QUANTUM FIELD THEORY

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Contents

Chapter 1

Introduction

1.1 Relativistic quantum mechanics

In *relativistic quantum field theory* the theories of quantum mechanics and special relativity are united. In quantum mechanics a special role is played by *Planck's constant* h, usually given divided by 2π,

$$\begin{aligned} \hbar \equiv h/2\pi &= 1.054\ 572\ 66\ (63) \times 10^{-34}\ \mathrm{J\ s} \\ &= 6.582\ 122\ 0\ (20) \times 10^{-22}\ \mathrm{MeV\ s}. \end{aligned} \tag{1.1}$$

In the limit that the action S is much larger than $\hbar$, $S \gg \hbar$, quantum effects do not play a role anymore and one is in the domain of classical mechanics. In special relativity a special role is played by *the velocity of light* c,

$$c = 299\ 792\ 458\ \mathrm{m\ s}^{-1}. \tag{1.2}$$

In the limit that $v \ll c$ one again reaches the domain of classical mechanics.

In the framework of quantum mechanics the position of a particle is a well-defined concept and the position coordinates can be used as *dynamical variables* in the description of the particles and their interactions. The position can in principle be determined at any time with any accuracy, so one can talk about states $|\boldsymbol{r}\rangle$ and the wave function $\psi(\boldsymbol{r}) = \langle \boldsymbol{r}||\psi\rangle$. In this representation the position operator simply acts as

$$\boldsymbol{r}_{op}\,\psi(\boldsymbol{r}) = \boldsymbol{r}\,\psi(\boldsymbol{r}). \tag{1.3}$$

The uncertainty principle tells us that in this representation the momentum cannot be fully determined. The corresponding position and momentum operators do not commute. They satisfy the well-known operator commutation relations

$$[\boldsymbol{r}_i, \boldsymbol{p}_j] = i\hbar\delta_{ij}, \tag{1.4}$$

where δ_{ij} is the Kronecker δ function. Indeed, the action of the momentum operator is in this representation not as simple. It is given by

$$\boldsymbol{p}_{op}\psi(\boldsymbol{r}) = -i\hbar\boldsymbol{\nabla}\psi(\boldsymbol{r}). \tag{1.5}$$

In the same way one can choose a representation in which the momenta of the particles are the dynamical variables. The corresponding states are $|\boldsymbol{p}\rangle$ and the wave functions $\tilde{\psi}(\boldsymbol{p}) = \langle \boldsymbol{p}||\psi\rangle$ are the Fourier transforms of the coordinate space wave functions,

$$\tilde{\psi}(\boldsymbol{p}) = \int d^3r\ \exp\left(-\frac{i}{\hbar}\boldsymbol{p}\cdot\boldsymbol{r}\right)\psi(\boldsymbol{r}), \tag{1.6}$$

and

$$\psi(\boldsymbol{r}) \;=\; \int \frac{d^3p}{(2\pi\hbar)^3}\; \exp\left(\frac{i}{\hbar}\,\boldsymbol{p}\cdot\boldsymbol{r}\right)\,\tilde{\psi}(\boldsymbol{p}). \tag{1.7}$$

However, the existence of a limiting velocity leads to new fundamental limitations on the possible measurements of physical quantities. Let us consider the measurement of the position of a particle. This position cannot be measured with infinite precision. Any device that wants to locate the position of say a particle within an interval Δx will contain momentum components $p \propto \hbar/\Delta x$. Therefore if we want $\Delta x \leq \hbar/mc$ (where m is the rest mass of the particle), momenta of the order $p \propto mc$ and energies of the order $E \propto mc^2$ are involved. It is then possible to create a particle - antiparticle pair and it is no longer clear of which particle we are measuring the position. As a result, we find that the original particle cannot be located better than within a distance $\hbar/mc$, its *Compton wavelength*,

$$\Delta x \geq \frac{\hbar}{mc}. \tag{1.8}$$

For a moving particle $mc^2 \longrightarrow E$ (or by considering the Lorentz contraction of length) one has $\Delta x \geq \hbar c/E$. If the particle momentum becomes relativistic, one has $E \approx pc$ and $\Delta x \geq \hbar/p$, which says that a particle cannot be located better than its *de Broglie wavelength*.
Thus the coordinates of a particle cannot act as dynamical variables (since these must have a precise meaning).

Some consequences are that only in cases where we restrict ourselves to distances $\gg \hbar/mc$, the concept of a wave function becomes a meaningful (albeit approximate) concept. For a massless particle one gets $\Delta x \gg \hbar/p = \lambda/2\pi$, i.e. the coordinates of a photon only become meaningful in cases where the typical dimensions are much larger than the wavelength.

For the momentum or energy of a particle we know that in a finite time Δt, the energy uncertainty is given by $\Delta E \geq \hbar/\Delta t$. This implies that the momenta of particles can only be measured exactly when one has an infinite time available. For a particle in interaction, the momentum changes with time and a measurement over a long time interval is meaningless. The only case in which the momentum of a particle can be measured exactly is when the particle is free. In this case the momentum is conserved and one can let Δt become infinitely large.

The result thus is that *the only observable quantities that can serve as dynamical coordinates are the momenta (and further the internal degrees of freedom like polarizations, ...) of free particles.* These are the particles in the initial and final state of a scattering process. The theory will not give an observable meaning to the time dependence of interaction processes. The description of such a process as occurring in the course of time is just as unreal as the classical paths are in non-relativistic quantum mechanics.

The main problem in Quantum Field Theory is to determine the probability amplitudes between well-defined initial and final states of a system of free particles. The set of such amplitudes is the *scattering matrix* or S-matrix.

Another point that needs to be emphasized is the meaning of particle in the above context. Actually, the better name might be 'degree of freedom'. If the energy is low enough to avoid excitation of internal degrees of freedom, an atom is a perfect example of a particle. In fact, it is the behavior under Poincaré transformations or in the limit $v \ll c$ Gallilei transformations that determine the description of a particle state, in particular the free particle state.

1.2 Units

It is important to choose an appropriate set of units when one considers a specific problem, because physical sizes and magnitudes only acquire a meaning when they are considered in relation to each other. This is true specifically for the domain of atomic, nuclear and high energy physics, where the typical numbers are difficult to conceive on a macroscopic scale. They are governed by a few fundamental units and constants, which have been discussed in the previous section, namely $\hbar$ and c.

Table 1.1: Physical quantities and their canonical dimensions d, determining units (energy)^d.

quantity	quantity in MKS	d
time t	$t/\hbar$	-1
length l	$l/\hbar c$	-1
energy E	E	1
momentum p	pc	1
mass m	mc^2	1
area A	$A/(\hbar c)^2$	-2
velocity v	v/c	0
force F	$F\,\hbar c$	2
charge^2	$\alpha = \frac{e^2}{4\pi\epsilon_0\,\hbar c}$	0

It turns out to be convenient to work with units such that $\hbar$ and c are set to one. All length, time and energy or mass units then can be expressed in one unit and powers thereof, for which one can use energy (see table 1.1). The elementary unit that is most relevant depends on the domain of applications, e.g. the eV for atomic physics the MeV or GeV for nuclear physics and the GeV or TeV for high energy physics. To convert to other units of length or time we use appropriate combinations of $\hbar$ and c, e.g. for lengths

$$\hbar c \;=\; 0.197\ 328\ 58\ (51)\ \text{GeV fm} \tag{1.9}$$

or (when $\hbar = c = 1$)

$$1\ \text{fm} \;=\; 10^{-15}\ m \;=\; 5.068\ \text{GeV}^{-1} \tag{1.10}$$

and

$$\hbar^2 c^2 \;=\; 0.389\ 385\ 7\ (20)\ \text{GeV}^2\ \text{mbarn} \tag{1.11}$$

(1 barn = 10^{-28} m^2 = 10^2 fm^2). For times one has

$$\hbar \;=\; 6.582\ 122\ 0\ (20) \times 10^{-22}\ \text{MeV s}. \tag{1.12}$$

or (when $\hbar = c = 1$)

$$1\ \text{s} \;=\; 1.519 \times 10^{21}\ \text{MeV}^{-1}. \tag{1.13}$$

Quantities that do not contain $\hbar$ or c are classical quantities, e.g. the mass of the electron m_e. Quantities that contain only $\hbar$ are expected to play a role in non-relativistic quantum mechanics, e.g. the Bohr radius, $a_\infty = 4\pi\epsilon_0\hbar^2/m_e e^2$. Quantities that only contain c occur in classical relativity, e.g. the electron rest mass $m_e c^2$ and the classical electron radius $r_e = e^2/4\pi\epsilon_0\ m_e c^2$. Quantities that contain both $\hbar$ and c play a role in relativistic quantum mechanics, e.g. the electron Compton wavelength $\lambda\!\!\!^-_e = \hbar/m_e c$ or the dimensionless fine structure constant $\alpha \equiv e^2/4\pi\epsilon_0\ \hbar c = 1/137.036\ 04$ (11).

1.3 Conventions

The three components of a vector $\boldsymbol{p}$ in 3-dimensional Euclidean space are indicated by an index i, thus p^i, which can be i = x, y, z, or i = 1, 2, 3. The inner product is indicated

$$\boldsymbol{p} \cdot \boldsymbol{q} = p^i q^i. \tag{1.14}$$

When a repeated index appears , such as on the right hand side of this equation, summation over this index is assumed (Einstein summation convention). The inner product of a vector with itself gives its length squared. For instance, *rotations* are those *real*, *linear* transformations that do not change the length of a vector. The following tensors are useful,

$$\delta^{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases} \tag{1.15}$$

$$\epsilon^{ijk} = \begin{cases} 1 & \text{if } ijk \text{ is an even permutation of } 123 \\ -1 & \text{if } ijk \text{ is an odd permutation of } 123 \\ 0 & \text{otherwise.} \end{cases} \tag{1.16}$$

They are used for instance in the scalar product of two vectors, $\boldsymbol{p} \cdot \boldsymbol{q} = p^i q^j \; \delta^{ij} = p^i q^i$ or the cross product of two vectors, $(\boldsymbol{p} \times \boldsymbol{q})^i = \epsilon^{ijk} \, p^j q^k$. Useful relations are

$$\epsilon^{ijk} \, \epsilon^{imn} = \delta^{jm} \, \delta^{kn} - \delta^{jn} \, \delta^{km}, \tag{1.17}$$

$$\epsilon^{ijk} \, \epsilon^{ijl} = 2 \, \delta^{kl}. \tag{1.18}$$

We note that for Euclidean vectors and tensors there exist only one type of indices. No difference is made between upper or lower. So we could have used all lower indices in the above equations. Because of our conventions in Minkowski space, however, it is convenient to stick to upper indices for three-vectors.

In special relativity we work in Minkowski space with four-vectors, e.g. $x^\mu = (t, \boldsymbol{x})$ or $p^\mu = (E, \boldsymbol{p})$. For four-vectors in Minkowski space we will use the notation

$$p^\mu = (E, \boldsymbol{p}) = (p^0, p^1, p^2, p^3). \tag{1.19}$$

The scalar product of two vectors p and q is denoted

$$p \cdot q = p^\mu q^\nu g_{\mu\nu}, \tag{1.20}$$

where the metric tensor $g_{\mu\nu}$ used to construct the scalar product is given by $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$ (the other components are zero). One thus has explicitly,

$$p \cdot q = p^0 q^0 - \boldsymbol{p} \cdot \boldsymbol{q}. \tag{1.21}$$

Because of the different signs occurring in $g_{\mu\nu}$, it is convenient to distinguish lower indices from upper indices. The lower indices are constructed in the following way,

$$p_\mu = g_{\mu\nu} p^\nu = (p_0, p_1, p_2, p_3) = (E, -\boldsymbol{p}), \tag{1.22}$$

and the scalar product of two vectors p and q is denoted

$$p \cdot q = p^\mu q^\nu g_{\mu\nu} = p^\mu q_\mu = p^0 q_0 + p^i q_i. \tag{1.23}$$

The length squared of a four-vector is defined by the scalar product of a vector by itself, e.g. the length of the momentum vector is

$$p^2 = p \cdot p = p^\mu p_\mu = E^2 - \boldsymbol{p}^2 = m^2, \tag{1.24}$$

where m is the mass of the system. The length of the position vector for a particle is

$$x^2 = x^\mu x_\mu = t^2 - \boldsymbol{x}^2 = \tau^2, \tag{1.25}$$

where τ is the eigentime. The *real*, *linear* transformations that do not change the length of a four-vector are the *Lorentz transformations* to be discussed in more detail in the next sections.

The derivative ∂_μ is defined

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = \left(\frac{\partial}{\partial t}, \boldsymbol{\nabla} \right). \tag{1.26}$$

It is easy to convince oneself that the derivative behaves as a four-vector with lower index, e.g. by realizing that it satisfies

$$\partial_\mu \, x^\nu = g_\mu^\nu. \tag{1.27}$$

Note that g_μ^ν with one upper and lower index is in essence a Kronecker delta, $g_0^0 = g_1^1 = g_2^2 = g_3^3 = 1$. The length squared of ∂_μ is the d'Alembertian operator, defined by

$$\Box \;=\; \partial^\mu \partial_\mu \;=\; \frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2. \tag{1.28}$$

The value of the antisymmetric tensor $\epsilon^{\mu\nu\rho\sigma}$ is determined in the same way as for ϵ^{ijk}, starting from

$$\epsilon^{0123} = 1. \tag{1.29}$$

(Note that here sometimes the opposite convention is used). The product of two tensors is given by

$$\epsilon^{\mu\nu\rho\sigma}\epsilon^{\mu'\nu'\rho'\sigma'} \;=\; -\begin{vmatrix} g^{\mu\mu'} & g^{\mu\nu'} & g^{\mu\rho'} & g^{\mu\sigma'} \\ g^{\nu\mu'} & g^{\nu\nu'} & g^{\nu\rho'} & g^{\nu\sigma'} \\ g^{\rho\mu'} & g^{\rho\nu'} & g^{\rho\rho'} & g^{\rho\sigma'} \\ g^{\sigma\mu'} & g^{\sigma\nu'} & g^{\sigma\rho'} & g^{\sigma\sigma'} \end{vmatrix}, \tag{1.30}$$

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_\mu^{\;\;\nu'\rho'\sigma'} \;=\; -\begin{vmatrix} g^{\nu\nu'} & g^{\nu\rho'} & g^{\nu\sigma'} \\ g^{\rho\nu'} & g^{\rho\rho'} & g^{\rho\sigma'} \\ g^{\sigma\nu'} & g^{\sigma\rho'} & g^{\sigma\sigma'} \end{vmatrix}, \tag{1.31}$$

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu}^{\;\;\;\;\rho'\sigma'} \;=\; -2\left(g^{\rho\rho'}g^{\sigma\sigma'} - g^{\rho\sigma'}g^{\sigma\rho'}\right), \tag{1.32}$$

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho}^{\;\;\;\;\;\;\sigma'} \;=\; -6g^{\sigma\sigma'}, \tag{1.33}$$

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho\sigma} \;=\; -24. \tag{1.34}$$

1.4 References

In these lectures I will follow for some part the book of Ryder [1]. Other text books of Quantum Field Theory that are useful are given in refs [2-8].

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1.5 Exercises

Exercise 1.1

(a) The (quantum mechanical) size of the Hydrogen atom is of the order of the Bohr radius, $a_\infty = 4\pi\epsilon_0\hbar^2/m_e e^2$. Reexpress this quantity in terms of the electron Compton wavelength $\lambda\!\!\!^{-}_e$ and the fine structure constant α.

(b) Similarly express the (relativistic) classical radius of the electron, $r_e = e^2/4\pi\epsilon_0\, m_e c^2$ in the Compton wavelength and the fine structure constant.

(c) Calculate the Compton wavelength of the electron and the quantities under (a) and (b) using the value of $\hbar c$, α and $m_e c^2 = 0.511$ MeV. This demonstrates how a careful use of units can save a lot of work. One does not need to know $\hbar$, c, ϵ_0, m_e, e, but only appropriate combinations.

Exercise 1.2
Prove the identity $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A} \cdot \boldsymbol{C})\, \boldsymbol{B}$ - $(\boldsymbol{A} \cdot \boldsymbol{B})\, \boldsymbol{C}$ using the properties of the tensor ϵ^{ijk} given in section 1.3.

Exercise 1.3
Prove the identities for the tensor $\epsilon^{\mu\nu\rho\sigma}$ in Eqs. 1.30 - 1.34. [Hint: for the first identity one can check e.g. $\epsilon^{0123}\,\epsilon^{0123}$ and then show that the relation remains valid for permutations].

Exercise 1.4
Proof the following relation

$$\epsilon^{\mu\nu\rho\sigma}\, g^{\alpha\beta} = \epsilon^{\alpha\nu\rho\sigma}\, g^{\mu\beta} + \epsilon^{\mu\alpha\rho\sigma}\, g^{\nu\beta} + \epsilon^{\mu\nu\alpha\sigma}\, g^{\rho\beta} + \epsilon^{\mu\nu\rho\alpha}\, g^{\sigma\beta}.$$

Exercise 1.5
Lightcone coordinates are defined as

$$a^\pm = (a^0 \pm a^3)/\sqrt{2}.$$

(a) Show that $g^{++} = g^{--} = 0$ and $g^{+-} = g^{-+} = 1$.

