which may be of exponential size, and augment along the cycles as many as possible. Similarly, the algorithm using Gröbner basis calculates the optimal flow by augmenting flows along the negative-cost cycles that correspond to the elements of the Gröbner basis. Thus, the cardinalities of reduced Gröbner bases may provide some time bounds for this algorithm. On the other hand, the standard pair approach for minimum cost flow problems first finds the set of standard pairs, and solves linear systems of equations for each standard pair until a non-negative integer solution is obtained. Therefore, the number of standard pairs, i.e., the arithmetic degree, indicates the complexity of this approach. Furthermore, the arithmetic degree for minimum cost flow problems gives the number of dual feasible bases.

We characterize Gröbner bases and standard pairs in terms of graphs, and give bounds for the number of dual and primal feasible bases of the minimum cost flow problems on the acyclic tournament graph with d vertices: for the number of dual feasible bases, the minimum and maximum are shown to be 1 and the Catalan number $\frac{1}{d}\binom{2(d-1)}{d-1}$, respectively, while for the number of primal feasible bases the lower bound is shown to be $\Omega(2^{\lfloor d/6 \rfloor})$. To analyze arithmetic degrees, we use two approaches: the computational algebraic duality between reduced Gröbner bases and standard pairs, and the results of combinatorics related with toric ideals, i.e., hypergeometric systems related with toric ideals.

Furthermore, we consider computational algebraic methods for primal and dual transportation problems. For transportation problems on bipartite graphs, the structures of feasible polyhedra, especially the number of vertices, have been studied for both primal [1, 11, 48, 58] and dual [4, 5] problems. We give computational algebraic proof via computations of normalized volumes for the following existing results:

- The maximum number of vertices for the feasible region of a transportation problem on $K_{2,n}$ is $(n - [n/2])\binom{n}{[n/2]}$ [48].
- The maximum number of vertices for the feasible region of a dual transportation problem on $K_{m,n}$ is $\binom{m+n-2}{m-1}$ [5].

For a network optimization problem, the duality between the reduced Gröbner basis and the set of standard pairs corresponds to the relation between circuits and dual feasible co-trees, dually, cutsets and primal feasible trees. As such relations have not been clarified, this computational algebraic duality may be an interesting method for the analysis of network problems.

1.4.2 Lawrence-Type Integer Programs

As the toric ideal of a matrix A is defined by linear dependencies of column vectors of A, the toric ideal of A (or $\Lambda(A)$) is related to the vector matroid $M[A]$ of A. Actually, the set of circuits of $M[A]$ is related to reduced Gröbner bases in the sense that the union of all possible reduced Gröbner bases is a superset of the set of binomials corresponding to the set of all circuits of $M[A]$. If A is unimodular, these two sets coincide. On the other hand, the relation between standard pairs for toric ideals of A (or $\Lambda(A)$) and $M[A]$ has not been characterized in detail.

This dissertation describes the relation between standard pairs for a toric ideal of the Lawrence-type matrix $\Lambda(A)$ and a vector matroid $M[A]$. In particular, we focus on standard pairs that correspond to dual feasible bases. As a toric ideal of $\Lambda(A)$ has information about the set of circuits of $M[A]$, a set of standard pairs may be related to the bases of $M[A]$. We describe a bijection between the set of bases of $M[A]$ and the set of standard pairs corresponding to dual feasible bases. In addition, we describe the matroidal structure of these standard pairs in the sense that the adjacency relation of standard pairs as facets in a regular triangulation corresponds to the adjacency of vertices in the base polyhedron [24] of $M[A]$, and each adjacency of vertices in the base polyhedron appears as adjacent facets for some regular triangulation.

As practical applications, we consider capacitated minimum cost flow problems on acyclic tournament graphs and multidimensional transportation problems of type $2 \times \cdots \times 2 \times M \times N$. For capacitated minimum cost flow problems on the acyclic tournament graph with d vertices, the number of dual feasible bases is shown to be d^{d-2}, where, as described in the previous section, for the uncapacitated minimum cost flow problem on the same graph the number is equal to or less than the Catalan number $\frac{1}{d}\binom{2(d-1)}{d-1}$. For multidimensional transportation problems of type $2 \times \cdots \times 2 \times M \times N$, the number of dual feasible bases is shown to be $M^{N-1}N^{M-1}\left(2^{(M-1)(N-1)}\right)^{s-3}$, where s is the dimension of the problem. On the other hand, for the number of primal feasible bases, we can analyze only the case $2 \times \cdots \times 2 \times 2 \times N$. These results may give analyses of the difficulty of statistical or combinatorial problems on multi-way contingency tables from the viewpoint of dualistic computational algebraic methods.

The structures of the feasible regions of general transportation problems and the structures of feasible polyhedra, especially the number of vertices, have been studied for primal [11] and dual [5] problems. On the other hand, for multidimensional transportation problems there have been few studies of the structures of feasible polyhedra [84]. Therefore, our results represent additional characterization of the structures of feasible polyhedra of primal and dual multidimensional transportation problems.

Enumerations of multi-way contingency tables with fixed marginal sums are equivalent to enumeration of feasible solutions for multidimensional transportation problems. When the table is of type $2\times\cdots\times 2\times M\times N$, as the coefficient matrix is unimodular, each vertex of the feasible polyhedron for the linear relaxation problem is integral. Therefore, the number of primal feasible bases for non-degenerate multidimensional transportation problems of type $2\times\cdots\times 2\times M\times N$ gives lower bounds for the number of multi-way contingency tables.

1.5 Organization of this Dissertation

This dissertation is organized as follows. Chapter 2 introduces some computational algebraic methods — Gröbner bases and standard pair decompositions — and their applications to integer programs. Chapter 3 and 4 represent the main part of the dissertation. Chapter 3 is concerned with unimodular integer programs. The maximum number of standard pairs is shown in terms of volume of a polytope, and we analyze the Hoçten-Thomas algorithm for unimodular integer programs from the viewpoint of computation of the reduced cost of linear programs. Computational algebraic methods for primal and dual minimum cost flow problems on acyclic tournament graphs are also studied. We give the algebraic difference of complexity between primal and dual problems in terms of arithmetic degree. Furthermore, we give computational algebraic proof for existing results on the numbers of primal and dual feasible bases of the transportation problem. Chapter 4 describes the study of Lawrence-type integer programs. We give a bijection between the set of standard pairs corresponding to dual feasible bases and the set of bases of the vector matroid, and also give the matroidal structure of such standard pairs. We also analyze capacitated minimum cost flow problems and primal and dual multidimensional transportation problems. The results in this section indicate the difficulty of analysis of statistical or combinatorial problems on multi-way contingency tables from the viewpoint of dualistic computational algebraic methods. Finally, Chapter 5 summarizes the findings described in this dissertation.

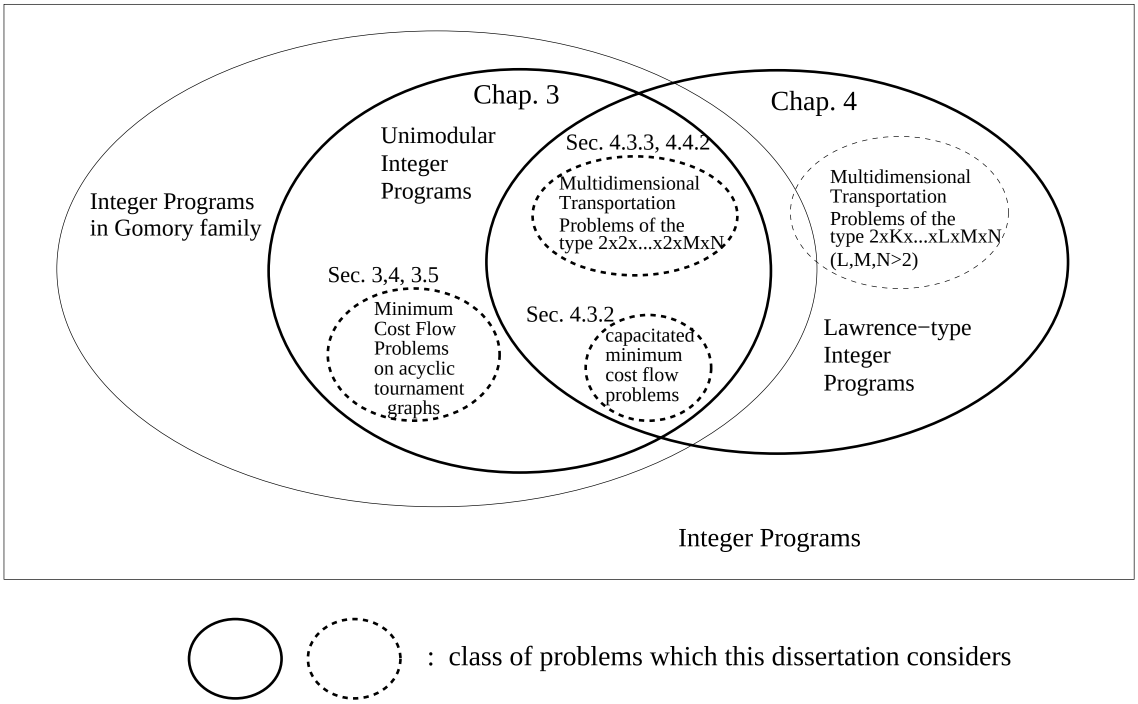

Figure 1.2: Gomory integer programs, unimodular integer programs and Lawrence-type integer programs.

Chapter 2

Algebraic Preliminaries: Computational Algebra and Integer Programs

This chapter provides the basic definitions of computational algebra theory for integer programming, which are used in subsequent sections of the dissertation. We refer to [16] for an introduction to Gröbner bases. For more details concerning the applications of computational algebra theory to integer programming, see [34, 71, 78].

2.1 Gröbner Bases

2.1.1 Definitions

Let k be a field and $k[x_1, \ldots, x_n]$ the polynomial ring with coefficients in k, and $\boldsymbol{x} = \{x_1, \ldots, x_n\}$ the set of variables of $k[x_1, \ldots, x_n]$. For an exponent vector $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$, we denote $\boldsymbol{x}^{\boldsymbol{a}} := x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$.

Definition 2.1 *A non-empty subset $I \subseteq k[x_1, \ldots, x_n]$ is an* ideal *if it satisfies the following conditions:*

1. *For any $f, g \in I$, $f + g \in I$, and,*

2. *For any $f \in I$ and $h \in k[x_1, \ldots, x_n]$, $fh \in I$.*

Proposition 2.2 *If* $f_1, \dots, f_s \in k[x_1, \dots, x_n]$ *be polynomials, Then*

$$\langle f_1, \dots, f_s \rangle := \left\{ \sum_{i=1}^{s} h_i f_i \;\middle|\; h_1, \dots, h_s \in k[x_1, \dots, x_n] \right\}$$

is an ideal of $k[x_1, \dots, x_n]$.

We call $\langle f_1, \dots, f_s \rangle$ the *ideal generated by* $f_1, \dots, f_s$, and $\{f_1, \dots, f_s\}$ a *generator* of $\langle f_1, \dots, f_s \rangle$. By the Hilbert Basis Theorem ([16, Chap. 2, §5, Theorem 4.]), every ideal I over $k[x_1, \dots, x_n]$ is finitely generated, i.e., there exist finite $f_1, \dots, f_s \in I$ such that $I = \langle f_1, \dots, f_s \rangle$.

To define a Gröbner basis of an ideal, it is first necessary to define a *term order*, i.e., a good total order of monomials.

Definition 2.3 *A total order* $\succ$ *on the set of monomials in* $k[x_1, \dots, x_n]$ *is a* term order *if it satisfies the following conditions:*

1. *If* $\boldsymbol{x}^{\boldsymbol{u}} \succ \boldsymbol{x}^{\boldsymbol{v}}$ *and* $\boldsymbol{w} \in \mathbb{N}^n$, *then* $\boldsymbol{x}^{\boldsymbol{u}}\boldsymbol{x}^{\boldsymbol{w}} \succ \boldsymbol{x}^{\boldsymbol{v}}\boldsymbol{x}^{\boldsymbol{w}}$, *and,*

2. *For any* $\boldsymbol{u} \in \mathbb{N}^n \setminus \{0\}$, $\boldsymbol{x}^{\boldsymbol{u}} \succ 1$.

The following lists some well-known examples of term orders.

Definition 2.4 *Fix variable ordering* $x_{i_1} > x_{i_2} > \cdots > x_{i_n}$. *A* lexicographic order *induced by this variable ordering is defined as follows:*

$$\boldsymbol{x}^{\boldsymbol{u}} \succ \boldsymbol{x}^{\boldsymbol{v}} \Longleftrightarrow \text{the leftmost non-zero element of } (u_{i_1} - v_{i_1}, \dots, u_{i_n} - v_{i_n}) \text{ is positive}.$$

Definition 2.5 *Fix variable ordering* $x_{i_1} > x_{i_2} > \cdots > x_{i_n}$. *A* degree lexicographic order *induced by this variable ordering is defined as follows:*

$$\boldsymbol{x}^{\boldsymbol{u}} \succ \boldsymbol{x}^{\boldsymbol{v}} \Longleftrightarrow \begin{array}{l} \text{the leftmost non-zero element of} \\ \left(\sum_{k=1}^{n}(u_k - v_k), u_{i_1} - v_{i_1}, \dots, u_{i_n} - v_{i_n}\right) \text{ is positive}. \end{array}$$

Definition 2.6 *Fix variable ordering* $x_{i_1} > x_{i_2} > \cdots > x_{i_n}$. *A* degree reverse lexicographic order *induced by this variable ordering is defined as follows:*

$$\boldsymbol{x}^{\boldsymbol{u}} \succ \boldsymbol{x}^{\boldsymbol{v}} \iff \begin{array}{l}\textit{the leftmost non-zero element of} \\ (\sum_{k=1}^{n}(u_k - v_k), v_{i_n} - u_{i_n}, \ldots, v_{i_1} - u_{i_1}) \textit{ is positive}.\end{array}$$

For a term order $\succ$ and a polynomial $f \in k[x_1, \ldots, x_n]$, we denote $in_\succ(f)$ the largest term of f with respect to $\succ$ and call the *initial term* of f. Then, for an ideal $I \subseteq k[x_1, \ldots, x_n]$, we define the *initial ideal* of I as $in_\succ(I) = \langle in_\succ(f) \mid f \in I\rangle$.

Definition 2.7 *Let I be an ideal over $k[x_1, \ldots, x_n]$ and $\succ$ a term order. A* Gröbner basis *of I with respect to $\succ$ is a finite set $\mathcal{G}_\succ = \{g_1, \ldots, g_s\} \subset I$ such that*

$$in_\succ(I) = \langle in_\succ(g_1), \ldots, in_\succ(g_s)\rangle.$$

A Gröbner basis $\mathcal{G}_\succ$ is reduced *if, for any i, j with $i \neq j$, no term of g_i is divisible by $in_\succ(g_j)$ and the coefficient of $in_\succ(g_i)$ is* 1 *for any i.*

A Gröbner basis is really a *basis* of an ideal.

Proposition 2.8 ([16, Chap. 2, §5, Corollary 6.]) *A Gröbner basis of an ideal I is a generator of I.*

One of the most useful properties of a Gröbner basis is a uniqueness of the remainder of a division.

Proposition 2.9 ([16, Chap. 2, §6. Proposition 1.]) *Let $\mathcal{G}_\succ = \{g_1, \ldots, g_s\}$ be a Gröbner basis for an ideal $I \subseteq k[x_1, \ldots, x_n]$ with respect to a term order $\succ$. Then, every $f \in k[x_1, \ldots, x_n]$ can be written as*

$$f = a_1 g_1 + \cdots + a_s g_s + r, \quad (a_1, \ldots, a_s, r \in k[x_1, \ldots, x_n]),$$

where either $r = 0$ or no term of r is divisible by any of $in_\succ(g_1), \ldots, in_\succ(g_s)$. In addition, r is unique, and called the normal form *of f by $\mathcal{G}_\succ$.*

2.1.2 Gröbner Bases in Integer Programs

We consider an integer program $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$:

$$IP_{A,\boldsymbol{c}}(\boldsymbol{b}) := minimize\ \{\boldsymbol{c}\cdot\boldsymbol{x} \mid A\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \in \mathbb{N}^n\},$$

where $A \in \mathbb{Z}^{d\times n}$, $\boldsymbol{b} \in \mathbb{Z}^d$, $\boldsymbol{c} \in \mathbb{R}^n$ and $\mathbb{N}$ is the set of non-negative integers. We define the family of integer programs $IP_{A,\boldsymbol{c}}$ as $IP_{A,\boldsymbol{c}} := \{IP_{A,\boldsymbol{c}}(\boldsymbol{b}) \mid \boldsymbol{b} = A\boldsymbol{u}$ for some $\boldsymbol{u} \in \mathbb{N}^n\}$. A cost vector $\boldsymbol{c}$ is *generic* if each program in $IP_{A,\boldsymbol{c}}$ has a unique optimal solution. In this dissertation, we assume that $\boldsymbol{c}$ is generic, and if not we use a generic $\boldsymbol{c}'$, which is obtained by perturbing $\boldsymbol{c}$ such that the optimal solution for $IP_{A,\boldsymbol{c}'}(\boldsymbol{b})$ is optimal for $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$.

For application of computational algebraic methods to integer programs, we consider the *toric ideal*.

Definition 2.10 *Let $A \in \mathbb{Z}^{d\times n}$. Then, we define the* toric ideal I_A *of A as*

$$I_A := \langle \boldsymbol{x}^{\boldsymbol{u}} - \boldsymbol{x}^{\boldsymbol{v}} \mid A\boldsymbol{u} = A\boldsymbol{v},\ \boldsymbol{u},\boldsymbol{v} \in \mathbb{N}^n\rangle \subseteq k[x_1,\dots,x_n].$$

Every vector $\boldsymbol{u} \in \mathbb{Z}^n$ can be written uniquely as $\boldsymbol{u} = \boldsymbol{u}^+ - \boldsymbol{u}^-$ where $\boldsymbol{u}^+$ and $\boldsymbol{u}^-$ are non-negative and have disjoint support, where the *support* of a vector $\boldsymbol{u}$ is $\{i \mid u_i \neq 0\}$.

Proposition 2.11 ([71, Corollary 4.4.]) *For every term order $\succ$, there is a finite set of vectors $\{\boldsymbol{u}_1,\dots,\boldsymbol{u}_s\} \subset \ker(A)\cap\mathbb{Z}^n$ such that the reduced Gröbner basis of I_A with respect to $\succ$ is equal to $\left\{\boldsymbol{x}^{\boldsymbol{u}_i^+} - \boldsymbol{x}^{\boldsymbol{u}_i^-} \;\middle|\; i = 1,\dots,s\right\}$.*

Example 2.12 *Let $A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$, the matrix obtained from the incidence matrix of the network shown in Figure 2.1 by deleting one row, and consider the minimum cost flow problem*

$$IP_{A,\boldsymbol{c}}(\boldsymbol{b}) = minimize\ \{\boldsymbol{c}\cdot\boldsymbol{x} \mid A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} = (x_{1,2}, x_{1,3}, x_{2,3}) \in \mathbb{N}^3\}.$$

Then, the toric ideal is $I_A = \langle x_{1,2}x_{2,3} - x_{1,3}\rangle$. □

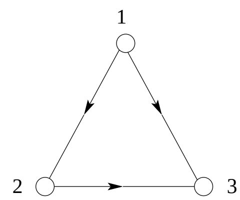

Figure 2.1: Acyclic tournament graph with 3 vertices.

For a fixed term order $\succ$, the *refinement* of $\boldsymbol{c}$ by $\succ$ is a total order $\succ_{\boldsymbol{c}}$ such that

$$\boldsymbol{x}^{\boldsymbol{u}} \succ_{\boldsymbol{c}} \boldsymbol{x}^{\boldsymbol{v}} \iff \boldsymbol{c} \cdot \boldsymbol{u} > \boldsymbol{c} \cdot \boldsymbol{v} \text{ or } (\boldsymbol{c} \cdot \boldsymbol{u} = \boldsymbol{c} \cdot \boldsymbol{v} \text{ and } \boldsymbol{x}^{\boldsymbol{u}} \succ \boldsymbol{x}^{\boldsymbol{v}}) .$$

If $\boldsymbol{c} \geq 0$, then $\succ_{\boldsymbol{c}}$ becomes a term order.

An ideal $I \subseteq k[x_1, \dots, x_n]$ is *homogeneous* with respect to the positive grading $\deg(x_i) = d_i > 0$ $(i = 1, \dots, n)$ if for any $f = f_1 + f_2 + \cdots + f_m \in I$ (f_i is the homogeneous component of degree i in f), $f_i \in I$ for any i. An ideal is homogeneous if and only if it is generated by homogeneous polynomials [16].

Proposition 2.13 ([71, Proposition 1.12.]) *If I is homogeneous with respect to some positive grading $\deg(x_i) = d_i > 0$, then $\succ_{\boldsymbol{c}}$ becomes a term order for any $\boldsymbol{c} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$, and the reduced Gröbner basis $\mathcal{G}_{\succ_{\boldsymbol{c}}}$ exists.*

In the rest of this chapter, we consider a cost vector $\boldsymbol{c}$ for which $\succ_{\boldsymbol{c}}$ becomes a term order for some term order $\succ$. Let $IP_{A,\succ_{\boldsymbol{c}}}(\boldsymbol{b})$ be the problem to find the unique minimal element in $\{\boldsymbol{x} \in \mathbb{N}^n \mid A\boldsymbol{x} = \boldsymbol{b}\}$ with respect to $\succ_{\boldsymbol{c}}$ and consider the family $IP_{A,\succ_{\boldsymbol{c}}}$ of integer programs same as $IP_{A,\boldsymbol{c}}$. Then, the solution $\boldsymbol{u}$ of $IP_{A,\succ_{\boldsymbol{c}}}(\boldsymbol{b})$ is one of the optimal solutions of $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$. Conti and Traverso [13] introduced an algorithm based on a Gröbner basis to solve $IP_{A,\succ_{\boldsymbol{c}}}(\boldsymbol{b})$. We describe two versions of the Conti-Traverso algorithm: the first is the original version (see [13]), and the other is a condensed version (see [71]).

The original version of the Conti-Traverso algorithm considers an ideal in a polynomial ring with $d+n+1$ variables. Thus, much more time is needed to calculate a Gröbner basis, although this version is more implementation-friendly than the condensed version, and the generator of the ideal involved is given automatically. The condensed version does not have this latter property.

In this algorithm, we consider an ideal in $k[x_1, \dots, x_n, t_0, t_1, \dots, t_d]$. Let $\succ'$ be any term order such that (i) every monomial that contains one of $t_0, t_1, \dots, t_d$ is greater than any monomial that does not, and (ii) $\succ'$ restricted to $k[x_1, \dots, x_n]$ induces the same total order as $\succ_{\boldsymbol{c}}$.

Algorithm 2.14 (Conti-Traverso Algorithm (The original version))

Input: *a matrix* $A \in \mathbb{Z}^{d\times n}$*, a cost vector* $\boldsymbol{c} \in \mathbb{R}^n$
a right hand side vector $\boldsymbol{b} \in \mathbb{Z}^d$*, a term order* $\succ$

Output: *whether* $IP_{A,\succ_{\boldsymbol{c}}}(\boldsymbol{b})$ *is feasible or not, and if feasible, the optimal solution* $\boldsymbol{u}$ *of* $IP_{A,\succ_{\boldsymbol{c}}}(\boldsymbol{b})$

Step 1. *Compute the reduced Gröbner basis* $\mathcal{G}$ *of an ideal* $J = \langle x_1\boldsymbol{t}^{\boldsymbol{a}_1^-} - \boldsymbol{t}^{\boldsymbol{a}_1^+}, \dots, x_n\boldsymbol{t}^{\boldsymbol{a}_n^-} - \boldsymbol{t}^{\boldsymbol{a}_n^+}, t_0t_1\cdots t_d - 1\rangle$*, where* $\boldsymbol{t} = \{t_1, \dots, t_d\}$*, with respect to* $\succ'$*.*

Step 2. *Let* $\beta := \max(\{|b_j| \mid b_j < 0\}, 0)$ *and* $\boldsymbol{e}_i$ *be the* i*-th unit vector in* $\mathbb{R}^d$*. For a monomial* $t_0^{\beta}\boldsymbol{t}^{\boldsymbol{b}+\beta(\boldsymbol{e}_1+\cdots+\boldsymbol{e}_d)}$*, compute its normal form* $t_0^{\gamma_0}\boldsymbol{t}^{\boldsymbol{\gamma}}\boldsymbol{x}^{\boldsymbol{u}}$ *by* $\mathcal{G}$*.*

Step 3. *If* $\gamma_0 = 0$ *and* $\boldsymbol{\gamma} = \mathbf{0}$*, then output "feasible" with the optimal solution* $\boldsymbol{u}$*. Otherwise output "infeasible".*

The condensed version of the Conti-Traverso algorithm is useful for highlighting the main computational step involved. This version is effective if a generator of I_A and one feasible solution of $IP_{A,\succ_{\boldsymbol{c}}}(\boldsymbol{b})$ are already known.

Algorithm 2.15 (Conti-Traverso Algorithm (The condensed version))

Input: *a matrix* $A \in \mathbb{Z}^{d\times n}$*, a cost vector* $\boldsymbol{c} \in \mathbb{R}^n$
a right hand side vector $\boldsymbol{b} \in \mathbb{Z}^d$*, a term order* $\succ$

Output: *the optimal solution* $\boldsymbol{u}$ *of* $IP_{A,\succ_{\boldsymbol{c}}}(\boldsymbol{b})$

Step 1. *Compute the reduced Gröbner basis* $\mathcal{G}$ *of* I_A *with respect to* $\succ_{\boldsymbol{c}}$*.*

Step 2. *For any feasible solution* $\boldsymbol{v}$ *of* $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$*, compute the normal form* $\boldsymbol{x}^{\boldsymbol{u}}$ *of* $\boldsymbol{x}^{\boldsymbol{v}}$ *by* $\mathcal{G}$*.*

Step 3. *Output* $\boldsymbol{u}$*.* $\boldsymbol{u}$ *is the optimal solution of* $IP_{A,\succ_{\boldsymbol{c}}}(\boldsymbol{b})$*.*

Thus, the reduced Gröbner basis $\mathcal{G}$ of I_A with respect to $\succ_{\boldsymbol{c}}$ is a *test set* for $IP_{A,\succ_{\boldsymbol{c}}}$ [76, 77].

Definition 2.16 *A set $\mathcal{T} \subseteq \{x \in \mathbb{Z}^n \mid Ax = 0\}$ is a* test set *for $IP_{A,\succ_{\boldsymbol{c}}}$ if $\mathcal{T}$ satisfies the following:*

1. *given any non-optimal solution α to any program in $IP_{A,\succ_{\boldsymbol{c}}}$, there exists $t \in \mathcal{T}$ such that $\alpha - t$ is a feasible solution to the same program and $\alpha \succ_{\boldsymbol{c}} \alpha - t$,*

2. *for the optimal solution β to a program in $IP_{A,\succ_{\boldsymbol{c}}}$, $\beta - t$ is infeasible for any $t \in \mathcal{T}$.*

Example 2.12 (continued.) *If $\boldsymbol{c} = (c_{1,2}, c_{1,3}, c_{2,3}) = (3,1,2)$, then $in_{\boldsymbol{c}}(I_A) = \langle x_{1,2}x_{2,3}\rangle$ and, for any term order $\succ$, the reduced Gröbner basis is $\mathcal{G}_{\succ_{\boldsymbol{c}}} = \{x_{1,2}x_{2,3} - x_{1,3}\}$. On the other hand, if $\boldsymbol{c} = (1,4,2)$, then $in_{\boldsymbol{c}}(I_A) = \langle x_{1,3}\rangle$ and, for any term order $\succ$, the reduced Gröbner basis is $\mathcal{G}_{\succ_{\boldsymbol{c}}} = \{x_{1,3} - x_{1,2}x_{2,3}\}$.*

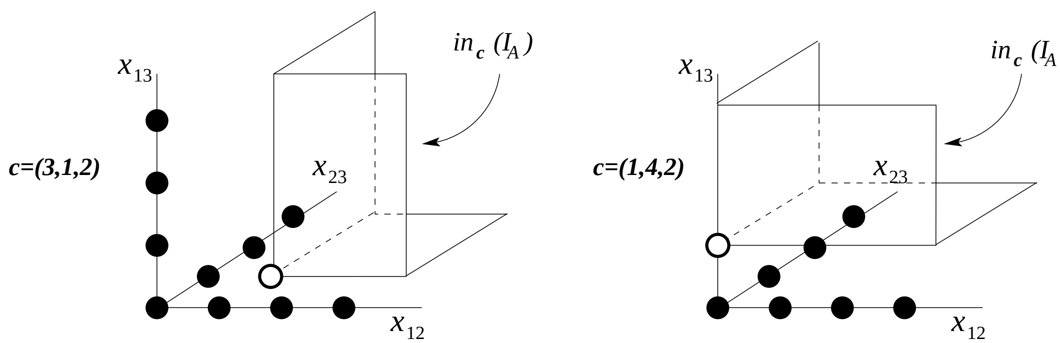

Figure 2.2: Two types of initial ideals for Example 2.12. A point (p,q,r) in the figure corresponds to the monomial $x_{1,2}^p x_{1,3}^q x_{2,3}^r$.

Let $\boldsymbol{c} = (3,1,2)$ and $\boldsymbol{b} = (4,5)$. For the original version of the Conti-Traverso algorithm, $J = \langle x_{1,2}t_2 - t_1, x_{1,3} - t_1, x_{2,3} - t_2, t_0t_1t_2 - 1\rangle$. Its reduced Gröbner basis for a suitable $\succ'$ is $\mathcal{G} = \{x_{1,2}x_{2,3} - x_{1,3}, t_2 - x_{2,3}, t_1 - x_{1,3}, t_0x_{1,3}x_{2,3} - 1, t_0x_{1,3}^2 - x_{1,2}\}$, and the normal form of $t_1^4t_2^5$ by $\mathcal{G}$ is $x_{1,3}^4x_{2,3}^5$. Thus, $IP_{A,\succ_{\boldsymbol{c}}}$ is feasible and the optimal solution is $(0,4,5)$.

For the condensed version of the Conti-Traverso algorithm, given a feasible solution $(4, 0, 9)$*, the normal form of* $x_{1,2}^4 x_{2,3}^9$ *by* $\mathcal{G} = \{x_{1,2}x_{2,3} - x_{1,3}\}$ *is* $x_{1,3}^4 x_{2,3}^5$*. Thus, the optimal solution of* $IP_{A,\succ_{\boldsymbol{c}}}(\boldsymbol{b})$ *is* $(0, 4, 5)$*.* □

Next we define three subsets of I_A: *circuits, universal Gröbner bases* and *Graver bases*.