(b) Show that with the help of the following vectors,

$$\begin{aligned} \hat{t}^\mu &= (1,0,0,0) \\ \hat{z}^\mu &= (0,0,0,1) \\ n_+^\mu &= (1,0,0,1)/\sqrt{2} \\ n_-^\mu &= (1,0,0,-1)/\sqrt{2} \end{aligned}$$

which satisfy $\hat{t}\cdot\hat{t} = -\hat{z}\cdot\hat{z} = 1$, $n_+\cdot n_+ = n_-\cdot n_- = 0$ and $n_+\cdot n_- = 1$, one has $p^0 = p\cdot\hat{t}$, $p^3 = -p\cdot\hat{z} = \boldsymbol{p}\cdot\hat{\boldsymbol{z}}$ and $p^\pm = p\cdot n_\mp$. Show that the projector on the transverse components (1 and 2) is given by[1]

$$g_T^{\mu\nu} = g^{\mu\nu} - \hat{t}^\mu\hat{t}^\nu + \hat{z}^\mu\hat{z}^\nu = g^{\mu\nu} - n_+^{\{\mu}\, n_-^{\nu\}}.$$

(c) Check that the antisymmetric tensor in the transverse space is given by

$$\epsilon_T^{\alpha\beta} = \epsilon^{01\alpha\beta} = \epsilon^{-+\alpha\beta}.$$

[1] The braces ({}) around the indices indicate symmetrization, i.e. $a^{\{\mu}\, b^{\nu\}} \equiv a^\mu b^\nu + a^\nu b^\mu$. Brackets ([]) are conventionally used for antisymmetrization of indices

Chapter 2

Relativistic wave equations

2.1 Special relativity

In special relativity we start with a four-dimensional real vector space E(1,3) with basis e_μ (μ = 0,1,2,3). Vectors are denoted $x = x^\mu e_\mu$. The coordinate x^0 is referred to as the time component, x^i are the three space components. The length of a vector is given by

$$x^2 = x \cdot x = (x^\mu e_\mu, x^\nu e_\nu) = x^\mu x^\nu (e_\mu, e_\nu) = x^\mu x^\nu g_{\mu\nu} = x^\mu x_\mu . \tag{2.1}$$

The quantity $g_{\mu\nu}$ is the metric tensor and can be written in matrix form

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.2}$$

in which case one can also write

$$x^2 = x^T G \, x \tag{2.3}$$

(Note that x is then considered a column vector, and its transposed x^T is a row vector). The invariant distance between two points in space-time is determined from $ds^\mu = x_1^\mu - x_2^\mu = (dt, dx, dy, dz)$ and is given by

$$ds^2 \;=\; ds^\mu ds_\mu \;=\; dt^2 - (dx^2 + dy^2 + dz^2). \tag{2.4}$$

The *real*, *linear* transformations that leave the length of a vector invariant are called *(homogeneous) Lorentz transformations*. The transformations that leave invariant the distance ds^2 between two points are called inhomogeneous Lorentz transformations or *Poincaré transformations*. The Poincaré transformations include Lorentz transformations and translations.

Unlike in Euclidean space, the invariant interval is not positive definite. One can distinguish:
• $ds^2 > 0$ (timelike intervals); in this case an inertial system exists in which the two points are at the same space point and ds^2 just represents the time difference $ds^2 = dt^2$;
• $ds^2 < 0$ (spacelike intervals); in this case an inertial system exists in which the two points are at the same time and $ds^2 = -d\boldsymbol{x}^2$;
• $ds^2 = 0$ (lightlike or null intervals); the points lie on the lightcone and they can be connected by a light signal.

2.2 The Klein-Gordon equation

In the remainder of this chapter, we just want to play a bit with equations and study their behavior under Lorentz transformations. The Schrödinger equation in quantum mechanics is the operator equation corresponding to the non-relativistic expression for the energy,

$$E = \frac{\boldsymbol{p}^2}{2m}, \tag{2.5}$$

under the substitution (in coordinate representation)

$$E \longrightarrow i\frac{\partial}{\partial t}, \quad \boldsymbol{p} \longrightarrow -i\boldsymbol{\nabla} \quad \text{or} \quad p_\mu \longrightarrow i\partial_\mu. \tag{2.6}$$

Thus acting on the wave function,

$$i\frac{\partial}{\partial t}\psi(\boldsymbol{r},t) = -\frac{\boldsymbol{\nabla}^2}{2m}\psi(\boldsymbol{r},t) \tag{2.7}$$

for a free particle. A covariant equation (the same in every frame of reference) can be obtained by starting from the invariant constructed from the energy and momentum four vector of a particle $p^\mu = (E,\boldsymbol{p})$,

$$p^2 = p^\mu p_\mu = E^2 - \boldsymbol{p}^2 = m^2, \tag{2.8}$$

where m is the particle mass. Substitution of operators gives the Klein-Gordon equation

$$\left(\Box + m^2\right)\phi(\boldsymbol{r},t) = \left(\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2 + m^2\right)\phi(\boldsymbol{r},t) = 0. \tag{2.9}$$

Although it is straightforward to find the solutions of this equation, namely plane waves characterized by a wave number $\boldsymbol{k}$,

$$\phi_k(\boldsymbol{r},t) = \exp(-iE_k t + i\boldsymbol{k}\cdot\boldsymbol{r}), \tag{2.10}$$

with $E_k^2 = \boldsymbol{k}^2 + m^2$, the interpretation of this equation as a single-particle equation in which ϕ is a complex wave function poses problems because the energy spectrum is not bounded from below and the probability is not positive definite.

- *The energy spectrum is not bounded from below*: considering the above stationary plane wave solutions one obtains

$$E_k = \pm\sqrt{\boldsymbol{k}^2 + m^2}, \tag{2.11}$$

  i.e. there are solutions with negative energy.

- *Probability is not positive*: in quantum mechanics one has the probability and probability current

$$\rho = \psi^*\psi \tag{2.12}$$

$$\boldsymbol{j} = -\frac{i}{2m}\left(\psi^*\boldsymbol{\nabla}\psi - (\boldsymbol{\nabla}\psi^*)\psi\right) \equiv -\frac{i}{2m}\psi^* \overleftrightarrow{\boldsymbol{\nabla}}\, \psi. \tag{2.13}$$

  They satisfy the continuity equation,

$$\frac{\partial\rho}{\partial t} = -\boldsymbol{\nabla}\cdot\boldsymbol{j}, \tag{2.14}$$

  which follows directly from the Schrödinger equation.
  The continuity equation can be written down covariantly using the four-current $j^\mu = (\rho,\boldsymbol{j})$,

$$\partial_\mu j^\mu = 0. \tag{2.15}$$

  Therefore, relativistically the density is not a scalar quantity, but rather the zero component of a four vector. It is not difficult to see that an appropriate current is given by

$$j^\mu = (\rho,\boldsymbol{j}) = i\,\phi^* \overleftrightarrow{\partial}{}^\mu \phi = \left(i\,\phi^* \overleftrightarrow{\partial}_0\,\phi, -i\,\phi^* \overleftrightarrow{\boldsymbol{\nabla}}\,\phi\right). \tag{2.16}$$

  It is easy to see that this current is conserved if ϕ (and ϕ^*) satisfy the Klein-Gordon equation. The Klein-Gordon equation, however, is a second order equation and ϕ and $\partial\phi/\partial t$ can be fixed arbitrarily at a given time, allowing negative densities.

As we will see both problems are related and have to do with the existence of particles and antiparticles, for which we need the interpretation of ϕ as a field that must be quantized.

We end this section by noting that an arbitrary solution can be written as a superposition of plane waves,

$$\phi(x) = \int \frac{d^4k}{(2\pi)^4}\, 2\pi\,\delta(k^2 - m^2)\, e^{-i\,k\cdot x}\,\tilde{\phi}(\boldsymbol{k}). \tag{2.17}$$

The integration over k-modes clearly is covariant and restricted to the 'mass'-shell (as required by Eq. 2.9). It is possible to rewrite it as an integration over positive energies only but this gives two terms,

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3\,2E_k}\,\left(e^{-i\,k\cdot x}\,\tilde{\phi}(\boldsymbol{k}) + e^{i\,k\cdot x}\,\tilde{\phi}(-\boldsymbol{k})\right). \tag{2.18}$$

Introducing $\tilde{\phi}(\boldsymbol{k}) \equiv a(\boldsymbol{k})$ and $\tilde{\phi}(-\boldsymbol{k}) \equiv b^*(\boldsymbol{k})$ one has

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3\,2E_k}\,\left(e^{-i\,k\cdot x}\,a(\boldsymbol{k}) + e^{i\,k\cdot x}\,b^*(\boldsymbol{k})\right). \tag{2.19}$$

This latter expression allows an easier distinction between the cases that ϕ is real ($a_k = b_k$) or complex (a_k and b_k are independent).

2.3 Exercises

Exercise 2.1
Show that for a conserved current ($\partial_\mu j^\mu = 0$) the charge in a finite volume,

$$Q_V \equiv \int_V d^3x\; j^0(x),$$

satisfies

$$\dot{Q}_V = -\int_S d\boldsymbol{s}\cdot\boldsymbol{j},$$

and thus for any normalized solution the full 'charge',

$$Q = \lim_{V\to\infty} Q_V,$$

is conserved, $\dot{Q} = 0$.

Exercise 2.2
Show that if ϕ en ϕ^* are solutions of the Klein-Gordon equation, that

$$j^\mu = i\,\phi^* \overleftrightarrow{\partial}^\mu \phi$$

is a conserved current (Note that $A \overleftrightarrow{\partial}_\mu B \equiv A\partial_\mu B - (\partial_\mu A)B$).

Exercise 2.3
Show that[1]

$$\frac{d^3k}{(2\pi)^3\,2E_k} = \frac{d^4k}{(2\pi)^4}\,2\pi\,\delta(k^2 - m^2)\,\theta(k^0).$$

[1] Use the following property of delta functions

$$\delta\left(f(x)\right) = \sum_{zeros\ x_n} \frac{1}{|f'(x_n)|}\,\delta(x - x_n).$$

Chapter 3

Groups and their representations

Since the Klein-Gordon equation expresses just the relativistic relation between energy and momentum, it also must hold for particles with spin. However, since the symmetry group describing rotations is embedded in the Lorentz group, we must study the representations of the Lorentz group. Particles with spin then will be described by certain spinors. The Klein-Gordon equation will hold in the sense that each component of these spinors will satisfy this equation.

Before proceeding with the Lorentz group we will first discuss the rotation group as an example of a Lie group with which we are familiar in ordinary quantum mechanics.

3.1 The rotation group and $SU(2)$

The rotation groups $SO(3)$ and $SU(2)$ are examples of *Lie groups*, that is groups characterized by a finite number of real parameters, in which the parameter space forms locally a Euclidean space. A general rotation — we will consider $SO(3)$ as an example — is of the form

$$\begin{pmatrix} V'_x \\ V'_y \\ V'_z \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} \tag{3.1}$$

for an (active lefthanded) rotation around the z-axis or shorthand $\boldsymbol{V}' = R(\theta, \hat{z})\boldsymbol{V}$. The parameter-space of $SO(3)$ is a sphere with radius π. Any rotation can be uniquely written as $R(\theta, \hat{\boldsymbol{n}})$ where $\hat{\boldsymbol{n}}$ is a unit vector and θ is the rotation angle, $0 \le \theta \le \pi$, provided we identify the antipodes, i.e.

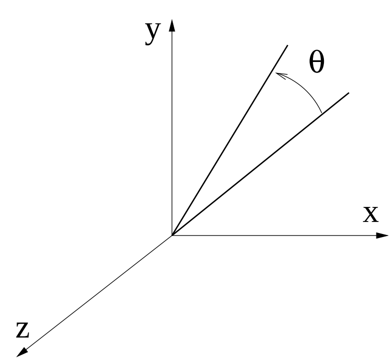


Figure 3.1: Rotation over an angle θ around the z-axis.

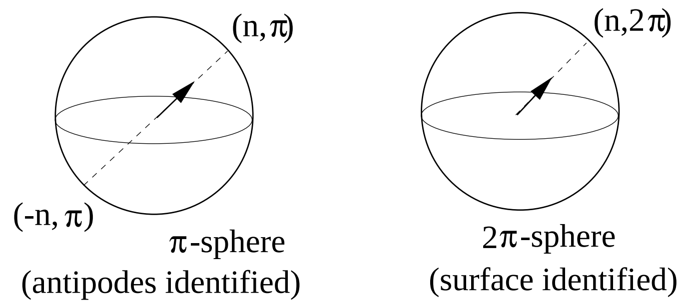


Figure 3.2: the parameter spaces of $SO(3)$ (left) and $SU(2)$ (right).

$R(\pi, \hat{\boldsymbol{n}}) \equiv R(\pi, -\hat{\boldsymbol{n}})$. Locally this parameter-space is 3-dimensional and correspondingly one has three generators. For an infinitesimal rotation around the z-axis one has

$$R(\theta, \hat{\boldsymbol{z}}) = 1 - i\ \delta\theta\ L^3 \tag{3.2}$$

with as generator

$$L^3 = \frac{1}{-i}\ \frac{\partial R(\theta, \hat{\boldsymbol{z}})}{\partial \theta}\bigg|_{\theta=0} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.3}$$

In the same way we can consider rotations around the x- and y-axes that are generated by

$$L^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad L^2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \tag{3.4}$$

or $(L^k)_{ij} = -i\ \epsilon^{ijk}$. It is straightforward to check that any rotation can be obtained from a combination of infinitesimal rotations,

$$R(\theta, \hat{\boldsymbol{z}}) = \lim_{N\to\infty} \left[R\left(\frac{\theta}{N}, \hat{\boldsymbol{z}}\right)\right]^N. \tag{3.5}$$

Rotations in general do not commute, which reflects itself in the noncommutation of the generators. They satisfy the commutation relations

$$[L^i, L^j] \ = \ i\ \epsilon^{ijk}\ L^k. \tag{3.6}$$

Summarizing, the rotations in $SO(3)$ can be generated from infinitesimal rotations that can be expressed in terms of a basis of three generators L^1, L^2 and L^3. These generators form a *three-dimensional Lie algebra* $\underline{SO(3)}$. With matrix commutation this algebra satisfies the requirements for a Lie algebra, namely that there exists a bilinear product $[,]$ that satisfies

- $\forall\ x, y \in \underline{A} \Rightarrow [x, y] \in \underline{A}$.
- $[x, x] = 0$ (thus $[x, y] = -[y, x]$).
- $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ (Jacobi identity).

Next, we turn to the group $SU(2)$ of special ($\det A = 1$) unitary ($A^\dagger = A^{-1}$) 2×2 matrices. These matrices can be defined as acting on 2-component spinors ($\chi \to A\chi$) or equivalently as acting on 2×2 matrices ($B \to ABA^\dagger$). It is straightforward to check that these conditions require

$$A \ = \ \begin{pmatrix} a^0 - i\,a^3 & -i\,a^1 - a^2 \\ -i\,a^1 + a^2 & a^0 + i\,a^3 \end{pmatrix} = a^0\,\mathbf{1} - i\,\boldsymbol{a}\cdot\boldsymbol{\sigma} \tag{3.7}$$

$$= \ a_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\,a^1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - i\,a^2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - i\,a^3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.8}$$

with real a's and $(a^0)^2 + \boldsymbol{a}^2 = 1$. One way of viewing the parameter space, thus is as the surface of a sphere in 4 dimensions. Locally this is a 3-dimensional Euclidean space and $SU(2)$, therefore, is a 3-dimensional Lie-group. Writing $a^0 = \cos(\phi/2)$ and $\boldsymbol{a} = \hat{\boldsymbol{n}}\,\sin(\phi/2)$ we have

$$\begin{aligned} A = A(\phi, \hat{n}) &= \mathbf{1}\,\cos\frac{\phi}{2} - i(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}})\sin\frac{\phi}{2} \\ &= \exp\left(-i\,\frac{\phi}{2}\,\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}\right) \end{aligned} \tag{3.9}$$

The parameter-space, thus, also can be considered as a *filled* 3-sphere with radius 2π, but with *all* points at the surface identified (see figure). The infinitesimal generators of $SU(2)$ are obtained by considering infinitesimal transformations, i.e. for fixed $\hat{\boldsymbol{n}}$,

$$A(\phi, \hat{\boldsymbol{n}}) \approx \mathbf{1} - i\phi\,\boldsymbol{J}\cdot\hat{\boldsymbol{n}}, \tag{3.10}$$

with

$$\boldsymbol{J}\cdot\hat{\boldsymbol{n}} \equiv \frac{1}{-i}\left.\frac{\partial A(\phi,\hat{\boldsymbol{n}})}{\partial\phi}\right|_{\phi=0} = \frac{\boldsymbol{\sigma}}{2}\cdot\hat{\boldsymbol{n}}. \tag{3.11}$$

Thus $\sigma^1/2$, $\sigma^2/2$ and $\sigma^3/2$ form the basis of the Lie-algebra $\underline{SU(2)}$. They satisfy

$$[\frac{\sigma^i}{2}, \frac{\sigma^j}{2}] = i\,\epsilon^{ijk}\,\frac{\sigma^k}{2}. \tag{3.12}$$

One, thus, immediately sees that the Lie algebras are identical, $\underline{SU(2)} \simeq \underline{SO(3)}$, i.e. one has a Lie algebra isomorphism that is linear and preserves the bilinear product.