Definition 2.17 *A binomial* $\boldsymbol{x}^{\boldsymbol{u}^+} - \boldsymbol{x}^{\boldsymbol{u}^-} \in I_A$ *is a* circuit *if the support of* $\boldsymbol{u}$ *is minimal with respect to inclusion and the elements of* $\boldsymbol{u}$ *are relatively prime. Let* $\mathcal{C}_A$ *denote a set of all circuits of* I_A*.*

Definition 2.18 *Let* $\mathcal{U}_A$ *be the union of all reduced Gröbner bases for* I_A *with respect to all term orders.* $\mathcal{U}_A$ *is the* universal Gröbner basis *of* I_A*.*

Definition 2.19 ([29]) *A binomial* $\boldsymbol{x}^{\boldsymbol{u}^+} - \boldsymbol{x}^{\boldsymbol{u}^-} \in I_A$ *is* primitive *if there exists no other binomial* $\boldsymbol{x}^{\boldsymbol{v}^+} - \boldsymbol{x}^{\boldsymbol{v}^-} \in I_A$ *such that* $\boldsymbol{x}^{\boldsymbol{v}^+}$ *divides* $\boldsymbol{x}^{\boldsymbol{u}^+}$ *and* $\boldsymbol{x}^{\boldsymbol{v}^-}$ *divides* $\boldsymbol{x}^{\boldsymbol{u}^-}$*. Let* Gr_A *denote the set of all primitive binomials of* I_A *and call the* Graver basis *of* I_A*.*

For algorithms that compute the universal Gröbner basis and the Graver basis, see [71, Algorithm 7.2. and Algorithm 7.6.].

Proposition 2.20 ([71, Proposition 4.11.]) *For every matrix* A*,* $\mathcal{C}_A \subseteq \mathcal{U}_A \subseteq Gr_A$*.*

Example 2.12 (continued.) *For this* A*,* $\mathcal{C}_A = \mathcal{U}_A = Gr_A = \{x_{1,2}x_{2,3} - x_{1,3}\}$*.* □

2.2 Standard Pair Decompositions

2.2.1 Definitions

An ideal $M \subseteq k[x_1, \ldots, x_n]$ is a *monomial ideal* if M is generated by monomials. For a monomial ideal M and a monomial $m \in k[x_1, \ldots, x_n]$, m is a *standard* monomial for M if $m \notin M$. A *standard pair decomposition* of M is a nice decomposition of the set of all standard monomials for M.

Let $[n] := \{1, \ldots, n\}$. For a monomial $\boldsymbol{x}^{\boldsymbol{u}} \in k[x_1, \ldots, x_n]$ and an index set $\sigma \subseteq [n]$, let $(\boldsymbol{x}^{\boldsymbol{u}}, \sigma)$ denote the set of monomials $\{\boldsymbol{x}^{\boldsymbol{u}} \cdot \boldsymbol{x}^{\boldsymbol{v}} \mid \boldsymbol{x}^{\boldsymbol{v}} \text{ is a monomial in } k[x_j \mid j \in \sigma]\}$.

Definition 2.21 *Let $M \subseteq k[x_1, \ldots, x_n]$ be a monomial ideal. Then, $(\boldsymbol{x^u}, \sigma)$ is a* standard pair *of M if it satisfies the following:*

(i) $\text{supp}(\boldsymbol{u}) \cap \sigma = \emptyset$*,*

(ii) *Every monomial in $(\boldsymbol{x^u}, \sigma)$ is a standard monomial for M, and*

(iii) $(\boldsymbol{x^u}, \sigma) \not\subset (\boldsymbol{x^v}, \tau)$ *for any other $(\boldsymbol{x^v}, \tau)$ that satisfies* (i) *and* (ii)*.*

Let $S(M)$ denote the set of all standard pairs of M. The cardinality of $S(M)$ is called the arithmetic degree *of M.*

The standard pairs of a monomial ideal M induce a unique covering for the set of monomials not in M, which we call the *standard pair decomposition* of M [34]. For algorithms that compute standard pairs of a monomial ideal, see [34, Algorithm 2.5] or [60, Algorithm 3.2.5]. The source code of the algorithm of Saito et al. for `Macaulay 2`, a mathematical software system, is reported in full in [32].

Example 2.12 (continued.) *For $\boldsymbol{c} = (3, 1, 2)$, the standard pairs of $in_{(3,1,2)}(I_A)$ are $(1, \{(1, 2), (1, 3)\})$ and $(1, \{(1, 3), (2, 3)\})$. Thus, the arithmetic degree of $in_{(3,1,2)}(I_A)$ is 2. On the other hand, for $\boldsymbol{c} = (1, 4, 2)$, the standard pair of $in_{(1,4,2)}(I_A)$ is $(1, \{(1, 2), (2, 3)\})$. Thus the arithmetic degree of $in_{(1,4,2)}(I_A)$ is* 1. □

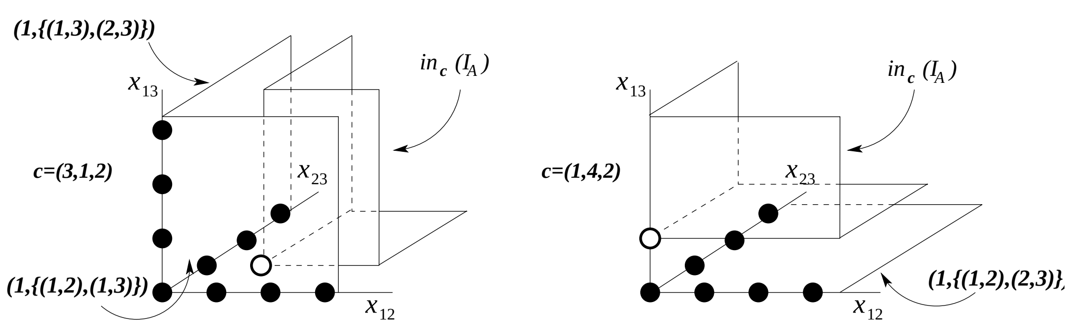

Figure 2.3: Two types of standard pair decompositions for Example 2.12.

In cases in which a monomial ideal is an initial ideal of a polynomial ring, the reduced Gröbner basis and the set of standard pairs are dual in the sense that standard pair decomposition is a nice decomposition for the complement of monomials in the initial ideal, generated by the initial terms of polynomials in the reduced Gröbner basis.

2.2.2 Standard Pair Decompositions in Integer Programs

In this section we apply standard pair decompositions to integer programs. We give the properties of standard pair decompositions for initial ideals of toric ideals.

For a polynomial $f \in I_A$, we define $in_{\boldsymbol{c}}(f)$ as the sum of terms of f that are maximal with respect to the inner product of $\boldsymbol{c}$ and the exponent vector, and $in_{\boldsymbol{c}}(I_A) := \langle in_{\boldsymbol{c}}(f) \mid f \in I_A \rangle$. Then, $in_{\boldsymbol{c}}(f)$ contains $in_{\succ_{\boldsymbol{c}}}(f)$ as a term. The assumption of the genericity of $\boldsymbol{c}$ implies that $in_{\boldsymbol{c}}(f) = in_{\succ_{\boldsymbol{c}}}(f)$ and $in_{\boldsymbol{c}}(I_A) = in_{\succ_{\boldsymbol{c}}}(I_A)$.

Let $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n\}$ be the column vectors of A and $\mathrm{cone}(A)$ the cone generated by $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$. For $\sigma \subseteq [n]$, let A_σ denote for the submatrix of A whose columns are indexed by σ.

Definition 2.22 *Let $\boldsymbol{c}$ be a cost vector.*

(i) *We define the* regular triangulation $\Delta_{\boldsymbol{c}}$ *of* $\mathrm{cone}(A)$ *as follows:* $\mathrm{cone}(A_\sigma)$ *is a* face *of* $\Delta_{\boldsymbol{c}}$ *if and only if there exists a vector* $\boldsymbol{y} \in \mathbb{R}^d$ *such that* $\boldsymbol{y} \cdot \boldsymbol{a}_j = c_j$ $(j \in \sigma)$ *and* $\boldsymbol{y} \cdot \boldsymbol{a}_j < c_j$ $(j \notin \sigma)$.

(ii) *If* $\mathrm{cone}(A_\sigma)$ *is a face of* $\Delta_{\boldsymbol{c}}$*, then* σ *is also called a* face *of* $\Delta_{\boldsymbol{c}}$.

(iii) σ *is called a* facet *of* $\Delta_{\boldsymbol{c}}$ *if* σ *is maximal face of* $\Delta_{\boldsymbol{c}}$ *with respect to inclusion.*

The genericity of $\boldsymbol{c}$ implies that $\Delta_{\boldsymbol{c}}$ is in fact a triangulation (i.e., each face of $\Delta_{\boldsymbol{c}}$ is simplicial) [72].

Example 2.12 (continued.) *For* $\boldsymbol{c} = (3, 1, 2)$, $\Delta_{(3,1,2)} = \{\{1, 2\}, \{2, 3\}, \{1\}, \{2\}, \{3\}, \emptyset\}$ *and facets are* $\{1, 2\}$ *and* $\{2, 3\}$ (Figure 2.4 left). *On the other hand, for* $\boldsymbol{c} = (1, 4, 2)$, $\Delta_{(1,4,2)} = \{\{1, 3\}, \{1\}, \{3\}, \emptyset\}$ *and only* $\{1, 3\}$ *is a facet* (Figure 2.4 right). □

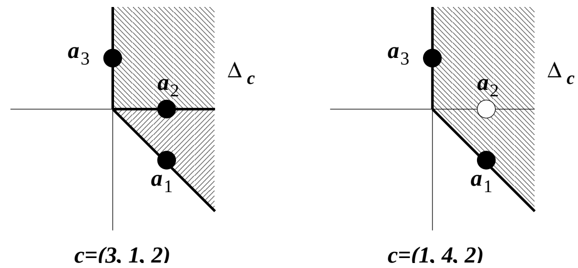

Figure 2.4: Two regular triangulations for Example 2.12.

When vertices of $\mathrm{conv}(A)$ are in the m-dimensional lattice $L \simeq \mathbb{Z}^m$, we define the *normalized volume* of a maximal face σ of $\Delta_{\boldsymbol{c}}$ by the volume of σ with normalization such that the volume of the convex hull of $\mathbf{0}, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_m$ is 1. Here, $\{\boldsymbol{e}_i\}_{1 \le i \le m}$ are the basis of the lattice L.

Lemma 2.23 ([71, 73])

(i) *If $in_{\boldsymbol{c}}(I_A)$ has $(*, \sigma)$ as a standard pair, then σ is a face of $\Delta_{\boldsymbol{c}}$.*

(ii) *$in_{\boldsymbol{c}}(I_A)$ has $(1, \sigma)$ as a standard pair if and only if σ is a maximal face of $\Delta_{\boldsymbol{c}}$.*

(iii) *If $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ span an affine hyperplane, then $\Delta_{\boldsymbol{c}}$ is the same as the regular triangulation of $\mathrm{conv}(A)$ with respect to $\boldsymbol{c}$, and the number of standard pairs $(*, \sigma)$ for a maximal face σ of $\Delta_{\boldsymbol{c}}$ equals the normalized volume of σ in $\Delta_{\boldsymbol{c}}$.*

Example 2.24 *Let* $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}$ *and $\boldsymbol{c} = (1, 3, 1, 1)$. Then the regular triangulation $\Delta_{\boldsymbol{c}}$ consists of $\{1, 3\}$, $\{3, 4\}$ and all of their subsets.*

The standard pair decomposition of $in_{\boldsymbol{c}}(I_A)$ is $(1, \{1, 3\})$, $(x_4, \{1, 3\})$, $(1, \{3, 4\})$ and $(x_2, \{1\})$. We remark that the normalized volume of the facet $\{1, 3\}$ in $\Delta_{\boldsymbol{c}}$ is 2 *and that of the facet $\{3, 4\}$ is* 1 (Figure 2.5). □

Hoçten and Thomas [34] introduced an algorithm based on a standard pair decomposition to solve $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$.

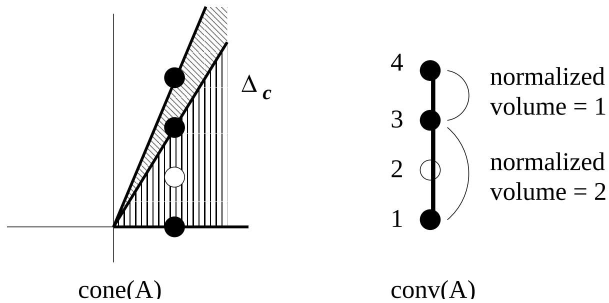

Figure 2.5: Regular triangulations of $\mathrm{cone}(A)$ and $\mathrm{conv}(A)$.

Let $\boldsymbol{u}$ be the optimal solution to $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$. As $\boldsymbol{x}^{\boldsymbol{u}} \notin in_{\boldsymbol{c}}(I_A)$ and standard pairs cover the set of monomials not in $in_{\boldsymbol{c}}(I_A)$, $\boldsymbol{x}^{\boldsymbol{u}}$ is covered by some standard pair $(\boldsymbol{x}^{\boldsymbol{a}}, \sigma)$. And then, $\boldsymbol{u} = \boldsymbol{a} + \sum_{i\in\sigma} k_i \boldsymbol{e}_i$ for some non-negative integers $\{k_i\}_{i\in\sigma}$, and $\boldsymbol{b} = A\boldsymbol{u} = A\left(\boldsymbol{a} + \sum_{i\in\sigma} k_i \boldsymbol{e}_i\right) = A\boldsymbol{a} + \sum_{i\in\sigma} k_i \boldsymbol{a}_i$. Lemma 2.23 implies that $\{\boldsymbol{a}_i\}_{i\in\sigma}$ are linearly independent. Therefore, $\{k_i\}_{i\in\sigma}$ is the unique solution to the linear system $\sum_{i\in\sigma} x_i \boldsymbol{a}_i = \boldsymbol{b} - A\boldsymbol{a}$. This observation induces an algorithm to solve $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$ using the standard pair decomposition of $in_{\boldsymbol{c}}(I_A)$.

Algorithm 2.25 (Hoçten-Thomas Algorithm [34])

Input: *a matrix* $A \in \mathbb{Z}^{d\times n}$, *a cost vector* $\boldsymbol{c} \in \mathbb{R}^n$
a right hand side vector $\boldsymbol{b} \in \mathbb{Z}^d$

Output: *the optimal solution* $\boldsymbol{u}$ *of* $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$

Step 1. *Compute the standard pair decomposition* $S(in_{\boldsymbol{c}}(I_A))$.

Step 2. *For each* $(\boldsymbol{x}^{\boldsymbol{a}}, \sigma) \in S(in_{\boldsymbol{c}}(I_A))$, *solve the linear system* $\sum_{i\in\sigma} x_i \boldsymbol{a}_i = \boldsymbol{b} - A\boldsymbol{a}$.
Let $\{k_i\}_{i\in\sigma}$ *be the solution.*

Step 3. *If* $\{k_i\}_{i\in\sigma}$ *are both integral and non-negative, output* $\boldsymbol{u} = \boldsymbol{a} + \sum_{i\in\sigma} k_i \boldsymbol{e}_i$ *as the optimal solution. Otherwise, return* **Step 2.** *for another standard pair.*

This algorithm solves at most $|S(in_{\boldsymbol{c}}(I_A))|$-many linear systems of equations. Therefore, the arithmetic degree $|S(in_{\boldsymbol{c}}(I_A))|$ is a measure of the complexity of $IP_{A,\boldsymbol{c}}$.

Example 2.12 (continued.) *Let* $\boldsymbol{c} = (3,1,2)$ *and* $\boldsymbol{b} = (4,5)$. *For the standard pair* $(1, \{(1,2),(1,3)\})$, *the linear system in* **Step 2.** *is* $\{x_{1,2} + x_{1,3} = 4,\ -x_{1,2} = 5\}$ *and its* *solution is* $x_{1,2} = -5$ *and* $x_{1,3} = 9$*, which does not satisfy the condition in* **Step 3.**. *In a similar way, for the standard pair* $(1, \{(1,3),(2,3)\})$*, the solution of the linear system in* **Step 2.** *is* $x_{1,3} = 4$ *and* $x_{2,3} = 5$*, which satisfies the condition. Thus, the optimal solution of* $IP_{A,\succ_{\boldsymbol{c}}}(\boldsymbol{b})$ *is* $(0,4,5)$. □

We next relate a standard pair decomposition of $in_{\boldsymbol{c}}(I_A)$ to the dual problem for the linear relaxation problem $LP_{A,\boldsymbol{c}}(\boldsymbol{b}) := minimize\ \{\boldsymbol{c} \cdot \boldsymbol{x} \mid A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$ of $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$.

Definition 2.26 *Let* $P \subset \mathbb{R}^n$ *be a polyhedron and* F *a face of* P*. The* normal cone *of* F *at* P *is the cone*

$$N_P(F) := \{\omega \in \mathbb{R}^n \mid \omega \cdot x' \geq \omega \cdot x \text{ for all } x' \in F \text{ and } x \in P\}.$$

The set of normal cones for all faces of P *is called the* normal fan *of* P*.*

Lemma 2.27 ([33]) $\Delta_{\boldsymbol{c}}$ *is the normal fan of the polyhedron* $P_{\boldsymbol{c}} := \{\boldsymbol{y} \in \mathbb{R}^d \mid \boldsymbol{y}A \leq \boldsymbol{c}\}$*.*

We remark that $P_{\boldsymbol{c}}$ is the dual polyhedron for the linear relaxation problem of $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$. When A is row-full rank, this lemma shows that there is a surjection from the dual feasible bases of $LP_{A,\boldsymbol{c}}(\boldsymbol{b})$ onto the maximal faces of $\Delta_{\boldsymbol{c}}$.

Example 2.12 (continued.) *The dual polyhedron* $P_{(3,1,2)}$ *is* $P_{(3,1,2)} = \{(y_1, y_2) \mid y_1 - y_2 \leq 3,\ y_1 \leq 1,\ y_2 \leq 2\}$. □

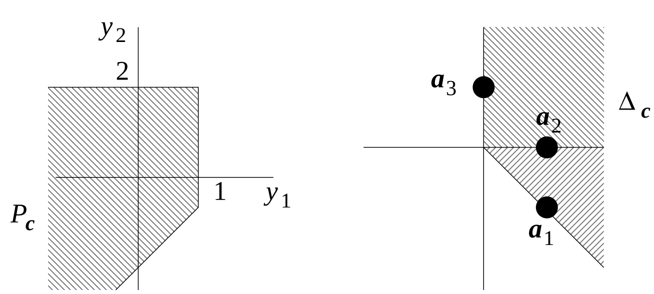

Figure 2.6: Dual polyhedron $P_{(3,1,2)}$ and regular triangulation $\Delta_{(3,1,2)}$.

Chapter 3

Dualistic Computational Algebraic Analyses of Unimodular Integer Programs

3.1 Introduction

Integer programs whose coefficient matrices are unimodular form a nice subclass in the sense that the system $yA \leq c$ becomes totally dual integral (TDI) for each $\boldsymbol{c}$. Then, the feasible polyhedron of the linear relaxation $LP_{A,\boldsymbol{c}}(\boldsymbol{b})$ of $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$ becomes integral (i.e., each vertex of the polyhedron is integral). Therefore, $LP_{A,\boldsymbol{c}}(\boldsymbol{b})$ also solves $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$. As mentioned in Section 1.2, if A is unimodular, each standard pair of $in_{\boldsymbol{c}}(I_A)$ is of type $(1, \sigma)$ for any cost vector $\boldsymbol{c}$. Furthermore, the set of circuits $\mathcal{C}_A$, the universal Gröbner basis $\mathcal{U}_A$ and the Graver basis Gr_A coincide. Thus, by considering unimodular matrices (or nice subclasses of unimodular matrices), the computational algebraic duality between Gröbner bases and standard pairs may become clear.

Minimum cost flow problems form a well-known subclass of unimodular integer programs that can be solved in polynomial time. The Conti-Traverso algorithm (Algorithm 2.14,2.15) for minimum cost flow problems is a variant of the cycle-canceling algorithm. Strongly polynomial time algorithms (for example, [27, 65, 66]) first find a feasible flow, use selection rules to choose the polynomial size of negative-cost cycles from the set of negative-cost cycles in the residual network, which may be of exponential size, and augment along the cycles as many as possible. The Conti-Traverso algorithm similarly calculates the optimal flow by augmenting flows along the negative-cost cycles that correspond to the elements of a Gröbner basis. Thus, the cardinalities of reduced Gröbner bases may provide some time bounds for this algorithm. On the other hand, the Hoçten-Thomas algorithm (Algorithm 2.25) for minimum cost flow problems first calculates the set of standard pairs, and solves linear systems of equations for each standard pair until a non-negative integer solution is obtained. For network optimization problems, the duality between the reduced Gröbner basis and the set of standard pairs corresponds to the relation between circuits and dual feasible co-trees, dually, cut-sets and primal feasible trees. As such relations have not been clarified in detail, the computational algebraic duality may be an interesting method for analysis of network problems.

This chapter deals first with standard pairs of unimodular integer programs. Via duality theory of linear programs, the Hoçten-Thomas algorithm for unimodular integer programs is shown to be equivalent to calculating the reduced cost of dual problems for each basis (Theorem 3.17). In addition, the maximum number of standard pairs is shown to be the normalized volume of the polytope defined by homogenizing the coefficient matrix (Theorem 3.15). This result gives a framework for computing the number of feasible bases via Gröbner bases and volume computations.

Computational algebraic methods for primal and dual minimum cost flow problems on acyclic tournament graphs are also studied. We characterize Gröbner bases and standard pairs in terms of graphs, and give bounds for the number of dual and primal feasible bases of the minimum cost flow problems: for the number of dual feasible bases, the minimum and maximum values are shown to be 1 (Theorem 3.28) and the Catalan number $\frac{1}{d}\binom{2(d-1)}{d-1}$ (Theorem 3.30), respectively, while for the number of primal feasible bases the lower bound is shown to be $\Omega(2^{\lfloor d/6 \rfloor})$ (Theorem 3.37). We use two approaches for analysis of arithmetic degrees: (1) the relation between reduced Gröbner bases and standard pairs, where the corresponding relation on the minimum cost flow — between a subset of circuits and dual feasible bases — has not been clarified; and (2) the results obtained in hypergeometric systems, using some types of differential equations, related with toric ideals.

Furthermore, we consider computational algebraic methods for primal and dual transportation problems. For transportation problems on bipartite graphs, the structures of feasible polyhedra, especially the number of vertices, have been studied for both primal [1, 11, 48, 58] and dual [4, 5] problems. We give computational algebraic proof for the following existing results:

- The maximum number of vertices for the feasible region of a transportation problem on $K_{2,n}$ is $(n-[n/2])\binom{n}{[n/2]}$ [48].
- The maximum number of vertices for the feasible region of a dual transportation problem on $K_{m,n}$ is $\binom{m+n-2}{m-1}$ [5].

The remainder of this section is organized as follows. In Section 3.2, we briefly review some basic definitions and related works concerning unimodular integer programs. In Section 3.3, we discuss standard pairs for unimodular programs and relate the Hoçten-Thomas algorithm with computing reduced costs. Section 3.4 describes the study of Gröbner bases and standard pairs in primal minimum cost flow problems on acyclic tournament graphs. Section 3.5 deals with dual minimum cost flow problems. In Section 3.6 we describe the study about primal and dual transportation problems. Finally, this chapter is summarized in Section 3.7.

This section describes work performed jointly with Hiroki Nakayama and Hiroshi Imai [44, 45, 46].

3.2 Unimodular Integer Programs: Definitions and Properties

This section briefly reviews some basic definitions and related works concerning unimodular integer programs. We refer to [33, 71, 78] for more details about unimodular integer programs and their computational algebraic properties.

Definition 3.1 *Let $A \in \mathbb{Z}^{d\times n}$ be a matrix with rank d. A is* unimodular *if each nonsingular submatrix of order d has determinant ± 1.*

Table 3.1: Dual algebraic approaches for primal and dual minimum cost flow problems

		Gröbner basis	Standard pair
Primal	Term on graph	Set of circuits	Set of spanning trees
	Algorithm	Variant of cycle-canceling	Enumeration of dual feasible bases
	On acyclic tournament graph with d vertices	$\min:\ d(d-1)/2$ $\max:$?	$\min:\ 1$ $\max:\ \frac{1}{d}\binom{2(d-1)}{d-1}$
Dual	Term on graph	Set of cutsets	Set of co-trees
	Algorithm	Variant of cutset-canceling	Enumeration of primal feasible bases
	On acyclic tournament graph with d vertices	$\min:\ d-1$ $\max:$?	Lower bound $\Omega(2^{\lfloor d/6 \rfloor})$

Definition 3.2 *Let $A \in \mathbb{Z}^{d \times n}$ be a matrix with rank d. A is* totally unimodular *if each square subdeterminant of A is 0, 1 or -1.*

Therefore, a totally unimodular matrix is unimodular.

The best known examples of (totally) unimodular matrices are incidence matrices of directed graphs.

Proposition 3.3 ([63]) *The vertex-arc incidence matrix A of any directed graph is totally unimodular.*

The most fundamental property for unimodular integer programs (integer programs whose coefficient matrices are unimodular) is the integrality for the linear relaxation.

Proposition 3.4 ([80],[63, Theorem 19.2.]) *Let A be an integral matrix of full row rank. Then, the polyhedron $\{\boldsymbol{x} \in \mathbb{R}^n \mid A\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}\}$ is integral (i.e., any vertex is integral) for each integral vector $\boldsymbol{b}$ if and only if A is unimodular.*

The polyhedron $\{\boldsymbol{x} \in \mathbb{R}^n \mid A\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}\}$ is the feasible region for the linear relaxation of $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$. Therefore, the optimal solution for the linear relaxation of $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$ is also the optimal solution to $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$.

A unimodular matrix also has good algebraic properties. The first concerns the inclusion of the set of circuits, the universal Gröbner basis, and the Graver basis in Proposition 2.20.

Proposition 3.5 ([71, Proposition 8.11.]) *If A is a unimodular matrix then $\mathcal{C}_A = \mathcal{U}_A = Gr_A$.*

Therefore, any reduced Gröbner basis for the toric ideal I_A of a unimodular matrix A is a subset of the set of circuits of A.

Proposition 3.6 ([71, Remark 8.10]) *A matrix A is unimodular if and only if all initial ideals of the toric ideal I_A are generated by square-free monomials.*

If column vectors of A span an affine hyperplane, the unimodularity implies good properties with regard to regular triangulations of $\mathrm{conv}(A)$.

Proposition 3.7 ([71, Corollary 8.9]) *A matrix A is unimodular if and only if, for any generic vector $\boldsymbol{c}$, the normalized volume of every maximal face in the regular triangulation $\Delta_{\boldsymbol{c}}$ of $\mathrm{conv}(A)$ becomes 1.*

Therefore, in this case, standard pair decompositions for any initial ideal of I_A can be completely described by regular triangulations.

Proposition 3.8 ([33]) *Let $\{m_1, \ldots, m_s\}$ be the minimal generators of $in_{\boldsymbol{c}}(I_A)$. If $m_1, \ldots, m_s$ are all square-free, then $S(in_{\boldsymbol{c}}(I_A)) = \{(1, \sigma) \mid \sigma$ is a maximal face of $\Delta_{\boldsymbol{c}}\}$.*

As mentioned in Section 1.2, there are additional related subclasses of integer programs that are involved in the integrality of feasible regions: i.e., the family of programs with *total dual integrality* [23] and the *Gomory family* [33].

Definition 3.9 *The system of* $\boldsymbol{y}A \leq \boldsymbol{c}$ *is* totally dual integral (TDI) *if the linear relaxation of* $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$ *has an integral optimal solution for each* $\boldsymbol{b} \in \operatorname{cone}(A) \cap \mathbb{Z}^d$.

Therefore, if A is unimodular, then the system $\boldsymbol{y}A \leq \boldsymbol{c}$ is TDI for *any* $\boldsymbol{c}$. The next proposition shows that the reverse is also true.

Proposition 3.10 ([33])

(i) *The system* $\boldsymbol{y}A \leq \boldsymbol{c}$ *is TDI if and only if* $in_{\boldsymbol{c}}(I_A)$ *is generated by square-free monomials.*

(ii) *The system* $\boldsymbol{y}A \leq \boldsymbol{c}$ *is TDI for* any $\boldsymbol{c}$ *if and only if* A *is unimodular.*

To define the *Gomory family* [33], we introduce a *group relaxation* [28], a classical relaxation method for integer programs. For $\tau \subseteq [n]$, let A_τ be the submatrix of A whose column vectors are indexed by τ, $\boldsymbol{c}_\tau$ the subvector of $\boldsymbol{c}$ whose elements are indexed by τ, and $\overline{\tau} = [n] \setminus \tau$. For a maximal face $\sigma \in \Delta_{\boldsymbol{c}}$, we define

$$\widetilde{\boldsymbol{c}}_{\overline{\sigma}} := \boldsymbol{c}_{\overline{\sigma}} - \boldsymbol{c}_\sigma A_\sigma^{-1} A_{\overline{\sigma}},$$

and, for any face $\tau \subseteq \sigma$, let $\widetilde{\boldsymbol{c}}_{\overline{\tau}}$ be the extension of $\widetilde{\boldsymbol{c}}_{\overline{\sigma}}$ to a vector in $\mathbb{R}^{|\overline{\tau}|}$ by adding zeros.

The group relaxation of $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$ with respect to τ is the relaxation program in which the non-negative constraints for x_i ($i \in \tau$) is relaxed.

Definition 3.11 *The* group relaxation *of* $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$ *with respect to the face* τ *of* $\Delta_{\boldsymbol{c}}$ *is the program*

$$G^\tau(\boldsymbol{b}) := minimize \, \{\widetilde{\boldsymbol{c}}_{\overline{\tau}} \cdot \boldsymbol{x}_{\overline{\tau}} \mid A_\tau \boldsymbol{x}_\tau + A_{\overline{\tau}} \boldsymbol{x}_{\overline{\tau}} = \boldsymbol{b}, \ \boldsymbol{x}_{\overline{\tau}} \geq 0, \ (\boldsymbol{x}_\tau, \boldsymbol{x}_{\overline{\tau}}) \in \mathbb{Z}^n\}.$$

Next we define the Gomory family.

Definition 3.12 *The family of integer programs* $IP_{A,\boldsymbol{c}}$ *is a* Gomory family *if, for* every $\boldsymbol{b} \in \{A\boldsymbol{x} \mid \boldsymbol{x} \in \mathbb{N}^n\}$, $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$ *is solved by a group relaxation* $G^\sigma(\boldsymbol{b})$ *where* σ *is a* maximal *face of* $\Delta_{\boldsymbol{c}}$.