There exists a corresponding mapping of the groups given by

$$\begin{array}{rcll} \mu: \quad SU(2) & \longrightarrow & SO(3) & \\ \exp(-i\,\phi\,\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}/2) & \longrightarrow & \exp(-i\,\phi\,\boldsymbol{L}\cdot\hat{\boldsymbol{n}}) & \\ & \text{or} & & \\ A(\phi,\hat{\boldsymbol{n}}) & \longrightarrow & R(\phi,\hat{\boldsymbol{n}}) & 0 \le \phi \le \pi \\ & \longrightarrow & R(2\pi-\phi,\hat{\boldsymbol{n}}) & \pi \le \phi \le 2\pi. \end{array}$$

It is easy to show that this satisfies the requirements of a homeomorphism using the relation

$$A(\boldsymbol{\sigma}\cdot\boldsymbol{a})A^{-1} = \boldsymbol{\sigma}\cdot R_A\boldsymbol{a} \tag{3.13}$$

The $SU(2) \to SO(3)$ mapping is a 2 : 1 mapping where both $A = \pm\mathbf{1}$ are mapped into $R = I$.

3.2 Representations of symmetry groups

The presence of symmetries simplify the description of a physical system. Suppose we have a system described by a hamiltonian H. The existence of symmetries means that there are operators g belonging to a symmetry group G that commute with the hamiltonian,

$$[g, H] = 0 \quad \text{for } g \in G. \tag{3.14}$$

For a Lie group, it is sufficient that the generators commute with H, since any finite rotation can be constructed from the infinitesimal ones, sometimes in more than one way (but this will be discussed later), i.e.

$$[\underline{g}, H] = 0 \quad \text{for } \underline{g} \in \underline{G}. \tag{3.15}$$

Representations Φ of a group are mappings of G into a finite dimensional vector space, preserving the group structure. In order to find local representations Φ of a Lie-group G, it is sufficient to consider the representations $\underline{\Phi}$ of the Lie-algebra $\underline{G}$. These are mappings from $\underline{G}$ into a finite dimensional vector space (its dimension is the dimension of the representation), which preserve the Lie-algebra structure, i.e. the commutation relations. Among the generators one looks for a maximal commuting set of operators (in this case consisting of the operator J^3 and the (quadratic Casimir) operator $\boldsymbol{J}^2$). Casimir operators commute with all the generators and the eigenvalue of $\boldsymbol{J}^2$ can be used to label the representation (j). Within the $(2j+1)$-dimensional representation space $V^{(j)}$ one can label the eigenstates $|j,m\rangle$ with eigenvalues of J^3. The other generators J^1 and J^2 (or $J_\pm \equiv J^1 \pm iJ^2$) then transform between the states in $V^{(j)}$.

Explicit representations using the basis states $|j,m\rangle$ with $m-values$ running from the heighest to the lowest, $m=j,j-1,\ldots,-j$ one has for $j=0$:

$$J^3=[0]\,,\ J_+=[0]\ J_-=[0]\,,$$

for $j=1/2$:

$$J^3=\begin{bmatrix}1/2 & 0\\ 0 & -1/2\end{bmatrix},\ J_+=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},\ J_-=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix},$$

for $j=1$:

$$J^3=\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{bmatrix},\ J_+=\begin{bmatrix}0 & \sqrt{2} & 0\\ 0 & 0 & \sqrt{2}\\ 0 & 0 & 0\end{bmatrix},\ J_-=\begin{bmatrix}0 & 0 & 0\\ \sqrt{2} & 0 & 0\\ 0 & \sqrt{2} & 0\end{bmatrix},$$

and for $j=3/2$:

$$J^3=\begin{bmatrix}3/2 & 0 & 0 & 0\\ 0 & 1/2 & 0 & 0\\ 0 & 0 & -1/2 & 0\\ 0 & 0 & 0 & 3/2\end{bmatrix},\ J_+=\begin{bmatrix}0 & \sqrt{2} & 0 & 0\\ 0 & 0 & \sqrt{3} & 0\\ 0 & 0 & 0 & \sqrt{2}\\ 0 & 0 & 0 & 0\end{bmatrix},\ J_-=\begin{bmatrix}0 & 0 & 0 & 0\\ \sqrt{2} & 0 & 0 & 0\\ 0 & \sqrt{3} & 0 & 0\\ 0 & 0 & \sqrt{2} & 0\end{bmatrix}.$$

If rotations leave H invariant, all states in the representation space have the same energy, or equivalently the Hilbert space can be written as a direct product space of spaces $V^{(j)}$.

For $j=1$ another commonly used representation starts with states:

$$|1,1\rangle \equiv \boldsymbol{\epsilon}_1 = -\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\\ 0\end{pmatrix},\quad |1,0\rangle \equiv \boldsymbol{\epsilon}_0 = \begin{pmatrix}0\\ 0\\ 1\end{pmatrix},\quad |1,-1\rangle \equiv \boldsymbol{\epsilon}_{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\\ 0\end{pmatrix}.$$

The spin matrices are:

$$J^1=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -i\\ 0 & i & 0\end{pmatrix},\ J^2=\begin{pmatrix}0 & 0 & -i\\ 0 & 0 & 0\\ i & 0 & 0\end{pmatrix},\ J^3=\begin{pmatrix}0 & -i & 0\\ i & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

From the (hermitean) representation $\underline{\Phi}(g)$ of $\underline{G}$, one obtains the unitary representations $\Phi(g) = \exp(-i\,\underline{\Phi}(g))$ of G. The matrix elements of the unitary representations are known as D-functions, for an element A of $SU(2)$ or R of $SO(3)$ parametrized with Euler angles, $U(\phi,\theta,\chi)=e^{-i\phi J^3}\,e^{-i\theta J^2}\,e^{-i\chi J^3}$,

$$\langle j,m'|U(\phi,\theta,\chi)|j,m\rangle = D^{(j)}_{m'm}(\phi,\theta,\chi) = e^{im'\phi}\,d^{(j)}_{m'm}(\theta)\,e^{-im\chi}. \tag{3.16}$$

Infinitesimally (around the identity) the D-functions for $SU(2)$ and $SO(3)$ are the same, e.g.

$$d^{(j)}_{m'm}(\theta) \approx \delta_{m'm} - i\theta(J^2)_{m'm}. \tag{3.17}$$

By moving through the parameter space the D-functions can be extended to global functions for all allowed angles. For those global representations, however, the topological structure of the group is important. If the group is *simply connected*, that is any closed curve in the parameter space can be contracted to a point, any point in the parameter space can be reached in a unique way and any local (infinitesimal) representation can be extended to a global one. Of all groups of which the Lie algebras are homeomorphic the simply connected group is called the *universal covering group*, i.e. $SU(2)$ is the covering group of $SO(3)$. The group $SO(3)$ is not simply connected, there are two different types of paths, contractable and paths that run from a point at the surface to its antipode. Given a point the extension of the representation from a local to a global must define a unique global representation. Since there are two paths to each point in the $SO(3)$ parameter space, we must require that they give the same D-functions. Thus, there exist the possibility that some representations of $SU(2)$ will not be representations of $SO(3)$. This is the case for all half-integer representations.

3.3 The Lorentz group

In the previous section spin has been introduced as a representation of the rotation group $SU(2)$ without worrying much about the rest of the symmetries of the world. We considered the generators and looked for representations in finite dimensional spaces, e.g. $\boldsymbol{\sigma}/2$ in a two-dimensional (spin 1/2) case. In this section we consider the Poincaré group, consisting of the Lorentz group and translations. Denoted in terms of column vectors and 4-dimensional matrices one has $x' = \Lambda\, x + a$. The Poincaré transformations are defined as transformations in Minkowski space,

$$\begin{aligned} x'^{\mu} &= (\Lambda)^{\mu\nu}\, x^{\nu} + a^{\mu} \\ &= \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu}. \end{aligned} \tag{3.18}$$

Invariance of the length of a vector requires for the Lorentz transformations

$$x'^2 = g_{\mu\nu} x'^{\mu} x'^{\nu} = g_{\mu\nu} \Lambda^{\mu}{}_{\rho} x^{\rho} \Lambda^{\nu}{}_{\sigma} x^{\sigma} = x^2 = g_{\rho\sigma} x^{\rho} x^{\sigma} \tag{3.19}$$

or

$$\Lambda^{\mu}{}_{\rho} g_{\mu\nu} \Lambda^{\nu}{}_{\sigma} = g_{\rho\sigma}, \tag{3.20}$$

which as a matrix equation with $(\Lambda)^{\mu\nu} = \Lambda^{\mu}{}_{\nu}$ and $(G)^{\mu\nu} = g_{\mu\nu}$ gives

$$(\Lambda^T)^{\rho\mu} (G)^{\mu\nu} (\Lambda)^{\nu\sigma} = (G)^{\rho\sigma}. \tag{3.21}$$

Thus one has the (pseudo orthogonality) relation.

$$\Lambda^T G \Lambda = G \;\Leftrightarrow\; G \Lambda^T G = \Lambda^{-1} \;\Leftrightarrow\; \Lambda G \Lambda^T = G \tag{3.22}$$

From this property, it is easy to derive some properties of the transformations Λ:

(i) $\det(\Lambda) = \pm 1$.
proof: $\det(\Lambda^T G \Lambda) = \det(G) \to (\det\, \Lambda)^2 = 1$.
($\det\, \Lambda = +1$ is called *proper*, $\det\, \Lambda = -1$ is called improper).

(ii) $|\Lambda^0{}_0| \geq 1$.
proof: $(\Lambda^T G \Lambda)^{00} = (G)^{00} = 1 \to \Lambda^{\mu}{}_0 g_{\mu\nu} \Lambda^{\nu}{}_0 = 1 \to (\Lambda^0{}_0)^2 - \sum_i (\Lambda^i{}_0)^2 = 1$.
Using (i) and (ii) the Lorentz transformations can be divided into 4 *classes* (with disconnected parameter spaces)

	det Λ	$\Lambda^0{}_0$	
$L_+^{\uparrow}$	+1	≥ 1	proper orthochrone
$L_+^{\downarrow}$	+1	$\leq$ -1	proper non-orthochrone
$L_-^{\uparrow}$	-1	≥ 1	improper orthochrone
$L_-^{\downarrow}$	-1	$\leq$ -1	improper non-orthochrone

(iii) $\sum_{i=1}^{3}(\Lambda^i_{\ 0})^2 = \sum_{i=1}^{3}(\Lambda^0_{\ i})^2$.
proof: use $\Lambda^T G \Lambda = G$ and $\Lambda G \Lambda^T = G$.

Note that Lorentz transformations generated from the identity must belong to $L_+^\uparrow$, since $I \in L_+^\uparrow$ and det Λ and $\Lambda^0_{\ 0}$ change continuously along a path from the identity. In $L_+^\uparrow$, one distinguishes rotations, e.g. rotations around the z-axis are given by

$$\begin{pmatrix} V^{0\prime} \\ V^{1\prime} \\ V^{2\prime} \\ V^{3\prime} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V^0 \\ V^1 \\ V^2 \\ V^3 \end{pmatrix}, \tag{3.23}$$

and boosts, e.g. a boost along the z-direction given by

$$\begin{pmatrix} V^{0\prime} \\ V^{1\prime} \\ V^{2\prime} \\ V^{3\prime} \end{pmatrix} = \begin{pmatrix} \cosh\phi & 0 & 0 & \sinh\phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\phi & 0 & 0 & \cosh\phi \end{pmatrix} \begin{pmatrix} V^0 \\ V^1 \\ V^2 \\ V^3 \end{pmatrix} \tag{3.24}$$

with $-\infty < \phi < \infty$. Note that the velocity $\beta = v = v/c$ and the Lorentz contraction factor $\gamma = (1-\beta^2)^{-1}$ corresponding to the boost are related to ϕ as $\gamma = \cosh\phi$, $\beta\gamma = \sinh\phi$.

Typical examples from each of the four classes are

$$I = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix} \in L_+^\uparrow \quad \text{(identity)} \tag{3.25}$$

$$I_t = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix} \in L_-^\downarrow \quad \text{(time inversion)} \tag{3.26}$$

$$I_s = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \in L_-^\uparrow \quad \text{(space inversion)} \tag{3.27}$$

$$I_s I_t = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \in L_+^\downarrow \quad \text{(space-time inversion)} \tag{3.28}$$

These four transformations form the Vierer group (group of Klein). Multiplying the proper orthochrone transformations with one of them gives all Lorentz transformations.

It is easy to check that the Lorentz transformations form a group ($\Lambda_1\Lambda_2$ and Λ^{-1} are again Lorentz transformations. Of the four parts only $L_+^\uparrow$ forms a group. This is a normal subgroup and the factor group is the Vierer group.

The extension to the Poincaré group is straightforward. Also this group can be divided into four parts, $P_+^\uparrow$ etc.

3.4 The generators of the Poincaré group

The infinitesimal transformations belong to $P_+^\uparrow$ and are written as

$$(\Lambda, a) = (I + \omega, \epsilon), \tag{3.29}$$

that explicitly read

$$
\begin{aligned}
x'^{\mu} &= (\Lambda)^{\mu\nu} x^{\nu} + a^{\mu} \overset{\text{inf}}{=} (\delta^{\mu\nu} + (\omega)^{\mu\nu})\, x^{\nu} + \epsilon^{\mu} \\
&= \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu} = (g^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu})\, x^{\nu} + \epsilon^{\mu}.
\end{aligned}
\tag{3.30}
$$

The condition $\Lambda^T G \Lambda = G$ yields

$$
\left(g^{\rho}{}_{\mu} + \omega^{\rho}{}_{\mu}\right) g_{\rho\sigma} \left(g^{\sigma}{}_{\nu} + \omega^{\sigma}{}_{\nu}\right) = g_{\mu\nu} \qquad \Longrightarrow \qquad \omega_{\nu\mu} + \omega_{\mu\nu} = 0,
\tag{3.31}
$$

thus $(\omega)^{ij} = -(\omega)^{ji}$ and $(\omega)^{0i} = (\omega)^{i0}$. We therefore have six generators, three of which only involve spatial coordinates (rotations) and three others involving time components (boosts).

Before coming to these transformations separately, we consider the covariant form of the six generators of the Lorentz transformations, which are obtained by writing the infinitesimal matrices $(\omega)^{\mu\nu}$ in terms of six antisymmetric matrices $(M^{\alpha\beta})^{\mu\nu}$. One immediately sees that

$$
(\omega)^{\mu\nu} = \omega^{\mu}{}_{\nu} = -\frac{i}{2}\, \omega_{\alpha\beta}\, (M^{\alpha\beta})^{\mu\nu},
\tag{3.32}
$$

$$
(M^{\alpha\beta})^{\mu\nu} = i\left(g^{\alpha\mu} g^{\beta}{}_{\nu} - g^{\alpha}{}_{\nu} g^{\beta\mu}\right)
\tag{3.33}
$$

The algebra of the generators of the Lorentz transformations can be obtained by an explicit calculation,

$$
[M^{\mu\nu}, M^{\rho\sigma}] = -i\left(g^{\mu\rho} M^{\nu\sigma} - g^{\nu\rho} M^{\mu\sigma}\right) - i\left(g^{\mu\sigma} M^{\rho\nu} - g^{\nu\sigma} M^{\rho\mu}\right).
\tag{3.34}
$$

Explicitly, we have for the (infinitesimal) rotations (around z-axis)

$$
\Lambda = I - i\,\theta^3\, J^3 = I - i\,\omega_{12}\, M^{12},
\tag{3.35}
$$

with

$$
J^3 = M^{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\tag{3.36}
$$

and for the (infinitesimal) boosts (along z-axis),

$$
\Lambda = I + i\,\phi^3\, K^3 = I + i\,\omega^{03}\, M^{03},
\tag{3.37}
$$

with

$$
K^3 = M^{03} = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},
\tag{3.38}
$$

The generators of rotations, $\boldsymbol{J}$, and those of the boosts, $\boldsymbol{K}$, satisfy the commutation relations

$$
\begin{aligned}
[J^i, J^j] &= i\,\epsilon^{ijk}\, J^k, \\
[J^i, K^j] &= i\,\epsilon^{ijk}\, K^k, \\
[K^i, K^j] &= -i\,\epsilon^{ijk}\, J^k.
\end{aligned}
$$

From the commutation relations one sees that the boosts or pure Lorentz transformations do not form a group, since the generators $\boldsymbol{K}$ do not form a closed algebra. the commutator of two boosts in different directions (e.g. the difference of first performing a boost in the y-direction and thereafter in the x-direction and the boosts in reversed order) contains a rotation (in the example around the z-axis). This is the origin of the Thomas precession.