The Gomory family is an extension of the total dual integrality in the following sense.

Proposition 3.13 ([33]) *If $\boldsymbol{y}A \le \boldsymbol{c}$ is TDI, then $IP_{A,\boldsymbol{c}}$ is a Gomory family.*

While TDI-ness guarantees a local integrality in the sense that the linear relaxation of $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$ has an integral optimum for every integral $\boldsymbol{b}$, for an integer program in the Gomory family, the linear optimum for its linear relaxation may not be integral.

The program $G^{\sigma}(\boldsymbol{b})$ where A_{σ} is the optimal basis of the linear relaxation is precisely a Gomory's group relaxation of $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$ [28]. Group relaxations $G^{\sigma}(\boldsymbol{b})$ as σ varies over the maximal faces of $\Delta_{\boldsymbol{c}}$ are the easiest to solve among all group relaxations of $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$, and the family of such group relaxations are called the *Gomory relaxations* of $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$.

3.3 Standard Pairs for Unimodular Integer Programs

Let $A \in \mathbb{Z}^{d\times n}$ be row-full rank and unimodular. Then, as described in Section 3.2, $in_{\boldsymbol{c}}(I_A)$ is minimally generated by square-free monomials (i.e., each exponent is 0 or 1) for any $\boldsymbol{c}$ [71], and all standard pairs are obtained from all maximal faces of $\Delta_{\boldsymbol{c}}$.

For a matrix $A \in \mathbb{Z}^{d\times n}$, the *homogenized matrix* $A' \in \mathbb{Z}^{(d+1)\times(n+1)}$ of A is

$$A' := \begin{pmatrix} 1 \;\; 1 \;\; \cdots \;\; 1 & 1 \\ & 0 \\ A & \vdots \\ & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ & & & & 0 \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_n & \vdots \\ & & & & 0 \end{pmatrix}. \tag{3.1}$$

Let $\boldsymbol{a}'_i = \binom{1}{\boldsymbol{a}_i}$ for $1 \le i \le n$ and $\boldsymbol{a}'_{n+1}$ be the $(n+1)$-th column vector of A'. We remark that $\boldsymbol{a}'_1, \ldots, \boldsymbol{a}'_n, \boldsymbol{a}'_{n+1}$ span an affine hyperplane.

We consider another integer program

$$IP_{A',(\boldsymbol{c},0)}(\boldsymbol{b},\beta) := minimize \left\{ \boldsymbol{c}\cdot\boldsymbol{x} \;\middle|\; A'\binom{\boldsymbol{x}}{x_{n+1}} = \binom{\beta}{\boldsymbol{b}}, \binom{\boldsymbol{x}}{x_{n+1}} \in \mathbb{N}^{n+1} \right\}$$

for $\beta \in \mathbb{Z}$, and the family $IP_{A',(\boldsymbol{c},0)}$ of integer programs $IP_{A',(\boldsymbol{c},0)}(\boldsymbol{b},\beta)$ as $\binom{\beta}{\boldsymbol{b}}$ varies in $\{A'\boldsymbol{u} \mid \boldsymbol{u} \in \mathbb{N}^{n+1}\}$. $(\boldsymbol{c}, 0)$ is generic if $\boldsymbol{c}$ is generic.

The next proposition is based on the report of Sturmfels et al. [73] regarding general ideals. We provide here another proof for the case of a toric ideal [41].

Proposition 3.14 ([73]) $(\boldsymbol{x}^{\boldsymbol{a}}, \sigma) \in S(in_{\boldsymbol{c}}(I_A))$ $(\boldsymbol{x}^{\boldsymbol{a}} \in k[x_1, \ldots, x_n],\ \sigma \subseteq [n])$ *if and only if* $(\boldsymbol{x}^{\boldsymbol{a}}, \sigma \cup \{n+1\}) \in S(in_{(\boldsymbol{c},0)}(I_{A'}))$.

Proof: We first show that any monomial in $(\boldsymbol{x}^{\boldsymbol{a}}, \sigma)$ is standard for $in_{\boldsymbol{c}}(I_A)$ if and only if any monomial in $(\boldsymbol{x}^{\boldsymbol{a}}, \sigma \cup \{n+1\})$ is standard for $in_{(\boldsymbol{c},0)}(I_{A'})$. Suppose that any monomial in $(\boldsymbol{x}^{\boldsymbol{a}}, \sigma)$ is standard for $in_{\boldsymbol{c}}(I_A)$ and choose any $\boldsymbol{x}^{\boldsymbol{u}} x_{n+1}^k \in (\boldsymbol{x}^{\boldsymbol{a}}, \sigma \cup \{n+1\})$. If there are any other $\binom{\boldsymbol{v}}{l} \in \mathbb{N}^{n+1}$ such that $A'\binom{\boldsymbol{v}}{l} = A'\binom{\boldsymbol{u}}{k}$ and $\binom{\boldsymbol{v}}{l} \neq \binom{\boldsymbol{u}}{k}$, then $A\boldsymbol{u} = A\boldsymbol{v}$, and $\boldsymbol{c} \cdot \boldsymbol{u} < \boldsymbol{c} \cdot \boldsymbol{v}$ as $\boldsymbol{x}^{\boldsymbol{u}} \notin in_{\boldsymbol{c}}(I_A)$. Therefore, $\binom{\boldsymbol{u}}{k}$ is the optimal solution to $IP_{A',(\boldsymbol{c},0)}(A\boldsymbol{u}, \sum_{i=1}^n u_i + k)$. If there is no such $\binom{\boldsymbol{v}}{l}$, then clearly $\binom{\boldsymbol{u}}{k}$ is optimal. This shows that any monomial in $(\boldsymbol{x}^{\boldsymbol{a}}, \sigma \cup \{n+1\})$ is standard for $in_{(\boldsymbol{c},0)}(I_{A'})$.

Conversely, suppose that any monomial in $(\boldsymbol{x}^{\boldsymbol{a}}, \sigma \cup \{n+1\})$ is standard for $in_{(\boldsymbol{c},0)}(I_{A'})$ and choose any $\boldsymbol{x}^{\boldsymbol{u}} \in (\boldsymbol{x}^{\boldsymbol{a}}, \sigma)$. If there exists some $\boldsymbol{v} \in \mathbb{N}^n$ such that $A\boldsymbol{v} = A\boldsymbol{u}$, then $A'\binom{\boldsymbol{u}}{p} = A'\binom{\boldsymbol{v}}{q}$ for any non-negative integer $p,\ q$ such that $p - q = \sum_{i=1}^n v_i - \sum_{i=1}^n u_i$. As $\boldsymbol{x}^{\boldsymbol{u}} x_{n+1}^p \in (\boldsymbol{x}^{\boldsymbol{a}}, \sigma \cup \{n+1\})$, $(\boldsymbol{c}, 0) \cdot \binom{\boldsymbol{u}}{p} < (\boldsymbol{c}, 0) \cdot \binom{\boldsymbol{v}}{q}$, which implies that $\boldsymbol{c} \cdot \boldsymbol{u} < \boldsymbol{c} \cdot \boldsymbol{v}$. Therefore, $\boldsymbol{u}$ is the optimal solution to $IP_{A,\boldsymbol{c}}(A\boldsymbol{u})$. If there is no such $\boldsymbol{v}$, then clearly $\boldsymbol{u}$ is optimal. Thus, any monomial in $(\boldsymbol{x}^{\boldsymbol{a}}, \sigma)$ is standard for $in_{\boldsymbol{c}}(I_A)$.

Next we show that $(\boldsymbol{x}^{\boldsymbol{a}}, \sigma) \in S(in_{\boldsymbol{c}}(I_A))$ if and only if $(\boldsymbol{x}^{\boldsymbol{a}}, \sigma \cup \{n+1\}) \in S(in_{(\boldsymbol{c},0)}(I_{A'}))$.

(if part) Let $(\boldsymbol{x}^{\boldsymbol{a}}, \sigma \cup \{n+1\}) \in S(in_{(\boldsymbol{c},0)}(I_{A'}))$. If $(\boldsymbol{x}^{\boldsymbol{a}}, \sigma) \subset (\boldsymbol{x}^{\boldsymbol{a}'}, \tau)$ for any other $(\boldsymbol{x}^{\boldsymbol{a}'}, \tau)$ which satisfies (i) and (ii) in Definition 2.21 for $in_{\boldsymbol{c}}(I_A)$, then $(\boldsymbol{x}^{\boldsymbol{a}}, \sigma \cup \{n+1\})$ must be contained in $\left(\boldsymbol{x}^{\boldsymbol{a}'}, \tau \cup \{n+1\}\right)$, which contradicts the assumption. Thus, $(\boldsymbol{x}^{\boldsymbol{a}}, \sigma) \in S(in_{\boldsymbol{c}}(I_A))$.

(only if part) If $(\boldsymbol{x}^{\boldsymbol{a}}, \sigma \cup \{n+1\}) \notin S(in_{(\boldsymbol{c},0)}(I_{A'}))$, then $(\boldsymbol{x}^{\boldsymbol{a}}, \sigma \cup \{n+1\})$ does not satisfy some condition in Definition 2.21 for $in_{(\boldsymbol{c},0)}(I_{A'})$. If $(\boldsymbol{x}^{\boldsymbol{a}}, \sigma \cup \{n+1\})$ does not satisfy (i), then $\mathrm{supp}(\boldsymbol{a}) \cap \sigma \neq \emptyset$ and $(\boldsymbol{x}^{\boldsymbol{a}}, \sigma) \notin S(in_{\boldsymbol{c}}(I_A))$. If $(\boldsymbol{x}^{\boldsymbol{a}}, \sigma \cup \{n+1\})$ does not satisfy (ii), then $(\boldsymbol{x}^{\boldsymbol{a}}, \sigma \cup \{n+1\})$ contains a non-standard monomial for $in_{(\boldsymbol{c},0)}(I_{A'})$, and the above discussion shows that $(\boldsymbol{x}^{\boldsymbol{a}}, \sigma)$ also contains a non-standard monomial for $in_{\boldsymbol{c}}(I_A)$, which implies $(\boldsymbol{x}^{\boldsymbol{a}}, \sigma) \notin S(in_{\boldsymbol{c}}(I_A))$. If $(\boldsymbol{x}^{\boldsymbol{a}}, \sigma \cup \{n+1\})$ does not satisfy only (iii), there exists some

$\left(\boldsymbol{x}^{\boldsymbol{a}'} x_{n+1}^k, \tau'\right)$, which contains $(\boldsymbol{x}^{\boldsymbol{a}}, \sigma \cup \{n+1\})$, and $\left(\boldsymbol{x}^{\boldsymbol{a}'} x_{n+1}^k, \tau'\right)$ satisfies (i) and (ii). Then, $n+1 \in \tau'$, and thus, $k = 0$. Therefore, $(\boldsymbol{x}^{\boldsymbol{a}'}, \tau' \setminus \{n+1\})$ contains $(\boldsymbol{x}^{\boldsymbol{a}}, \sigma)$ and satisfies (i) and (ii) for $in_{\boldsymbol{c}}(I_A)$. Thus, $(\boldsymbol{x}^{\boldsymbol{a}}, \sigma) \notin S(in_{\boldsymbol{c}}(I_A))$.

□

Example 2.12 (continued.) *For this A, the homogenized matrix A' is*

$$A' = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}.$$

We consider $I_{A'} \subset k[x_{1,2}, x_{1,3}, x_{2,3}, x_4]$. For $\boldsymbol{c} = (3,1,2)$, the standard pairs of $in_{(3,1,2,0)}(I_{A'})$ are $(1, \{(1,2),(1,3),4\})$ and $(1, \{(1,3),(2,3),4\})$. On the other hand, for $\boldsymbol{c} = (1,4,2)$, the standard pairs of $in_{(1,4,2,0)}(I_{A'})$ are $(1, \{(1,2),(1,3),(2,3)\})$ and $(1, \{(1,2),(2,3),4\})$. In this case, the only standard pair $(1, \{(1,2),(2,3),4\})$ satisfies the condition in Proposition 3.14, which corresponds to the standard pair $(1, \{(1,2), (2,3)\})$ of $in_{(1,4,2)}(I_A)$. □

As $\boldsymbol{a}'_1, \ldots, \boldsymbol{a}'_{n+1}$ span an affine hyperplane, the normalized volume of $\operatorname{conv}(A')$ gives the number of standard pairs of $in_{(\boldsymbol{c},k)}(I_{A'})$, which correspond to the maximal faces of $\Delta'_{(\boldsymbol{c},k)}$ by Lemma 2.23 (iii) for any $k \in \mathbb{R}$ such that $(\boldsymbol{c}, k)$ becomes homogeneous. Therefore, using Proposition 3.14, the maximum arithmetic degree of $in_{\boldsymbol{c}}(I_A)$ can be obtained via $\operatorname{conv}(A')$.

Theorem 3.15 ([30]) *If A is a unimodular matrix, then the maximum arithmetic degree of $in_{\boldsymbol{c}}(I_A)$ equals the normalized volume of $\operatorname{conv}(A')$.*

Proof: For any $\boldsymbol{c}$, the set of standard pairs of $in_{\boldsymbol{c}}(I_A)$ is $\{(1,\sigma) \mid \sigma \text{ is a maximal face of } \Delta_{\boldsymbol{c}}\}$, and each $(1,\sigma)$ corresponds to the standard pair $(1, \sigma \cup \{n+1\})$ of $in_{(\boldsymbol{c},0)}(I_{A'})$.

Especially, $\sigma \cup \{n+1\}$ is a maximal face of $\Delta'_{(\boldsymbol{c},0)}$. Therefore,

$$\begin{aligned} arith\text{-}deg\,(in_{\boldsymbol{c}}(I_A)) &= |\{(1,\sigma) \in S\,(in_{\boldsymbol{c}}(I_A))\}| \\ &= \left|\left\{(1,\sigma \cup \{n+1\}) \in S\left(in_{(\boldsymbol{c},0)}(I_{A'})\right)\right\}\right| \\ &\leq \left|\left\{(*,\tau) \in S\left(in_{(\boldsymbol{c},0)}(I_{A'})\right) \mid \tau : \text{ maximal face of } \Delta'_{(\boldsymbol{c},0)}\right\}\right| \\ &= \text{normalized volume of } \operatorname{conv}(A'). \end{aligned}$$

Let $I_A \subset k[x_1,\dots,x_n]$ and $I_{A'} \subset k[x_1,\dots,x_n,x_{n+1}]$. Then, $\boldsymbol{x}^{\boldsymbol{a}} - \boldsymbol{x}^{\boldsymbol{b}} x_{n+1}^k \in I_{A'}$ ($\boldsymbol{x}^{\boldsymbol{a}},\ \boldsymbol{x}^{\boldsymbol{b}} \in k[x_1,\dots,x_n]$) if and only if $\sum_{i=1}^n (a_i - b_i) = k$ and $\boldsymbol{x}^{\boldsymbol{a}} - \boldsymbol{x}^{\boldsymbol{b}} \in I_A$. We consider that $\boldsymbol{c} = (1,1,\dots,1)$ and $\succ$ is any degree reverse lexicographic term order such that x_{n+1} is the smallest variable. Then, for any g in the reduced Gröbner basis $\mathcal{G}$ for $I_{A'}$ with respect to $\succ_{(\boldsymbol{c},0)}$, $in_{\succ_{(\boldsymbol{c},0)}}(g)$ does not contain x_{n+1} by the definition of the term order, and $in_{\succ_{(\boldsymbol{c},0)}}(g)$ is square-free as $\{in_{\succ_{(\boldsymbol{c},0)}}(g) \mid g \in \mathcal{G}\}$ minimally generates $in_{\succ'_{\boldsymbol{c}}}(I_A)$ for some term order $\succ'$. Thus, the corresponding triangulation $\Delta'_{\succ_{(\boldsymbol{c},0)}}$ is unimodular [71], and each facet of $\Delta'_{\succ_{(\boldsymbol{c},0)}}$ corresponds to a standard pair of $in_{\boldsymbol{c}}(I_A)$ injectively. Then, the arithmetic degree of $in_{\boldsymbol{c}}(I_A)$ is equal to the number of facets of $\Delta'_{\succ_{(\boldsymbol{c},0)}}$, which is the normalized volume of $\operatorname{conv}(A')$. □

If column vectors of A themselves span an affine hyperplane, then the normalized volume of $\operatorname{conv}(A')$ is equal to that of $\operatorname{conv}(A)$, and Proposition 3.8 implies that this number gives the arithmetic degree of $in_{\boldsymbol{c}}(I_A)$ for any cost vector $\boldsymbol{c}$.

Let $B \subset [n]$ be a basis of $LP_{A,\boldsymbol{c}}(\boldsymbol{b})$, i.e., $|B| = d$ and $A_B := \{\boldsymbol{a}_i \mid i \in B\}$ is a non-singular matrix, and $A_N := \{\boldsymbol{a}_j \mid j \notin B\}$. We consider the dictionary of $LP_{A,\boldsymbol{c}}(\boldsymbol{b})$ for a basis B:

$$P_{B,\widetilde{\boldsymbol{c}}}(\widetilde{\boldsymbol{b}}) := maximize\ \{(-\widetilde{\boldsymbol{c}}_N)\cdot \boldsymbol{x}_N \mid M\boldsymbol{x}_N + I_d \boldsymbol{x}_B = \widetilde{\boldsymbol{b}}_B,\ \boldsymbol{x}_B, \boldsymbol{x}_N \geq \boldsymbol{0}\},$$

and its dual problem

$$D_{B,\widetilde{\boldsymbol{b}}}(\widetilde{\boldsymbol{c}}) := minimize\ \{\widetilde{\boldsymbol{b}}_B \cdot \boldsymbol{y}_B \mid I_{n-d}\boldsymbol{y}_N - M^{\mathrm{T}}\boldsymbol{y}_B = \widetilde{\boldsymbol{c}}_N,\ \boldsymbol{y}_B, \boldsymbol{y}_N \geq \boldsymbol{0}\},$$

where $M := A_B^{-1}A_N \in \mathbb{Z}^{d\times(n-d)}$, $\widetilde{\boldsymbol{b}} := (\widetilde{\boldsymbol{b}}_B, \widetilde{\boldsymbol{b}}_N) = ((A_B)^{-1}\boldsymbol{b}, \boldsymbol{0})$, I_d (resp. I_{n-d}) is $d \times d$ (resp. $(n-d)\times(n-d)$) identity matrices, and $\widetilde{\boldsymbol{c}} := (\widetilde{\boldsymbol{c}}_B, \widetilde{\boldsymbol{c}}_N) = (\boldsymbol{0}, \widetilde{\boldsymbol{c}}_N)$, $\widetilde{\boldsymbol{c}}_N$ is the *reduced cost* of $LP_{A,\boldsymbol{c}}(\boldsymbol{b})$ for B.

Definition 3.16 *Let B be a basis of $LP_{A,\boldsymbol{c}}(\boldsymbol{b})$, $\boldsymbol{c}_B := (c_i)_{i\in B}$ and $\boldsymbol{c}_N := (c_i)_{i\notin B}$. The* reduced cost $\widetilde{\boldsymbol{c}} = (\widetilde{\boldsymbol{c}}_B, \widetilde{\boldsymbol{c}}_N)$ *of $LP_{A,\boldsymbol{c}}(\boldsymbol{b})$ for B is defined as*

$$\widetilde{\boldsymbol{c}}_N := \boldsymbol{c}_N - A_N^{\mathrm{T}}(A_B^{-1})^{\mathrm{T}}\boldsymbol{c}_B.$$

As $P_{B,\widetilde{\boldsymbol{c}}}(\widetilde{\boldsymbol{b}})$ is equivalent to $P_{A,\boldsymbol{c}}(\boldsymbol{b})$, $in_{\boldsymbol{c}}(I_A) = in_{\widetilde{\boldsymbol{c}}}(I_{(M\ I)})$. For any standard pair $(1,\sigma)$ of $in_{\boldsymbol{c}}(I_A)$, $\overline{\sigma} := \{1,\dots,n\}\setminus\sigma$ forms a basis of the dual problem $D_{B,\widetilde{\boldsymbol{b}}}(\widetilde{\boldsymbol{c}})$.

Theorem 3.17 *When we apply Algorithm 2.25 to $P_{B,\widetilde{\boldsymbol{c}}}(\widetilde{\boldsymbol{b}})$, the solution of the linear system in* **Step. 2** *for a standard pair $(1,\sigma)$ is the reduced cost of $D_{B,\widetilde{\boldsymbol{b}}}(\widetilde{\boldsymbol{c}})$ for the basis $\overline{\sigma}$.*

Proof: Let $\sigma_1 := ([n]\setminus B)\cap\sigma$, $\sigma_2 := B\cap\sigma$, $\overline{\sigma_1} := ([n]\setminus B)\cap\overline{\sigma}$, $\overline{\sigma_2} := B\cap\overline{\sigma}$ (Figure 3.1). We remark that all elements in $\widetilde{\boldsymbol{b}}_{\sigma_1}$ and $\widetilde{\boldsymbol{b}}_{\overline{\sigma_1}}$ are zero.

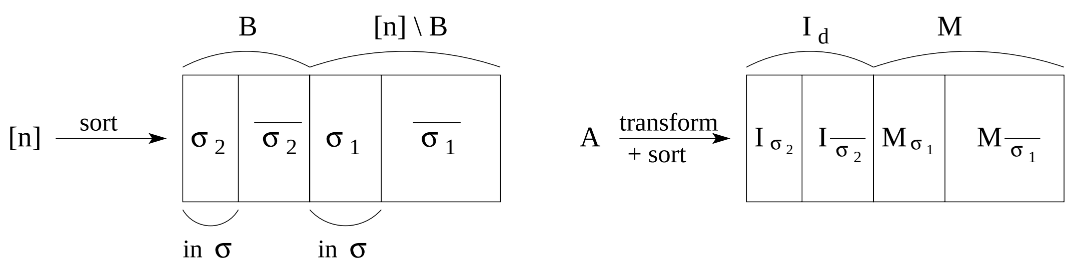

Figure 3.1: Definition of σ_1 and σ_2.

Then the reduced cost of $D_{B,\widetilde{\boldsymbol{b}}}(\widetilde{\boldsymbol{c}})$ for $\overline{\sigma}$ is $\widetilde{\boldsymbol{b}}'_\sigma = \widetilde{\boldsymbol{b}}_\sigma - N_1^{\mathrm{T}}(B_1^{-1})^{\mathrm{T}}\widetilde{\boldsymbol{b}}_{\overline{\sigma}}$, where $B_1 = (I_{\overline{\sigma_1}}\ (-M^{\mathrm{T}})_{\overline{\sigma_2}}) \in \mathbb{Z}^{(n-d)\times(n-d)}$ and $N_1 = (I_{\sigma_1}\ (-M^{\mathrm{T}})_{\sigma_2}) \in \mathbb{Z}^{(n-d)\times d}$.

We show that $\widetilde{\boldsymbol{b}}'_\sigma$ is a solution of the linear system in **Step. 2** of Algorithm 2.25 for $(1,\sigma)$, i.e., $(M_{\sigma_1}\ I_{\sigma_2})\widetilde{\boldsymbol{b}}'_\sigma = \widetilde{\boldsymbol{b}}_B$. Because $M = MI = M_{\sigma_1}(I_{\sigma_1})^{\mathrm{T}} + M_{\overline{\sigma_1}}(I_{\overline{\sigma_1}})^{\mathrm{T}}$ and $-M = I(-M) = I_{\sigma_2}((-M^{\mathrm{T}})_{\sigma_2})^{\mathrm{T}} + I_{\overline{\sigma_2}}((-M^{\mathrm{T}})_{\overline{\sigma_2}})^{\mathrm{T}}$,

$$\begin{aligned}
(M_{\sigma_1}\ I_{\sigma_2})N_1^{\mathrm{T}}(B_1^{-1})^{\mathrm{T}} &= (M_{\sigma_1}(I_{\sigma_1})^{\mathrm{T}} + I_{\sigma_2}((-M^{\mathrm{T}})_{\sigma_2})^{\mathrm{T}})(B_1^{-1})^{\mathrm{T}} \\
&= \left\{(M - M_{\overline{\sigma_1}}(I_{\overline{\sigma_1}})^{\mathrm{T}}) + (-M - I_{\overline{\sigma_2}}((-M^{\mathrm{T}})_{\overline{\sigma_2}})^{\mathrm{T}})\right\}(B_1^{-1})^{\mathrm{T}} \\
&= -(M_{\overline{\sigma_1}}\ I_{\overline{\sigma_2}})(I_{\overline{\sigma_1}}\ (-M^{\mathrm{T}})_{\overline{\sigma_2}})^{\mathrm{T}}(B_1^{-1})^{\mathrm{T}} \\
&= -(M_{\overline{\sigma_1}}\ I_{\overline{\sigma_2}})B_1^{\mathrm{T}}(B_1^{-1})^{\mathrm{T}} \\
&= -(M_{\overline{\sigma_1}}\ I_{\overline{\sigma_2}}).
\end{aligned}$$

Therefore, $(M_{\sigma_1}\ I_{\sigma_2})\widetilde{\boldsymbol{b}}'_{\sigma} = (M_{\sigma_1}\ I_{\sigma_2})\widetilde{\boldsymbol{b}}_{\sigma} - (M_{\sigma_1}\ I_{\sigma_2})N_1^{\mathrm{T}}(B_1^{-1})^{\mathrm{T}}\widetilde{\boldsymbol{b}}_{\overline{\sigma}} = (M_{\sigma_1}\ I_{\sigma_2})\widetilde{\boldsymbol{b}}_{\sigma} + (M_{\overline{\sigma_1}}\ I_{\overline{\sigma_2}})\widetilde{\boldsymbol{b}}_{\overline{\sigma}} = I_{\sigma_2}\widetilde{\boldsymbol{b}}_{\sigma_2} + I_{\overline{\sigma_2}}\widetilde{\boldsymbol{b}}_{\overline{\sigma_2}} = \widetilde{\boldsymbol{b}}_B$. □

Example 2.12 (continued.) *Let* $\boldsymbol{c} = (3,1,2)$ *and* $\boldsymbol{b} = (4,5)$. *Then the primal and dual problem which corresponds to the basis* $\{(1,2),(2,3)\}$ *are the following.*

$$\begin{array}{ll} \max & 4x_{1,3} \\ \text{s.t.} & \left(\begin{array}{c|cc} 1 & 1 & 0 \\ 1 & 0 & 1 \end{array}\right)\begin{pmatrix} x_{1,3} \\ x_{1,2} \\ x_{2,3} \end{pmatrix} = \begin{pmatrix} 4 \\ 9 \end{pmatrix} \\ & x_{1,2}, x_{1,3}, x_{2,3} \geq 0 \end{array} \qquad \begin{array}{ll} \min & 4y_{1,2} + 9y_{2,3} \\ \text{s.t.} & \left(\begin{array}{c|cc} 1 & -1 & -1 \end{array}\right)\begin{pmatrix} y_{1,3} \\ y_{1,2} \\ y_{2,3} \end{pmatrix} = -4 \\ & y_{1,2}, y_{1,3}, y_{2,3} \geq 0 \end{array}$$

For the standard pair $(1,\{(1,2),(1,3)\})$, $\sigma_1 = \{(1,3)\},\ \sigma_2 = \{(1,2)\},\ \overline{\sigma_1} = \emptyset,\ \overline{\sigma_2} = \{(2,3)\}$, *and the reduced cost of the dual problem for the basis* $\{(2,3)\}$ *is*

$$\begin{pmatrix} \widetilde{b}_{1,3} \\ \widetilde{b}_{1,2} \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix}(-1)\cdot 9 = \begin{pmatrix} -5 \\ 9 \end{pmatrix}.$$

In Algorithm 2.25, for the above standard pair we solve

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_{1,3} \\ x_{1,2} \end{pmatrix} = \begin{pmatrix} 4 \\ 9 \end{pmatrix}$$

in **Step. 2**, *whose solution is* $\begin{pmatrix} x_{1,3} \\ x_{1,2} \end{pmatrix} = \begin{pmatrix} -5 \\ 9 \end{pmatrix}$. □

3.4 Analyses of Primal Minimum Cost Flow Problems

This section deals with the study of the Gröbner bases and standard pairs in primal minimum cost flow problems on acyclic tournament graphs.

The study described in Section 3.4.1 is from [38] and [40].

3.4.1 Gröbner Bases and Their Size

Let G_d be the acyclic tournament graph with vertices $1, 2, \ldots, d$ and $n = \binom{d}{2}$ arcs, where each arc (i,j) $(i < j)$ is directed from i to j. We consider the following minimum cost flow problem $P_{A,\boldsymbol{c}}(\boldsymbol{b})$:

$$P_{A,\boldsymbol{c}}(\boldsymbol{b}) := minimize \ \{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \mid A\boldsymbol{x} = \boldsymbol{b}, \ \boldsymbol{x} \geq \boldsymbol{0}\},$$

where $A \in \mathbb{Z}^{d\times n}$ is the vertex-arc incidence matrix of G_d.

A *walk* in G_d is a sequence $(v_1, v_2, \ldots, v_p)$ of vertices such that (v_i, v_{i+1}) or (v_{i+1}, v_i) is an arc of G_d for each $1 \leq i < p$. A *cycle* is a walk $(v_1, v_2, \ldots, v_p, v_1)$. A *circuit* is a cycle $(v_1, v_2, \ldots, v_p, v_1)$ such that $v_i \neq v_j$ for any $i \neq j$.

Definition 3.18 *Let C be a circuit in G_d and fix a direction of C. If C passes an arc (i,j) u_{ij}^+ times in the forward direction and u_{ij}^- times in the backward direction, then we define $\boldsymbol{u}_C^+ = (u_{ij}^+)_{1\leq i<j\leq d}, \boldsymbol{u}_C^- = (u_{ij}^-)_{1\leq i<j\leq d} \in \mathbb{R}^n$. The vector $\boldsymbol{u}_C := \boldsymbol{u}_C^+ - \boldsymbol{u}_C^-$ is called the* incidence vector *of C. We identify a cycle C of G_d with the binomial $f_C := \boldsymbol{x}^{\boldsymbol{u}_C^+} - \boldsymbol{x}^{\boldsymbol{u}_C^-} \in I_A$.*

Let $\mathcal{C}_A = \{f_C \mid C$ is a circuit of $G_d\}$. As described in Section 3.2, A is totally unimodular. Therefore, the following proposition holds.

Proposition 3.19 ([71]) *$\mathcal{C}_A = \mathcal{U}_A = Gr_A$. In particular, any reduced Gröbner basis of I_A is square-free, and the number of elements in $\mathcal{U}_A$ is of exponential order with respect to d.*

Next, we demonstrate the existence of the reduced Gröbner basis for I_A for any $\boldsymbol{c} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$.

Proposition 3.20 *I_A is not homogeneous for the grading $\deg(x_{i,j}) = 1$, but is homogeneous for the grading $\deg(x_{i,j}) = j - i$.*

Proof: For any d, $x_{1,2}x_{2,3} - x_{1,3} \in I_A$ and $x_{1,2}x_{2,3} \notin I_A$. This implies that I_A is not homogeneous for the grading $\deg(x_{i,j}) = 1$.

Let $v_1, v_2, \ldots, v_p, v_1$ be a circuit in G_d, $C^+ := \{k \mid v_k < v_{k+1}\}$ and $C^- := \{k \mid v_k > v_{k+1}\}$ (we set $v_{p+1} := v_1$). The binomial f_C corresponding to C is $f_C = \prod_{k\in C^+} x_{v_k v_{k+1}} -$

$\prod_{k\in C^-} x_{v_{k+1}v_k}$. Then, f_C is homogeneous for the grading $\deg(x_{i,j}) = j - i$ because

$$\deg\left(\prod_{k\in C^+} x_{v_k v_{k+1}}\right) - \deg\left(\prod_{k\in C^-} x_{v_{k+1} v_k}\right) = \sum_{k\in C^+}(v_{k+1} - v_k) - \sum_{k\in C^-}(v_k - v_{k+1})$$
$$= \sum_{k=1}^{p}(v_{k+1} - v_k) = 0.$$

□

Thus, the reduced Gröbner basis exists for any $\boldsymbol{c} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ by Proposition 2.13.

Now we give two specific reduced Gröbner bases in terms of graphs. As a corollary, we can show that there exist term orders for which reduced Gröbner bases remain in polynomial order. For other applications of the Gröbner bases found in this section, see [39].

Proposition 3.21 *Let $\succ$ be the purely lexicographic order induced by the variable ordering such that $x_{i,j} \succ x_{k,l}$ if and only if $i < k$ or $(i = k$ and $j < l)$. Then, the reduced Gröbner basis for I_A with respect to $\succ$ is $\{g_{ijk} := x_{i,j}x_{j,k} - x_{i,k} \mid i < j < k\} \cup \{g_{ijkl} := x_{i,k}x_{j,l} - x_{i,l}x_{j,k} \mid i < j < k < l\}$. In particular, the number of elements in this Gröbner basis is equal to $\binom{d}{3} + \binom{d}{4}$.*

The set $\{g_{ijk} \mid i < j < k\}$ corresponds to all of the circuits of length three, and $\{g_{ijkl} \mid i < j < k < l\}$ corresponds to some circuits of length four uniquely determined for each of the four vertices i, j, k, l (Figure 3.2).

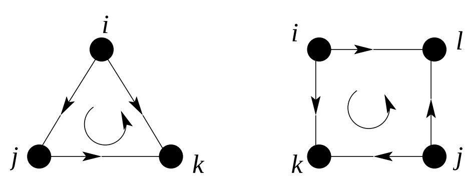

Figure 3.2: Circuit corresponding to g_{ijk} (left) and circuit corresponding to g_{ijkl} (right).

Proof: By Proposition 3.19, it suffices to show that any binomial that corresponds to a circuit in G_d is g_{ijk}, g_{ijkl} or whose initial term is divisible by some $in_\succ(g_{ijk})$ or $in_\succ(g_{ijkl})$.

Any binomial corresponding to a circuit of length 3 is contained in $\{g_{ijk}\}$.

The circuits defined by four vertices $i < j < k < l$ are $C_1 := (i,j,k,l,i)$, $C_2 := (i,j,l,k,i)$, $C_3 := (i,k,j,l,i)$ and their opposites. The binomial that corresponds to C_1 or its opposite is $\pm(x_{i,j}x_{j,k}x_{k,l} - x_{i,l})$, whose initial term $x_{i,j}x_{j,k}x_{k,l}$ is divisible by $in_\succ(g_{ijk})$. Similarly, the initial term of the binomial corresponding to C_2 or its opposite is divisible by $in_\succ(g_{ijl})$. The binomial that corresponds to C_3 or its opposite is g_{ijkl}.

Let C be a circuit of length more than five. Let v_1 be the vertex whose label is minimum in C, and $C := (v_1, v_2, \ldots, v_p, v_1)$. Without loss of generality, we set $v_2 < v_p$. Let f_C be the binomial corresponding to C, then $in_\succ(f_C)$ is the product of all variables whose associated arcs have the same direction as (v_1, v_2) on C. If $v_2 < v_3$, then (v_1, v_2) and (v_2, v_3) have the same direction on C. Thus, both x_{v_1,v_2} and x_{v_2,v_3} appear in $in_\succ(f_C)$, and $in_\succ(f_C)$ is divisible by $in_\succ(g_{v_1v_2v_3})$. If $v_2 > v_3$, then as $v_3 < v_2 < v_p$, there exists some k $(3 \leq k \leq p-1)$ such that $v_1 < v_k < v_2 < v_{k+1}$. Then, both x_{v_1,v_2} and $x_{v_k,v_{k+1}}$ appear in $in_\succ(f_C)$, and $in_\succ(f_C)$ is divisible by $in_\succ(g_{v_1v_kv_2v_{k+1}})$. □

Theorem 3.22 *Let $\succ$ be any term order and $\boldsymbol{c} = (c_{1,2}, \ldots, c_{1,d}, c_{2,3}, \ldots, c_{d-1,d}) \in \mathbb{R}^n$ satisfy $c_{i,j} + c_{j,k} > c_{i,k}$ for any $i < j < k$ and $c_{i,k} + c_{j,l} > c_{i,l} + c_{j,k}$ for any $i < j < k < l$. Then, the reduced Gröbner basis for I_A with respect to $\succ_{\boldsymbol{c}}$ is the same as the basis in Proposition 3.21.*

Proposition 3.23 *Let $\succ$ be the purely lexicographic order induced by the variable ordering such that $x_{i,j} \succ x_{k,l}$ if and only if $i < k$ or $(i = k$ and $j > l)$. Then, the reduced Gröbner basis for I_A with respect to $\succ$ is $\{g_{ij} := x_{i,j} - x_{i,i+1}x_{i+1,i+2}\cdots x_{j-1,j} \mid i < j-1\}$. In particular, the number of elements in this Gröbner basis is equal to $\binom{d}{2} - (d-1)$.*

The set $\{g_{ij} \mid i < j-1\}$ corresponds to all of the fundamental circuits of G_d for the spanning tree $T := \{(i, i+1) \mid 1 \leq i < d\}$.

Proof: Let C be a circuit that is not a fundamental circuit of T. Let v_1 be the vertex whose label is minimum in C, and $C := (v_1, v_2, \ldots, v_p, v_1)$. Without loss of generality, we set $v_2 < v_p$. Then, the variable x_{v_1,v_p} appears in the initial term of the associated binomial f_C. Thus, $in_\succ(f_C)$ is divisible by $in_\succ(g_{v_1 v_p})$. □

Theorem 3.24 *Let $\succ$ be any term order and $\boldsymbol{c} = (c_{1,2}, \ldots, c_{1,d}, c_{2,3}, \ldots, c_{d-1,d}) \in \mathbb{R}^n$ satisfy $c_{i,j} > c_{i,i+1} + c_{i+1,i+2} + \cdots + c_{j-1,j}$ for any $i < j-1$. Then, the reduced Gröbner basis for I_A with respect to $\succ_{\boldsymbol{c}}$ is the same as the basis in Proposition 3.23.*

Proof: Let $\succ'$ be the term order defined in Proposition 3.23. Then $in_{\succ_{\boldsymbol{c}}}(g_{ij}) = x_{i,j} = in_{\succ'}(g_{ij})$ as $c_{i,j} > c_{i,i+1} + c_{i+1,i+2} + \cdots + c_{j-1,j}$. Thus, $in_{\succ_{\boldsymbol{c}}}(I_A) = in_{\succ'}(I_A)$, which implies that the reduced Gröbner bases for I_A with respect to $\succ_{\boldsymbol{c}}$ and $\succ'$ are the same. □

Generally, the degree of any reduced Gröbner basis for a toric ideal is of exponential order with respect to the number of rows in the matrix [69], but the cardinality is not well understood. For the case of the toric ideals of acyclic tournament graphs, as these vertex-arc incidence matrices are unimodular, the cardinalities of the reduced Gröbner bases may be bounded.

Proposition 3.25 *The minimum cardinality of the reduced Gröbner bases for I_A is $\binom{d}{2} - (d-1)$. The basis shown in Proposition 3.23 achieves this cardinality.*

Proof: As the reduced Gröbner basis forms a basis for I_A, the cardinality of the reduced Gröbner basis is greater than that of the basis for I_A. As I_A corresponds to the cycle space of G_d, the cardinality of the basis for I_A is equal to the dimension of the cycle space, which is $\binom{d}{2} - (d-1)$. □

To analyze the upper bounds for the cardinalities of the reduced Gröbner bases, we calculate all reduced Gröbner bases for small d using TiGERS [35]. Table 3.2 is the result for $d = 4, 5, 6, 7$.

For the case of $d = 7$, the number of reduced Gröbner bases and the maximum of the cardinality are both too large, so we cannot determine the correct values. For $d \leq 5$, the reduced Gröbner basis in Proposition 3.21 achieves maximum cardinality, but for $d \geq 6$ the maximum cardinality is slightly larger than the cardinality of the Gröbner

Table 3.2: Number of reduced Gröbner bases, maximum and minimum of cardinality.

d	# GB	max cardinality	min cardinality
4	10	5	3
5	211	15	6
6	48312	37	10
7	$\geq$ 37665	$\geq$ 75	15

basis in Proposition 3.21. For $d = 6$, some Gröbner bases of size 36 are the bases with respect to purely lexicographic orders, but all Gröbner bases of size 37 are not with respect to purely lexicographic orders. Thus the reduced Gröbner bases which achieve the maximum number of elements seem to be complicated and difficult to characterize.

We show an example of a reduced Gröbner basis for $d = 6$ with respect to a purely lexicographic order whose number of elements is 36.

Example 3.26 ([56]) *Let $d = 6$. The reduced Gröbner basis with respect to the purely lexicographic order induced by the variable ordering*

$$x_{12} \succ x_{13} \succ x_{23} \succ x_{45} \succ x_{46} \succ x_{56} \succ x_{14} \succ x_{25}$$
$$\succ x_{36} \succ x_{15} \succ x_{16} \succ x_{24} \succ x_{26} \succ x_{34} \succ x_{35}$$

has 36 binomials as the following:

$$\{\underline{x_{12}x_{23}} - x_{13},\ \underline{x_{12}x_{24}} - x_{14},\ \underline{x_{12}x_{25}} - x_{15},\ \underline{x_{12}x_{26}} - x_{16},\ \underline{x_{13}x_{24}} - x_{14}x_{23},$$
$$\underline{x_{13}x_{25}} - x_{15}x_{23},\ \underline{x_{13}x_{26}} - x_{16}x_{23},\ \underline{x_{13}x_{34}} - x_{14},\ \underline{x_{13}x_{35}} - x_{15},\ \underline{x_{13}x_{36}} - x_{16},$$
$$\underline{x_{14}x_{25}} - x_{15}x_{24},\ \underline{x_{14}x_{26}} - x_{16}x_{24},\ \underline{x_{14}x_{35}} - x_{15}x_{34},\ \underline{x_{14}x_{36}} - x_{16}x_{34},$$
$$\underline{x_{14}x_{45}} - x_{15},\ \underline{x_{14}x_{46}} - x_{16},\ \underline{x_{15}x_{36}} - x_{16}x_{35},\ \underline{x_{15}x_{56}} - x_{16},\ \underline{x_{16}x_{25}} - x_{15}x_{26},$$
$$\underline{x_{16}x_{45}} - x_{15}x_{46},\ \underline{x_{23}x_{34}} - x_{24},\ \underline{x_{23}x_{35}} - x_{25},\ \underline{x_{23}x_{36}} - x_{26},\ \underline{x_{24}x_{36}} - x_{26}x_{34},$$
$$\underline{x_{24}x_{45}} - x_{25},\ \underline{x_{24}x_{46}} - x_{26},\ \underline{x_{25}x_{34}} - x_{24}x_{35},\ \underline{x_{25}x_{36}} - x_{26}x_{35},\ \underline{x_{25}x_{56}} - x_{26},$$
$$\underline{x_{26}x_{45}} - x_{25}x_{46},\ \underline{x_{34}x_{45}} - x_{35},\ \underline{x_{34}x_{46}} - x_{36},\ \underline{x_{35}x_{56}} - x_{36},\ \underline{x_{36}x_{45}} - x_{35}x_{46},$$
$$\underline{x_{45}x_{56}} - x_{46},\ \underline{x_{15}x_{26}x_{34}} - x_{16}x_{24}x_{35}\}$$

3.4.2 Standard Pair Decompositions and Their Size

As mentioned in Section 2.1.2, we assume that $\boldsymbol{c}$ is generic. As one constraint of $P_{A,\boldsymbol{c}}(\boldsymbol{b})$ is redundant, we can consider the problem $P_{\overline{A},\boldsymbol{c}}(\overline{\boldsymbol{b}})$, which is obtained from $P_{A,\boldsymbol{c}}(\boldsymbol{b})$ by deleting the last constraint. Then $in_{\boldsymbol{c}}(I_A) = in_{\boldsymbol{c}}(I_{\overline{A}})$, and $\overline{A}$ is row-full rank. In addition, the regular triangulation of $\mathrm{cone}(A)$ and that of $\mathrm{cone}(\overline{A})$ by $\boldsymbol{c}$ are the same as a simplicial complex. Thus, Let $\Delta_{\boldsymbol{c}}$ denote both triangulations.

As any initial ideal $in_{\boldsymbol{c}}(I_A)$ is generated by square-free monomials (Proposition 3.19), the set of standard pairs $S(in_{\boldsymbol{c}}(I_A))$ are $(1, \sigma)$ where σ ranges among all maximal faces of $\Delta_{\boldsymbol{c}}$.

The arcs in the optimum flow of uncapacitated minimum cost flow problems form a forest [2]. Therefore, as the dimension of $\mathrm{cone}(A)$ equals $d-1$, the next proposition is implied by Lemma 2.23, Proposition 3.8 and Proposition 3.19.

Proposition 3.27 *$(\boldsymbol{x}^{\boldsymbol{a}}, \sigma)$ is a standard pair of $in_{\boldsymbol{c}}(I_A)$ if and only if $\boldsymbol{x}^{\boldsymbol{a}} = 1$ and σ is a spanning tree of G_d such that $\boldsymbol{x}^{\sigma} \notin in_{\boldsymbol{c}}(I_A)$.*

The results shown in Section 3.3 indicate that there is a one-to-one correspondence between the standard pairs $(1, *)$ of $in_{\boldsymbol{c}}(I_A)$ and the dual feasible bases of $P_{\overline{A},\boldsymbol{c}}(\overline{\boldsymbol{b}})$. Therefore, the Hoçten-Thomas algorithm for the minimum cost flow problem $P_{A,\boldsymbol{c}}(\boldsymbol{b})$ is a variant of the enumeration of dual feasible bases.

The Gröbner bases shown in the previous section give upper and lower bounds for the arithmetic degree (i.e., bounds for the number of vertices of the dual polyhedron). The genericity of $\boldsymbol{c}$ implies that the arithmetic degree of $in_{\boldsymbol{c}}(I_A)$ is equal to or greater than 1.

Theorem 3.28 *The minimum arithmetic degree of $in_{\boldsymbol{c}}(I_A)$ in which $\boldsymbol{c}$ varies all generic cost vectors equals* 1.

Proof: For a cost vector $\boldsymbol{c}$ as in Theorem 3.24, $in_{\boldsymbol{c}}(I_A) = \langle x_{i,j} \mid j - i > 1 \rangle$. Then $\boldsymbol{x}^{\boldsymbol{a}} \notin in_{\boldsymbol{c}}(I_A)$ if and only if $a_{i,j} = 0$ for any (i,j) such that $j - i > 1$. The set of all such monomials is equal to $(1, \{(1,2), (2,3), \ldots, (d-1, d)\})$. Thus, only this pair is a standard pair of $in_{\boldsymbol{c}}(I_A)$. $\square$

To show the upper bound, we use the next result based on the study of hypergeometric systems on unipotent matrices reported by Gelfand et al. [26].

Lemma 3.29 ([26]) *Let A' be the homogenized matrix* (3.1) *for the incidence matrix A of the acyclic tournament graph with d vertices, and $\operatorname{conv}(A')$ be the convex hull of $\boldsymbol{a}'_1, \ldots, \boldsymbol{a}'_{n+1}$. Then, the normalized volume of $\operatorname{conv}(A')$ equals the $(d-1)$-th Catalan number C_{d-1}.*

Therefore, by Theorem 3.15, we obtain the upper bound for the arithmetic degree.

Theorem 3.30 *The maximum arithmetic degree of $in_{\boldsymbol{c}}(I_A)$ in which $\boldsymbol{c}$ varies all generic cost vectors equals $C_{d-1} := \frac{1}{d}\binom{2(d-1)}{d-1}$, which is the $(d-1)$-th Catalan number.*

The Catalan number equals $\frac{4^n}{\sqrt{\pi}n^{3/2}}\left(1+O\left(\frac{1}{n}\right)\right)$ (e.g., see [15]).

We show an example of a cost vector which achieve the maximum arithmetic degree in Theorem 3.30.

Theorem 3.31 *For the cost vector as in Theorem 3.22, the arithmetic degree of $in_{\boldsymbol{c}}(I_A)$ is the $(d-1)$-th Catalan number.*

Proof: Because of Theorem 3.22, $(1, \sigma)$ is a standard pair of $in_{\boldsymbol{c}}(I_A)$ if and only if σ is a spanning tree of the acyclic tournament graph that satisfies the following two conditions:

(a) there are no $1 \leq i < j < k \leq d$ such that both (i, j) and (j, k) are arcs in σ, and

(b) there are no $1 \leq i < j < k < l \leq d$ such that both (i, k) and (j, l) are arcs in σ.

The number of such spanning trees is the $(d-1)$-th Catalan number (e.g., see [67]). □

3.5 Analyses of Dual Minimum Cost Flow Problems

In this section, we describe the study of Gröbner bases and standard pairs for dual minimum cost flow problems on acyclic tournament graphs.

This section is from [44, 45, 46] and is an expansion of [52].

3.5.1 Gröbner Bases and Their Size

As in Section 3.3, we study the problem that is equivalent to $P_{A,\boldsymbol{c}}(\boldsymbol{b})$:

$$P_{B,\widetilde{\boldsymbol{c}}}(\widetilde{\boldsymbol{b}}) := maximize\ \{(-\widetilde{\boldsymbol{c}}_N)\cdot \boldsymbol{x}_N \mid M\boldsymbol{x}_N + I_d\boldsymbol{x}_B = \widetilde{\boldsymbol{b}}_B,\ \boldsymbol{x}_B, \boldsymbol{x}_N \geq \boldsymbol{0}\},$$

which corresponds to the basis $B := \{(1,2),(2,3),\ldots,(d-1,d)\}$, and its dual problem

$$D_{B,\widetilde{\boldsymbol{b}}}(\widetilde{\boldsymbol{c}}) := minimize\ \{\widetilde{\boldsymbol{b}}_B \cdot \boldsymbol{y}_B \mid I_{n-d}\boldsymbol{y}_N - M^{\mathrm{T}}\boldsymbol{y}_B = \widetilde{\boldsymbol{c}}_N,\ \boldsymbol{y}_B, \boldsymbol{y}_N \geq \boldsymbol{0}\},$$

where $(M\ I)$ (resp. $(I\ -M^{\mathrm{T}})$) is the fundamental cutset (resp. fundamental circuit) matrix that corresponds to the spanning tree $\{(1,2),(2,3),\ldots,(d-1,d)\}$, $\widetilde{\boldsymbol{b}} = (\widetilde{\boldsymbol{b}}_B, \widetilde{\boldsymbol{b}}_N) = (\widetilde{b}_{ij})_{1\leq i<j\leq d}$, $\widetilde{\boldsymbol{b}}_B = (\widetilde{b}_{i,i+1})_{1\leq i<d}$, $\widetilde{\boldsymbol{b}}_N = (\widetilde{b}_{i,j})_{i<j-1} = \boldsymbol{0}$, and

$$\boldsymbol{x} = (\boldsymbol{x}_B, \boldsymbol{x}_N),\ \boldsymbol{x}_B = (x_{1,2}, x_{2,3}, \ldots, x_{d-1,d}),\ \boldsymbol{x}_N = (x_{1,3}\ldots, x_{1,d}, x_{2,4}, \ldots, x_{d-2,d}),$$
$$\boldsymbol{y} = (\boldsymbol{y}_B, \boldsymbol{y}_N),\ \boldsymbol{y}_B = (y_{1,2}, y_{2,3}, \ldots, y_{d-1,d}),\ \boldsymbol{y}_N = (y_{1,3}\ldots, y_{1,d}, y_{2,4}, \ldots, y_{d-2,d}).$$

Then, $P_{B,\widetilde{\boldsymbol{c}}}(\widetilde{\boldsymbol{b}})$ has $d-1$ constraints (i.e., $(M\ I) \in \mathbb{Z}^{(d-1)\times n}$), $D_{B,\widetilde{\boldsymbol{b}}}(\widetilde{\boldsymbol{c}})$ has $n-d+1$ constraints (i.e., $(I\ -M^{\mathrm{T}}) \in \mathbb{Z}^{(n-d+1)\times n}$).

Let $G_d = (V, E)$. $D \subseteq E$ is a *cutset* in G_d if there exists a partition (V_1, V_2) of V (i.e., $V_1 \cap V_2 = \emptyset,\ V_1 \cup V_2 = V$) such that $D = \{(i,j) \in E \mid i \in V_1$ and $j \in V_2,$ or $i \in V_2$ and $j \in V_1\}$.

Definition 3.32 *Let D be a cutset in G_d, which corresponds to $V = (V^+, V^-)$. We define the vector $\boldsymbol{u}_D \in \mathbb{R}^n$ as*

$$(\boldsymbol{u}_D)_{ij} := \begin{cases} 1 & (i \in V^+ \textit{ and } j \in V^-) \\ -1 & (i \in V^- \textit{ and } j \in V^+) \\ 0 & (\textit{otherwise}) \end{cases}.$$

The vector $\boldsymbol{u}_D$ is called the incidence vector *of D.*

We identify a cutset D that corresponds to (V^+, V^-) with the binomial $f_D := \boldsymbol{x}^{\boldsymbol{u}_D^+} - \boldsymbol{x}^{\boldsymbol{u}_D^-}$. As the rank of the fundamental circuit matrix $(I\ -M^{\mathrm{T}})$ is $n-d+1$ and each row vector of the fundamental cutset matrix $(M\ I)$ is in $\ker((I\ -M^{\mathrm{T}}))$, the set of row vectors of the fundamental cutset matrix $(M\ I)$ forms a basis of $\ker(I\ -M^{\mathrm{T}})$.

For the fundamental circuit matrix $(I\ -M^{\mathrm{T}})$, the set of circuits $\mathcal{C}_{(I\ -M^{\mathrm{T}})}$ corresponds to the set of all cutsets of G_d. As the fundamental circuit matrix $(I\ -M^{\mathrm{T}})$ is totally unimodular (e.g,. see [74]), $\mathcal{C}_{(I\ -M^{\mathrm{T}})} = \mathcal{U}_{(I\ -M^{\mathrm{T}})}$.

Proposition 3.33 *For a cost vector $\widetilde{\boldsymbol{b}}$ such that the linear system $(M\ I)\boldsymbol{x} = \widetilde{\boldsymbol{b}}_B$ has a non-negative solution, $I_{(I\ -M^{\mathrm{T}})}$ has a reduced Gröbner basis with respect to $\widetilde{\boldsymbol{b}}$.*

Proof: Let $\boldsymbol{a} \geq \mathbf{0}$ be a solution of $(M\ I)\boldsymbol{x} = \widetilde{\boldsymbol{b}}_B$. Let $\boldsymbol{r}_i$ denote the i-th row of $(M\ I)$, i.e., the row that corresponds to the fundamental cutset for the arc $(i, i+1)$. For each cutset D corresponding to $(V^+, V \setminus V^+)$ ($V^+ \subseteq \{1, \dots, d-1\}$), as $\boldsymbol{u}_D = \sum_{i \in V^+,\ i+1 \notin V^+} \boldsymbol{r}_i - \sum_{i \notin V^+,\ i+1 \in V^+} \boldsymbol{r}_i$,

$$\begin{aligned}
\boldsymbol{a} \cdot \boldsymbol{u}_D &= \sum_{i \in V^+,\ i+1 \notin V^+} \boldsymbol{a} \cdot \boldsymbol{r}_i - \sum_{i \notin V^+,\ i+1 \in V^+} \boldsymbol{a} \cdot \boldsymbol{r}_i \\
&= \sum_{i \in V^+,\ i+1 \notin V^+} \widetilde{b}_{i,i+1} - \sum_{i \notin V^+,\ i+1 \in V^+} \widetilde{b}_{i,i+1} \\
&= \widetilde{\boldsymbol{b}} \cdot \boldsymbol{u}_D.
\end{aligned}$$

Thus, $in_{\boldsymbol{a}}(f_D) = in_{\widetilde{\boldsymbol{b}}}(f_D)$ for any cutset D, and $in_{\boldsymbol{a}}(I_{(I\ -M^{\mathrm{T}})}) = in_{\widetilde{\boldsymbol{b}}}(I_{(I\ -M^{\mathrm{T}})})$. As $\boldsymbol{a} \geq \mathbf{0}$, $I_{(I\ -M^{\mathrm{T}})}$ has a reduced Gröbner basis with respect to $\widetilde{\boldsymbol{b}}$. □

Example 2.12 (continued.) *Let $\boldsymbol{c} = (3, 1, 2)$ and $\boldsymbol{b} = (4, 5, -9)$ and consider the primal and dual problem which corresponds to the spanning tree $\{(1,2), (2,3)\}$:*

$$\begin{array}{ll}
\max & 4x_{1,3} \\
\textit{s.t.} & \left(\begin{array}{c|cc} 1 & 1 & 0 \\ 1 & 0 & 1 \end{array}\right) \begin{pmatrix} x_{1,3} \\ x_{1,2} \\ x_{2,3} \end{pmatrix} = \begin{pmatrix} 4 \\ 9 \end{pmatrix} \\
& x_{1,2}, x_{1,3}, x_{2,3} \geq 0
\end{array}
\qquad
\begin{array}{ll}
\min & 4y_{1,2} + 9y_{2,3} \\
\textit{s.t.} & \left(\begin{array}{c|cc} 1 & -1 & -1 \end{array}\right) \begin{pmatrix} y_{1,3} \\ y_{1,2} \\ y_{2,3} \end{pmatrix} = -4 \\
& y_{1,2}, y_{1,3}, y_{2,3} \geq 0
\end{array}$$

Then $I_{(1,-1,-1)} = \langle x_{1,2} - x_{2,3},\ x_{1,2}x_{1,3} - 1,\ x_{1,3}x_{2,3} - 1 \rangle$ and reduced Gröbner basis for $\widetilde{\boldsymbol{b}} = (4, 0, 9)$ is $\{x_{2,3} - x_{1,2},\ x_{1,2}x_{1,3} - 1\}$. □

As for primal problems, the elements in a reduced Gröbner basis to some specific term order can be given in terms of graphs.

Theorem 3.34 *Let $\widetilde{\boldsymbol{b}}$ be the cost vector that satisfies the condition in Proposition 3.33, $\widetilde{b}_{i,i+1} > \widetilde{b}_{j,j+1}$ $(1 \leq {}^\forall i < {}^\forall j \leq d)$ and $\widetilde{b}_{i,j} = 0$ (${}^\forall i, j$ such that $j > i+1$). Then, the reduced Gröbner basis for $I_{(I\ -M^{\mathrm{T}})}$ with respect to $\widetilde{\boldsymbol{b}}$ is*

$$\left\{ g_i := \prod_{j<i} x_{j,i} - \prod_{k>i} x_{i,k} \mid i = 2, 3, \ldots, d \right\}.$$

In particular, the number of elements in this Gröbner basis is equal to $d-1$.

Each g_i is an incidence vector of the cutset that corresponds to $(V \setminus \{i\}, \{i\})$.

Proof: For a cutset D, which corresponds to (V^+, V^-) such that $1 \in V^+$, we define $P^+ := \{i \in V^+ \mid i \neq d,\ i+1 \in V^-\}$ and $P^- := \{i \in V^- \mid i \neq d,\ i+1 \in V^+\}$. Let $P^+ = \{i_1, \ldots, i_p\}$ $(i_1 < i_2 < \cdots < i_p)$ and $P^- = \{j_1, \ldots, j_q\}$ $(j_1 < j_2 < \cdots < j_q)$. Then, $p = q$ or $p = q+1$, and $i_1 < j_1 < i_2 < j_2 < \cdots < i_k < j_k < i_{k+1} < j_{k+1} < \cdots$. As $\widetilde{\boldsymbol{b}} \cdot \boldsymbol{u}_D^+ = \sum_{r=1}^{p} \widetilde{b}_{i_r, i_r+1} > \sum_{r=1}^{q} \widetilde{b}_{j_r, j_r+1} = \widetilde{\boldsymbol{b}} \cdot \boldsymbol{u}_D^-$, $in_{\widetilde{\boldsymbol{b}}}(f_D) = \boldsymbol{x}^{\boldsymbol{u}_D^+}$. As $in_{\widetilde{\boldsymbol{b}}}(g_{i_1+1}) = \prod_{j \leq i_1} x_{j, i_1+1}$, $in_{\widetilde{\boldsymbol{b}}}(f_D)$ can be reduced by $in_{\widetilde{\boldsymbol{b}}}(g_{i_1+1})$. $\square$

Proposition 3.35 *The minimum cardinality of the reduced Gröbner bases for $I_{(I\ -M^{\mathrm{T}})}$ is $d-1$. The basis we have shown in Theorem 3.34 is the example achieving this cardinality.*

Proof: As the reduced Gröbner basis forms a basis for $I_{(I\ -M^{\mathrm{T}})}$, the cardinality of the reduced Gröbner basis is greater than that of the basis for $I_{(I\ -M^{\mathrm{T}})}$, which is $d-1$. $\square$

To analyze the upper bounds for the cardinalities of the reduced Gröbner bases, we calculate all reduced Gröbner bases for small d using TiGERS [35]. Table 3.3 is the result for $d = 4, 5, 6, 7$.

The reduced Gröbner bases that achieve maximum cardinality are complicated and difficult to characterize, and we could not determine what cost vectors produce the Gröbner bases of maximum cardinality.