Up to this point we have only used the four-dimensional defining representation of the Lorentz transformations to find the algebra for the Lorentz transformations, which will be the starting point for the search of representations acting in linear vector spaces. These are of the form $U(\Lambda,0)$ with the infinitesimal form

$$U(I+\omega,0) = 1 - \frac{i}{2}\,\omega_{\alpha\beta}\,M^{\alpha\beta}, \tag{3.39}$$

but $M^{\alpha\beta}$ acting no longer exclusively in Minkowski space. Before looking for representations, we first extend the algebra to the full Poincaré group. For this we will use a slightly different method than above, namely we just require Poincaré invariance for the generators themselves. The generator of the translations is the (momentum) operator P^μ, i.e. the infinitesimal form of the unitary representations $U(I,a)$ is

$$U(I,\epsilon) = 1 - i\,\epsilon_\alpha\,P^\alpha. \tag{3.40}$$

The requirement that P^μ transforms as a four-vector (for which we know the explicit behavior from the defining four-dimensional representation) gives

$$U(\Lambda,a)P^\mu U^{-1}(\Lambda,a) = \Lambda^\mu{}_\nu P^\nu, \tag{3.41}$$

or infinitesimal

$$(1 - i\epsilon_\gamma P^\gamma - \frac{i}{2}\,\omega_{\alpha\beta}M^{\alpha\beta})P^\mu(1 + i\epsilon_\delta P^\delta + \frac{i}{2}\,\omega_{\rho\sigma}M^{\rho\sigma}) = P^\mu + \omega^\mu{}_\nu P^\nu, \tag{3.42}$$

from which one obtains the following commutation relations by equating the coefficients of ϵ_μ and $\omega_{\mu\nu}$,

$$[P^\mu, P^\nu] = 0, \tag{3.43}$$
$$[M^{\mu\nu}, P^\rho] = -i\,(g^{\mu\rho}P^\nu - g^{\nu\rho}P^\mu)\,. \tag{3.44}$$

Note that the commutation relations among the generators $M^{\mu\nu}$ in Eq. 3.34 could have been obtained in the same way. They just state that $M^{\mu\nu}$ transforms as a tensor with two Lorentz indices. Explicitly, writing the generator $P^\mu = (H/c, \boldsymbol{P})$ in terms of the *hamiltonian* and the *three-momentum* operators, the tensor $M^{\mu\nu}$ in terms of boosts $cK^i = M^{0i}$ and rotations $J^i = \frac{1}{2}\,\epsilon^{ijk}M^{jk}$, one obtains

$$[P^i, P^j] = [P^i, H] = [J^i, H] = 0, \tag{3.45}$$
$$[J^i, J^j] = i\,\epsilon^{ijk}J^k,\ [J^i, P^j] = i\,\epsilon^{ijk}P^k,\ [J^i, K^j] = i\,\epsilon^{ijk}K^k, \tag{3.46}$$
$$[K^i, H] = i\,P^i,\ [K^i, K^j] = -i\,\epsilon^{ijk}J^k/c^2,\ [K^i, P^j] = i\,\delta^{ij}H/c^2. \tag{3.47}$$

We have here reinstated c, because one then sees that by letting $c \to \infty$ the commutation relations of the *Galilei group*, known from non-relativistic quantum mechanics are obtained.

In quantum mechanics the coordinates and momenta of the particles become operators in a Hilbert space, satisfying the canonical commutation relations. For non-relativistic quantum mechanics it is straightforward to express the generators of the Gallilei group in terms of the CM position, momentum and spin operators,

$$\begin{aligned} H &= \frac{\boldsymbol{P}^2}{2M} + H_{int}, \\ \boldsymbol{P} &= \boldsymbol{P}, \\ \boldsymbol{J} &= \boldsymbol{R}\times\boldsymbol{P} + \boldsymbol{S}, \\ \boldsymbol{K} &= M\,\boldsymbol{R} - t\,\boldsymbol{P}, \end{aligned} \tag{3.48}$$

For instance, for two particles one finds with $M = m_1 + m_2$ and $\mu = m_1\, m_2/M$,

$$\begin{aligned}
\boldsymbol{P} &= \boldsymbol{p}_1 + \boldsymbol{p}_2, \\
\boldsymbol{p} &= \frac{m_2}{M}\boldsymbol{p}_1 - \frac{m_1}{M}\boldsymbol{p}_2, \\
\boldsymbol{R} &= \frac{m_1}{M}\boldsymbol{r}_1 + \frac{m_2}{M}\boldsymbol{r}_2, \\
\boldsymbol{r} &= \boldsymbol{r}_1 - \boldsymbol{r}_2, \\
\boldsymbol{S} &= \boldsymbol{r}\times\boldsymbol{p} + \boldsymbol{s}_1 + \boldsymbol{s}_2, \\
H_{int} &= \frac{\boldsymbol{p}^2}{2\mu} + V(\boldsymbol{r}, \boldsymbol{s}_1, \boldsymbol{s}_2). \qquad (3.49)
\end{aligned}$$

The correct commutation relations follow from $[r^i, p^j] = i\delta^{ij}$ and $[s^i, s^j] = i\epsilon^{ijk}\, s^k$ for each particle. This shows that quantum mechanics can consistently be set up with spin decoupled from the dynamics.

Relativistic dynamics requires as symmetry group the Poincaré group. For a free particle or the CM, the generators of the Poincaré group can in the same way as above be expressed as

$$\begin{aligned}
H &= \sqrt{\boldsymbol{P}^2\, c^2 + M^2\, c^4}, \\
\boldsymbol{P} &= \boldsymbol{P}, \\
\boldsymbol{J} &= \boldsymbol{R}\times\boldsymbol{P} + \boldsymbol{S}, \\
\boldsymbol{K} &= \frac{1}{2c^2}\left(\boldsymbol{R}H + H\boldsymbol{R}\right) - t\boldsymbol{P} + \frac{\boldsymbol{P}\times\boldsymbol{S}}{H + Mc^2}. \qquad (3.50)
\end{aligned}$$

A consistent relativistic quantum mechanical treatment in terms of relative coordinates, however, requires great care [See e.g. L.L. Foldy, Phys. Rev. 122 (1961) 275 and H. Osborn, Phys. Rev. 176 (1968) 1514] and is possible in an expansion in $1/c$. Interaction terms, however, enter not only in the hamiltonian, but also in the boost operators.

3.5 The Lorentz group and $SL(2,C)$

Instead of the generators $\boldsymbol{J}$ and $\boldsymbol{K}$ of the homogeneous Lorentz transformations we can use the (non-hermitean) combinations

$$\boldsymbol{A} = \frac{1}{2}(\boldsymbol{J} + i\,\boldsymbol{K}), \qquad (3.51)$$

$$\boldsymbol{B} = \frac{1}{2}(\boldsymbol{J} - i\,\boldsymbol{K}), \qquad (3.52)$$

which satisfy the commutation relations

$$[A^i, A^j] = i\,\epsilon^{ijk} A^k, \qquad (3.53)$$

$$[B^i, B^j] = i\,\epsilon^{ijk} B^k, \qquad (3.54)$$

$$[A^i, B^j] = 0. \qquad (3.55)$$

This shows that the Lie algebra of the Lorentz group is identical to that of $SU(2)\otimes SU(2)$. This tells us how to find the representations of the group. They will be labeled by two angular momenta corresponding to the A and B groups, respectively, (j, j'). Special cases are the following representations:

$$\textit{Type I}: \quad (j,0) \qquad \boldsymbol{K} = -i\,\boldsymbol{J} \quad (\boldsymbol{B} = 0), \qquad (3.56)$$

$$\textit{Type II}: \quad (0,j) \qquad \boldsymbol{K} = i\,\boldsymbol{J} \quad (\boldsymbol{A} = 0). \qquad (3.57)$$

From the considerations above, it also follows directly that the Lorentz group is homeomorphic with the group $SL(2,C)$, similarly as the homeomorphism between $SO(3)$ and $SU(2)$. The group

$SL(2,C)$ is the group of complex 2×2 matrices with determinant one. It is simply connected and forms the covering group of $L_+^\uparrow$. It is easy to see that these matrices can be written as

$$M = \exp\left(-\frac{i}{2}\,\boldsymbol{\theta}\cdot\boldsymbol{\sigma} \pm \frac{1}{2}\boldsymbol{\phi}\cdot\boldsymbol{\sigma}\right), \tag{3.58}$$

with $\boldsymbol{\phi} = \phi\hat{\boldsymbol{n}}$ and $\boldsymbol{\theta} = \theta\hat{\boldsymbol{n}}$, where we restrict (for fixed $\hat{\boldsymbol{n}}$ the parameters $0 \le \theta \le 2\pi$ and $0 \le \phi < \infty$). With this choice of parameter-spaces the plus and minus signs are actually relevant. They precisely correspond to the two types of representations that we have seen before, becoming the defining representations of $SL(2,C)$:

$$\text{Type I (denoted } M\text{):} \qquad \boldsymbol{J} = \frac{\boldsymbol{\sigma}}{2}, \quad \boldsymbol{K} = -i\,\frac{\boldsymbol{\sigma}}{2}, \tag{3.59}$$

$$\text{Type II (denoted } \overline{M}\text{):} \qquad \boldsymbol{J} = \frac{\boldsymbol{\sigma}}{2}, \quad \boldsymbol{K} = +i\,\frac{\boldsymbol{\sigma}}{2}, \tag{3.60}$$

Let us investigate the defining (two-dimensional) representations of $SL(2,C)$. One defines spinors ξ and η transforming similarly under unitary rotations ($U^\dagger = U^{-1}$, $\overline{U} \equiv (U^\dagger)^{-1} = U$)

$$\begin{aligned} &\xi \to U\xi, \qquad \eta \to U\eta, \\ &U(\boldsymbol{\theta}) = \exp(-i\,\boldsymbol{\theta}\cdot\boldsymbol{\sigma}/2) \end{aligned} \tag{3.61}$$

but differently under hermitean boosts ($H^\dagger = H$, $\overline{H} \equiv (H^\dagger)^{-1} = H^{-1}$), namely

$$\xi \longrightarrow H\xi, \qquad \eta \to \overline{H}\eta,$$

$$H(\boldsymbol{\phi}) = \exp(\boldsymbol{\phi}\cdot\boldsymbol{\sigma}/2), \tag{3.62}$$

$$\overline{H}(\boldsymbol{\phi}) = \exp(-\boldsymbol{\phi}\cdot\boldsymbol{\sigma}/2). \tag{3.63}$$

A practical boost for the spinors is the one transforming from the rest frame to the frame with momentum $\boldsymbol{p}$. With $E = M\gamma = M\cosh(\phi)$ and $\boldsymbol{p} = M\beta\gamma\hat{\boldsymbol{n}} = M\sinh(\phi)\,\hat{\boldsymbol{n}}$ it is given by

$$H(\boldsymbol{p}) = \exp\left(\frac{\boldsymbol{\phi}\cdot\boldsymbol{\sigma}}{2}\right) = \cosh\left(\frac{\phi}{2}\right) + i\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}\,\sinh\left(\frac{\phi}{2}\right) = \frac{M+E+\boldsymbol{\sigma}\cdot\boldsymbol{p}}{\sqrt{2M(E+M)}}, \tag{3.64}$$

Also useful is the relation $H^2(\boldsymbol{p}) = \tilde{\sigma}^\mu p_\mu/M = (E + \boldsymbol{\sigma}\cdot\boldsymbol{p})/M$.

For the construction of the explicit mapping between the groups $L_+^\uparrow$ and $SL(2,C)$ the following definitions are useful

$$\sigma^\mu \equiv (\mathbf{1}, \boldsymbol{\sigma}), \qquad \tilde{\sigma}^\mu \equiv (\mathbf{1}, -\boldsymbol{\sigma}), \tag{3.65}$$

These matrices satisfy $\mathrm{Tr}(\sigma^\mu\,\tilde{\sigma}^\nu) = -2\mathrm{g}^{\mu\nu}$ and $\mathrm{Tr}(\sigma^\mu\,\sigma^\nu) = 2\mathrm{g}^\mu{}_\nu = 2\delta^{\mu\nu}$ (and are thus not covariant!). The mapping is given by

$$\begin{array}{rccc} \mu: & SL(2,C) & \longrightarrow & L_+^\uparrow \\ & M & \longrightarrow & \Lambda \end{array}$$

with the relations:

$$(\Lambda)^{\mu\nu} = \Lambda^\mu{}_\nu(M) = \frac{1}{2}\,Tr\left(\sigma^\mu M \sigma^\nu M^\dagger\right), \tag{3.66}$$

and furthermore

$$\pm M = \frac{1}{N}\left[\mathrm{Tr}(\Lambda)\,\sigma^0 + ((\Lambda)^{\mathrm{j}0} + (\Lambda)^{0\mathrm{j}})\sigma^{\mathrm{j}} - \mathrm{i}\epsilon^{\mathrm{klm}}(\Lambda)^{\mathrm{kl}}\sigma^{\mathrm{m}}\right], \tag{3.67}$$

where the normalization N is fixed by $\det(\pm M) = 1$.

We end this section with a note on the equivalence of representations. One might wonder if type I and type II representations are not equivalent, i.e. if there does not exist a unitary matrix S, such that $\overline{M} = SMS^{-1}$. In fact there exist the matrix

$$\epsilon = i\sigma^2, \qquad (\epsilon^* = \epsilon;\ \epsilon^{-1} = \epsilon^\dagger = \epsilon^T = -\epsilon), \tag{3.68}$$

that can be used to relate e.g.

$$\tilde{\sigma}^{\mu *} = \epsilon^\dagger \sigma^\mu \epsilon = -\epsilon \sigma^\mu \epsilon, \tag{3.69}$$

$$U^* = \epsilon^\dagger U \epsilon = -\epsilon\, U \epsilon, \tag{3.70}$$

$$H^* = \epsilon^\dagger \bar{H} \epsilon = -\epsilon\, \bar{H} \epsilon, \tag{3.71}$$

Considering just $SU(2)$, Eq. 3.70 shows that the conjugate representation with spinors χ^* transforming according to $-\boldsymbol{\sigma}^*/2$ is equivalent with the ordinary two-dimensional representation ($\epsilon\chi^*$ transforms according to $\boldsymbol{\sigma}/2$). This is not true for the two-component spinors in $SL(2,C)$ or $L_+^\uparrow$. Eqs 3.70 and 3.71 show that $\epsilon\xi^*$ transforms as a type-II (η) spinor. This shows that $M^* \simeq \overline{M}$. Note that the Lorentz transformations $\Lambda(\overline{M})$ and $\Lambda(M^*)$ corresponding to the equivalent $SL(2,C)$ representations can be connected by a Lorentz transformation as $\epsilon \in SL(2,C)$ and thus $\Lambda(\epsilon) \in L_+^\uparrow$. The Lorentz transformations $\Lambda(M)$ and $\Lambda(M^*)$, however, cannot be connected by a Lorentz transformation belonging to $L_+^\uparrow$ and form inequivalent representations. In fact

$$\Lambda(M^*) = I_2\, \Lambda(M)\, I_2^{-1}, \tag{3.72}$$

where

$$I_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.73}$$

The matrix I_2 does not belong to $L_+^\uparrow$, but to $L_-^\uparrow$.

3.6 Representations of the Poincaré group

In order to label the states in an irreducible representation we construct a maximal set of commuting operators. These define states with specified quantum numbers that are eigenvalues of these generators. For instance, the generators $\boldsymbol{J}^2$ and J^3 in the case of the rotation group. Taking any one of the states in an irreducible representation, other states in that representation are obtained by the action of operators outside the maximal commuting set. For instance, the generators $J_\pm$ in the case of the rotation group.