3.5.2 Standard Pair Decompositions and Their Size

Similar to Section 3.4.2, we assume here that $\widetilde{\boldsymbol{b}}$ is generic. As any initial ideal $in_{\widetilde{\boldsymbol{b}}}(I_{(I\ -M^{\mathrm{T}})})$ is generated by square-free monomials, any standard pair in $S(in_{\widetilde{\boldsymbol{b}}}(I_{(I\ -M^{\mathrm{T}})}))$

Table 3.3: Number of reduced Gröbner bases of dual problems, maximum and minimum of cardinality.

d	# GB	max cardinality	min cardinality
4	7	5	3
5	48	10	4
6	820	20	5
7	44288	39	6

is of the form $(1,*)$. Moreover, as the support of each optimal solution of $D_{(I\ -M^{\mathrm{T}}),\widetilde{\boldsymbol{c}}}(\widetilde{\boldsymbol{b}})$ does not include a cutset, taken together with the fact that $\dim \mathrm{cone}((I\ \ -M^{\mathrm{T}})) = n-d+1$, the following proposition is implied by Lemma 2.23 and Proposition 3.8.

Proposition 3.36 *$(\boldsymbol{x}^{\boldsymbol{a}},\sigma)$ is a standard pair of $in_{\widetilde{\boldsymbol{b}}}(I_{(I\ -M^{\mathrm{T}})})$ if and only if $\boldsymbol{x}^{\boldsymbol{a}}=1$ and σ is a co-tree of G_d such that $\boldsymbol{x}^{\sigma}\notin in_{\widetilde{\boldsymbol{b}}}(I_{(I\ -M^{\mathrm{T}})})$.*

Example 2.12 (continued.) *For $\boldsymbol{c}=(3,1,2)$ and $\boldsymbol{b}=(4,5,-9)$, the initial ideal $in_{(4,0,9)}(I_{(1|-1,-1)})=\langle x_{2,3},\ x_{1,2}x_{1,3}\rangle$ has two standard pairs $(1,\{(1,2)\})$ and $(1,\{(1,3)\})$.* □

Theorem 3.37 *For any $\widetilde{\boldsymbol{b}}$ that satisfies the condition in Proposition 3.33, there exists $S\subset\{1,\dots,d-1\}$ with $|S|\geq\lfloor(d-1)/6\rfloor$ such that, for any $\sigma\subseteq S$, there exists a spanning tree T_σ of G_d that satisfies the following:*

(A) *T_σ contains the arc set $\{(i,i+1)\mid i\in S\setminus\sigma\}$ and does not contain any arc in $\{(j,j+1)\mid j\in\sigma\}$, and*

(B) *$(1,\overline{T_\sigma})$ is a standard pair of $in_{\widetilde{\boldsymbol{b}}}(I_{(I\ -M^{\mathrm{T}})})$, where $\overline{T_\sigma}:=E\setminus T_\sigma$ is a co-tree of T_σ.*

In particular, as $T_\sigma\neq T_\tau$ for any $\sigma,\tau\subseteq S$ $(\sigma\neq\tau)$, $in_{\widetilde{\boldsymbol{b}}}(I_{(I\ -M^{\mathrm{T}})})$ has at least $\Omega(2^{\lfloor d/6\rfloor})$ standard pairs for any generic $\boldsymbol{b}$ that satisfies the condition in Proposition 3.33.

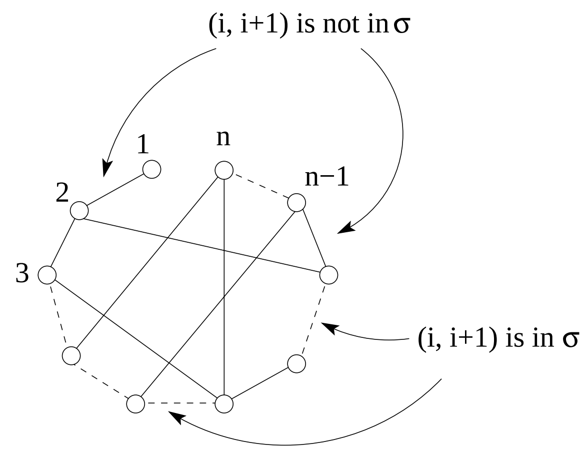

Figure 3.3: The condition (A) of the definition of T_σ.

For a subset S of arcs in G_d, we set $\boldsymbol{x}^S := \prod_{(i,j)\in S} x_{i,j}$.

Proof: We divide $\{1, \dots, d-1\}$ into the following subsets.

$$
\begin{aligned}
M_0 &:= \{i \in \{1, \dots, d-1\} \mid x_{i,i+1} \in in_{\tilde{\boldsymbol{b}}}(I_{(I\ -M^{\mathrm{T}})})\} \\
M_1 &:= \{i \in \{1, \dots, d-1\} \mid i \notin M_0,\ i \equiv 0 \pmod 3\} \\
M_2 &:= \{i \in \{1, \dots, d-1\} \mid i \notin M_0,\ i \equiv 1 \pmod 3\} \\
M_3 &:= \{i \in \{1, \dots, d-1\} \mid i \notin M_0,\ i \equiv 2 \pmod 3\}
\end{aligned}
$$

Lemma 3.38 $|M_0| \le \lceil (d-1)/2 \rceil$.

Proof of Lemma 3.38: We consider a cutset D which corresponds to (V^+, V^-) such that f_D contains $x_{i,j}$ as a term of degree 1. Without loss of generality, we set $i \in V^+$. Assuming that $j - i > 1$, for any k $(i < k < j)$, if $k \in V^+$ then f_D contains $x_{k,j}$ and $x_{i,j}$ in the same term, otherwise f_D contains $x_{i,k}$ and $x_{i,j}$ in the same term, which contradicts that $x_{i,j}$ is a term of f_D of degree 1. Thus, $j = i + 1$. In addition, $k \in V^-$ for any $k < i$ and $k \in V^+$ for any $k > i + 1$. Therefore, $V^+ = \{i, i+2, i+3, \dots, d\}$ and $V^- = \{1, \dots, i-1, i+1\}$.

We consider that $in_{\tilde{\boldsymbol{b}}}(f) = x_{i,i+1}$ for some $f \in I_{(I\ -M^{\mathrm{T}})}$. If $x_{i-1,i} \in in_{\tilde{\boldsymbol{b}}}(I_{(I\ -M^{\mathrm{T}})})$, then f can be reduced by the binomial corresponding to the cutset between $\{i-1\ , i+$

$1,\ldots,d\}$ and $\{1,\ldots,i-2,i\}$ to

$$f' \;:=\; x_{i,i+1} - \left\{ \left(\prod_{k\le i-2} x_{k,i} \right) \left(\prod_{k\ge i+2} x_{i+1,k} \right) \right.$$
$$\left. \left(\prod_{\substack{k\le i-1,\\ l\ge i+2}} x_{k,l} \right) \left(\prod_{k\le i-2} x_{k,i-1} \right) \left(\prod_{k\ge i+1} x_{i,k} \right) \left(\prod_{\substack{k\le i-2,\\ l\ge i+1}} x_{k,l} \right) \right\},$$

and its initial term is $x_{i,i+1}$. As both terms of this binomial contain $x_{i,i+1}$, this implies that $in_{\widetilde{\boldsymbol{b}}}(f'/x_{i,i+1}) = 1$. As $\widetilde{\boldsymbol{b}}$ defines a term order by Proposition 3.33, this is a contradiction.

Similarly, $x_{i+1,i+2} \notin in_{\widetilde{\boldsymbol{b}}}(I_{(I\ -M^{\mathrm{T}})})$. Thus, $|M_0| \le \lceil (d-1)/2 \rceil$. $\square$

Thus, at least one of $M_1,\ M_2,\ M_3$ has at least $\lfloor (d-1)/6 \rfloor$ elements. Let S be one such M_i $(i=1,2,3)$. For any $\sigma := \{i_1 > i_2 > \cdots > i_r\} \subseteq S$, we construct the desired spanning trees $T_\emptyset,\ T_{\{i_1\}}, T_{\{i_1,i_2\}},\ \ldots\ ,\ T_\sigma$ inductively.

· **Initial step:**

Let $T_\emptyset := \{(1,2),(2,3),\ldots,(d-1,d)\}$. Clearly, $T_\emptyset$ is a spanning tree. As the reduced Gröbner basis corresponds to a subset of cutsets, the initial term of any element of the reduced Gröbner basis contains a variable $x_{i,i+1}$ for some i. Thus, $\boldsymbol{x}^{\overline{T_\emptyset}} \notin in_{\widetilde{\boldsymbol{b}}}(I_{(I\ -M^{\mathrm{T}})})$.

· **Induction step:**

Let $T_{\sigma\setminus\{i_r\}}$ be the desired spanning tree for $\sigma \setminus \{i_r\}$. We define two edge sets

$$T^1 := \left\{ T_{\sigma\setminus\{i_r\}} \setminus \{(i_r, i_r+1)\} \right\} \cup \{(i_r, i_r+2)\},$$
$$T^2 := \{ T^1 \setminus \{(i_r+1, i_r+2)\} \} \cup \{(i_r-1, i_r+1)\}.$$

Then, both T^1 and T^2 are spanning trees and satisfy the condition (A). We show here that either T^1 or T^2 satisfies the condition (B).

(a) The case where T^1 satisfies the condition (B).

Then, T^1 is the desired spanning tree T_σ.

(b) The case where T^1 does not satisfy the condition (B).

In this case, $\boldsymbol{x}^{\overline{T^1}} \in in_{\widetilde{\boldsymbol{b}}}(I_{(I\ -M^{\mathrm{T}})})$. Let $\mathcal{G}$ be the reduced Gröbner basis for $I_{(I\ -M^{\mathrm{T}})}$ with respect to $\widetilde{\boldsymbol{b}}$. Then, $\boldsymbol{x}^{\overline{T^1}}$ can be reduced by some binomial $g \in \mathcal{G}$, and such g is of the following form (See Figure 3.5).

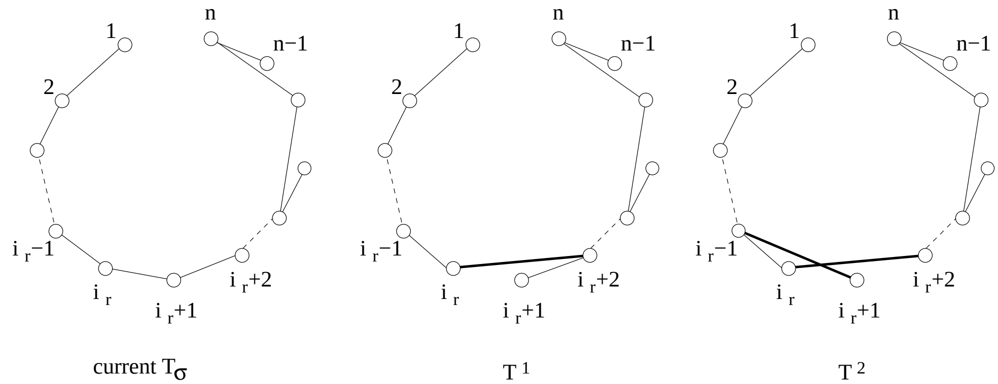

Figure 3.4: Two trees T^1 (middle) and T^2 (right) for the current spanning tree (left).

(i) $g^{(1)}_{(p)}$, which corresponds to the cutset for (V^+, V^-), $V^+ = \{p, p+1, \dots, i_r, i_r + 2, i_r + 3, \dots, d\}$ and $V^- = \{1, 2, \dots, p-1, i_r+1\}$ for some $p \leq i_r$, and its initial term is a product of variables corresponding to arcs from V^+ to V^-, or

(ii) (The case of $r > 1$) $g^{(2)}_{(p,t)}$, which corresponds to the cutset for (V^+, V^-), $V^- = \{1, 2, \dots, p-1, i_r+1, i_{q(1)}+1, \dots, i_{q(t)}+1\}$ and $V^+ = V \setminus V^-$ for $1 \leq {}^{\exists}q(t) < \cdots < {}^{\exists}q(1) < r$ such that $(i_{q(k)}+1, i_{q(k)}+2) \in T_{\sigma \setminus \{i_r\}}$ for $k = 1, \dots, t$ and $1 \leq {}^{\exists}p \leq i_r$, and its initial term is a product of variables corresponding to arcs from V^+ to V^-.

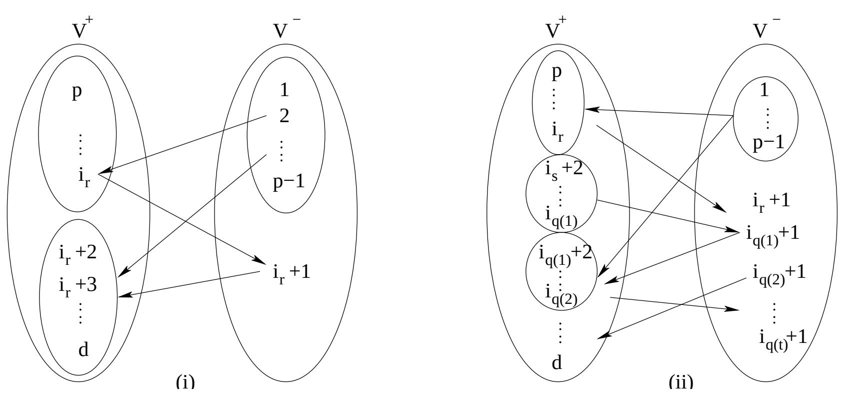

Figure 3.5: Cutsets corresponding to binomials $g^{(1)}_{(p)}$ (left) and $g^{(2)}_{(p,t)}$ (right).

Lemma 3.39 *$g^{(1)}_{(p)} \in \mathcal{G}$ for some p and $\boldsymbol{x}^{\overline{T^1}}$ can be reduced by $g^{(1)}_{(1)}$, i.e., the initial term of $g^{(1)}_{(1)}$ corresponds to the set of arcs $\{(k, i_r+1) : k \leq i_r\}$.*

Proof of Lemma 3.39: The case of $r = 1$ is trivial.

We suppose that $r > 1$ and $\boldsymbol{x}^{\overline{T^1}}$ cannot be reduced by any $g^{(1)}_{(p)}$. Then, $\boldsymbol{x}^{\overline{T^1}}$ can be reduced by some $g^{(2)}_{(p,t)}$, which is an element of $\mathcal{G}$, and $\boldsymbol{x}^{\overline{T^1}}$ can also be reduced by $g^{(2)}_{(1,t)}$ (otherwise, $g^{(2)}_{(p,t)}$ is reduced by $g^{(2)}_{(1,t)}$ and $g^{(2)}_{(p,t)}$ cannot be an element of $\mathcal{G}$).

Suppose that $\boldsymbol{x}^{\overline{T^1}}$ can be reduced by $g^{(2)}_{(1,t)}$ with $t = 1$. Let m_1 be the monomial obtained by reducing $\boldsymbol{x}^{\overline{T^1}}$ by $g^{(2)}_{(1,t)}$. Then m_1 can be reduced to the monomial m_2 by $g^{(1)}_{(1)}$ (the initial term of $g^{(1)}_{(1)}$ is a product of variables corresponding to arcs from V^- to V^+ by assumption).

Table 3.4: Divided and multiplied variables while reducing by $g^{(2)}_{(1,1)}$ and $g^{(1)}_{(1)}$.

reduce by $g^{(2)}_{(1,1)}$		*reduce by $g^{(1)}_{(1)}$*	
divided variables	multiplied variables	divided variables	multiplied variables
$\{x_{k,i_r+1} : k \leq i_r\}$, $\{x_{k,i_{q(1)}+1} : k \leq i_{q(1)}, k \neq i_r+1\}$	$\{x_{i_r+1,l} : l \geq i_r+2, l \neq i_{q(1)}+1\}$, $\{x_{i_{q(1)}+1,l} : l \geq i_{q(1)}+2\}$	$\{x_{i_r+1,l} : l \geq i_r+2\}$	$\{x_{k,i_r+1} : k \leq i_r\}$

For a binomial $f_D \in I_{(I\ -M^{\mathrm{T}})}$, which corresponds to the cutset D for (V^+_D, V^-_D) such that $V^-_D = \{i_{q(1)}+1\}$ and $V^+_D = V \setminus V^-_D$, $in_{\tilde{\boldsymbol{b}}}(f_D)$ corresponds to arcs from V^-_D to V^+_D (otherwise, $\boldsymbol{x}^{\overline{T_{\sigma\setminus\{i_r\}}}}$ can be reduced by f_D, which contradicts the assumption of the induction). Then, m_2 can be reduced by f_D, and the resulting monomial is $\boldsymbol{x}^{\overline{T^1}}$ (see Table 3), which contradicts the definition of term order by $\tilde{\boldsymbol{b}}$.

Similarly, in the case that in which $\boldsymbol{x}^{\overline{T^1}}$ can be reduced by $g^{(2)}_{(1,t)}$ for some $t > 1$, using $f_D \in I_{(I\ -M^{\mathrm{T}})}$, which corresponds to the cutset D for (V^+_D, V^-_D) such that $V^-_D = \{i_{q(1)}+1, i_{q(2)}+1, \ldots, i_{q(t)}+1\}$, and $V^+_D = V \setminus V^-_D$, we can show a contradiction. Thus, there exists some p such that $g^{(1)}_{(p)} \in \mathcal{G}$.

If $\boldsymbol{x}^{\overline{T1}}$ cannot be reduced by $g^{(1)}_{(1)}$, i.e., the initial term of $g^{(1)}_{(1)}$ corresponds to the set of arcs $\{(i_r+1, l) : \; l \geq i_r+2\}$, then $g^{(1)}_{(p)}$ can be reduced by $g^{(1)}_{(1)}$, which contradicts that $g^{(1)}_{(p)}$ is an element of reduced Gröbner basis $\mathcal{G}$. Thus, the second statement follows. $\square$

If $\boldsymbol{x}^{\overline{T1}} \in in_{\tilde{\boldsymbol{b}}}(I_{(I\ -M^{\mathrm{T}})})$, then $\boldsymbol{x}^{\overline{T2}}$ cannot be reduced by any binomial in $\mathcal{G}$. If $\boldsymbol{x}^{\overline{T2}}$ can be reduced by some $g \in \mathcal{G}$, then g is of the following form.

(i) the binomial $g^{(1)}_{(i_r)}$, and its initial term is x_{i_r, i_r+1},

(ii) any binomial that corresponds to the cutset for (V^+, V^-) such that $i_r+1 \in V^+$ and $1, 2, \ldots, i_r, i_r+2 \in V^-$, and its initial term is a product of variables corresponding to arcs from V^+ to V^-, or

(iii) (The case of $r > 1$**)** $g^{(2)}_{(i_r,t)}$, and its initial term is a product of variables corresponding to arcs from V^+ to V^-.

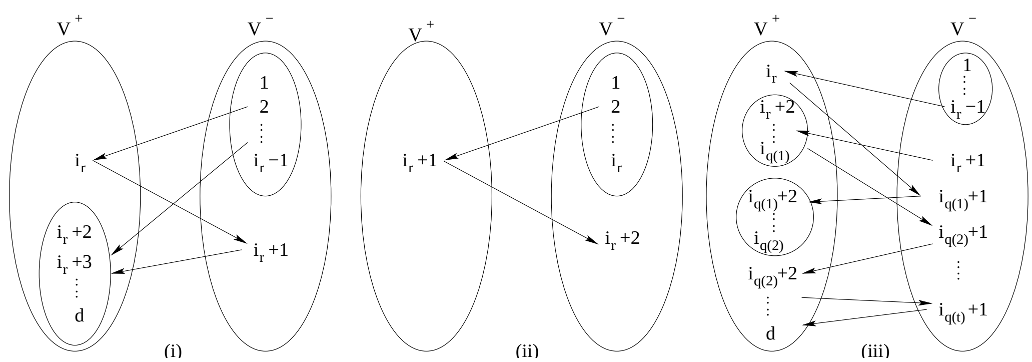

Figure 3.6: Cutsets corresponding to binomials (i), (ii) and (iii).

If case (i) occurs, the initial term of $g^{(1)}_{(i_r)}$ is x_{i_r,i_r+1}, which contradicts $i_r \notin M(0)$. On the other hand, a binomial of type (ii) can be reduced by $g^{(1)}_{(1)}$ by the above lemma, and cannot be contained in $\mathcal{G}$.

Let us consider that case (iii) occurs. If $\boldsymbol{x}^{\overline{T2}}$ can be reduced by $g^{(2)}_{(i_r,t)}$ with $t = 1$, then the monomial to which $\boldsymbol{x}^{\overline{T2}}$ is reduced by $g^{(2)}_{(i_r,1)}$ can be reduced by a binomial $f_D \in I_{(I\ -M^{\mathrm{T}})}$, for the cutset D which corresponds to (V^+_D, V^-_D) where $V^+_D = \{1, 2, \ldots, i_r -$

$1, i_r + 1\}$ and $V_D^- = V \setminus V_D^+$, to some monomial m (the initial term of f_D is a product of variables corresponding to arcs from V_D^+ to V_D^- since $i_r \notin M(0)$).

Table 3.5: Divided and multiplied variables while reducing by $g^{(2)}_{(i_r,1)}$ and f_D.

reduce by $g^{(2)}_{(i_r,1)}$		*reduce by* f_D	
divided variables	multiplied variables	divided variables	multiplied variable
$x_{i_r,i_r+1}, x_{i_r,i_{q(1)}+1}$, $x_{i_r+2,i_{q(1)}+1}$, $x_{i_r+3,i_{q(1)}+1}$, $\dots, x_{i_{q(1)},i_{q(1)}+1}$	$\{x_{k,i_r} :\ k \leq i_r - 1\}$, $\{x_{k,l} : k \leq i_r + 1,\ k \neq i_r,$ $l \geq i_r + 2, l \neq i_{q(1)} + 1\}$, $\{x_{i_{q(1)}+1,l} :\ l \geq i_{q(1)} + 2\}$	$\{x_{k,i_r} : k \leq i_r - 1\}$, $\{x_{k,l} : k \leq i_r + 1,$ $k \neq i_r, l \geq i_r + 2\}$	x_{i_r,i_r+1}

For a binomial $f_{D'} \in I_{(I\ -M^{\top})}$, which corresponds to the cutset D' for $(V_{D'}^+, V_{D'}^-)$ such that $V_{D'}^- = \{i_{q(1)} + 1\}$ and $V_{D'}^+ = V \setminus V_{D'}^-$, $in_{\tilde{\boldsymbol{b}}}(f_{D'})$ corresponds to arcs from $V_{D'}^-$ to $V_{D'}^+$ (otherwise, $\boldsymbol{x}^{\overline{T_{\sigma \setminus \{i_r\}}}}$ can be reduced by $f_{D'}$, which contradicts the assumption of the induction). Then, m can be reduced by $f_{D'}$, and the resulting monomial is $\boldsymbol{x}^{\overline{T^2}}$ (see Table 4), which contradicts the definition of a term order by $\tilde{\boldsymbol{b}}$.

Similarly, in the case in which $\boldsymbol{x}^{\overline{T^2}}$ can be reduced by $g^{(2)}_{(i_r,t)}$ for some $t > 1$, using the same f_D and $f_{D'} \in I_{(I\ -M^{\top})}$, which corresponds to the cutset D' for $(V_{D'}^+, V_{D'}^-)$ such that $V_{D'}^- = \{i_{q(1)}+1, i_{q(2)}+1, \dots, i_{q(t)}+1\}$, and $V_{D'}^+ = V \setminus V_{D'}^-$, we can show a contradiction.

Therefore, $\boldsymbol{x}^{\overline{T^2}} \notin in_{\tilde{\boldsymbol{b}}}(I_{(I\ -M^{\top})})$, and T^2 is the desired spanning tree T_σ. $\square$

3.6 Analyses of Transportation Problems

This section deals with the number of vertices for transportation polytopes of type $2 \times N$ and dual transportation polyhedra. The maximum number of vertices have been reported for primal transportation polyhedron [48] and dual transportation polyhedra [5]. We give computational algebraic proof for their results using results in previous sections.

3.6.1 Number of Vertices of Dual Transportation Polyhedra

Let A be the incidence matrix of the bipartite graph $K_{m,n}$. Then $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$ is the transportation problem on $K_{m,n}$:

$$P_{A,\boldsymbol{c}}(\boldsymbol{b}) := minimize\ \{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \mid A\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}\}.$$

Our results in previous sections give another proof for the following result by Balinski and Russakoff [5].

Theorem 3.40 ([5]) *The maximum number of vertices for the dual polyhedron of the transportation problem on $K_{m,n}$ is equal to $\binom{m+n-2}{m-1}$.*

As column vectors of A span an affine hyperplane, the normalized volume of $\mathrm{conv}(A')$ is equal to that of $\mathrm{conv}(A)$.

To show the theorem, we show a unimodular triangulation of $\mathrm{conv}(A)$, i.e., a triangulation such that the normalized volume of any facet is 1.

Lemma 3.41 ([71]) *Let $\succ$ be the reverse lexicographic order induced from the variable ordering*

$$x_{1,1} \prec x_{1,2} \prec \cdots \prec x_{1,n} \prec x_{2,1} \prec \cdots x_{m,n}.$$

The the reduced Gröbner basis of I_A with respect to $\succ$ equals

$$\mathcal{G}_{\succ} = \{\underline{x_{i,l}x_{j,k}} - x_{i,k}x_{j,l} \mid 1 \leq i < j \leq m,\ 1 \leq k < l \leq n\},$$

where underlined term is the initial term.

Corollary 3.42 ([71]) *Let $\boldsymbol{c} \in \mathbb{R}^{mn}$ be a cost vector that satisfies*

$$c_{i,l} + c_{j,k} > c_{i,k} + c_{j,l} \quad \textit{for any}\ 1 \leq i < j \leq m,\ 1 \leq k < l \leq n.$$

Then $\sigma \subset \{(i,j) \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$ is a facet of the regular triangulation $\Delta_{\boldsymbol{c}}$ of $\mathrm{conv}(A)$ if and only if, for any pair (p,q) and (r,s) in σ, $p \leq r$ and $q \leq s$. Furthermore, the normalized volume of σ is 1.

Let us consider the table of size $m \times n$ as below.

$(1,1)$	$(1,2)$	$\cdots$	$(1,n)$
$(2,1)$	$(2,2)$	$\cdots$	$(2,n)$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
$(m,1)$	$(m,2)$	$\cdots$	(m,n)

Then Corollary 3.42 implies that each path from $(1,1)$ to (m,n) of length $m+n-2$ corresponds to a facet of $\Delta_{\boldsymbol{c}}$. Therefore, the normalized volume of $\mathrm{conv}(A)$ is equal to the total number of such paths, which is $\binom{m+n-2}{m-1}$.

3.6.2 Number of Vertices of Transportation Polytopes on $K_{2,n}$

In this section, we consider the dual problem of the transportation problem on $K_{2,n}$. Our results in previous sections give another proof for the following result by Klee and Witzgall [48].

Theorem 3.43 ([48]) *The maximum number of vertices for the feasible region of the transportation problem on $K_{2,n}$ is equal to $(n-[n/2])\binom{n}{[n/2]}$.*

As one constraint of the transportation problem $P_{A,\boldsymbol{c}}(\boldsymbol{b})$ on $K_{2,n}$ is redundant, we can consider the problem $P_{\overline{A},\boldsymbol{c}}(\overline{\boldsymbol{b}})$, which is obtained from $P_{A,\boldsymbol{c}}(\boldsymbol{b})$ by deleting the second constraint.

For the basis $B := \{(1,n),(2,1),(2,2),\ldots,(2,n)\}$, the coefficient matrix of the dictionary of $P_{\overline{A},\boldsymbol{c}}(\overline{\boldsymbol{b}})$ is

$$\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 1 & 0 \\ -1 & -1 & \cdots & -1 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

and the coefficient matrix D for its dual problem is

$$D := \begin{pmatrix} -1 & -1 & 0 & \cdots & 0 & 1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & -1 & \cdots & 0 & 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & -1 & 1 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Therefore, $\mathrm{conv}(D')$ is a linear transformation of the convex hull of $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n-1}, -\boldsymbol{e}_1, \ldots, -\boldsymbol{e}_{n-1}, \mathbf{1}, -\mathbf{1}, \mathbf{0}\}$, where $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n-1}$ are unit vectors of $\mathbb{R}^{n-1}$, $\mathbf{1} \in \mathbb{R}^{n-1}$ is the vector all of whose elements are 1, and $\mathbf{0} \in \mathbb{R}^{n-1}$ is the origin. Let P_n denote the convex hull of $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n-1}, -\boldsymbol{e}_1, \ldots, -\boldsymbol{e}_{n-1}, \mathbf{1}, -\mathbf{1}, \mathbf{0}\}$.

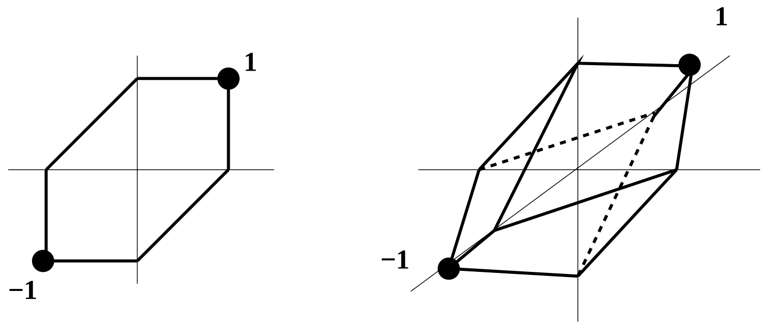

Figure 3.7: Polytope P_3 and P_4.

Discussion of the above theorem requires some background in polytope theory.

Definition 3.44 *Let $P \subset \mathbb{R}^n$ be a polytope.*

(i) *A maximal face of P is called a* facet *of P.*

(ii) *Let F be a facet of P. The* supporting hyperplane *of F is a hyperplane $a_1x_1 + \cdots + a_nx_n = a_0$ such that P is contained in the region $a_1x_1 + \cdots + a_nx_n < a_0$ and F is on the hyperplane $a_1x_1 + \cdots + a_nx_n = a_0$.*

(iii) *Let $\boldsymbol{p} = (p_1, \ldots, p_n) \in \mathbb{R}^n$ be a point outside P, and F a facet of P whose supporting hyperplane is $a_1x_1 + \cdots + a_nx_n = a_0$. Then F is* visible *from $\boldsymbol{p}$ if $a_1p_1 + \cdots + a_np_n > a_0$.*

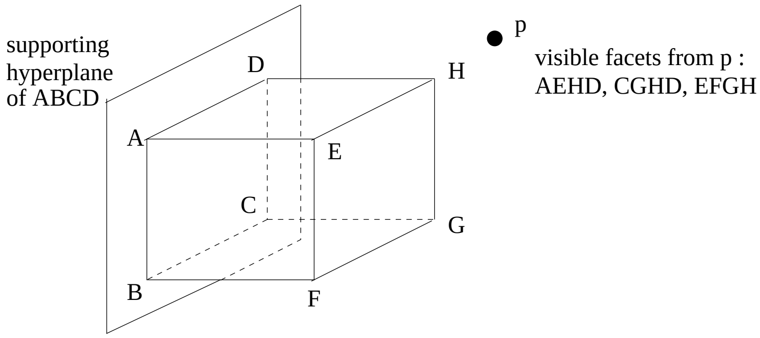

Figure 3.8: Supporting hyperplane and visible facets.