Of the generators of the Poincaré group we choose first of all the generators P^μ, that commute among themselves, as part of the set. The eigenvalues of these will be the four-momentum p^μ of the state,

$$P^\mu |p, \ldots\rangle = p^\mu |p, \ldots\rangle, \tag{3.74}$$

where p^μ is a set of four arbitrary real numbers. To find other generators that commute with P^μ we look for Lorentz transformations that leave the four vector p^μ invariant. These form a group called the little group associated with that four vector.

$$\begin{aligned} \Lambda^\mu{}_\nu p^\nu &= p^\mu + \omega^\mu{}_\nu p^\nu = p^\mu \\ \Rightarrow \omega_{\mu\nu} p^\nu &= 0 \\ \Rightarrow \omega_{\mu\nu} &= \epsilon_{\mu\nu\rho\sigma} p^\rho s^\sigma, \end{aligned} \tag{3.75}$$

where s^ρ is an arbitrary (spacelike) vector of which the length and the component along p^σ are irrelevant. The elements of the little group thus are

$$\begin{aligned} U(\Lambda(p)) &= \exp\left(-\frac{i}{2}\omega_{\mu\nu} M^{\mu\nu}\right) \\ &= \exp\left(-\frac{i}{2}\epsilon_{\mu\nu\rho\sigma} M^{\mu\nu} p^\rho s^\sigma\right) \end{aligned} \tag{3.76}$$

and

$$U(\Lambda(p))|p,\ldots\rangle = \exp\left(-\frac{i}{2}\epsilon_{\mu\nu\rho\sigma}M^{\mu\nu}p^\rho s^\sigma\right)|p,\ldots\rangle \tag{3.77}$$

$$= \exp\left(-is_\mu W^\mu\right)|p,\ldots\rangle \tag{3.78}$$

where the Pauli-Lubanski operators W^μ are given by

$$W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}M^{\nu\rho}P^\sigma. \tag{3.79}$$

The following properties follow from the fact that W^μ is a four-vector by construction, of which the components generate the little group of p^μ or (for the third of the following relations) from explicit calculation

$$[M_{\mu\nu}, W_\alpha] = -i\left(g_{\mu\alpha}W_\nu - g_{\nu\alpha}W_\mu\right), \tag{3.80}$$

$$[W_\mu, P_\nu] = 0, \tag{3.81}$$

$$[W_\mu, W_\nu] = i\,\epsilon_{\mu\nu\rho\sigma}W^\rho P^\sigma. \tag{3.82}$$

This will enable us to pick a suitable commuting 'spin' operator. From the algebra of the generators one finds that

$$P^2 = P_\mu P^\mu \qquad \text{and} \qquad W^2 = W_\mu W^\mu \tag{3.83}$$

commute with all generators and therefore are invariants under Poincaré transformations. These operators are the Casimir operators of the algebra and can be used to define the representations of $P_+^\uparrow$, for which we distinguish the following cases

$p^2 = m^2 > 0$ $\quad$ $p^0 > 0$
$p^2 = 0$ $\quad$ $p^0 > 0$
$p^\mu \equiv 0$
$p^2 = m^2 > 0$ $\quad$ $p^0 < 0$
$p^2 = 0$ $\quad$ $p^0 < 0$
$p^2 < 0$

Only the first two cases correspond to physical states, the third case represents the *vacuum*, while the others have no physical significance.

3.6.1 Massive particles: $p^2 = m^2 > 0$, $p^0 > 0$.

Given the momentum four vector p^μ we choose a tetrade consisting of three orthogonal spacelike unit vectors $n^{i\,\mu}(p)$, satisfying

$$g^{\mu\nu}n^i_\mu(p)n^j_\nu(p) = -\delta_{ij}, \tag{3.84}$$

$$n^i_\mu(p)p^\mu = 0, \tag{3.85}$$

$$\epsilon_{\mu\nu\rho\sigma}p^\mu n^{1\nu}n^{2\rho}n^{3\sigma} = M. \tag{3.86}$$

We can write

$$W_\mu(p) = \sum_{i=1}^{3} W^i(p)n^i_\mu(p) \tag{3.87}$$

(i.e. $W^i(p) = -W\cdot n^i(p)$).

Having made a covariant decomposition, it is sufficient to choose a particular frame to investigate the coefficients in Eq. 3.87 The best insight in the meaning of the operators W^μ is to sit in the rest frame of the particle and put $P^\mu = (M, \mathbf{0})$. In that case the vectors $n^i(p)$ are just the space directions and

$$W^\mu = \left(0, \frac{M}{2}\epsilon^{ijk} M^{jk}\right) = (0, M\boldsymbol{J}). \tag{3.88}$$

The commutation relations

$$[W^i, W^j] = i\,M\,\epsilon^{ijk} W^k, \tag{3.89}$$

show that $\boldsymbol{W}/M$ can be interpreted as the intrinsic spin.

More generally (fully covariant) one can proceed by defining $S^i(p) \equiv W^i(p)/M$ we obtain

$$\begin{aligned} [S^i(p), S^j(p)] &= -i\frac{n^{i\mu}(p) n^{j\nu}(p)}{M^2}\epsilon_{\mu\nu\rho\sigma} W^\rho P^\sigma \\ &= i\epsilon^{ijk} S^k(p), \end{aligned} \tag{3.90}$$

i.e. the $S^i(p)$ form the generators of an $SU(2)$ subgroup that belongs to the maximal set of commuting operators. Noting that

$$\sum_i \left(S^i(p)\right)^2 = \frac{1}{M^2} W^\mu(p) W^\nu(p) \sum_i n^i_\mu(p) n^i_\nu(p), \tag{3.91}$$

and using the completeness relation

$$\sum_i n^i_\mu(p) n^i_\nu(p) = -\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{M^2}\right), \tag{3.92}$$

one sees that

$$\sum_i \left(S^i(p)\right)^2 = -\frac{W^2}{M^2}. \tag{3.93}$$

Thus W^2 has the eigenvalues $-M^2\, s(s+1)$ with $s = 0, \frac{1}{2}, 1, \ldots$. Together with the four momentum states thus can be labeled as

$$|M, s; \boldsymbol{p}, m_s\rangle, \tag{3.94}$$

where $E = \sqrt{\boldsymbol{p}^2 + M^2}$ and m_s is the z-component of the spin $-s \le m_s \le +s$ (in steps of one).

The explicit construction of the wave function can be done using the D-functions (analogous as the rotation functions). We will not do this but construct them as solutions of a wave equation to be discussed explicitly in section 4

3.6.2 Massless particles: $p^2 = m^2 = 0$, $p^0 > 0$.

In this case a set of four independent vectors is chosen to be in a reference frame where $p \propto p^{(0)}$ in the following way:

$$\begin{aligned} p^{(0)} &= (1, 0, 0, 1), && (3.95)\\ n^1(p^{(0)}) &\equiv (0, 1, 0, 0), && (3.96)\\ n^2(p^{(0)}) &\equiv (0, 0, 1, 0), && (3.97)\\ s(p^{(0)}) &\equiv (s^0, 0, 0, s^3) \quad \text{with} \quad s^0 > |s^3| && (3.98) \end{aligned}$$

such that in an arbitrary frame where the four vectors p, $n^1(p)$, $n^2(p)$ and $s(p)$ are obtained by a Lorentz boost from the reference frame one has the property

$$\epsilon_{\mu\nu\rho\sigma} s^\mu n^{1\nu} n^{2\rho} p^\sigma = p \cdot s > 0. \tag{3.99}$$

The vector $W^\mu(p)$ now can be expanded

$$W_\mu(p) = W^1(p) n^1_\mu(p) + W^2(p) n^2_\mu(p) + \lambda(p) p_\mu + W^s(p) s_\mu(p), \tag{3.100}$$

where $W \cdot p = 0$ implies that $W^s(p) = 0$. The algebra of W^1, W^2 and λ is derived from the general commutation relations for $[W_\mu, W_\nu]$, and gives

$$[W^1, W^2] = 0, \tag{3.101}$$
$$[W^1, \lambda] = i\,W^2, \tag{3.102}$$
$$[W^2, \lambda] = -i\,W^1. \tag{3.103}$$

This algebra corresponds to the one generated by rotations and translations in a 2-dimensional Euclidean plane. The eigenvalues of W^1 and W^2 (corresponding to the translations) can in principle take any continuous values, implying continuous values for the spin. Such states are not physical, however, and the eigenvalues of W^1 and W^2 are set to zero (corresponding to the limit $m \to 0$ for the $W^\mu W_\mu$ eigenvalues of $M^2\,s(s+1) \to 0$). Thus we have

$$P^2 = 0, \quad P_\mu, \quad W_\mu(p) = \lambda(p)\,P_\mu, \quad W^2 = 0, \tag{3.104}$$

as a commuting set. The meaning of $\lambda(p)$ is most easily seen by comparing

$$W^0(p) = \lambda(p)\,|\boldsymbol{p}|$$

with

$$W^0(p) = -\frac{1}{2}\epsilon^{0ijk}M_{ij}p_k = \boldsymbol{J}\cdot\boldsymbol{p}.$$

Thus

$$\lambda(p) = \frac{\boldsymbol{J}\cdot\boldsymbol{p}}{|\boldsymbol{p}|}, \tag{3.105}$$

which is called the helicity. Note that angular momentum does not contribute to helicity as $\boldsymbol{L}\cdot\boldsymbol{p} = 0$. A massless particle, thus, is characterized by its momentum and the helicity,

$$|\boldsymbol{p}, \lambda\rangle, \tag{3.106}$$

which can take any *integer* or *half-integer* value.

The vacuum: $p^\mu = 0$.

This state is in physical applications nondegenerate and is denoted by

$$|0\rangle. \tag{3.107}$$

One has $P_\mu|0\rangle = W_\mu|0\rangle = 0$ and the state is left invariant by Lorentz transformations, $U(\Lambda, a)|0\rangle = |0\rangle$.

3.7 Excersises

Exercise 3.1
Show that if U is a unitary operator ($U = \exp[-i\alpha_k J_k]$) with real coefficients α_k, the operators J_k are hermitean. Show that if $\det(U) = 1$ the trace of the operators J_k is zero (Convince yourself that $\det(e^A) = e^{TrA}$ after diagonalizing A; note that $e^A \equiv \sum_{n=0}^{\infty} A^n/n!$).

Exercise 3.2
Show the relation

$$A(\boldsymbol{\sigma}\cdot\boldsymbol{a})A^{-1} = \boldsymbol{\sigma}\cdot R_A\boldsymbol{a}.$$

It it is sufficient to consider $\boldsymbol{a} = \hat{\boldsymbol{z}}$ (why?).

Exercise 3.3
Show that the operators

$$\begin{aligned} H &= \sqrt{\boldsymbol{p}^2\, c^2 + m^2\, c^4}, \\ \boldsymbol{P} &= \boldsymbol{p}, \\ \boldsymbol{J} &= \boldsymbol{r} \times \boldsymbol{p} + \boldsymbol{s}, \\ \boldsymbol{K} &= \frac{1}{2c^2}\,(\boldsymbol{r}\,H + H\,\boldsymbol{r}) - t\,\boldsymbol{p} + \frac{\boldsymbol{p} \times \boldsymbol{s}}{H + mc^2}. \end{aligned}$$

satisfy the commutation relations of the Poincaré group if the position, momentum and spin operators satisfy the canonical commutation relations, $[r^i, p^j] = i\delta^{ij}$ and $[s^i, s^j] = i\epsilon^{ijk}\, s^k$.
Hint: for the hamiltonian, show first the operator identity $[\boldsymbol{r}, f(\boldsymbol{p})] = i\,\boldsymbol{\nabla}_p f(\boldsymbol{p})$.

Excercise 3.4
Give a representation of the algebra in section 3.6.2 in the space of functions on the two-dimensional Euclidean plane. (consider rotations and translations in the plane).

Excercise 3.5
Show that the Lie-algebra representations $\boldsymbol{J} = \boldsymbol{\sigma}/2$ and $\boldsymbol{J} = -\boldsymbol{\sigma}^*/2$ are equivalent, i.e. show that a matrix ϵ exist such that $\epsilon^\dagger \boldsymbol{\sigma} \epsilon = -\boldsymbol{\sigma}^*$.

Chapter 4

The Dirac equation

4.1 Spin 1/2 representations of the Lorentz group

Both representations $(0,\frac{1}{2})$ and $(\frac{1}{2},0)$ of $SL(2,C)$ are suitable for representing spin 1/2 particles. The angular momentum operators J^i are represented by $\sigma^i/2$, that have the correct commutation relations. Although these operators cannot be used to label the representations, but rather the operators $W^i(p)$ should be used, we have seen that in the rest frame $W^i(p)/M \to J^i$, and the angular momentum in the rest frame is what we are familiar with as the spin of a particle. We also have seen that the representations $(0,\frac{1}{2})$ and $(\frac{1}{2},0)$ are inequivalent, i.e. they cannot be connected by a unitary transformation. They can be connected, however, by a transformation belonging to the class $P_-^\uparrow$, as we have seen in section 3.3. The representing member of the class $P_-^\uparrow$ is the parity or space inversion operator I_s. Indeed we know that under a parity transformation P:

$$\begin{aligned}
\boldsymbol{r} &\longrightarrow -\boldsymbol{r} \quad \text{(vector)},\\
t &\longrightarrow t \quad \text{(scalar)},\\
\boldsymbol{p} &\longrightarrow -\boldsymbol{p} \quad \text{(vector)},\\
H &\longrightarrow H \quad \text{(scalar)},\\
\boldsymbol{J} &\longrightarrow \boldsymbol{J} \quad \text{(axial vector)},\\
\lambda(p) &\longrightarrow -\lambda(p) \quad \text{(pseudoscalar)},\\
\boldsymbol{K} &\longrightarrow -\boldsymbol{K} \quad \text{(vector)},
\end{aligned}$$

and thus from the definition of the representations $(0,\frac{1}{2})$ and $(\frac{1}{2},0)$ one sees that under parity

$$(0,\frac{1}{2}) \longrightarrow (\frac{1}{2},0). \tag{4.1}$$

In nature parity turns (often) out to be a good quantum number for elementary particle states. For the spin 1/2 representations of the Poincaré group including parity we, therefore, must combine the representations, i.e. consider the four component spinor that transforms under a Lorentz transformation as

$$u = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \longrightarrow \begin{pmatrix} M(\Lambda) & 0 \\ 0 & \overline{M}(\Lambda) \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \tag{4.2}$$

where $\bar{M}(\Lambda) = \epsilon M^*(\Lambda)\epsilon^{-1}$ with ϵ given in Eq. 3.68. For a particle at rest only angular momentum is important and we can choose $\xi(\mathbf{0},m) = \eta(\mathbf{0},m) = \chi_m$, the well-known two-component spinor for a spin 1/2 particle. Taking $M(\Lambda) = H(\boldsymbol{p})$, the boost in Eq. 3.64, we obtain for the two components of

u which we will refer to as *chiral right* and *chiral left* components,

$$u(\boldsymbol{p}, m) = \begin{pmatrix} u_R \\ \\ u_L \end{pmatrix} = \begin{pmatrix} H(\boldsymbol{p}) & 0 \\ \\ 0 & \overline{H}(\boldsymbol{p}) \end{pmatrix} \begin{pmatrix} \chi_m \\ \\ \chi_m \end{pmatrix}, \tag{4.3}$$

with

$$H(\boldsymbol{p}) = \frac{E + M + \boldsymbol{\sigma} \cdot \boldsymbol{p}}{\sqrt{2M(E+M)}}, \tag{4.4}$$

$$\bar{H}(\boldsymbol{p}) = \frac{E + M - \boldsymbol{\sigma} \cdot \boldsymbol{p}}{\sqrt{2M(E+M)}} = H^{-1}(\boldsymbol{p}). \tag{4.5}$$

It is straightforward to eliminate χ_m and obtain the following constraint on the components of u,

$$\begin{pmatrix} 0 & H^2(p) \\ \\ H^{-2}(p) & 0 \end{pmatrix} \begin{pmatrix} u_R \\ \\ u_L \end{pmatrix} = \begin{pmatrix} u_R \\ \\ u_L \end{pmatrix}, \tag{4.6}$$

or explicitly

$$\begin{pmatrix} -M & E + \sigma \cdot \boldsymbol{p} \\ \\ E - \sigma \cdot \boldsymbol{p} & -M \end{pmatrix} \begin{pmatrix} u_R \\ \\ u_L \end{pmatrix} = 0 \qquad \text{(Weyl)}, \tag{4.7}$$

which is an explicit realization of the (momentum space) Dirac equation, which in general is a linear equation in p^μ,

$$(p_\mu \gamma^\mu - M)\, u \quad \equiv \quad (\not{p} - M)\, u = 0, \tag{4.8}$$

where γ^μ are 4×4 matrices called the Dirac matrices.

As in section 2 we can use $p_\mu = i\partial_\mu$ to get the Dirac equation in coordinate space,

$$(i\gamma^\mu \partial_\mu - M)\, \psi(x) = 0, \tag{4.9}$$

which is a covariant (linear) first order differential equation. The general requirements for the γ matrices are easily obtained. Applying $(i\gamma^\mu \partial_\mu + M)$ from the left gives

$$\left(\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + M^2\right) \psi(x) = 0. \tag{4.10}$$

Since $\partial_\mu \partial_\nu$ is symmetric, this can be rewritten

$$\left(\frac{1}{2}\left\{\gamma^\mu, \gamma^\nu\right\} \partial_\mu \partial_\nu + M^2\right) \psi(x) = 0. \tag{4.11}$$

The energy-momentum relation for each component of ψ must be satisfied, i.e. $\left(\Box + M^2\right) \psi(x) = 0$. From this it follows that the Clifford algebra for the Dirac matrices,

$$\left\{\gamma^\mu, \gamma^\nu\right\} = 2\, g^{\mu\nu}. \tag{4.12}$$

The explicit realization in Eq. 4.7 is known as the Weyl representation. We will discuss another explicit realization of this algebra in the next section.

We first want to investigate if the Dirac equation solves the problem with the negative densities and negative energies. The first indeed is solved. To see this consider the equation for the hermitean conjugate spinor $\psi^\dagger$ (noting that $\gamma_0^\dagger = \gamma_0$ and $\gamma_i^\dagger = -\gamma_i$),

$$\psi^\dagger \left(-i\gamma^0 \overleftarrow{\partial}_0 + i\gamma^i \overleftarrow{\partial}_i - M\right) = 0. \tag{4.13}$$

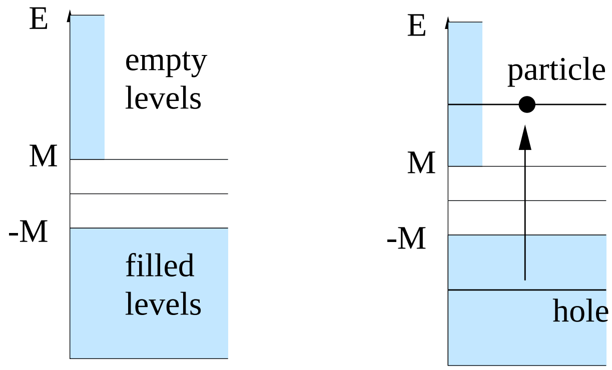


Figure 4.1: The Dirac sea of negative energy states and antiparticles.

Multiplying with γ^0 on the right and pulling it through one finds

$$\overline{\psi}(x)\left(i\gamma^\mu \overleftarrow{\partial}_\mu + M\right) = 0, \tag{4.14}$$

for the *adjoint* spinor $\overline{\psi} \equiv \psi^\dagger\gamma_0$. From the equations for ψ and $\overline{\psi}$ one immediately sees that

$$j^\mu = \overline{\psi}\gamma^\mu\psi \tag{4.15}$$

is a conserved probability current. The density

$$j^0 = \overline{\psi}\gamma^0\psi = \psi^\dagger\psi \tag{4.16}$$

is indeed positive and j^0 can serve as a probability density.

The second problem, with the negative energy states remains, as can be easily seen in the particle rest frame, where

$$\gamma^0 p_0\,\psi = M\psi \quad \longrightarrow \quad E\psi = M\gamma^0\psi. \tag{4.17}$$

Using the explicit form of γ^0 in the Weyl representation (see Eq. 4.7), one sees that there are two positive and two negative eigenvalues, $E = +M$ (twice) and $E = -M$ (twice),

This problem was solved by Dirac through the introduction of a negative energy sea. Relying on the Pauli exclusion principle for spin 1/2 particles, Dirac supposed that the negative energy states were already completely filled and the exclusion principle prevents any more particles to enter the sea of negative energy states. The Dirac sea forms the vacuum.

From the vacuum a particle can be removed. This hole forms an *antiparticle* with the same mass, but with properties such that it can be annihilated by the particle (e.g. opposite charge). We will see how this is implemented when quantizing the spin 1/2 field.

4.2 General representations of γ matrices and Dirac spinors

The general algebra for the Dirac matrices is

$$\{\gamma^\mu, \gamma^\nu\} = 2\,g^{\mu\nu}. \tag{4.18}$$

Two often used explicit representations are the following: The *standard* representation:

$$\gamma^0 = \rho^3 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \qquad \gamma^k = i\rho^2\sigma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}; \tag{4.19}$$

The Weyl representation:

$$\gamma^0 = \rho^1 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \qquad \gamma^k = -i\rho^2\sigma^k = \begin{pmatrix} 0 & -\sigma^k \\ \sigma^k & 0 \end{pmatrix}. \tag{4.20}$$

Different representations can be related to each other by unitary transformations,

$$\gamma_\mu \longrightarrow S\gamma_\mu S^{-1}, \tag{4.21}$$
$$\psi \longrightarrow S\psi. \tag{4.22}$$

We note that the explicit matrix taking us from the Weyl representation to the standard representation, $(\gamma_\mu)_{S.R.} = S(\gamma_\mu)_{W.R.}S^{-1}$, is

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & -\mathbf{1} \end{pmatrix}. \tag{4.23}$$

For all representations one has

$$\gamma_\mu^\dagger = \gamma_0\gamma_\mu\gamma_0, \tag{4.24}$$

and an adjoint spinor defined by

$$\overline{\psi} = \psi^\dagger\gamma_0. \tag{4.25}$$

Another matrix which is often used is γ_5 defined as

$$\gamma_5 = i\,\gamma^0\gamma^1\gamma^2\gamma^3 = -i\gamma_0\gamma_1\gamma_2\gamma_3 = \frac{i\epsilon_{\mu\nu\rho\sigma}}{4!}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma. \tag{4.26}$$

Explicitly one has

$$(\gamma_5)_{S.R.} = \rho^1 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \qquad (\gamma_5)_{W.R.} = \rho^3 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}. \tag{4.27}$$

Therefore

$$P_{R/L} = \frac{1}{2}\,(1 \pm \gamma_5) \tag{4.28}$$

are projection operators, that project out chiral right/left states, which in the case of the Weyl representation are just the upper and lower components.

There are a number of symmetries in the Dirac equation, e.g. parity. It is easy to convince oneself that if $\psi(x)$ is a solution of the Dirac equation (Eq. 4.9), then also

$$(i\partial\!\!\!/ - M)\,\psi^p(x) = 0, \tag{4.29}$$

with $\psi^p(x) \equiv \eta_p\,\gamma_0\,\psi(\tilde{x})$ (where $\tilde{x} = (t, -\boldsymbol{x})$). This is proven by simply applying space-inversion (Exercise 4.4).

The existence of positive and negative energy solutions implies another symmetry in the Dirac equation. This symmetry does not change the spin 1/2 nature, but it does, for instance, reverse the charge of the particle. As with parity we look for a transformation, called *charge conjugation*, that brings $\psi \to \psi^c$, which is again a solution of the Dirac equation, i.e.

$$(i\partial\!\!\!/ - M)\,\psi^c(x) = 0. \tag{4.30}$$

We note that by conjugating and transposing the Dirac equation one obtains

$$\left(i\gamma^{\mu T}\partial_\mu + M\right)\overline{\psi}^T = 0. \tag{4.31}$$

In any representation a matrix C exist, such that

$$C\gamma_\mu^T C^{-1} = -\gamma_\mu, \tag{4.32}$$

e.g.

$$(C)_{S.R.} \;=\; i\gamma^2\gamma^0 = -i\rho_1\sigma^2 = \begin{pmatrix} 0 & -i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix}, \tag{4.33}$$

$$(C)_{W.R.} \;=\; i\gamma^2\gamma^0 = -i\rho^3\sigma^2 = \begin{pmatrix} -i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix}. \tag{4.34}$$

Therefore, one can identify

$$\psi^c = \eta_c\, C\overline{\psi}^T, \tag{4.35}$$

where η_c is an arbitrary (unobservable) phase, usually to be taken unity.

4.3 Plane wave solutions

For a massive particle, the best representation to describe particles at rest is the standard representation, in which γ^0 is diagonal (see discussion of negative energy states in section 4.1). The explicit Dirac equation in the standard representation reads

$$\begin{pmatrix} i\frac{\partial}{\partial t} - M & i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla} \\ -i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla} & -i\frac{\partial}{\partial t} - M \end{pmatrix} \psi(x) = 0. \tag{4.36}$$

Looking for positive energy solutions $\propto \exp(-iEt)$ one finds two solutions, $\psi(x) = u^{\pm s}(p)\, e^{-i\,p\cdot x}$, with $E = E_p = +\sqrt{\boldsymbol{p}^2 + M^2}$, where u satisfies

$$\begin{pmatrix} E_p - M & -\boldsymbol{\sigma}\cdot\boldsymbol{p} \\ \boldsymbol{\sigma}\cdot\boldsymbol{p} & -(E_p + M) \end{pmatrix} u(p) = 0, \qquad \Leftrightarrow \qquad (\not{p} - M)\, u(p) = 0. \tag{4.37}$$

There are also two negative energy solutions, $\psi(x) = v^{\pm s}(p)\, e^{i\,p\cdot x}$, where v satisfies

$$\begin{pmatrix} -(E_p + M) & \boldsymbol{\sigma}\cdot\boldsymbol{p} \\ -\boldsymbol{\sigma}\cdot\boldsymbol{p} & (E_p - M) \end{pmatrix} v(p) = 0, \qquad \Leftrightarrow \qquad (\not{p} + M)\, v(p) = 0. \tag{4.38}$$

Explicit solutions in the Standard representation are

$$u^{\pm s}(p) = \sqrt{E_p + M} \begin{pmatrix} \chi_\pm(\hat{\boldsymbol{s}}) \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E_p+M}\chi_\pm(\hat{\boldsymbol{s}}) \end{pmatrix}, \qquad v^{\pm s}(p) = \sqrt{E_p + M} \begin{pmatrix} \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E_p+M}\bar{\chi}_\pm(\hat{\boldsymbol{s}}) \\ \bar{\chi}_\pm(\hat{\boldsymbol{s}}) \end{pmatrix}, \tag{4.39}$$

where $\chi_\pm(\hat{\boldsymbol{s}})$ are two-component spinors which are eigenfunctions of $\boldsymbol{\sigma}\cdot\hat{\boldsymbol{s}}/2$. Note that the spinors for the negative energy modes (antiparticles) are the equivalent spin representation with states $\bar{\chi} = \epsilon\,\chi^*$ and spin matrices $\bar{\boldsymbol{\sigma}} = -\epsilon\,\boldsymbol{\sigma}^*\,\epsilon^{-1}$. The solutions are normalized to

$$\bar{u}^s(p)u^{s'}(p) \;=\; 2M\,\delta_{ss'}, \tag{4.40}$$

$$\bar{v}^s(p)v^{s'}(p) \;=\; -2M\,\delta_{ss'}, \tag{4.41}$$

$$u^{s\dagger}(p)u^{s'}(p) \;=\; v^{s\dagger}(p)v^{s'}(p) = 2E_p\,\delta_{ss'}. \tag{4.42}$$

We will sometimes use the notations $u(p,s)$ and $v(p,s)$ for the spinors, which may in some cases be more convenient.

An arbitrary spin 1/2 field can be expanded in the independent solutions. After separating positive and negative energy solutions as done in the case of the scalar field one has

$$\psi(x) = \sum_s \int \frac{d^3k}{(2\pi)^3\, 2E_k} \left(u^s(k)\, e^{-i\,k\cdot x}\, b(\boldsymbol{k},s) + v^s(k)\, e^{i\,k\cdot x}\, d^*(\boldsymbol{k},s)\right). \tag{4.43}$$

It is straightforward to find projection operators for the positive and negative energy states

$$P_+ \;=\; \sum_s \frac{u^s(p)\bar{u}^s(p)}{2M} = \frac{\not p + M}{2M}, \tag{4.44}$$

$$P_- \;=\; -\sum_s \frac{v^s(p)\bar{v}^s(p)}{2M} = \frac{-\not p + M}{2M}. \tag{4.45}$$

In order to project out spin states, the spin polarization vector in the rest frame is the starting point. It is a spacelike unit vector $s^\mu = (0, \hat{\boldsymbol{s}})$. In an arbitrary frame it is obtained by a Lorentz transformation. It is easy to check that

$$P_s = \frac{1 \pm \gamma_5 \not s}{2} \tag{4.46}$$

projects out spin $\pm 1/2$ states (check in restframe for $\hat{\boldsymbol{s}} = \hat{\boldsymbol{z}}$).

Note that for solutions of the massless Dirac equation $\not p\psi = 0$. Therefore, $\gamma_5 \not p \psi = 0$ but also $\not p \gamma_5 \psi = -\gamma_5 \not p \psi = 0$. This means that in the solution space for massless fermions the chirality states,

$$\psi_{R/L} \equiv \frac{1}{2}\,(I \pm \gamma_5)\,\psi, \tag{4.47}$$

are independent solutions. In principle massless fermions can be described by two-component spinors. The chirality projection operators replace the spin projection operators which are not defined (by lack of a rest frame).

Explicit examples of spinors are useful to illustrate spin eigenstates, helicity states, chirality, etc. For instance with the z-axis as spin quantization axis, one has in Standard representation:

$$u^{+z}(p) = \frac{1}{\sqrt{E+M}} \begin{pmatrix} E+M \\ 0 \\ p^3 \\ p^1 + i\,p^2 \end{pmatrix}, \qquad u^{-z}(p) = \frac{1}{\sqrt{E+M}} \begin{pmatrix} 0 \\ E+M \\ p^1 - i\,p^2 \\ -p^3 \end{pmatrix}, \tag{4.48}$$

$$v^{+z}(p) = \frac{1}{\sqrt{E+M}} \begin{pmatrix} p^3 \\ p^1 + i\,p^2 \\ E+M \\ 0 \end{pmatrix}, \qquad v^{-z}(p) = \frac{1}{\sqrt{E+M}} \begin{pmatrix} -p^1 + i\,p^2 \\ p^3 \\ 0 \\ -(E+M) \end{pmatrix}. \tag{4.49}$$

Helicity states ($\boldsymbol{p}$ along $\hat{\boldsymbol{z}}$) in Standard representation are:

$$u^{\lambda=+}(p) = \begin{pmatrix} \sqrt{E+M} \\ 0 \\ \sqrt{E-M} \\ 0 \end{pmatrix}, \qquad u^{\lambda=-}(p) = \begin{pmatrix} 0 \\ \sqrt{E+M} \\ 0 \\ -\sqrt{E-M} \end{pmatrix}, \tag{4.50}$$

$$v^{\lambda=+}(p) = \begin{pmatrix} \sqrt{E-M} \\ 0 \\ \sqrt{E+M} \\ 0 \end{pmatrix}, \qquad v^{\lambda=-}(p) = \begin{pmatrix} 0 \\ \sqrt{E-M} \\ 0 \\ -\sqrt{E+M} \end{pmatrix}. \tag{4.51}$$

By writing the helicity states in Weyl representation it is easy to project out righthanded (upper components) and lefthanded (lower components). One finds for the helicity states in Weyl representation:

$$u^{\lambda=+}(p) = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{E+M}+\sqrt{E-M} \\ 0 \\ \sqrt{E+M}-\sqrt{E-M} \\ 0 \end{pmatrix}, \quad u^{\lambda=-}(p) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \sqrt{E+M}-\sqrt{E-M} \\ 0 \\ \sqrt{E+M}+\sqrt{E-M} \end{pmatrix},$$

$$v^{\lambda=+}(p) = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{E+M}+\sqrt{E-M} \\ 0 \\ -\sqrt{E+M}+\sqrt{E-M} \\ 0 \end{pmatrix}, \quad v^{\lambda=-}(p) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -\sqrt{E+M}+\sqrt{E-M} \\ 0 \\ \sqrt{E+M}+\sqrt{E-M} \end{pmatrix}.$$

We note that for high energy the positive helicity ($\lambda = +$) is in essence righthanded, while the negative helicity ($\lambda = -$) is in essence lefthanded. The ratio of the components is (directly or inversely) proportional to

$$\frac{\sqrt{E+M}-\sqrt{E-M}}{\sqrt{E+M}+\sqrt{E-M}} = \frac{2M}{(\sqrt{E+M}+\sqrt{E-M})^2},$$

which vanishes in the ultra-relativistic limit $E \gg M$ or in the massless case.

4.4 γ gymnastics

(An excellent overview can be found in the appendix of for instance Itzykson and Zuber) Some properties are listed below:

Properties of products of γ-matrices:
(1) $\gamma^\mu\gamma_\mu = 4$
(2) $\gamma^\mu\,\gamma^\nu\,\gamma_\mu = -2\,\gamma^\nu$ or $\gamma^\mu \not{a} \gamma_\mu = -2\,\not{a}$.
(3) $\gamma^\mu\,\gamma^\rho\gamma^\sigma\,\gamma_\mu = 4\,g^{\rho\sigma}$ or $\gamma^\mu \not{a}\not{b}\gamma_\mu = 4\,a\cdot b$.
(4) $\gamma^\mu\,\gamma^\nu\gamma^\rho\gamma^\sigma\,\gamma_\mu = -2\,\gamma^\sigma\gamma^\rho\gamma^\nu$.
(5) $\gamma^\mu\,\gamma^\lambda\gamma^\nu\gamma^\rho\gamma^\sigma\,\gamma_\mu = 2\left[\gamma^\sigma\gamma^\lambda\gamma^\nu\gamma^\rho + \gamma^\rho\gamma^\nu\gamma^\lambda\gamma^\sigma\right]$.
(6) $\gamma_5 = i\,\gamma^0\gamma^1\gamma^2\gamma^3 = -i\,\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma/4!$.
(7) $\gamma_\mu\gamma_5 = -\gamma_5\gamma_\mu = i\,\epsilon_{\mu\nu\rho\sigma}\gamma^\nu\gamma^\rho\gamma^\sigma/3!$.
(8) $\sigma_{\mu\nu} \equiv (i/2)[\gamma_\mu,\gamma_\nu]$
(9) $\sigma_{\mu\nu}\gamma_5 = \gamma_5\sigma_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}\gamma^\rho\gamma^\sigma/2$.
(10) $\gamma^\mu\gamma^\nu = g^{\mu\nu} - i\,\sigma^{\mu\nu}$
(11) $\gamma^\mu\gamma^\nu\gamma^\rho = S^{\mu\nu\rho\sigma}\gamma_\sigma + i\,\epsilon^{\mu\nu\rho\sigma}\gamma_\sigma\gamma_5$ with $S^{\mu\nu\rho\sigma} = (g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$.