Proposition 3.45 ([10]) *Let $P \subset \mathbb{R}^n$ be a polytope and $\boldsymbol{p} \in \mathbb{R}^n$ a point outside P. Then,*

$$\mathrm{conv}(P \cup \{\boldsymbol{p}\}) = P \cup \{\mathrm{conv}(F \cup \{\boldsymbol{p}\}) \mid F \textit{ is visible from } \boldsymbol{p}\}.$$

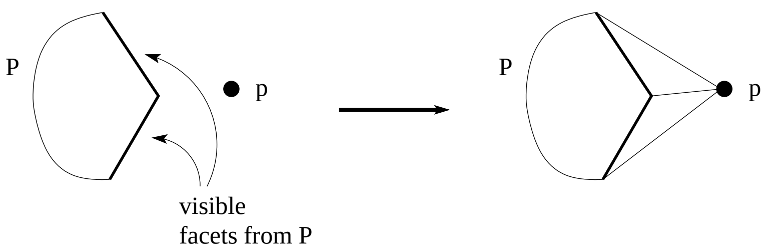

Figure 3.9: $\mathrm{conv}(P \cup \{\boldsymbol{p}\})$ is equal to the union of P and the set of simplices $\mathrm{conv}(F \cup \{\boldsymbol{p}\})$ that F is visible from $\boldsymbol{p}$.

Proof of Theorem 3.43: We calculate the normalized volume of P_n. We decompose P_n to $V_n := \mathrm{conv}(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n-1}, -\boldsymbol{e}_1, \ldots, -\boldsymbol{e}_{n-1})$ and its outside.

First, we show that the normalized volume of V_n is equal to 2^{n-1} by induction for n. Clearly, the normalized volume of V_2 is 2. As V_n is the convex hull of V_{n-1}, $\boldsymbol{e}_{n-1}$ and $-\boldsymbol{e}_{n-1}$, the normalized volume of V_n is twice that of $\mathrm{conv}(V_{n-1}, \boldsymbol{e}_{n-1})$. By the hypothesis of induction, the normalized volume of $\mathrm{conv}(V_{n-1}, \boldsymbol{e}_{n-1})$ is 2^{n-2}, and that of V_n is shown to be 2^{n-1}.

We next calculate the normalized volume of outside of V_n. As V_n is a $(n-1)$-dimensional *crosspolytope*, a hyperplane $a_1x_1 + \cdots + a_{n-1}x_{n-1} = a_0$ is a supporting hyperplane of some facet of V_n if and only if $a_0 = 1$ and $|a_i| = 1$ for any $i = 1, \ldots, n-1$ [85]. Let $F(a_1, \ldots, a_{n-1})$ be a facet of V_n whose supporting hyperplane is $a_1x_1 + \cdots + a_{n-1}x_{n-1} = 1$. Then, if $F(a_1, \ldots, a_{n-1})$ is visible from **1**, $F(-a_1, \ldots, -a_{n-1})$ is visible from $-\mathbf{1}$, and the sets of facets of V_n that are visible from **1** and that are visible from $-\mathbf{1}$ are disjoint. Therefore, by Proposition 3.45, we need consider only the facets that are visible from **1**.

$F(a_1, \ldots, a_{n-1})$ is visible from **1** if and only if $a_1 + \cdots + a_{n-1} > 1$. We compute the normalized volume of $\mathrm{conv}(F \cup \{\mathbf{1}\})$ for a facet F of P that is visible from **1**. As vertices of P_n lie on the lattice generated by $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n-1}$, the normalized volume of $\mathrm{conv}(F \cup \{\mathbf{1}\})$ is k if the Euclidean distance between **1** and the hyperplane of F is $k/\sqrt{n-1}$. Thus, for a facet $F(a_1, \ldots, a_{n-1})$ of V_n visible from **1**,

$$\begin{array}{l}\text{the normalized volume of}\\ \mathrm{conv}(F(a_1, \ldots, a_{n-1}) \cup \{\mathbf{1}\}) = p-1\end{array} \iff |\{i \mid a_i = 1\}| - |\{i \mid a_i = -1\}| = p.$$

Therefore, for $k = 0, \ldots, \lfloor \frac{n-2}{2} \rfloor$, the number of facets $F(a_1, \ldots, a_{n-1})$ such that the normalized volume of $\mathrm{conv}(F(a_1, \ldots, a_{n-1}) \cup \{\mathbf{1}\}) = n-2k-2$ is equal to $\binom{n-1}{k}$, and the normalized volume of P_n is equal to $2^{n-1} + 2\sum_{k=0}^{\lfloor (n-2)/2 \rfloor} \binom{n-1}{k}(n-2k-2) = (n-[n/2])\binom{n}{[n/2]}$. $\square$

3.7 Summary

In this section, we characterized standard pairs of unimodular integer programs, which are the most fundamental subclass of integer programs.

We presented an interpretation of the Hoçten-Thomas algorithm for unimodular integer programs in terms of the reduced cost of linear programs. In this case, the maximum arithmetic degree, which indicates the complexity of the Hoçten-Thomas algorithm, was given by the normalized volume of polytopes. This result gives a framework for computing the number of feasible bases via Gröbner bases and volume computations.

Next, by applying this algorithm to primal and dual minimum cost flow problems, we described the algebraic differences in complexity (i.e., arithmetic degrees) between primal and dual problems by giving the number of primal and dual feasible bases: for the number of dual feasible bases, arithmetic degrees range between 1 and the Catalan number, and for the number of primal feasible bases even the lower bound is of exponential order. We also gave computational algebraic proof for existing results on the numbers of primal [48] and dual [5] feasible bases of the transportation problem.The results presented in this section also indicated computational algebraic duality between Gröbner bases and standard pair decompositions. For network optimization problems, this duality corresponds to the relation between circuits and dual feasible co-trees, dually, cutsets and primal feasible trees. As this relation has not been clarified previously, the computational algebraic duality is of interest for the development of novel methods of analysis for network problems.

Several interesting open problems remain:

- Are the cardinalities of reduced Gröbner bases for primal and dual minimum cost flow problems of polynomial order with respect to the number of vertices?
- Is there any relation with arithmetic degree for a pair of primal and dual problems for fixed $\boldsymbol{b}$ and $\boldsymbol{c}$?

Chapter 4

Standard Pair Decompositions of Lawrence-Type Integer Programming Problems

4.1 Introduction

Lawrence-type integer programs, i.e., integer programs whose coefficient matrices are of the Lawrence type, arise in many types of combinatorial optimization problems. For example, the capacitated integer program

$$\mathit{minimize}\ \{\boldsymbol{c}\cdot\boldsymbol{x} \mid A\boldsymbol{x}=\boldsymbol{b},\ 0\leq\boldsymbol{x}\leq\boldsymbol{u}\}$$

is equivalent to the following program

$$\mathit{minimize}\ \left\{\boldsymbol{c}\cdot\boldsymbol{x}\ \middle|\ \Lambda(A)\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{s}\end{pmatrix}=\begin{pmatrix}\boldsymbol{b}\\ \boldsymbol{u}\end{pmatrix},\ \boldsymbol{x},\boldsymbol{s}\geq 0\right\}$$

with the introduction of slack variables $\boldsymbol{s}$. In multidimensional transportation problems [84], the coefficient matrices of certain types of problem can be replaced by Lawrence-type matrices. As mentioned in Section 3.1, for general transportation problems on bipartite graphs, the structures of feasible polyhedra, especially the number of vertices, have been studied for both primal [1, 11, 48, 58] and dual [4, 5] problems. On the other hand, there have been few studies concerning the structures of feasible polyhedra for multidimensional transportation problems [59, 84].

As the toric ideal I_A of a matrix A is defined via linear dependencies of column vectors of A, I_A (or $I_{\Lambda(A)}$) is related to the vector matroid $M[A]$ of A, or the oriented matroid defined by A. The Graver basis Gr_A of A contains the set of circuits $\mathcal{C}_A$, which corresponds to the set of circuits of $M[A]$, and, for any term order, there exists a bijection between Gr_A and the set of elements in the reduced Gröbner basis for $I_{\Lambda(A)}$ [71]. In particular, when A is unimodular, as $Gr_A = \mathcal{C}_A$, circuits of $M[A]$ and the reduced Gröbner basis for $I_{\Lambda(A)}$ are related bijectively. On the other hand, the relation between standard pairs and $M[A]$ has not been characterized in detail.

Lawrence-type integer programs also have interesting properties with regard to dual problems. The convex hull of column vectors of $\Lambda(A)$ is called a *Lawrence polytope*, which has been studied in detail in the oriented matroid theory [6, 7, 8, 61]. Lawrence polytopes have good properties in that their *Gale diagram*, another type of polytope that gives an oriented matroid duality for a polytope, is centrally symmetric. In terms of integer programs, this duality implies that coefficient matrices of dual problems of Lawrence-type integer programs are $(D, -D)$, which correspond to integer programs without non-negative constraints.

In this section, we describe the relation between standard pairs for toric ideals for Lawrence-type integer programs and vector matroids $M[A]$. We focus on standard pairs of type $(1, \sigma)$, which correspond to dual feasible bases. As an initial ideal of $I_{\Lambda(A)}$ has information about the set of circuits of $M[A]$, a set of standard pairs may be related to bases of $M[A]$. We show the bijection between the set of bases of $M[A]$ and the set of standard pairs of type $(1, \sigma)$ (Theorem 4.4). In addition, we show the matroidal structure of such standard pairs in the sense that an adjacency relation of standard pairs as facets in a regular triangulation corresponds to the adjacency of vertices in a base polyhedron [24] of $M[A]$, and each adjacency of vertices in the base polyhedron appears as adjacent facets for *some* regular triangulation (Proposition 4.6).

As a corollary of our results, the number of dual feasible bases for the capacitated minimum cost flow problem on acyclic tournament graphs with d vertices is shown to be d^{d-2} (Proposition 4.8), where for the uncapacitated minimum cost flow problem on the same graph the number is equal to or less than the Catalan number $\frac{1}{d}\binom{2(d-1)}{d-1}$, as mentioned in Section 3.4.2. In addition, we analyze the number of dual feasible bases for the multidimensional transportation problem of type $2 \times \cdots \times 2 \times M \times N$. This number also gives the number of vertices of the feasible polyhedron for the dual problem of the multidimensional transportation problem. We show that the number of dual feasible bases for s-dimensional transportation problems of type $2 \times \cdots \times 2 \times M \times N$ becomes $M^{N-1} N^{M-1} \left(2^{(M-1)(N-1)}\right)^{s-3}$ (Theorem 4.14), using the relation between bases for a matrix and those for its Lawrence lifting.

We also study standard pairs for dual Lawrence-type integer programs. As mentioned above, column vectors of the coefficient matrix of a dual Lawrence-type integer program are centrally symmetrical. Based on these results and those of several previous reports in oriented matroid theory [6, 8, 61], we show that the maximum number of primal feasible bases for the multidimensional transportation problem of type $2 \times \cdots \times 2 \times 2 \times N$ becomes $(N - [N/2])\binom{N}{[N/2]}$ (Theorem 4.18), which is same as general transportation problem on $K_{2,N}$.

The remainder of this chapter is organized as follows. Section 4.2 presents a brief review of some definitions and related studies concerning Lawrence-type matrices. In Section 4.3.1, we give the relation between standard pairs for primal Lawrence-type integer programs and vector matroids $M[A]$. Sections 4.3.2 and 4.3.3 discuss their applications to minimum cost flow problems with upper bound constraints and to multidimensional transportation problems of type $2 \times \cdots \times 2 \times M \times N$, respectively, and the number of their dual feasible bases are discussed. In Section 4.4.1, we describe the relation between standard pairs for dual Lawrence-type integer programs and vector matroids $M[A]$. Then, we present the maximum number of primal feasible bases for multidimensional transportation problems of type $2 \times \cdots \times 2 \times 2 \times N$ in Section 4.4.2. Finally, this chapter is summarized in Section 4.5.

Lawrence-type matrices, or *r-th Lawrence lifting* [62] is also used in statistical analyses as a multi-way contingency table [3, 20, 62]. The problem of counting the number of contingency tables for fixed marginal sums was shown to be #P-complete for two-dimensional [22] and three-dimensional [18] tables. On the other hand, polynomial-time Markov Chain Monte Carlo methods for sampling tables of type $2 \times N$ [21] and type

$2 \times \cdots \times 2 \times N$ [51] have also been studied. Section 4.3.3 describes analysis of both cases of type $2 \times \cdots \times 2 \times 2 \times N$ and $2 \times \cdots \times 2 \times M \times N$ ($M \geq 3$). In Section 4.4.2 we describe analysis of cases of type $2 \times \cdots \times 2 \times 2 \times N$. Therefore, this section describes the difficulties in analyzing statistical and combinatorial problems concerning multi-way contingency tables from the viewpoint of dualistic computational algebraic methods.

This section describes work performed jointly with Hiroshi Imai [42, 43].

Table 4.1: Dual algebraic approaches for primal and dual multidimensional transportation problems of several types.

	$2 \times \cdots \times 2 \times 2 \times N$	$2 \times \cdots \times 2 \times M \times N$
Primal	$N \cdot 2^{(N-1)(s-2)}$	$M^{N-1}N^{M-1}2^{(s-3)(M-1)(N-1)}$
Dual	$\max:$ $(N-[N/2])\binom{N}{[N/2]}$	?

Table 4.2: Vector matroids $M[A]$ of A and computational algebraic methods for $I_{\Lambda(A)}$. Each inclusion in the second column becomes equal if A is unimodular.

Gröbner basis $\mathcal{G}$ for $I_{\Lambda(A)}$	$\supseteq$	subset of $\mathcal{G}$	$\longleftrightarrow$	set of circuits of $M[A]$
$\updownarrow$				$\updownarrow$
standard pairs for $I_{\Lambda(A)}$	$\supseteq$	standard pairs corresponding to dual feasible bases	$\longleftrightarrow$	set of bases of $M[A]$

4.2 Lawrence-type Integer Programs: Definitions and Properties

In this section, we briefly review some basic definitions and related work concerning Lawrence-type integer programs. We refer to [71] for further details about Lawrence-type integer programs and their computational algebraic properties, and [6, 8, 61, 68, 85] for details of their relations to the theory of convex polytopes and oriented matroids.

Definition 4.1 *A matrix of the form* $\Lambda(A) := \begin{pmatrix} A & O \\ I & I \end{pmatrix} \in \mathbb{Z}^{(d+n)\times 2n}$, *where* $A \in \mathbb{Z}^{d\times n}$, I *is the* $n \times n$ *identity matrix and* O *is the* $d \times n$ *zero matrix is said to be of* Lawrence-type, *or to be the* Lawrence lifting *of* A.

As mentioned in Section 1.3, Lawrence-type matrices were originally proposed by Lawrence in 1980 from the viewpoint of oriented matroid theory.

The matrices of A and $\Lambda(A)$ have isomorphic kernels: $\ker(\Lambda(A)) = \{(\boldsymbol{u}, -\boldsymbol{u}) \mid \boldsymbol{u} \in \ker(A)\}$. Therefore the toric ideal of $\Lambda(A)$ is $I_{\Lambda(A)} = \langle \boldsymbol{x}^{\boldsymbol{u}^+}\boldsymbol{y}^{\boldsymbol{u}^-} - \boldsymbol{x}^{\boldsymbol{u}^-}\boldsymbol{y}^{\boldsymbol{u}^+} \mid \boldsymbol{u} \in \ker(A)\rangle$ in the polynomial ring $k[x_1, \ldots, x_n, y_1, \ldots, y_n]$ [71]. For a binomial $f = \boldsymbol{x}^{\boldsymbol{p}}\boldsymbol{y}^{\boldsymbol{q}} - \boldsymbol{x}^{\boldsymbol{r}}\boldsymbol{y}^{\boldsymbol{s}} \in I_{\Lambda(A)}$, we can rewrite f to $f = \boldsymbol{x}^{\boldsymbol{v}}\boldsymbol{y}^{\boldsymbol{w}}(\boldsymbol{x}^{\boldsymbol{u}^+}\boldsymbol{y}^{\boldsymbol{u}^-} - \boldsymbol{x}^{\boldsymbol{u}^-}\boldsymbol{y}^{\boldsymbol{u}^+})$, where $v_i = \min(p_i, r_i)$, $w_i = \min(q_i, s_i)$ and $\boldsymbol{u} = \boldsymbol{p} - \boldsymbol{r}$.

Lemma 4.2 ([71]) *For a Lawrence-type matrix* $\Lambda(A)$, *its Graver basis is* $Gr_{\Lambda(A)} = \{\boldsymbol{x}^{\boldsymbol{u}^+}\boldsymbol{y}^{\boldsymbol{u}^-} - \boldsymbol{x}^{\boldsymbol{u}^-}\boldsymbol{y}^{\boldsymbol{u}^+} \mid \boldsymbol{x}^{\boldsymbol{u}^+} - \boldsymbol{x}^{\boldsymbol{u}^-} \in Gr_A\}$. *Moreover, the following sets of binomials coincide:*

(i) *the Graver basis of* $\Lambda(A)$,

(ii) *the universal Gröbner basis of* $\Lambda(A)$,

(iii) *any reduced Gröbner basis of* $I_{\Lambda(A)}$,

(iv) *any minimal generating set of* $I_{\Lambda(A)}$ (*up to scalar multiples*).

Lawrence-type integer programs arise in many situations in combinatorial optimization. For example, the capacitated integer program

$$minimize\ \{\boldsymbol{c} \cdot \boldsymbol{x} \mid A\boldsymbol{x} = \boldsymbol{b},\ 0 \le \boldsymbol{x} \le \boldsymbol{u}\}$$

is equivalent to the problem

$$minimize\ \left\{\boldsymbol{c} \cdot \boldsymbol{x}\ \middle|\ \Lambda(A)\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{s}\end{pmatrix} = \begin{pmatrix}\boldsymbol{b}\\ \boldsymbol{u}\end{pmatrix},\ \begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{s}\end{pmatrix} \ge 0\right\}$$

with introduction of slack variables, $\boldsymbol{s}$, to transform all capacity constraints to equality constraints.

4.3 Characterizations of Standard Pair Decompositions of Lawrence-Type Matrices

This section describes the relation between standard pairs for primal Lawrence-type integer programs and vector matroids. Their application to several Lawrence-type integer programs — capacitated minimum cost flow problems and certain multidimensional transportation problems — is also discussed.

4.3.1 Vector Matroids and Standard Pair Decompositions of Lawrence-Type Matrices

In the rest of this chapter, a matrix A is row-full rank, and vector matroids are considered over the field $\mathbb{R}$ of real numbers. For fundamental definitions of matroids, see Appendix A.

Let $M[A]$ be the vector matroid of A, and $\mathcal{B}(M[A])$ (resp. $\mathcal{C}(M[A])$) the set of bases (resp. circuits) of $M[A]$, i.e., $\mathcal{B}(M[A])$ is the family of a set of indices corresponding to maximal independent column vectors, and $\mathcal{C}(M[A])$ is the family of a set of indices corresponding to minimal dependent column vectors. We assume that $M[A]$ does not have a loop (i.e., A does not have a column zero vector). Any circuit $C \in \mathcal{C}(M[A])$ corresponds to a binomial $\boldsymbol{x}^{\boldsymbol{u}^+} - \boldsymbol{x}^{\boldsymbol{u}^-} \in \mathcal{C}_A$ via the minimal linear dependence $\sum_{i \in C} u_i \boldsymbol{a}_i = 0$, and we define $f_C := \boldsymbol{x}^{\boldsymbol{u}^+}\boldsymbol{y}^{\boldsymbol{u}^-} - \boldsymbol{x}^{\boldsymbol{u}^-}\boldsymbol{y}^{\boldsymbol{u}^+} \in I_{\Lambda(A)}$. For any $B \in \mathcal{B}(M[A])$ and $i \notin B$, $B \cup \{i\}$ contains a unique circuit $C_i \in \mathcal{C}(M[A])$, which is the fundamental circuit of i with respect to B, and f_{C_i} is denoted by f_i^B.

Let $[n, \overline{n}] := \{1, \ldots, n, \overline{1}, \ldots, \overline{n}\}$ be the index set of $k[x_1, \ldots, x_n, y_1, \ldots, y_n]$, where i is the index of x_i and $\overline{i}$ is the index of y_i, and fix a generic cost vector $\boldsymbol{c} = (c_1, \ldots, c_n, c_{\overline{1}}, \ldots, c_{\overline{n}})$. For each basis $B \in \mathcal{B}(M[A])$, we define $\sigma_B \subset [n, \overline{n}]$ as follows:

1. If $i \in B$, then $i, \overline{i} \in \sigma_B$, and

2. if $j \notin B$, we set the minimal linear dependence on C_j as $\sum_{i \in C_j} u_i \boldsymbol{a}_i = 0$ with $u_j \geq 0$, and

$$\begin{cases} \overline{j} \in \sigma_B,\ j \notin \sigma_B & (\text{if } in_{\boldsymbol{c}}(f_j^B) \text{ contains } x_j) \\ j \in \sigma_B,\ \overline{j} \notin \sigma_B & (\text{if } in_{\boldsymbol{c}}(f_j^B) \text{ contains } y_j). \end{cases}$$

Example 4.3 *Let* $A = (1\ 2\ 4)$. *Then,* $\mathcal{C}_A = \{x_1^2 - x_2,\ x_1^4 - x_3,\ x_2^2 - x_3\}$, $Gr_A = \mathcal{C}_A \cup \{x_3 - x_1^2 x_2\}$ *and* $\mathcal{B}(M[A]) = \{\{1\},\ \{2\},\ \{3\}\}$. *For* $\boldsymbol{c} = (2, 2, 1, 1, 1, 1)$ *and* $B := B_2 = \{2\}$, $in_{\boldsymbol{c}}(f_1^B) = x_1^2 y_2$ *and* $in_{\boldsymbol{c}}(f_3^B) = x_2^2 y_3$, *and therefore,* $\sigma_{B_2} = \{2, 3, \overline{1}, \overline{2}\}$. *Similarly, for* $B_1 = \{1\}$ *and* $B_3 = \{3\}$, $\sigma_{B_1} = \{1, 2, 3, \overline{1}\}$ *and* $\sigma_{B_3} = \{3, \overline{1}, \overline{2}, \overline{3}\}$. □

Theorem 4.4 $(1, \sigma)$ *is a standard pair of* $in_{\boldsymbol{c}}(I_{\Lambda(A)})$ *if and only if there exists a basis* $B \in \mathcal{B}(M[A])$ *such that* $\sigma = \sigma_B$.

Proof:

(if part) Suppose that $in_{\boldsymbol{c}}(f) = \boldsymbol{x}^{\boldsymbol{u}^+} \boldsymbol{y}^{\boldsymbol{u}^-} \in (1, \sigma_B)$ for some $f = \boldsymbol{x}^{\boldsymbol{u}^+} \boldsymbol{y}^{\boldsymbol{u}^-} - \boldsymbol{x}^{\boldsymbol{u}^-} \boldsymbol{y}^{\boldsymbol{u}^+} \in I_{\Lambda(A)}$. Then, there exist some variables in $\boldsymbol{x}^{\boldsymbol{u}^-} \boldsymbol{y}^{\boldsymbol{u}^+}$ that are not in σ_B. Let x_i be such variable (the case in which such a variable is y_j is analogous). $f \in I_{\Lambda(A)}$ implies that $\boldsymbol{x}^{k\boldsymbol{u}^+} \boldsymbol{y}^{k\boldsymbol{u}^-} - \boldsymbol{x}^{k\boldsymbol{u}^-} \boldsymbol{y}^{k\boldsymbol{u}^+} \in I_{\Lambda(A)}$ for any $k \in \mathbb{N} \setminus \{0\}$ and its initial term is $\boldsymbol{x}^{k\boldsymbol{u}^+} \boldsymbol{y}^{k\boldsymbol{u}^-}$. We take $m_i \in (1, \sigma_B)$ and $k \in \mathbb{N} \setminus \{0\}$ such that the exponent of x_i in f_i^B equals $-ku_i (> 0)$ and $in_{\boldsymbol{c}}(f_i^B)$ divides $m_i \boldsymbol{x}^{k\boldsymbol{u}^-} \boldsymbol{y}^{k\boldsymbol{u}^+}$. Then, by reducing $m_i(\boldsymbol{x}^{k\boldsymbol{u}^+} \boldsymbol{y}^{k\boldsymbol{u}^-} - \boldsymbol{x}^{k\boldsymbol{u}^-} \boldsymbol{y}^{k\boldsymbol{u}^+})$ by f_i^B, we obtain the binomial of the form $m(\boldsymbol{x}^{\boldsymbol{v}^+} \boldsymbol{y}^{\boldsymbol{v}^-} - \boldsymbol{x}^{\boldsymbol{v}^-} \boldsymbol{y}^{\boldsymbol{v}^+})$ for some monomial $m \in (1, \sigma_B)$ and $\boldsymbol{v} \in \mathbb{Z}^n$. Let $b_1 := \boldsymbol{x}^{\boldsymbol{v}^+} \boldsymbol{y}^{\boldsymbol{v}^-} - \boldsymbol{x}^{\boldsymbol{v}^-} \boldsymbol{y}^{\boldsymbol{v}^+}$. Then, $in_{\boldsymbol{c}}(b_1) = \boldsymbol{x}^{\boldsymbol{v}^+} \boldsymbol{y}^{\boldsymbol{v}^-} \in (1, \sigma_B)$, and the number of variables in b_1 that are not in σ_B is one less than that in f.

Repeating the above procedure as long as possible yields a sequence of binomials, $f \to b_1 \to b_2 \to \cdots \to b_{s-1} \to b_s$. Then, $b_s = 0$ (otherwise $b_s \in I_{\Lambda(A)}$ only contains the variables in B, which contradicts $B \in \mathcal{B}(M[A])$). Let $b_{s-1} = \boldsymbol{x}^{\boldsymbol{w}^+} \boldsymbol{y}^{\boldsymbol{w}^-} - \boldsymbol{x}^{\boldsymbol{w}^-} \boldsymbol{y}^{\boldsymbol{w}^+}$ with $in_{\boldsymbol{c}}(b_{s-1}) = \boldsymbol{x}^{\boldsymbol{w}^+} \boldsymbol{y}^{\boldsymbol{w}^-}$. If the last reduction is done by f_j^B, then $\boldsymbol{x}^{k\boldsymbol{w}^+} \boldsymbol{y}^{k\boldsymbol{w}^-} - \boldsymbol{x}^{k\boldsymbol{w}^-} \boldsymbol{y}^{k\boldsymbol{w}^+} = -f_j^B$ for some $k \in \mathbb{N} \setminus \{0\}$ and its initial term is $\boldsymbol{x}^{k\boldsymbol{w}^+} \boldsymbol{y}^{k\boldsymbol{w}^-}$. However, as $in_{\boldsymbol{c}}(f_j^B) = \boldsymbol{x}^{k\boldsymbol{w}^-} \boldsymbol{y}^{k\boldsymbol{w}^+}$, this is a contradiction.

(only if part) Let $(1, \sigma)$ be a standard pair of $in_{\boldsymbol{c}}(I_{\Lambda(A)})$ and $B \subseteq \{1, \dots, n\}$ be the set of indices such that $i, \bar{i} \in \sigma$. Then, $|B| \geq r$ (r is the rank of $M[A]$) as $|\sigma| = n + r$. By Lemma 2.23 (ii), $(1, \sigma)$ is a standard pair of $in_{\boldsymbol{c}}(I_{\Lambda(A)})$ only if there is no $f \in I_{\Lambda(A)}$ such that $\sigma = \mathrm{supp}(in_{\boldsymbol{c}}(f))$ [71]. Therefore, any monomial in $(1, \sigma)$ is not divisible by $in_{\boldsymbol{c}}(f_C)$ for any $C \in \mathcal{C}(M[A])$, which implies that B does not contain any circuit of $M[A]$. Hence, $B \in \mathcal{B}(M[A])$. In addition, for any $j \notin B$, $j \notin \sigma$ and $\overline{j} \in \sigma$ if $in_{\boldsymbol{c}}(f_j^B)$ contains x_j, and $j \in \sigma$ and $\overline{j} \notin \sigma$ otherwise. Therefore, $\sigma = \sigma_B$.

$\square$

Example 4.3 (continued.) *The set of standard pairs for* $in_{\boldsymbol{c}}(I_{\Lambda(A)})$ *consists of* $(1, \sigma_{B_1})$, $(1, \sigma_{B_2})$, $(1, \sigma_{B_3})$ *and six other pairs* (x_1, σ_{B_2}), (x_1, σ_{B_3}), $(x_1 x_2, \sigma_{B_3})$, (x_2, σ_{B_3}), $(x_1^2, \{3, \overline{1}, \overline{3}\})$, $(x_1^3, \{3, \overline{1}, \overline{3}\})$. $\square$

Corollary 4.5 *If A is unimodular, for any cost vector* $\boldsymbol{c}$ *the arithmetic degree of* $in_{\boldsymbol{c}}(I_{\Lambda(A)})$ *equals the cardinality of* $\mathcal{B}(M[A])$.

Proof: If A is unimodular, then $\Lambda(A)$ is also unimodular. Then, for any term order $\boldsymbol{c}$ all standard pairs of $in_{\boldsymbol{c}}(I_{\Lambda(A)})$ are of type $(1, \sigma)$, and the above theorem shows there is a one-to-one correspondence between $\mathcal{B}(M[A])$ and the standard pairs of type $(1, \sigma)$ of $in_{\boldsymbol{c}}(I_{\Lambda(A)})$. $\square$

Let $M = (E, \rho)$ be a matroid on E with a rank function ρ. The *matroid polyhedron* $R(\rho)$ and the *base polyhedron* $B(\rho)$ of M are determined by

$$R(\rho) := \left\{ \boldsymbol{x} \in \mathbb{R}^E \mid \textstyle\sum_{i \in X} x_i \leq \rho(X) \ ({}^{\forall} X \subseteq 2^E) \right\}$$
$$B(\rho) := \left\{ \boldsymbol{x} \in R(\rho) \mid \textstyle\sum_{i \in E} x_i = \rho(E) \right\}.$$

Then, each vertex of $B(\rho)$ corresponds to a basis of M, and two vertices corresponding to bases B_1 and B_2 are adjacent if and only if there exist $e_1 \in B_1 \setminus B_2$ and $e_2 \in B_2 \setminus B_1$ such that $B_2 = (B_1 \setminus \{e_1\}) \cup \{e_2\}$ [24].