Properties of traces of γ-matrices
(1) $Tr(\gamma^{\mu_1}\cdots\gamma^{\mu_n}) = 0$ if n is odd.
(Use $(\gamma_5)^2 = 1$ and pull one γ_5 through.)
(2) $Tr(1) = 4$.
(3) $Tr(\gamma^\mu\gamma^\nu) = 4\,g^{\mu\nu}$ or $Tr(\not{a}\not{b}) = 4\,a\cdot b$.
(4) $Tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4\,S^{\mu\nu\rho\sigma}$.
(5) $Tr(\gamma^{\mu_1}\cdots\gamma^{\mu_n}) = g^{\mu_1\mu_2}Tr(\gamma^{\mu_3}\cdots\gamma^{\mu_n}) - g^{\mu_1\mu_3}Tr(\gamma^{\mu_2}\gamma^{\mu_4}\cdots\gamma^{\mu_n}) + \cdots$.
(6) $Tr\gamma_5 = 0$.
(7) $Tr(\gamma_5\gamma^\mu\gamma^\nu) = 0$ or $Tr(\gamma_5\not{a}\not{b}) = 0$.
(8) $Tr(\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = -4i\,\epsilon^{\mu\nu\rho\sigma}$.
(9) $Tr(\gamma^{\mu_1}\cdots\gamma^{\mu_{2n}}) = Tr(\gamma^{\mu_{2n}}\cdots\gamma^{\mu_1})$.
(Use matrices C to transpose the matrices in the expression under the trace and use $Tr\,A^T = Tr\,A$.)

In order to show the use of the above relations, we will give one example, namely the calculation of the quantity

$$L^{\mu\nu}(k,k') = \frac{1}{2}\sum_{s,s'} \left(\bar{u}(k,s)\gamma^\mu u(k',s')\right)^* \left(\bar{u}(k,s)\gamma^\nu u(k',s')\right), \tag{4.52}$$

which appears quantum electrodynamics calculations for the emission of a photon from an electron changing its momentum from k to k' ($k^2 = k'^2 = m^2$). In principle one can take a representation and just calculate the quantity $\bar{u}(k,s)\gamma^\mu u(k',s')$, etc. It is, however, more convenient to use the projection operators introduced earlier. For this we first have to proof (do this) that

$$\left(\bar{u}(k,s)\gamma^\mu u(k',s')\right)^* = \bar{u}(k',s')\gamma^\mu u(k,s), \tag{4.53}$$

This leads with explicit Dirac indices to

$$\begin{aligned} L^{\mu\nu} &= \frac{1}{2}\sum_{s,s'} \bar{u}_i(k',s')(\gamma^\mu)_{ij} u_j(k,s)\bar{u}_k(k,s)(\gamma^\nu)_{kl} u_l(k',s') \\ &= \frac{1}{2}\sum_{s,s'} u_l(k',s')\bar{u}_i(k',s')(\gamma^\mu)_{ij} u_j(k,s)\bar{u}_k(k,s)(\gamma^\nu)_{kl} \\ &= \frac{1}{2}\sum_{s,s'} (\not{k}' + m)_{li}(\gamma^\mu)_{ij}(\not{k} + m)_{jk}(\gamma^\nu)_{kl} \\ &= \frac{1}{2} Tr\left[(\not{k}' + m)\gamma^\mu(\not{k} + m)\gamma^\nu\right]. \end{aligned}$$

The trace is linear and can be split up in parts containing traces with up to four gamma-matrices, of which only the traces of four and two gamma-matrices are nonzero. They lead to

$$\begin{aligned} L^{\mu\nu} &= \frac{1}{2} Tr\left[(\not{k}' + m)\gamma^\mu(\not{k} + m)\gamma^\nu\right] \\ &= 2\left[k^\mu k'^\nu + k^\nu k'^\mu - g^{\mu\nu}(k\cdot k' - m^2)\right] \\ &= 2\,k^\mu k'^\nu + 2\,k^\nu k'^\mu + (k-k')^2\, g^{\mu\nu}. \end{aligned} \tag{4.54}$$

4.5 Exercises

Exercise 4.1
Show that $j^\mu = \overline{\psi}\gamma^\mu\psi$ is a conserved current if $\overline{\psi}$ en ψ are solutions of the Dirac equation

Exercise 4.2
Show that

$$P_+ \equiv \sum_{s=\pm} \frac{u^s(p)\bar{u}^s(p)}{2M} = \frac{\not{p} + M}{2M}$$

and show that P_+ is a projection operator.

Exercise 4.3
Show starting from the relation $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$ that

$$\begin{aligned} \gamma_\mu \not{a} \gamma^\mu &= -2\,\not{a}, \\ Tr\not{a}\not{b} &= 4\; a\cdot b, \\ Tr(\gamma^{\mu_1}\cdots\gamma^{\mu_n}) &= 0 \quad \text{if n is odd.} \end{aligned}$$

Excercise 4.4

Show using space-inversion applied to the Dirac equation that the spinor $\psi^p(\boldsymbol{x},t) = \gamma_0 \psi(-\boldsymbol{x},t)$ is also a solution of the Dirac equation.

Exercise 4.5
Show explicitly what the effect is of charge conjugation on a free spinor with positive energy, i.e. what is $u^c(\boldsymbol{p},\uparrow)$.

Exercise 4.6

(a) Show that (in standard representation)

$$\psi(\boldsymbol{x},t) = N \begin{pmatrix} j_0(kr)\chi_m \\ i\,\frac{k}{E+M}\,\sigma\cdot\hat{\boldsymbol{r}}\; j_1(kr)\chi_m \end{pmatrix} e^{-i\,Et},$$

is a solution of the (free) Dirac equation in a spherical cavity with radius R with $k^2 = E^2 - M^2$ and χ_m a two-component spinor.

(b) A condition for confining the fermion in the cavity is $i\,\not{n}\psi = \psi$ (determine the relation for $\overline{\psi}$ and show that this implies $n_\mu j^\mu = \overline{\psi}\not{n}\psi = 0$, i.e. no current is flowing through the surface). What is the condition for the lowest eigenmode in the spherical cavity for which $n^\mu = (0, \hat{\boldsymbol{r}})$. Calculate it for $M = 0$ and $M = \infty$. Plot for $M = 0$ the lower and upper component.

Chapter 5

Maxwell equations

With the electromagnetic field tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{5.1}$$

$$= \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}, \tag{5.2}$$

where A^μ is the four-vector potential

$$A^\mu = (\phi, \boldsymbol{A}), \tag{5.3}$$

and the (conserved) current $j^\mu = (\rho, \boldsymbol{j})$ the Maxwell equations read

$$\partial_\mu F^{\mu\nu} = j^\nu. \tag{5.4}$$

Furthermore, the antisymmetry of $F^{\mu\nu}$ implies that

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \qquad \text{or} \qquad \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0, \tag{5.5}$$

where $\tilde{F}^{\mu\nu} \equiv -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ is the tensor where $\boldsymbol{E} \to \boldsymbol{B}$ and $\boldsymbol{B} \to -\boldsymbol{E}$. It states the absence of magnetic charges and Faraday's law.

For electrodynamics one has the freedom of gauge transformations. Under a gauge transformation

$$A_\mu \longrightarrow A_\mu + \partial_\mu \chi, \tag{5.6}$$

the electric and magnetic fields are unchanged,

$$F_{\mu\nu} \longrightarrow F_{\mu\nu} + (\partial_\mu\partial_\nu\chi - \partial_\nu\partial_\mu\chi) = F_{\mu\nu}. \tag{5.7}$$

The equations of motion for the fields A_μ become

$$\Box A_\mu - \partial_\mu(\partial_\nu A^\nu) = j_\mu. \tag{5.8}$$

This equation is not affected by a gauge transformation. The gauge freedom can be restricted by imposing the Lorentz condition

$$\partial_\mu A^\mu = 0, \tag{5.9}$$

in which case one has

$$\Box A_\mu = j_\mu, \tag{5.10}$$

of which the solutions give the Liénard-Wiechert potentials. The equation in vacuum, $\Box A_\mu = 0$, moreover, shows that the electromagnetic fields correspond to massless particles.

Although the Lorentz condition is a constraint, it does not eliminate the freedom of gauge transformations but they are now restricted to $A_\mu \to A_\mu + \partial_\mu \chi$ with $\Box \chi = 0$. The Lorentz condition can be incorporated by changing the equations of motion to

$$\Box A_\mu + (\lambda - 1)\partial_\mu(\partial_\nu A^\nu) = j_\mu. \tag{5.11}$$

Taking the divergence one has for a conserved current and $\lambda \neq 0$ the condition

$$\Box \partial_\mu A^\mu = 0. \tag{5.12}$$

Hence, if $\partial_\mu A^\mu = 0$ for large times $|t|$, it will vanish at all times. This is not changed if in Eq. 5.11 the d'Alembertian is replaced by $\Box + M^2$, corresponding to a massive field.

The solutions of $(\Box + M^2)A_\mu = 0$ are

$$A_\mu(x) = \epsilon_\mu(k)\, e^{\pm ik\cdot x}. \tag{5.13}$$

The Lorentz condition $\partial_\mu A^\mu = 0$ implies that $k^\mu\, \epsilon_\mu(k) = 0$. In the restframe there are three independent possibilities for $\epsilon_\mu(k)$, namely $(0, \boldsymbol{\epsilon}_\lambda)$, where the vectors $\boldsymbol{\epsilon}_\lambda$ are the ones discussed in section 3.2, forming the basis for a spin 1 representation. The vectorfield is therefore suited to describe spin 1. In an arbitrary frame $\epsilon_\mu^{(\lambda)}(k)$ with $\lambda = 0, \pm(1)$ are obtained by boosting the restframe vectors. The (real) field $A_\mu(x)$, expanded in modes is given by

$$A_\mu(x) = \sum_{\lambda=0,\pm} \int \frac{d^3k}{(2\pi)^3\, 2E} \left(\epsilon_\mu^{(\lambda)}(k)\, e^{-ik\cdot x}\, c(\boldsymbol{k},\lambda) + \epsilon_\mu^{(\lambda)*}(k)\, e^{ik\cdot x}\, c^*(\boldsymbol{k},\lambda)\right). \tag{5.14}$$

Since the vectors $\epsilon_\mu^{(\lambda)}(k)$ together with the momentum k^μ form a complete set of four-vectors, one has

$$\sum_{\lambda=0,\pm} \epsilon_\mu^{(\lambda)}(k)\, \epsilon_\nu^{(\lambda)*}(k) = -g_{\mu\nu} + \frac{k_\mu k_\nu}{M^2}. \tag{5.15}$$

For $M = 0$, there is still a gauge freedom left ($A_\mu \to A_\mu + \partial_\mu \chi$ with $\Box\chi = 0$). One possible choice is the gauge

$$A_0 = 0 \tag{5.16}$$

(radiation, transverse or Coulomb gauge). In that case one also has $\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$ or $k_i\, \epsilon_i(k) = 0$. Therefore if

$$k^\mu = (|\boldsymbol{k}|, 0, 0, |\boldsymbol{k}|), \tag{5.17}$$

only two independent vectors remain

$$\epsilon^{(\pm)\,\mu}(k) = (0, \mp 1, -i, 0), \tag{5.18}$$

corresponding to the helicity states $\lambda = \pm 1$ of a massless photon.

Chapter 6

Classical lagrangian field theory

6.1 Euler-Lagrange equations

In the previous chapter we have seen the equations for scalar fields (Klein-Gordon equation), Dirac fields (Dirac equation) and massless vector fields (Maxwell equations) and corresponding to these fields conserved currents describing the probability and probability current. These equations can be obtained from a lagrangian using the action principle.

As an example, recall classical mechanics with the action

$$S = \int_{t_1}^{t_2} dt\ L(x, \dot{x}) \tag{6.1}$$

and as an example the lagrangian

$$L(x, \dot{x}) = K - V = \frac{1}{2} m \dot{x}^2 - V(x). \tag{6.2}$$

The principle of minimal action looks for a stationary action under variations in the coordinates and time, thus

$$t' = t + \delta\tau, \tag{6.3}$$

$$x'(t) = x(t) + \delta x(t), \tag{6.4}$$

and the total change

$$x'(t') = x(t) + \Delta x(t), \tag{6.5}$$

with $\Delta x(t) = \delta x(t) + \dot{x}(t)\,\delta\tau$. The requirement $\delta S = 0$ with fixed boundaries $x(t_1) = x_1$ and $x(t_2) = x_2$ leads to

$$\begin{aligned} \delta S &= \int_{t_1}^{t_2} dt\ \left\{ \frac{\partial L}{\partial x}\,\delta x + \frac{\partial L}{\partial \dot{x}} \delta \dot{x} \right\} + L\,\delta\tau \Big|_{t_1}^{t_2} \\ &= \int_{t_1}^{t_2} dt\ \left\{ \frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) \right\} \delta x + \left(\frac{\partial L}{\partial \dot{x}}\,\delta x + L\,\delta\tau \right) \Big|_{t_1}^{t_2}. \end{aligned} \tag{6.6}$$

The first term leads to the Euler-Lagrange equations,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}\ . \tag{6.7}$$

The quantity $\partial L/\partial \dot{x}$ plays a special role and is known as the *canonical momentum*,

$$p = \frac{\partial L}{\partial \dot{x}}\ . \tag{6.8}$$

For the lagrangian specified above this leads directly to Newton's law $\dot{p} = m\ddot{x} = -\partial V/\partial x$. The second term can be rewritten as

$$\delta S = \ldots + \left.\left(\frac{\partial L}{\partial \dot{x}}\,\Delta x - H\,\delta\tau\right)\right|_{t_1}^{t_2}, \tag{6.9}$$

which is done because the first term (multiplying Δx, which in classical mechanics vanishes at the boundary) does not play a role. The *hamiltonian* H is defined by

$$H(p,x) \equiv p\,\dot{x} - L\ . \tag{6.10}$$

One sees that invariance under time translations requires that $H(t_1) = H(t_2)$, i.e. H is a conserved quantity.

In classical field theory one proceeds in complete analogy but using functions depending on space and time (classical fields, think for instance of a temperature or density distribution or of an electromagnetic field). Consider a lagrangian density $\mathcal{L}$ which depends on these functions, their derivatives and possibly on the position, $\mathcal{L}(\phi(x), \partial_\mu\phi(x), x)$ and an action

$$\begin{aligned} S = \int_{t_1}^{t_2} dt\ L \quad &= \quad \int dt\, d^3x\ \ \mathcal{L}(\phi(x), \partial_\mu\phi(x)) && (6.11)\\ &= \quad \int_R d^4x\, \mathcal{L}(\phi(x), \partial_\mu\phi(x)). && (6.12) \end{aligned}$$

Here R indicates a space-time volume bounded by (R^3, t_1) and $R^3, t_2)$, also indicated by ∂R (a more general volume in four-dimensional space-time with some boundary ∂R can also be considered). Variations in the action can come from the coordinates or the fields, indicated as

$$\begin{aligned} x'^\mu &= x^\mu + \delta x^\mu, && (6.13)\\ \phi'(x) &= \phi(x) + \delta\phi(x) && (6.14) \end{aligned}$$

or combined

$$\phi'(x') = \phi(x) + \Delta\phi(x), \tag{6.15}$$

with

$$\Delta\phi(x) = \delta\phi(x) + (\partial_\mu\phi)\delta x^\mu. \tag{6.16}$$

The resulting variation of the action is

$$\delta S = \int_R d^4x'\ \ \mathcal{L}(\phi', \partial_\mu\phi', x') - \int_R d^4x\ \ \mathcal{L}(\phi, \partial_\mu\phi, x). \tag{6.17}$$

The change in variables $x \to x'$ in the integration volume involves a surface variation of the form

$$\int_{\partial R} d\sigma_\mu\ \ \mathcal{L}\ \delta x^\mu.$$

Note for the specific choice of the surface for constant times t_1 and t_2,

$$\int_{\partial R} d\sigma_\mu\ \ \ldots = \int_{(R^3,t_2)} d^3x\ \ \ldots - \int_{(R^3,t_1)} d^3x\ \ \ldots . \tag{6.18}$$

Furthermore the variations $\delta\phi$ and $\delta\partial_\mu\phi$ contribute to δS, giving[1]

$$\begin{aligned} \delta S &= \int_R d^4x \left[\frac{\delta\,\mathcal{L}}{\delta(\partial_\mu\phi)}\,\delta(\partial_\mu\phi) + \frac{\delta\,\mathcal{L}}{\delta\phi}\,\delta\phi\right] + \int_{\partial R} d\sigma_\mu\ \ \mathcal{L}\ \delta x^\mu\\ &= \int_R d^4x \left[\frac{\delta\,\mathcal{L}}{\delta\phi} - \partial_\mu\left(\frac{\delta\,\mathcal{L}}{\delta(\partial_\mu\phi)}\right)\right]\delta\phi + \int_{\partial R} d\sigma_\mu\ \left[\frac{\delta\,\mathcal{L}}{\delta(\partial_\mu\phi)}\,\delta\phi + \mathcal{L}\ \delta x^\mu\right]. && (6.19) \end{aligned}$$

[1] Taking a functional derivative, indicated with $\delta F[\phi]/\delta\phi$ should pose no problems. We will come back to it in a bit more formal way in section 9.2.