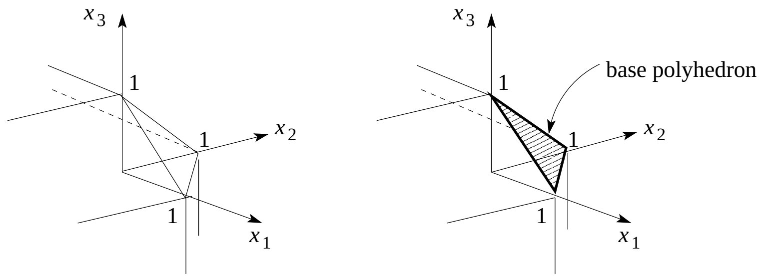

Figure 4.1: Matroid polyhedron (left) and Base polyhedron (right) for Example 4.3.

Example 4.3 (continued.) *For the matroid $M[A]$, $R(\rho)$ and $B(\rho)$ are as drawn in Figure 4.1. Each vertex $\boldsymbol{e}_i$ corresponds to a basis $\{i\}$ of $M[A]$.* □

Let $\Delta_{\boldsymbol{c}}$ be the regular triangulation of $\mathrm{cone}(\Lambda(A))$ with respect to a cost vector $\boldsymbol{c}$. The set of standard pairs of $in_{\boldsymbol{c}}(I_{\Lambda(A)})$ of type $(1, \sigma)$ has a matroidal structure in the sense that the adjacency of facets in $\Delta_{\boldsymbol{c}}$ corresponds to that of vertices of $B(\rho)$, and conversely any adjacency of vertices of $B(\rho)$ appears as that of facets in $\Delta_{\boldsymbol{c}}$ for some $\boldsymbol{c}$.

Proposition 4.6

(i) *Suppose that $(1, \sigma_{B_1})$ and $(1, \sigma_{B_2})$ are standard pairs of $in_{\boldsymbol{c}}(I_{\Lambda(A)})$. If σ_{B_1} and σ_{B_2} share a facet in $\Delta_{\boldsymbol{c}}$, then vertices of the base polyhedron $B(\rho)$ that correspond to B_1 and B_2 are adjacent.*

(ii) *For any two bases B_1 and B_2 such that corresponding vertices are adjacent in $B(\rho)$, there exists a cost vector $\mathbf{c}$ such that σ_{B_1} and σ_{B_2} share a facet in $\Delta_{\boldsymbol{c}}$.*

Proof: (i): σ_{B_1} and σ_{B_2} share a facet in $\Delta_{\boldsymbol{c}}$ if and only if there exist $i \in \sigma_{B_1} \setminus \sigma_{B_2}$ and $j \in \sigma_{B_2} \setminus \sigma_{B_1}$ such that $\sigma_{B_2} = \sigma_{B_1} \setminus \{k\} \cup \{l\}$ ($k = i$ or $\bar{i}$, $l = j$ or $\bar{j}$). As for the definition of σ_B, $i \neq j$, and $B_2 = B_1 \setminus \{i\} \cup \{j\}$. Therefore, two vertices corresponding to B_1 and B_2 are adjacent in $B(\rho)$.

(ii): Suppose that $B_2 = B_1 \setminus \{i\} \cup \{j\}$ where $i \in B_1 \setminus B_2$ and $j \in B_2 \setminus B_1$. Let $\boldsymbol{c}$ be a generic cost vector obtained by perturbing $\boldsymbol{c}'$, such as

$$c'_k = \begin{cases} 1 & \text{if } k = i,\ j, \text{ or } k \notin B_1 \cup B_2 \\ 0 & \text{otherwise} \end{cases} \qquad c'_{\overline{k}} = \begin{cases} 1 & \text{if } k = i,\ j \\ 0 & \text{otherwise} \end{cases} \qquad (1 \leq k \leq n).$$

Then, $\overline{e} \in \sigma_{B_1} \cap \sigma_{B_2}$ and $e \notin \sigma_{B_1} \cup \sigma_{B_2}$ for any $e \notin B_1 \cup B_2$. Furthermore, either j or $\overline{j}$ is an element of σ_{B_1} and $i, \overline{i} \in \sigma_{B_1}$, and similarly either i or $\overline{i}$ is an element of σ_{B_2} and $j, \overline{j} \in \sigma_{B_2}$. Therefore, $|\sigma_{B_1} \setminus \sigma_{B_2}| = |\sigma_{B_2} \setminus \sigma_{B_1}| = 1$, which implies that σ_{B_1} and σ_{B_2} share a facet in $\Delta_{\boldsymbol{c}}$. □

4.3.2 Application to Minimum Cost Flow Problems with Upper Bound Constraints

When A is the incidence matrix of a directed graph G, $M[A]$ is a graphic matroid, and $\mathcal{B}(M[A])$ is the set of spanning trees of G. In this case, $IP_{\Lambda(A),(\boldsymbol{c},0,\dots,0)}(\boldsymbol{b})$ (in this subsection, we assume that $\boldsymbol{c} \in \mathbb{R}^n$) is equivalent to a capacitated minimum cost flow problem on G:

$$minimize\ \{\boldsymbol{c} \cdot \boldsymbol{x} \mid A\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{0} \leq \boldsymbol{x} \leq \boldsymbol{u}\}.$$

As A is unimodular, $\Lambda(A)$ is also unimodular and all standard pairs of $in_{\boldsymbol{c}}(I_{\Lambda(A)})$ are of type $(1, \sigma)$. There are several algorithms to calculate the number of all spanning trees of undirected graphs in polynomial time (e.g., see [82]). Thus the arithmetic degree of $in_{\boldsymbol{c}}(I_{\Lambda(A)})$ can be calculated in polynomial time.

Proposition 4.7 *Let* $\boldsymbol{c}$ *be a cost vector. If* A *is the incidence matrix of a directed graph, then the arithmetic degree of* $in_{\boldsymbol{c}}(I_{\Lambda(A)})$ *can be calculated in polynomial time.*

If G is a tournament graph with d vertices, the number of spanning trees is equal to d^{d-2}, which implies the following proposition.

Proposition 4.8 *If* A *is the incidence matrix of a tournament graph with* d *vertices, then the arithmetic degree of* $in_{\boldsymbol{c}}(I_{\Lambda(A)})$ *is* d^{d-2} *for any generic cost vector* $\boldsymbol{c}$*.*

If G is the acyclic tournament graph with d vertices, the arithmetic degree of $in_{\boldsymbol{c}}(I_A)$ is less than the $(d-1)$-th Catalan number $\frac{1}{d}\binom{2(d-1)}{d-1} = O(4^d)$ (Theorem 3.30). Therefore, for the minimum cost flow problem on G, the number of dual feasible bases for a capacitated problem is much larger than that for an uncapacitated problem.

4.3.3 Application to Multidimensional Transportation Problems of Type $2 \times \cdots \times 2 \times M \times N$

Let $M_1, M_2, \ldots, M_s \in \mathbb{N}$. A *multidimensional transportation problem* (*MTP*) [84, 59] of type $M_1 \times M_2 \times \cdots \times M_s$ is the following problem:

$$\begin{aligned}
\min \quad & \sum_{i_1=1}^{M_1} \sum_{i_2=1}^{M_2} \cdots \sum_{i_s=1}^{M_s} c_{i_1 i_2 \cdots i_s} x_{i_1 i_2 \cdots i_s} \\
\text{s.t.} \quad & \sum_{i_1=1}^{M_1} x_{i_1 i_2 \cdots i_s} = p_1(i_2, i_3, \cdots, i_s) \quad ({}^{\forall} i_2, i_3, \ldots, i_s) \\
& \sum_{i_2=1}^{M_2} x_{i_1 i_2 \cdots i_s} = p_2(i_1, i_3, \cdots, i_s) \quad ({}^{\forall} i_1, i_3, \ldots, i_s) \\
& \qquad \vdots \\
& \sum_{i_s=1}^{M_s} x_{i_1 i_2 \cdots i_s} = p_s(i_1, i_2, \cdots, i_{s-1}) \quad ({}^{\forall} i_1, i_2, \ldots, i_{s-1}) \\
& x_{i_1 i_2 \ldots i_s} \geq 0,
\end{aligned} \tag{4.1}$$

where $p_j(i_1 \ldots, i_{j-1}, i_{j+1}, \ldots, i_s)$'s are non-negative and satisfy

$\sum_{i_j=1}^{M_j} p_k(i_1, i_2, \cdots, i_{k-1}, i_{k+1}, \cdots, i_s) = \sum_{i_k=1}^{M_k} p_j(i_1, i_2, \cdots, i_{j-1}, i_{j+1}, \cdots, i_s)$
for any $1 \leq j < k \leq s$.

Example 4.9 *Consider the MTP of type $2 \times 2 \times 3$ for transportation of goods for two days between two factories and three shops. Let $x_{i_1 i_2 i_3}$ $(i_1, i_2 \in \{1, 2\}, i_3 \in \{1, 2, 3\})$ be the flow of an edge (i_2, i_3) on the i_1-th day, and $c_{i_1 i_2 i_3}$ its cost. Supply, demand, and cost values vary for the first and second days. For each edge, the total amount of goods that pass the edge in two days is also given by* (*as $p_1(i_2, i_3)$ for an edge (i_2, i_3)*). *Then, the*

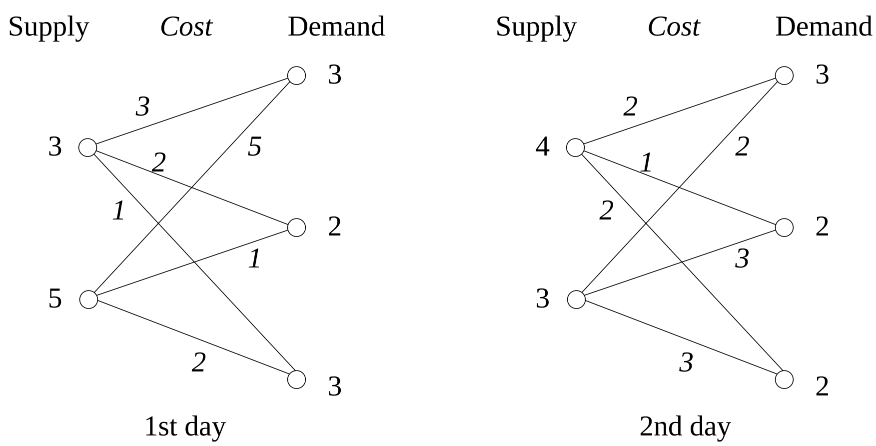

Figure 4.2: Multidimensional transportation problem of the type $2 \times 2 \times 3$.

MTP is to find the flow $x = (x_{i_1 i_2 i_3})$ *of goods that minimizes the total cost:*

$$\begin{aligned}
\min \quad & \sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \sum_{i_s=1}^{3} c_{i_1 i_2 i_3} x_{i_1 i_2 i_3} \\
\textit{s.t.} \quad & \sum_{i_1=1}^{2} x_{i_1 i_2 i_3} = p_1(i_2, i_3) \quad ({}^{\forall} i_2, i_3) \\
& \sum_{i_2=1}^{2} x_{i_1 i_2 i_3} = p_2(i_1, i_3) \quad ({}^{\forall} i_1, i_3) \\
& \sum_{i_3=1}^{3} x_{i_1 i_2 i_3} = p_3(i_1, i_2) \quad ({}^{\forall} i_1, i_2) \\
& x_{i_1 i_2 i_3} \geq 0.
\end{aligned}$$

□

The coefficient matrix of the MTP of type $M_1 \times M_2 \times \cdots \times M_s$ is also used in statistics for sampling or counting the multi-way contingency table of the same size for fixed marginal sums [3, 19, 20, 62]. The problem of counting the number of contingency tables for fixed marginal sums was shown to be #P-complete for two-dimensional [22] and three-dimensional [18] tables. On the other hand, for tables of types $2 \times N$ [21] and $2 \times \cdots \times 2 \times N$ [51], polynomial-time Markov Chain Monte Carlo methods have been studied. Therefore, matrices for MTPs of type $2 \times \cdots \times 2 \times N$ may have some good properties.

Sturmfels [71] showed that the MTP of type $2 \times M_2 \times M_3$ can be reformulated using the Lawrence lifting of the incidence matrix of the bipartite graph K_{M_2,M_3}. We can naturally extend this reformulation for the MTP of type $2 \times M_2 \times \cdots \times M_s$.

Proposition 4.10 *The MTP* (4.1) *with* $M_1 = 2$ *is equivalent to the following problem:*

$$\begin{array}{ll} \min & \displaystyle\sum_{i_1=1}^{2}\sum_{i_2=1}^{M_2}\cdots\sum_{i_s=1}^{M_s} c_{i_1 i_2 \cdots i_s} x_{i_1 i_2 \cdots i_s} \\ \textit{s.t.} & \displaystyle\sum_{i_1=1}^{2} x_{i_1 i_2 \cdots i_s} = p_1(i_2, i_3, \cdots, i_s) \quad ({}^{\forall} i_2, i_3, \ldots, i_s) \\ & \displaystyle\sum_{i_2=1}^{M_2} x_{1 i_2 \cdots i_s} = p_2(1, i_3, \cdots, i_s) \quad ({}^{\forall} i_3, \ldots, i_s) \\ & \quad\vdots \\ & \displaystyle\sum_{i_s=1}^{M_s} x_{1 i_2 \cdots i_s} = p_s(1, i_2, \cdots, i_{s-1}) \quad ({}^{\forall} i_2, \ldots, i_{s-1}) \\ & x_{i_1 i_2 \ldots i_s} \geq 0. \end{array} \tag{4.2}$$

We remark that the coefficient matrix of (4.2) is the Lawrence lifting of that for the MTP of the type $M_2 \times \cdots \times M_s$.

Proof: As any feasible solution of (4.1) clearly satisfies (4.2), we only need to show that any feasible solution of (4.2) satisfies (4.1).

Let $\boldsymbol{u} = (u_{i_1 i_2 \ldots i_s})$ be a feasible solution of (4.2). Then, $u_{2 i_2 \cdots i_s} = p_1(i_2, i_3, \cdots, i_s) - u_{1 i_2 \cdots i_s}$ $({}^{\forall} i_2, \ldots, i_s)$, and therefore, for any $j \geq 2$,

$$\begin{aligned} \sum_{i_j=1}^{M_j} u_{2 i_2 \cdots i_s} &= \sum_{i_j=1}^{M_j} \left(p_1(i_2, i_3, \cdots, i_s) - u_{1 i_2 \cdots i_s}\right) \\ &= \sum_{i_1=1}^{2} p_j(i_1, i_2, \cdots, i_{j-1}, i_{j+1}, \cdots, i_s) - p_j(1, i_2, \cdots, i_{j-1}, i_{j+1}, \cdots, i_s) \\ &= p_j(2, i_2, \cdots, i_{j-1}, i_{j+1}, \cdots, i_s), \end{aligned}$$

which implies that $\boldsymbol{u}$ is feasible for (4.1). □

Sturmfels [71] showed that the coefficient matrix of the three-dimensional MTP of type $K \times M \times N$ is unimodular if and only if $K = 2$. Actually, several computational results [9, 71] have shown that a certain reduced Gröbner basis is not square-free, and this induces non-unimodularity. As Lawrence lifting preserves unimodularity, this implies that the coefficient matrix of the (more than three-dimensional) MTP is unimodular if and only if it is of type $2 \times 2 \times \cdots \times 2 \times M \times N$.

We apply the results described in Section 4.3.1 to MTPs of type $2 \times M_2 \times \cdots \times M_s$. In particular, MTPs of the type $2 \times 2 \times \cdots \times 2 \times M_{s-1} \times M_s$ can be analyzed via the incidence matrix $A_{M_{s-1}M_s}$ of the bipartite graph $K_{M_{s-1}M_s}$ by repeating Lawrence lifting $s-2$ times (Table 4.3).

Table 4.3: The number of dual feasible bases of MTPs and that of bases of vector matroids. $A_{M,N}$ is the incidence matrix of the bipartite graph $K_{M,N}$. $M := M_{s-1}$, $N := M_s$ and $\Lambda^k(A) := \Lambda(\Lambda^{k-1}(A))$.

Matrix A	Number of dual feasible bases of $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$	Number of bases of $M[A]$
$A_0 := A_{M,N}$	$\binom{M+N-2}{M-1}$ [5]	X_1
$\Lambda(A_0)$	X_1	X_2
$\Lambda^2(A_0)$	X_2	X_3
$\vdots$	$\vdots$	$\vdots$
$\Lambda^{s-2}(A_0)$	X_{s-2}	X_{s-1}

Example 4.8 (continued.) *This MTP can be rewritten as*

$$\begin{array}{ll} \min & \sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\sum_{i_s=1}^{3} c_{i_1i_2i_3}x_{i_1i_2i_3} \\ \textit{s.t.} & \sum_{i_1=1}^{2} x_{i_1i_2i_3} = p_1(i_2,i_3) \quad ({}^{\forall}i_2,i_3) \\ & \sum_{i_2=1}^{2} x_{1i_2i_3} = p_2(1,i_3) \quad ({}^{\forall}i_1,i_3) \\ & \sum_{i_3=1}^{3} x_{1i_2i_3} = p_3(1,i_2) \quad ({}^{\forall}i_1,i_2) \\ & x_{i_1i_2i_3} \geq 0. \end{array}$$

We remark that the coefficient matrix is the Lawrence lifting of that for the general transportation problem on $K_{2,3}$. □

Next the relation between X_{k-1} and X_k ($k = 1, 2, \dots$) in Table 4.3 is shown.

Definition 4.11 *Let B be a subset of $[n]$, and $F \subseteq B$. We denote by*

$$_FB := (B \setminus F) \cup \{\bar{i} \mid i \in F\} \subseteq [n, \overline{n}],$$

and call the reorientation *of B at F.*

The next lemma due to Santos [61] characterizes the set of bases of $M[\Lambda(A)]$ by that of $M[A]$ using a reorientation. The proof is via the duality of oriented matroids. We also presented another proof using only linear algebra [43].

Lemma 4.12 ([61]) *Let $M[A]$ be the vector matroid of A and $M[\Lambda(A)]$ that of $\Lambda(A)$. Then,*

$$\mathcal{B}(M[\Lambda(A)]) = \left\{ {}_F([n] \setminus B) \cup \{i, \bar{i} \mid i \in B\} \,\middle|\, B \in \mathcal{B}(M[A]),\ F \subseteq [n] \setminus B \right\}. \quad (4.3)$$

Proof: Let $A = \{\boldsymbol{a}_1, \dots, \boldsymbol{a}_n\}$ and $\Lambda(A) = \{\boldsymbol{b}_1, \dots, \boldsymbol{b}_n, \boldsymbol{b}_{\overline{1}}, \dots, \boldsymbol{b}_{\overline{n}}\}$. For a basis $B = \{i_1, \dots, i_r\} \subseteq [n]$ of $M[A]$, $\{i_1, \dots, i_r, \overline{1}, \dots, \overline{n}\}$ is a basis of $M[\Lambda(A)]$, which implies that the rank of $\Lambda(A)$ is $n + r$.

From B, we construct a subset $B' \subseteq [n, \overline{n}]$ as follows:

- $i, \bar{i} \in B'$ for any $i \in B$, and
- just one of j or $\bar{j}$ is in B' for any $j \notin B$.

We show that B' is a basis of $M[\Lambda(A)]$. Suppose that $\sum_{j \in B'} \alpha_j \boldsymbol{b}_j + \sum_{\bar{j} \in B'} \alpha_{\bar{j}} \boldsymbol{b}_{\bar{j}} = 0$, then $\sum_{j \in B} \alpha_j \boldsymbol{a}_j = 0$, $\alpha_j + \alpha_{\bar{j}} = 0$ for $j \in B$ and $\alpha_j,\ \alpha_{\bar{j}} = 0$ for $j \notin B$. The independence of B implies that $\alpha_j = \alpha_{\bar{j}} = 0$ for $j \in B$. Therefore, $B' \in \mathcal{B}[M(\Lambda(A))]$. Now let $F := \{j \in [n] \mid j \notin B,\ \bar{j} \in B'\}$, then $B' = {}_F([n] \setminus B) \cup \{i, \bar{i} \mid i \in B\}$.

Conversely, we show that any basis B' of $M[\Lambda(A)]$ can be obtained by the above procedure. We define $B \subseteq [n]$ such that $B := \{i \in \ \mid i, \bar{i} \in B'\}$, and $F \subseteq [n] \setminus B$ as $F := \{i \in [n] \setminus B \mid \bar{i} \in B'\}$. Then, $|B| \geq r$. Furthermore, the linear independence of B' implies that B is also linearly independent. Therefore, B is a basis of $M[A]$, and $B' = {}_F([n] \setminus B) \cup \{i, \bar{i} \mid i \in B\}$. □

As a corollary, the relation of the number of bases of $M[\Lambda(A)]$ and that of $M[A]$ is induced.

Corollary 4.13 *Let r be the rank of $A \in \mathbb{Z}^{d \times n}$ and α the number of bases of $M[A]$. Then, the number of bases of $M[\Lambda(A)]$ is equal to $\alpha \cdot 2^{n-r}$.*

Proof: In (4.3), we can choose 2^{n-r} F's for a fixed $B \in \mathcal{B}(M[A])$, which implies that there are 2^{n-r} bases of $M[\Lambda(A)]$ per basis. In addition, for two bases B, B' of $M[A]$, the set of bases of $M[\Lambda(A)]$ defined by B and that defined by B' are disjoint. Therefore, the number of bases of $M[\Lambda(A)]$ is equal to $\alpha \cdot 2^{n-r}$.

□

Finally, we can analyze the number of dual feasible bases for MTPs of type $2 \times 2 \times \cdots \times 2 \times M_{s-1} \times M_s$. Let $M := M_{s-1}$ and $N := M_s$.

Theorem 4.14 *The number of dual feasible bases for the s-dimensional MTP of type $2 \times 2 \times \cdots \times 2 \times M \times N$ is equal to $M^{N-1} N^{M-1} 2^{(s-3)(M-1)(N-1)}$.*

Proof: As the corank of $\Lambda(A)$ is equal to that of A for any matrix A, the number $n - r$ in Lemma 4.13 for $A = \Lambda^k(A_{M,N})$, which is the corank of A, is the same for any $k \geq 0$.

For $k = 0$ this number equals the number of edges in $K_{M,N}$ that are not edges of a spanning tree, i.e., $(M-1)(N-1)$. The number of bases for $A_{M,N}$ is that of spanning trees of $K_{M,N}$, which is $M^{N-1}N^{M-1}$. Therefore, the number of dual feasible bases for the MTP of type $2 \times 2 \times \cdots \times 2 \times M \times N$ is equal to $M^{N-1}N^{M-1}2^{(s-3)(M-1)(N-1)}$. □

For MTPs of type $2 \times 2 \times \cdots \times 2 \times M \times N$, numbers of dual feasible bases are shown in Table 4.4.

Table 4.4: The number of dual feasible bases of MTPs and that of bases of vector matroids. $A_{M,N}$ is the incidence matrix of the bipartite graph $K_{M,N}$. $M := M_{s-1}$, $N := M_s$ and $\Lambda^k(A) := \Lambda(\Lambda^{k-1}(A))$.

Matrix A	Number of dual feasible bases of $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$	Number of bases of $M[A]$
$A_0 := A_{M,N}$	$\binom{M+N-2}{M-1}$ [5]	$M^{N-1}N^{M-1}$
$\Lambda(A_0)$	$M^{N-1}N^{M-1}$	$M^{N-1}N^{M-1}(2^{(M-1)(N-1)})$
$\Lambda^2(A_0)$	$M^{N-1}N^{M-1}2^{(M-1)(N-1)}$	$M^{N-1}N^{M-1}2^{2(M-1)(N-1)}$
$\vdots$	$\vdots$	$\vdots$
$\Lambda^{s-2}(A_0)$	$M^{N-1}N^{M-1}2^{(s-3)(M-1)(N-1)}$	$M^{N-1}N^{M-1}2^{(s-2)(M-1)(N-1)}$

4.4 Standard Pair Decompositions of Dual Problems of Lawrence-Type Integer Programs

In this section, we discuss the standard pairs for dual problems of Lawrence-type integer programs.

4.4.1 Vector Matroids and Dual Problems of Lawrence-Type Integer Programs

In this section, we discuss the standard pairs for dual problems of (a linear relaxation of) Lawrence-type integer programs

$$IP_{\Lambda(A),\boldsymbol{c}}(\boldsymbol{b}) := minimize\ \{\boldsymbol{c}\cdot\boldsymbol{x} \mid \Lambda(A)\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x}\in\mathbb{Z}^n\}.$$

Let $B \in \mathcal{B}(M[A])$. Then, the linear problem

$$minimize\ \{\boldsymbol{c}'\cdot\boldsymbol{x} \mid A\boldsymbol{x} = \boldsymbol{b}',\ \boldsymbol{x}\geq\boldsymbol{0}\}$$

is equivalent to the problem

$$maximize\ \{(-\widetilde{\boldsymbol{c}}'_N)^{\mathrm{T}}\boldsymbol{x}_N \mid M\boldsymbol{x}_N + I_d\boldsymbol{x}_B = \widetilde{\boldsymbol{b}}'_B,\ \boldsymbol{x}_B,\boldsymbol{x}_N\geq\boldsymbol{0}\},$$

where $\widetilde{\boldsymbol{c}}'_N$ is the reduced cost for B.

For this B, $B' := B\cup\{\overline{1},\ldots,\overline{n}\}$ is a basis of $M[\Lambda(A)]$ (Lemma 4.12). We consider the dictionary of the linear relaxation problem

$$LP_{\Lambda(A),\boldsymbol{c}}(\boldsymbol{b}) := minimize\ \{(\boldsymbol{c}\cdot\boldsymbol{x} \mid \Lambda(A)\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x}\geq\boldsymbol{0}\},$$

of $IP_{\Lambda(A),\boldsymbol{c}}(\boldsymbol{b})$ for a basis B'. As $LP_{\Lambda(A),\boldsymbol{c}}(\boldsymbol{b})$ is equivalent to the problem

$$minimize\ \left\{\boldsymbol{c}_1\cdot\boldsymbol{x}\ \middle|\ \begin{pmatrix} M & I_d & O & O\\ I_{n-d} & O & I_{n-d} & O\\ O & I_d & O & I_d\end{pmatrix}\boldsymbol{x} = \begin{pmatrix}\widetilde{\boldsymbol{b}}'\\ \boldsymbol{b}_1\end{pmatrix}\right\} \tag{4.4}$$

for certain $\boldsymbol{c}_1$ and $\boldsymbol{b}_1$, the dictionary of $LP_{\Lambda(A),\boldsymbol{c}}(\boldsymbol{b})$ is

$$maximize\ \left\{-\widetilde{\boldsymbol{c}}_N\cdot\boldsymbol{x}_N\ \middle|\ \begin{pmatrix} M & I_d & O & O\\ I_{n-d} & O & I_{n-d} & O\\ -M & O & O & I_d\end{pmatrix}\begin{pmatrix}\boldsymbol{x}_N\\ \boldsymbol{x}_B\end{pmatrix} = \widetilde{\boldsymbol{b}}_B\right\}.$$

Therefore, the dual problem of $LP_{\Lambda(A),\boldsymbol{c}}(\boldsymbol{b})$ is

$$D_{B',\widetilde{\boldsymbol{b}}}(\widetilde{\boldsymbol{c}}) := minimize\ \left\{\widetilde{\boldsymbol{b}}_B^{\mathrm{T}}\boldsymbol{y}_B\ \middle|\ \begin{pmatrix} I_{n-d} & -M^{\mathrm{T}} & -I_{n-d} & M^{\mathrm{T}}\end{pmatrix}\begin{pmatrix}\boldsymbol{y}_N\\ \boldsymbol{y}_B\end{pmatrix} = \widetilde{\boldsymbol{c}}_N,\ \boldsymbol{y}_B,\boldsymbol{y}_N\geq\boldsymbol{0}\right\}.$$

Example 4.3 (continued.) *For this A and a basis $\{1\}$, the coefficient matrix of the dual problem of $LP_{\Lambda(A),\boldsymbol{c}}(\boldsymbol{b})$ is* $D = \begin{pmatrix} 1 & 0 & -2 & -1 & 0 & 2 \\ 0 & 1 & -4 & 0 & -1 & 4 \end{pmatrix}$. □

Let D be the coefficient matrix of $D_{B',\widetilde{\boldsymbol{b}}}(\widetilde{\boldsymbol{c}})$. $D = (D_0, -D_0)$ where D_0 is the coefficient matrix of (4.4). Thus, the set of column vectors of D is centrally symmetrical, which corresponds to the central symmetricity for the *Gale transform* (e.g., see [85]) of the Lawrence polytope, which is combinatorially equivalent to the convex hull of $\Lambda(A)$.

Now we consider standard pair decompositions for I_D.

Lemma 4.15 ([57]) *$M[D]$ is the dual matroid of $M[\Lambda(A)]$. Especially,*

(i) *$B \subseteq [n, \overline{n}] \in \mathcal{B}(M[D])$ if and only if $[n, \overline{n}] \setminus B \in \mathcal{B}(M[\Lambda(A)])$.*

(ii) *$F \subseteq [n, \overline{n}]$ is an independent set of $M[D]$ if and only if $[n, \overline{n}] \setminus B$ contains a basis of $M[\Lambda(A)]$.*

Generally, if $(\boldsymbol{x}^{\boldsymbol{u}}, \sigma)$ is a standard pair for a toric ideal I_A for some matrix A, then column vectors of A_σ, the submatrix of A whose indices of column vectors are σ, is independent. Thus the next corollary follows.

Corollary 4.16 *If $(\boldsymbol{x}^{\boldsymbol{u}}\boldsymbol{y}^{\boldsymbol{v}}, \sigma)$ ($\sigma \subseteq [n, \overline{n}]$) is a standard pair of I_D, $[n, \overline{n}] \setminus \sigma$ contains a basis of $M[\Lambda(A)]$.*

Example 4.3 (continued.) *For this D and $\widetilde{\boldsymbol{b}} = (1, 1, 3, 1, 3, 5)$, standard pairs for $in_{\widetilde{\boldsymbol{b}}}(I_D)$ are*

$$(y_2y_3, \{\overline{1}\}),\ (y_2y_3, \{1\}),\ (1, \{2, \overline{1}\}),\ (x_3, \{2, \overline{1}\}),\ (x_3^2, \{2, \overline{1}\}),\ (y_3, \{2, \overline{1}\}),$$
$$(1, \{1, 2\}),\ (x_3, \{1, 2\}),\ (x_3^2, \{1, 2\}),\ (y_3, \{1, 2\}),\ (1, \{\overline{1}, \overline{2}\}),\ (x_3, \{\overline{1}, \overline{2}\}),$$
$$(x_3^2, \{\overline{1}, \overline{2}\}),\ (x_3^3, \{\overline{1}, \overline{2}\}),\ (1, \{1, \overline{2}\}),\ (x_3, \{1, \overline{2}\}),\ (x_3^2, \{1, \overline{2}\}),\ (x_3^3, \{1, \overline{2}\}),$$

□

4.4.2 Application to Dual Problems of Multidimensional Transportation Problems of Type $2 \times \cdots \times 2 \times 2 \times N$

In this section, we analyze the number of standard pairs for dual problems of MTPs of type $2 \times \cdots \times 2 \times 2 \times N$.

Proposition 4.17 *For the dual problem of the s-dimensional MTP of type $2\times\cdots\times2\times2\times N$, the set of column vectors of the coefficient matrix $D_s(N)$ is $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_{N-1}, -\boldsymbol{e}_1, \dots, -\boldsymbol{e}_{N-1}, \mathbf{1}, -\mathbf{1}\}$, where $\boldsymbol{e}_1, \dots, \boldsymbol{e}_{N-1}$ are unit vectors of $\mathbb{R}^{N-1}$ and $\mathbf{1} \in \mathbb{R}^{N-1}$ is the vector all of whose elements are 1. Especially, the set of column vectors does not depend on s.*

Proof: Let $A_s(N)$ be the coefficient matrix for the s-dimensional MTP of type $2 \times \cdots \times 2 \times 2 \times N$. Then, $A_s(N)$ is the Lawrence lifting of $A_{s-1}(N)$. Thus, $D_s(N) = (D_{s-1}(N)\ -D_{s-1}(N))$. As we showed in Section 3.6.2, $D_2(N) = (I_{N-1},\ -\mathbf{1},\ -I_{N-1},\ \mathbf{1})$. Therefore, the set of column vectors of $D_s(N)$ is $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_{N-1}, -\boldsymbol{e}_1, \dots, -\boldsymbol{e}_{N-1}, \mathbf{1}, -\mathbf{1}\}$. $\square$

Therefore, the maximum number of primal feasible bases for the MTP of type $2\times\cdots\times2\times2\times N$ is the normalized volume of P_N as in Section 3.6.2, which is $(N-[N/2])\binom{N}{[N/2]}$ (see Theorem 3.43).

Theorem 4.18 *The maximum number of primal feasible bases for the MTP of type $2 \times \cdots \times 2 \times 2 \times N$ is equal to*

$$(N - [N/2])\binom{N}{[N/2]}.$$

4.5 Summary

In this section, we characterized the standard pairs of Lawrence-type integer programs, which is a subclass often encountered in combinatorial optimization.

For computational algebraic properties for toric ideals of Lawrence-type matrices, only the relation between Gröbner bases and vector matroids has been characterized in detail. We provided another relation between standard pairs corresponding to dual feasible bases and the set of bases of vector matroids. Furthermore, we described the matroidal structure of such standard pairs: characterization for adjacency of standard pairs in regular triangulations and that of vertices of the base polyhedron.

We also studied the primal and dual multidimensional transportation problems of certain types. Multidimensional transportation problems are related to enumeration of multi-way contingency tables with fixed marginal sums, as each feasible solution of a multidimensional transportation problem corresponds to one multi-way contingency table of the same size. However, the problem of counting the number of two-dimensional [22] and three-dimensional [18] contingency tables with fixed marginal sums has been shown to be #P-complete [18]. For a sampling of tables, there exist polynomial-time Markov Chain Monte Carlo methods for tables of type $2 \times N$ [21] and type $2 \times \cdots \times 2 \times N$ [51]. For multidimensional transportation problems, we calculated the number of dual feasible bases for problems of type $2 \times \cdots \times 2 \times M \times N$, and the maximum number of primal feasible bases, which gave the lower bounds for the number of tables, for problems of type $2 \times \cdots \times 2 \times 2 \times N$. Therefore, the results presented in this chapter indicated the difficulty of algebraic analysis of statistical or combinatorial problems on multi-way contingency tables.

Several interesting problems remain:

- Is it possible to analyze arithmetic degrees for multidimensional transportation problems of type $2 \times \cdots \times 2 \times M \times N$ ($M \geq 3$)?

- Can polynomial-time Markov Chain Monte Carlo methods [21, 51] be interpreted in terms of computational algebraic methods?

- Is it possible to analyze arithmetic degrees for other types of multidimensional transportation problems? Recently, Santos and Sturmfels [62] studied the size of Markov bases (the set of moves of a Markov chain that connects the chain) and Graver bases for general *three*- dimensional transportation problems.

Chapter 5

Concluding Remarks

This dissertation focused on computational algebraic duality for several common subclasses of integer programs: unimodular integer programs and Lawrence-type integer programs.

For unimodular integer programs, we gave an interpretation of the Hoçten-Thomas algorithm in terms of the reduced cost of linear programs, and calculated the maximum arithmetic degree, which indicates the complexity of the Hoçten-Thomas algorithm. Furthermore, by applying this algorithm to primal and dual minimum cost flow problems, we presented the algebraic difference of complexity between primal and dual problems in terms of arithmetic degree. We also gave computational algebraic proof for existing results on the numbers of primal and dual feasible bases of the transportation problem. These results were obtained using the computational algebraic duality of Gröbner bases and standard pair decompositions, which is not a well-known relation in classical network optimization. Therefore, these results are important, in that they suggest the effectiveness of computational algebraic duality for analyzing integer programs.

For Lawrence-type integer programs, we gave a bijection between the set of standard pairs corresponding to dual feasible bases and the set of bases of the vector matroid. This bijection also indicated the matroidal structure of such standard pairs, in the sense that an adjacency relation of standard pairs as facets in a regular triangulation corresponds to an adjacency of vertices in a base polyhedron, and each adjacency of vertices in a base polyhedron appears as adjacent facets for some regular triangulation. Furthermore, we studied primal and dual multidimensional transportation problems of certain types, for which polynomial-time Markov Chain Monte Carlo methods for such types have already been reported. For primal multidimensional transportation problems, we calculated the number of dual feasible bases for problems of types $2 \times \cdots \times 2 \times M \times N$. On the other hand, for dual multidimensional transportation problems, we can only analyze the case $2 \times \cdots \times 2 \times 2 \times N$. These results indicate the difficulty of analysis of statistical or combinatorial problems on multi-way contingency tables from the viewpoint of dualistic computational algebraic methods.

Several interesting problems remain regarding approaches of general integer programs using computational algebraic methods:

- Does a standard pair for a toric ideal that is not of type $(1, \sigma)$ provide some information about an integer program?
- Under what conditions for a face σ of a regular triangulation does (m, σ) become a standard pair?
- Is it possible to characterize the upper or lower bounds of arithmetic degrees for any toric ideal?

Appendix A

Matroids and Oriented Matroids

Here we briefly review matroids and oriented matroids. For more details, we refer to [57] about matorids and [8] about oriented matroids.

First, we define a matroid.

Definition A.1 *A* matroid *is an ordered pair* $M = (E, \mathcal{I})$ *of a finite set* E *and a collection* $\mathcal{I}$ *of subsets of* E *satisfying the following three conditions:*

(I1) $\emptyset \in \mathcal{I}$.

(I2) *If* $I \in \mathcal{I}$ *and* $I' \subseteq I$, *then* $I' \in \mathcal{I}$.

(I3) *If* $I_1,\ I_2 \in \mathcal{I}$ *and* $|I_1| < |I_2|$, *then there is an element* $e \in I_2 \setminus I_1$ *such that* $I_1 \cup \{e\} \in \mathcal{I}$.

An element of $\mathcal{I}$ *is called an* independent set *of* M, *and* E *is called the* ground set *of* M.

A subset of E which is not in $\mathcal{I}$ is called *dependent*.

There are several important objects which also define matroids, i.e. define independent sets of matroids. We give some of such objects. For the way to construct independent sets from each object, see [57].

Definition A.2 *Let* M *be a matroid with a ground set* E.

Circuits *A minimal dependent set of M is called a* circuit *of M. The set of circuits $\mathcal{C}$ of M determines independent sets of M.*

Bases *A maximal independent set of M is called a* basis *of M. The set of bases $\mathcal{B}$ of M determines independent sets of M.*

Rank *The cardinality of a maximal independent set of $X \subseteq E$ is called the* rank *of X and denoted by $\rho(X)$. The rank function $\rho : 2^E \to \mathbb{Z}$ of M determines independent sets of M.*

Closure *For $X \subseteq E$, the* closure *of X is defined as $cl(X) := \{x \in E \mid \rho(X \cup \{x\}) = \rho(X)\}$. The closure operator of M determines independent sets of M.*

A *loop* of a matroid $M = (E, \mathcal{I})$ is an element $e \in E$ such that $\{e\}$ is a dependent set of M.

Proposition A.3 *Let B be a basis of the matroid $M = (E, \mathcal{I})$ and $i \notin B$. Then $B \cup \{i\}$ contains a unique circuit, which is called the* fundamental circuit *of i with respect to B.*

We give two famous examples of matroids: vector matroids and graphic matroids.

Proposition A.4 *Fix a matrix A with n columns and a field k. Let $E = \{1, \ldots, n\}$ and $\mathcal{I}$ be the set of subsets X of E such that the set of column vectors of A labeled by X is linearly independent in the vector space k^d. Then $M[A] := (E, \mathcal{I})$ becomes a matroid, which is called the* vector matroid *of A over k.*

$B \subseteq E$ is a basis of $M[A]$ if and only if $|B|$ is equal to the rank of A and column vectors labeled by B form a basis for the linear space spanned by column vectors of A.

Example A.5 *Let A be the matrix*

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

over the field $\mathbb{R}$. Then $E = \{1, 2, 3, 4, 5\}$ and $\mathcal{I} = \{\emptyset, \{1\}, \{2\}, \{4\}, \{5\}, \{1, 2\}, \{1, 5\}, \{2, 4\}, \{2, 5\}, \{4, 5\}\}$, and the set of circuits is $\mathcal{C} = \{\{3\}, \{1, 4\}, \{1, 2, 5\}, \{2, 4, 5\}\}$. □

Proposition A.6 *Fix an undirected graph G. Let E be the set of edges of G and $\mathcal{I}$ the family of sets of edges which does not contain a cycle of G. Then $M(G) := (E, \mathcal{I})$ becomes a matroid, which is called the* graphic matroid *of G.*

Then $B \subseteq E$ is a basis of $M(G)$ if and only if B forms a spanning tree of G, and $C \subseteq E$ is a circuit of $M(G)$ if and only if C forms a circuit of G.

Example A.7 *Let G be the graph as below. Then since $\mathcal{I} = \{\emptyset, \{1\}, \{2\}, \{4\}, \{5\},$*

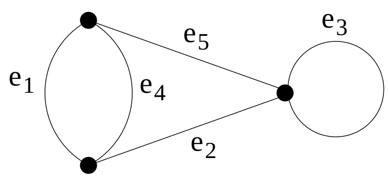

Figure A.1: The graphic matroid of this graph is same to $M[A]$ in Example A.5.

$\{1, 2\}, \{1, 5\}, \{2, 4\}, \{2, 5\}, \{4, 5\}\}$*, $M(G)$ is same as $M[A]$ in Example A.5.* $\square$

Next we define an oriented matroid. A *signed vector* on a finite set E is a vector in $\{-, 0, +\}^E$. For a signed vector X, we denote $X^\sigma := \{e \in E \mid X_e = \sigma\}$ for $\sigma \in \{-, 0, +\}$. The *support* of a signed vector X is the union $\underline{X} := X^+ \cup X^-$.

Definition A.8 *An* oriented matroid *is an ordered pair $\mathcal{M} = (E, \mathcal{C})$ of a finite set E and a set $\mathcal{C}$ of signed vectors on E satisfying the following four conditions:*

(1) $\emptyset \notin \mathcal{C}$.

(2) *If $X \in \mathcal{C}$, then $-X \in \mathcal{C}$.*

(3) *If $X, Y \in \mathcal{C}$ and $\underline{X} \subseteq \underline{Y}$, then $X = Y$ or $X = -Y$.*

(4) *For all $X, Y \in \mathcal{C}$, $X \neq -Y$ and $e \in X^+ \cap Y^-$, there exists $Z \in \mathcal{C}$ such that*

$$Z^+ \subseteq (X^+ \cup Y^+) \setminus \{e\} \text{ and } Z^- \subseteq (X^- \cup Y^-) \setminus \{e\}.$$

An element of $\mathcal{C}$ is called a circuit *of $\mathcal{M}$.*

We give two examples of oriented matroids.

Proposition A.9 *Fix a matrix A with column vectors $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n$ and a field k. Let $E = \{1, \dots, n\}$. For $X \subseteq E$ such that $\sum_{i \in X} c_i \boldsymbol{a}_i = 0$ ($0 \neq c_i \in k$ for $i \in X$) be a minimal linear dependence, we consider the signed vector $X = (X^+, X^-)$ given by $X^+ = \{i \mid c_i > 0\}$ and $X^- = \{i \mid c_i < 0\}$. Then for the set $\mathcal{C}$ of such signed vectors for all the minimal dependences, $\mathcal{M}[A] := (E, \mathcal{C})$ becomes an oriented matroid.*

If A does not have a column zero-vector, then, for a circuit $\boldsymbol{x}^{\boldsymbol{c}^+} - \boldsymbol{x}^{\boldsymbol{c}^-} \in \mathcal{C}_A$ defined in Section 2.1, $\boldsymbol{c} = \boldsymbol{c}^+ - \boldsymbol{c}^-$ corresponds to a circuit of $\mathcal{M}[A]$.

Example A.10 *Let A be the matrix*

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

over the field $\mathbb{R}$. Then $E = \{1, 2, 3, 4, 5\}$ and $\mathcal{C} = \{\pm(0, 0, +, 0, 0), \pm(+, 0, 0, -, 0), \pm(+, +, 0, 0, -), \pm(0, +, 0, +, -)\}$. $\square$

Proposition A.11 *Fix a directed graph G with an edge set E. For a circuit C of G, together with an orientation of each cycle, we define C^+ to be the set of forward edges and C^- the set of backward edges. Let $\mathcal{C}$ be the set of such signed vectors for all the circuits of G. Then $\mathcal{M}(G)$ becomes an oriented matroid.*

Example A.12 *Let G be the directed graph as below. Then $E = \{1, 2, 3, 4, 5, 6\}$ and*

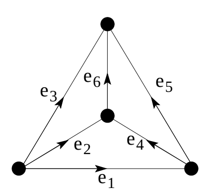

Figure A.2: An acyclic orientation of K_4.

$\mathcal{C} = \{\pm(+, -, 0, +, 0, 0), \pm(+, 0, -, 0, +, 0), \pm(0, +, -, 0, 0, +), \pm(0, 0, 0, +, -, +), \pm(+, -, 0, 0, +, -), \pm(+, 0, -, +, 0, +), \pm(0, +, -, -, +, 0)\}$. $\square$

References

[1] Joachim. H. Ahrens. A conjecture of E. D. Bolker. *Journal of Combinatorial Theory, Series B*, 31:1–8, 1981.

[2] Ravindra K. Ahuja, Thomas L. Magnanti, and James B. Orlin. *Network Flows: Theory, Algorithms, and Applications*. Prentice Hall, New Jersey, 1993.

[3] Satoshi Aoki and Akimichi Takemura. Minimal basis for connected Markov chain over $3 \times 3 \times K$ contingency tables with fixed two-dimensional marginals. *Australian and New Zealand Journal of Statistics*. To appear.
Preliminary version appeared in *METR Technical Report*, METR 2002-02, The University of Tokyo, February 2002.

[4] Michel L. Balinski. The Hirsh conjecture for dual transportation polyhedra. *Mathematics of Operations Research*, 9:629–633, 1984.

[5] Michel L. Balinski and Andrew Russakoff. Faces of dual transportation polyhedra. *Mathematical Programming Study*, 22:1–8, 1984.

[6] Margaret Bayer and Bernd Sturmfels. Lawrence polytopes. *Canadian Journal of Mathematics*, 42:62–79, 1990.

[7] Louis J. Billera and Beth Spellman Munson. Polarity and inner products in oriented matroids. *European Journal of Combinatorics*, 5:293–308, 1984.

[8] Anders Björner, Michel Las Vergnas, Bernd Sturmfels, Neil White, and Günter M. Ziegler. *Oriented Matroids*, volume 46 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second edition, 1999.

[9] Giandomenico Boffi and Fabio Rossi. Gröbner bases related to 3-dimensional transportation problems. Technical Report II, 482, Quaderni matematici Univ, Trieste, 2000.

[10] Jean-Daniel Boissonnat and Mariette Yvinec. *Algorithmic Geometry*. Cambridge University Press, Cambridge, 1998. Translated by Hervé Brönnimann.

[11] Ethan D. Bolker. Transportation polytopes. *Journal of Combinatorial Theory, Series B*, 13:251–262, 1972.

[12] Bruno Buchberger. *An Algorithm for Finding a Basis for the Residue Class Ring of a Zero-dimensional Polynomial Ideal*. PhD thesis, University of Innsbruck, 1965. (in German).

[13] Pasqualina Conti and Carlo Traverso. Buchberger algorithm and integer programming. In *Proceedings of the 9th International Symposium on Applied Algebra, Algebraic Algorithms and Error-Correcting Codes (AAECC-9)*, volume 539 of *Lecture Notes in Computer Science*, pages 130–139, Berlin, 1991. Springer.

[14] William Cook, Albertus M. H. Gerards, Alexander Schrijver, and Éva Tardos. Sensitivity theorems in integer linear programming. *Mathematical Programming*, 34:251–264, 1986.

[15] Thomas H. Cormen, Charles E. Leiserson, and Ronald L. Rivest. *Introduction to Algorithms*. MIT Press, Cambridge, 1990.

[16] David Cox, John Little, and Donal O'Shea. *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*. Springer-Verlag, New York, second edition, 1996.

[17] David Cox, John Little, and Donal O'Shea. *Using Algebraic Geometry*, volume 185 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998.

[18] Jesús A. De Loera and Shmuel Onn. The complexity of three-way statistical tables. Los Alamos e-print archive, math.CO/0207200, July 2002.

[19] Jesús A. De Loera and Bernd Sturmfels. Algebraic unimodular counting. Los Alamos e-print archive, math.CO/0104286, April 2001.

[20] Persi Diaconis and Bernd Sturmfels. Algebraic algorithms for sampling from conditional distributions. *Annals of Statistics*, 26:363–397, 1998.

[21] Martin Dyer and Catherine Greenhill. Polynomial-time counting and sampling of two-rowed contingency tables. *Theoretical Computer Science*, 246:265–278, 2000.

[22] Martin Dyer, Ravi Kannan, and John Mount. Sampling contingency tables. *Random Structures and Algorithms*, 10:487–506, 1997.

[23] Jack Edmonds and Rick Giles. A min-max relation for submodular functions on graphs. *Annals of Discrete Mathematics*, 1:185–204, 1977.

[24] Satoru Fujishige. *Submodular Functions and Optimization*, volume 47 of *Annals of Discrete Mathematics*. North-Holland, Amsterdam, 1991.

[25] Karin Gatermann. *Computer Algebra for Equivariant Dynamical Systems*, volume 1728 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2000.

[26] Israel M. Gelfand, Mark I. Graev, and Alexander Postnikov. Combinatorics of hypergeometric functions associated with positive roots. In *Arnold-Gelfand Mathematical Seminars: Geometry and Singularity Theory*, pages 205–221. Birkhäuser, Boston, 1996.

[27] Andrew Goldberg and Robert Tarjan. Finding minimum-cost circulations by cancelling negative cycles. *Journal of the ACM*, 36:873–886, 1989.

[28] Ralph E. Gomory. On the relation between integer and non-integer solutions to linear programs. *Proceedings of the National Academy of Science*, 53:260–265, 1965.

[29] Jack E. Graver. On the foundations of linear and integer programming I. *Mathematical Programming*, 8:207–226, 1975.

[30] Takayuki Hibi and Hidefumi Ohsugi. Private communication, February 2002.

[31] Serkan Hoçten. *Degrees of Gröbner Bases of Integer Programs*. PhD thesis, Cornell University, 1997.

[32] Serkan Hoçten and Gregory G. Smith. Monomial ideals. In David Eisenbud, Daniel R. Grayson, Michael Stillman, and Bernd Sturmfels, editors, *Computations in Algebraic Geometry with Macaulay 2*, volume 8 of *Algorithms and Computation in Mathematics*, pages 73–100. Springer-Verlag, Berlin, 2001.

[33] Serkan Hoçten and Rekha R. Thomas. Gomory integer programs. *Mathematical Programming*. To appear.
Preliminary version is available at Los Alamos e-print archive, math.OC/0106031, 2001.

[34] Serkan Hoçten and Rekha R. Thomas. Standard pairs and group relaxations in integer programming. *Journal of Pure and Applied Algebra*, 139:133–157, 1999.

[35] Birkett Huber and Rekha R. Thomas. Computing Gröbner fans of toric ideals. *Experimental Mathematics*, 9:321–331, 2000. Software: TiGERS is available at `http://www.math.tamu.edu/~rekha.thomas/programs.html`.

[36] Daisuke Ikegami. A maximum likelihood decoding algorithm using Gröbner bases. *Sūrikaisekikenkyūsho Kōkyūroku*, 1289:110–121, 2002. (in Japanese).

[37] Robert W. Irving and Mark R. Jerrum. Three dimensional statistical data security problems. *SIAM Journal on Computing*, 23:170–184, 1994.

[38] Takayuki Ishizeki. Analysis of Gröbner Bases for Toric Ideals of Acyclic Tournament Graphs. Master's thesis, The University of Tokyo, 2000.

[39] Takayuki Ishizeki and Hiroshi Imai. Gröbner bases of acyclic tournament graphs and hypergeometric systems on the group of unipotent matrices. *Sūrikaisekikenkyūsho Kōkyūroku*, 1171:7–20, 2000.

[40] Takayuki Ishizeki and Hiroshi Imai. Gröbner bases of acyclic directed graphs and minimum cost flow problems. In *Proceedings of the 2nd Japanese-Hungarian Symposium on Discrete Mathematics and Its Applications*, pages 82–91, April 2001.

[41] Takayuki Ishizeki and Hiroshi Imai. Standard pair decompositions of toric ideals and minimum cost flow problems. In *Electrical Proceedings of the 4th Symposium on Algebra and Computation*, November 2001. `ftp://tnt.math.metro-u.ac.jp/pub/ac01/ishizeki/ishizeki.pdf`.

[42] Takayuki Ishizeki and Hiroshi Imai. Computational algebraic analyses of multidimensional transportation problems of type $2^k \times M \times N$. In *The Second Japanese-Sino Optimization Meeting*, September 2002.

[43] Takayuki Ishizeki and Hiroshi Imai. Standard pairs for Lawrence-type matrices and their applications to several Lawrence-type integer programs. Submitted., 2002.

[44] Takayuki Ishizeki, Hiroki Nakayama, and Hiroshi Imai. Analyses of primal and dual minimum cost flow problems using computational algebraic methods. *Sūrikaisekikenkyūsho Kōkyūroku*, 1289:122–140, 2002. (in Japanese).

[45] Takayuki Ishizeki, Hiroki Nakayama, and Hiroshi Imai. Numbers of primal and dual bases of network flow and unimodular integer programs. Submitted. Los Alamos e-print archive, math.CO/0206159, June 2002.

[46] Takayuki Ishizeki, Hiroki Nakayama, and Hiroshi Imai. Computational algebraic analyses of unimodular integer programming problems. In *Proceedings of the 3rd Hungarian-Japanese Symposium on Discrete Mathematics and Its Applications*, pages 330–339, January 2003.

[47] Ravi Kannan. Lattice translates of a polytope and the frobenius problem. *Combinatorica*, 12:161–177, 1992.

[48] Victor Klee and Christoph Witzgall. Facets and vertices of transportation polytopes. *Lecture Notes in Applied Mathematics*, 11:257–282, 1968.

[49] John B. Little. Applications to coding theory. In *Applications of Computational Algebraic Geometry*, volume 53 of *Proceedings of Symposia in Applied Mathematics*, pages 143–167. American Mathematical Society, 1998.

[50] Masaki Maruyama. *Gröbner Bases and its Applications*. Gendai Sūgaku no Chouryū. Kyōritsu Shuppan, 2002. (in Japanese).

[51] Tomomi Matsui, Yasuko Matsui, and Yoko Ono. Random generation of $2 \times \cdots \times 2 \times J$ contingency tables. Technical Report METR 2003-03, The University of Tokyo, January 2003.

[52] Hiroki Nakayama. The Analysis of Structure on Toric Ideal and Standard Pair about Dual Problem of Minimum Cost Flow. Bachelor thesis, The University of Tokyo, 2002.

[53] George L. Nemhauser and Laurence A. Wolsey. *Integer and Combinatorial Optimization*. John Wiley & sons, New York, 1988.

[54] Masayuki Noro. Gröbner bases — theory, more efficient calculation and applications. *Sūrikaisekikenkyūsho Kōkyūroku*, 1138:127–171, 2000. (in Japanese).

[55] Toshinori Oaku. *Gröbner Bases and Systems of Linear Partial Differential Equations — An introduction to computational algebraic analysis —*, volume 38 of *Sophia Kokyuroku in Mathematics*. Department of Mathematics, Sophia University, Tokyo, 1994.

[56] Hidefumi Ohsugi. Private communication, January 2000.

[57] James G. Oxley. *Matroid Theory*. Oxford University Press, Oxford, 1992.

[58] Igor Pak. On the number of faces of certain transportation polytopes. *European J. Combinatorics*, 21:689–694, 2000.

[59] Maurice Queyranne and Frits Spieksma. Multi-index transportation problems. In Christodoulos A. Floudas and Panos M. Pardalos, editors, *Encyclopedia of Op-*

timization, volume III, pages 450–456. Kluwer Academic Publishers, Dordrecht, 2001.

[60] Mutsumi Saito, Bernd Sturmfels, and Nobuki Takayama. *Gröbner Deformations of Hypergeometric Differential Equations*, volume 6 of *Algorithms and Computation in Mathematics*. Springer-Verlag, Berlin, 2000.

[61] Francisco Santos. *Triangulations of Oriented Matroids*, volume 156 of *Memoirs of the American Mathematical Society*. American Mathematical Society, 2002.

[62] Francisco Santos and Bernd Sturmfels. Higher Lawrence configurations. Los Alamos e-print archive, math.CO/0209326, September 2002.

[63] Alexander Schrijver. *Theory of Linear and Integer Programming*. John Wiley & Sons, Chichester, 1986.

[64] Thomas W. Sederberg. Applications to computer aided geometric design. In David A. Cox and Bernd Sturmfels, editors, *Applications of Computational Algebraic Geometry*, volume 53 of *Proceedings of Symposia in Applied Mathematics*, pages 67–89. American Mathematical Society, Providence, 1998.

[65] Maiko Shigeno, Satoru Iwata, and S. Thomas McCormick. Relaxed most negative cycle and most positive cut canceling algorithms for minimum cost flow. *Mathematics of Operations Research*, 25:76–104, 2000.

[66] Palaniswamy T. Sokkalingam, Ravindra K. Ahuja, and James B. Orlin. New polynomial-time cycle-canceling algorithms for minimum cost flows. *Networks*, 36:53–63, 2000.

[67] Richard P. Stanley. *Enumerative Combinatorics*, volume 2. Cambridge University Press, Cambridge, 1999.

[68] Bernd Sturmfels. Some applications of affine Gale diagrams to polytopes with few vertices. *SIAM Journal on Discrete Mathematics*, 1:121–133, 1988.

[69] Bernd Sturmfels. Gröbner bases of toric varieties. *Tohoku Mathematical Journal*, 43:249–261, 1991.

[70] Bernd Sturmfels. *Algorithms in Invariant Theory*. Texts and Monographs in Symbolic Computation. Springer-Verlag, Vienna, 1993.

[71] Bernd Sturmfels. *Gröbner Bases and Convex Polytopes*, volume 8 of *University Lecture Series*. American Mathematical Society, Providence, 1996.

[72] Bernd Sturmfels and Rekha R. Thomas. Variation of cost functions in integer programming. *Mathematical Programming*, 77:357–387, 1997.

[73] Bernd Sturmfels, Ngô Viêt Trung, and Wolfgang Vogel. Bounds on degrees of projective schemes. *Mathematische Annalen*, 302:417–432, 1995.

[74] Iwaro Takahashi and Satoru Fujishige. *Discrete Mathematics*. Iwanami Shoten, 1981. (in Japanese).

[75] Naoki Terai. Degrees and arithmetic degrees of Stanley-Reisner rings. *Sūrikaisekikenkyūsho Kōkyūroku*, 1289:35–40, 2002. (in Japanese).

[76] Rekha R. Thomas. A geometric Buchberger algorithm for integer programming. *Mathematics of Operations Research*, 20:864–884, 1995.

[77] Rekha R. Thomas. Applications to integer programming. In David A. Cox and Bernd Sturmfels, editors, *Applications of Computational Algebraic Geometry*, volume 53 of *Proceedings of Symposia in Applied Mathematics*, pages 119–141. American Mathematical Society, Providence, 1998.

[78] Rekha R. Thomas. The structure of group relaxations, 2001. to appear in Handbook of Discrete Optimization (Karen Aardal, George Nemhauser, and Robert Weismantel, eds.), Elsevier.

[79] Regina Urbaniak, Robert Weismantel, and Günter M. Ziegler. A variant of Buchberger's algorithm for integer programming. *SIAM Journal on Discrete Mathematics*, 10:96–108, 1997.

[80] Arthur F. Veinott Jr. and George B. Dantzig. Integral extreme points. *SIAM Review*, 10:371–372, 1968.

[81] Dominic J. A. Welsh. The computational complexity of some classical problems from statistical physics. *Disorder in Physical Systems*, pages 307–321, 1990.

[82] Dominic. J. A. Welsh. *Complexity: Knots, Colourings and Counting*, volume 186 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1993.

[83] Franz Winkler. *Polynomial Algorithms in Computer Algebra*. Texts and Monographs in Symbolic Computation. Springer-Verlag, Vienna, 1996.

[84] Vladimir A. Yemelichev, Michail M. Kovalev, and M. K. Kratsov. *Polytopes: Graphs and Optimization*. Cambridge University Press, Cambridge, 1984.

[85] Günter M. Ziegler. *Lectures on Polytopes*, volume 152 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[86] Günter M. Ziegler. Gröbner bases and integer programming. In Arjeh M. Cohen, Hans Cuypers, and Hans Sterk, editors, *Some Tapas of Computer Algebra*, chapter 7, pages 168–183. Springer-Verlag, Berlin, 1999.