With for the situation of classical fields all variations of the fields and coordinates at the surface vanishing, the second term is irrelevant. The integrand of the first term must vanish, leading to the Euler-Lagrange equations,

$$\partial_\mu \left(\frac{\delta\,\mathcal{L}}{\delta(\partial_\mu \phi)} \right) = \frac{\delta\,\mathcal{L}}{\delta\phi}. \tag{6.20}$$

6.2 Lagrangians for spin 0, 1/2 and 1 fields

By an appropriate choice of lagrangian density the equations of motion discussed in previous chapters for the scalar field (spin 0), the Dirac field (spin 1/2) and the vector field (spin 1) can be recovered,

6.2.1 The scalar field

It is straightforward to derive the equations of motion for a real scalar field ϕ from the lagrangian density

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\,\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\,M^2\,\phi^2 & (6.21)\\ &= -\frac{1}{2}\,\phi\left(\partial_\mu\partial^\mu\phi + \frac{1}{2}\,M^2\right)\phi, & (6.22)\end{aligned}$$

leading to

$$(\Box + M^2)\phi(x) = 0. \tag{6.23}$$

For the complex scalar field one conventionally uses

$$\begin{aligned} \mathcal{L} &= \partial_\mu\phi^*\partial^\mu\phi - M^2\,\phi^*\phi & (6.24)\\ &= -\phi^*\left(\partial_\mu\phi^*\partial^\mu\phi + M^2\right)\phi, & (6.25)\end{aligned}$$

which can be considered as the sum of the lagrangian densities for two scalar fields ϕ_1 and ϕ_2 with $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$. One easily obtains

$$\begin{aligned} (\Box + M^2)\phi(x) &= 0, & (6.26)\\ (\Box + M^2)\phi^*(x) &= 0. & (6.27)\end{aligned}$$

6.2.2 The Dirac field

The appropriate lagrangian from which to derive the equations of motion is

$$\begin{aligned} \mathcal{L} &= \frac{i}{2}\overline{\psi}\overset{\leftrightarrow}{\partial\!\!\!/}\,\psi - M\,\overline{\psi}\psi = \frac{i}{2}\overline{\psi}\overset{\rightarrow}{\partial\!\!\!/}\,\psi - \frac{i}{2}\overline{\psi}\overset{\leftarrow}{\partial\!\!\!/}\,\psi - M\,\overline{\psi}\psi & (6.28)\\ &= \overline{\psi}\,(i\partial\!\!\!/ - M)\,\psi. & (6.29)\end{aligned}$$

Using the variations in $\overline{\psi}$,

$$\begin{aligned} \frac{\delta\,\mathcal{L}}{\delta(\partial_\mu\overline{\psi})} &= -\frac{i}{2}\gamma^\mu\psi \\ \frac{\delta\,\mathcal{L}}{\delta\overline{\psi}} &= \frac{i}{2}\,\overset{\rightarrow}{\partial\!\!\!/}\,\psi - M\,\psi, \end{aligned}$$

one obtains immediately

$$\left(i\,\overset{\rightarrow}{\partial\!\!\!/} - M\right)\,\psi = 0, \tag{6.30}$$

and similarly from the variation with respect to ψ

$$\overline{\psi}\,\left(i\,\overset{\leftarrow}{\partial\!\!\!/} + M\right) = 0. \tag{6.31}$$

6.2.3 Vector field

From the lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2} M^2 V_\mu V^\mu - \frac{\lambda}{2} (\partial_\mu V^\mu)^2 \tag{6.32}$$

$$= \frac{1}{2} V^\mu \left[(\partial^2 + M^2)g_{\mu\nu} - (1-\lambda)\partial_\mu\partial_\nu\right] V^\nu, \tag{6.33}$$

one immediately obtains

$$\frac{\delta\,\mathcal{L}}{\delta(\partial_\mu V_\nu)} = -\partial^\mu V^\nu + \partial^\nu V^\mu - \lambda\, g^{\mu\nu}\,(\partial_\rho V^\rho),$$

leading to the equations of motion

$$\left(\Box + M^2\right)\; V^\mu - (1-\lambda)\partial^\mu(\partial_\nu V^\nu) = 0, \tag{6.34}$$

implying

$$\partial_\mu F^{\mu\nu} + M^2\, V^\mu V_\mu = 0 \qquad \text{and} \qquad \partial_\mu V^\mu = 0. \tag{6.35}$$

6.2.4 General lagrangian for spin 1/2 fields

The most general lagrangian density for spin 1/2 representations invariant under boosts and rotations written down in terms of *Weyl spinors* $\xi(x)$ and $\eta(x)$ of type I and II respectively, is

$$\begin{aligned}\mathcal{L} = {} & \frac{1}{2}\xi^\dagger(i\,\sigma^\mu \overleftrightarrow{\partial}_\mu)\xi + \frac{1}{2}\eta^\dagger(i\,\tilde\sigma^\mu \overleftrightarrow{\partial}_\mu)\eta - \frac{1}{2}\left(M_D\,\eta^\dagger\xi - M_D^*\,\eta^T\xi^*\right) - \frac{1}{2}\left(M_D^*\,\xi^\dagger\eta - M_D\,\xi^T\eta^*\right) \\ & - \frac{1}{2}\left(M_M^*\,\xi^\dagger\epsilon\xi^* - M_M\,\xi^T\epsilon\xi\right) - \frac{1}{2}\left(M_M\,\eta^\dagger\epsilon\eta^* - M_M^*\,\eta^T\epsilon\eta\right),\end{aligned} \tag{6.36}$$

where M_M and M_D are referred to as Majorana and Dirac masses[2]. As we have seen the conjugate spin 1/2 spinors ξ^* and η^* are not appropriate type I or type II spinors. Proper type I spinors are ξ and $\epsilon\eta^*$, proper type II spinors are η and $\epsilon\xi^*$.

The equations of motion can be obtained easily using the Euler-Lagrange equations, e.g.

$$i\,\tilde\sigma^\mu\partial_\mu\,\eta - M_M\,\epsilon\,\eta^* - M_D^*\,\xi = 0, \tag{6.37}$$

$$i\,\sigma^\mu\partial_\mu\,\xi - M_M\,\epsilon\,\xi^* - M_D\eta = 0, \tag{6.38}$$

$$i\,\tilde\sigma^\mu\partial_\mu\,\epsilon\,\xi^* - M_M^*\,\xi + M_D\epsilon\,\eta^* = 0, \tag{6.39}$$

implying also $(\partial^2 + |M_M|^2 + |M_D|^2)\,\xi = 0$ and similarly for η. Thus the solutions are plane waves, but the (linear) equations impose conditions. Writing

$$\xi = \xi(p)\,e^{-i\,p\cdot x} \qquad \text{with} \quad \xi(p) = H(p)\xi_0, \tag{6.40}$$

$$\eta = \eta(p)\,e^{-i\,p\cdot x} \qquad \text{with} \quad \eta(p) = \bar H(p)\eta_0, \tag{6.41}$$

one obtains with $M = \sqrt{|M_M|^2 + |M_D|^2}$

$$\begin{aligned} M\,\xi_0 - M_M\,\epsilon\xi_0^* - M_D\,\eta_0 &= 0, \\ M\,\eta_0 - M_M\,\epsilon\eta_0^* - M_D^*\,\xi_0 &= 0. \end{aligned} \tag{6.42}$$

[2] This lagrangian is written down with ξ and η being Grassmann numbers for which

$$\alpha\beta = -\beta\alpha,$$
$$(\alpha\beta)^* = \beta^*\alpha^* = -\alpha^*\beta^*,$$

and thus $(\beta^*\alpha)^* = \alpha^*\beta = -\beta\alpha^*$. For (ordinary) complex-valued functions the minus signs in the pairwise grouped mass-terms become plus signs, while actually the Majorana-terms do not appear. The reasons for Grassmann variables will become clear in the next chapter.

Special cases are:

Massless fermions

For massless fermions, $M_D = M_M = 0$ one has definite helicity states

$$\sigma^\mu p_\mu\,\xi(p) = 0 \longrightarrow (E - \sigma\cdot\boldsymbol{p})\,\xi(p) = 0 \longrightarrow \xi(p) = \xi_+(p), \tag{6.43}$$

$$\tilde{\sigma}^\mu p_\mu\,\eta(p) = 0 \longrightarrow (E + \sigma\cdot\boldsymbol{p})\,\eta(p) = 0 \longrightarrow \eta(p) = \eta_-(p). \tag{6.44}$$

Dirac fermion

For a *Dirac fermion* one considers the case $M_M = 0$. Writing M_D as $|M_D|\,e^{i\phi}$, one sees from the constraint in Eq. 6.42 that it requires $\eta_0 = e^{-i\phi}\,\xi_0$. In order to incorporate parity, the fields ξ and η in the Lagrangian

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\,\xi^\dagger(i\,\sigma^\mu \overleftrightarrow{\partial}_\mu)\xi + \frac{1}{2}\,\eta^\dagger(i\,\tilde{\sigma}^\mu \overleftrightarrow{\partial}_\mu)\eta - M_D\,\eta^\dagger\xi - M_D^*\,\xi^\dagger\eta \\ &= \begin{pmatrix}\xi^\dagger & \eta^\dagger\end{pmatrix}\begin{pmatrix}\frac{1}{2}\,i\sigma^\mu \overleftrightarrow{\partial}_\mu & -M_D^* \\ -M_D & \frac{1}{2}\,i\tilde{\sigma}^\mu \overleftrightarrow{\partial}_\mu\end{pmatrix}\begin{pmatrix}\xi \\ \eta\end{pmatrix} = \begin{pmatrix}\eta^\dagger & \xi^\dagger\end{pmatrix}\begin{pmatrix}-M_D & \frac{1}{2}\,i\tilde{\sigma}^\mu \overleftrightarrow{\partial}_\mu \\ \frac{1}{2}\,i\sigma^\mu \overleftrightarrow{\partial}_\mu & -M_D^*\end{pmatrix}\begin{pmatrix}\xi \\ \eta\end{pmatrix}\end{aligned} \tag{6.45}$$

are written in terms of a four-component *Dirac spinor*

$$\psi \equiv \begin{pmatrix}\xi \\ \eta\end{pmatrix} \tag{6.46}$$

and we get

$$\mathcal{L} = \overline{\psi}\left(\frac{1}{2}\,i\gamma^\mu \overleftrightarrow{\partial}_\mu - M - iM'\gamma_5\right)\psi \tag{6.47}$$

with

$$\gamma^0 = \begin{pmatrix}0 & \boldsymbol{I} \\ \boldsymbol{I} & 0\end{pmatrix}, \qquad \gamma^i = \begin{pmatrix}0 & -\sigma^i \\ \sigma^i & 0\end{pmatrix}, \tag{6.48}$$

$\overline{\psi} \equiv \psi^\dagger\,\gamma^0$ and $M = \text{Re}\ M_D$, $M' = \text{Im}\ M_D$. Under parity and charge conjugation one has (omitting the space-time arguments which also change)

$$\psi = \begin{pmatrix}\xi \\ \eta\end{pmatrix} \xrightarrow{\mathcal{P}} \begin{pmatrix}\eta \\ \xi\end{pmatrix} = \gamma^0\,\psi \tag{6.49}$$

$$\psi = \begin{pmatrix}\xi \\ \eta\end{pmatrix} \xrightarrow{\mathcal{C}} \begin{pmatrix}-\epsilon\,\eta^* \\ \epsilon\,\xi^*\end{pmatrix} = C\,\overline{\psi}^T \qquad \text{with } C = i\gamma^2\gamma^0 = \begin{pmatrix}-i\sigma^2 & 0 \\ 0 & i\sigma^2\end{pmatrix}. \tag{6.50}$$

For a Dirac fermion parity is a good symmetry if $\xi_0 = \eta_0 = \chi$, which means M_D is real or $M' = 0$. Plane wave solutions have been considered in section 4.3. Note that the Dirac Lagrangian split up into lefthanded and righthanded fields, $\psi_{R/L} = \frac{1}{2}(1 \pm \gamma_5)\psi$, is (of course) precisely of the form of the $\xi - \eta$ lagrangian,

$$\mathcal{L} = \overline{\psi}_R\frac{1}{2}\,i\gamma^\mu \overleftrightarrow{\partial}_\mu\ \psi_R + \overline{\psi}_L\frac{1}{2}\,i\gamma^\mu \overleftrightarrow{\partial}_\mu\ \psi_L - M\,(\overline{\psi}_R\,\psi_L + \overline{\psi}_L\,\psi_R). \tag{6.51}$$

Majorana fermion

For the case $M_D = 0$ we have (taking just the lefthanded species)

$$\mathcal{L} = \frac{1}{4}\,\eta^\dagger(i\,\tilde{\sigma}^\mu \overleftrightarrow{\partial}_\mu)\eta - \frac{1}{4}\,\eta^T\,\epsilon(i\,\sigma^\mu \overleftrightarrow{\partial}_\mu)\epsilon\,\eta^* - \frac{M}{2}\,\left(\eta^\dagger\epsilon\eta^* - \eta^T\epsilon\eta\right), \tag{6.52}$$

where the mass can be chosen real.

We can again introduce a Dirac field by combining η and $-\epsilon\,\eta^*$ into one four-component *Majorana spinor*,

$$\psi \equiv \begin{pmatrix} -\epsilon\,\eta^* \\ \eta \end{pmatrix} \tag{6.53}$$

for which one finds the familiar lagrangian,

$$\mathcal{L} = \overline{\psi}\left(\frac{1}{2}\,i\gamma^\mu \overset{\leftrightarrow}{\partial_\mu} - M_\xi\right)\psi \tag{6.54}$$

Under charge conjugation the Majorana spinor gives

$$\psi \xrightarrow{\mathcal{C}} C\overline{\psi}^T = \psi, \tag{6.55}$$

i.e. we have a self-conjugate spinor (which can be considerd the definition).

For Majorana fermions one can have mass mixing. To investigate this case, it is convenient to rewrite the fields ξ and η as $\eta = \chi_1$ and $\xi = \epsilon\,\chi_2^*$. allowing for a Dirac mass and two different Majorana masses. This leads to the lagrangian

$$\begin{aligned} \mathcal{L} \;=\; & \frac{1}{2}\,\chi_1^\dagger(i\,\tilde{\sigma}^\mu \overset{\leftrightarrow}{\partial}_\mu)\chi_1 + \frac{1}{2}\,\chi_2^\dagger(i\,\tilde{\sigma}^\mu \overset{\leftrightarrow}{\partial}_\mu)\chi_2 \\ & - \frac{1}{2}\,\left(M_D\,\chi_1^\dagger\epsilon\,\chi_2^* - M_D^*\,\chi_1^T\epsilon\,\chi_2\right) - \frac{1}{2}\,\left(M_D\,\chi_2^\dagger\epsilon\,\chi_1^* - M_D^*\,\chi_2^T\epsilon\,\chi_1\right) \\ & - \frac{1}{2}\,\left(M_1\,\chi_1^\dagger\epsilon\chi_1^* - M_1^*\,\chi_1^T\epsilon\chi_1\right) - \frac{1}{2}\,\left(M_2\,\chi_2^\dagger\epsilon\chi_2^* - M_2^*\,\chi_2^T\epsilon\chi_2\right). \end{aligned} \tag{6.56}$$

This is the lagrangian for two lefthanded fermions with a mass matrix

$$M = \begin{pmatrix} M_1 & M_D \\ M_D & M_2 \end{pmatrix} = \begin{pmatrix} M_1 & |M_D|\,e^{i\phi} \\ |M_D|\,e^{i\phi} & M_2 \end{pmatrix}, \tag{6.57}$$

assuming in the last expression that M_1 and M_2 are real and non-negative. This choice is possible without loss of generality because the phases can be absorbed into χ_1 and χ_2.
This is a mixing problem (with an anti-hermitean mass matrix) leading to two (real) mass eigenstates. Assuming the mass matrix to be diagonal for $\phi = U\chi$ one sees that $U^*\,M\,U^\dagger = U_0$. This implies $U\,M^\dagger\,U^T = M_0$ and thus a 'normal' diagonalization of the (hermitean) matrix $M^\dagger M$,

$$U\,(M^\dagger M)\,U^\dagger = M_0^2, \tag{6.58}$$

Thus one obtains from

$$M^\dagger M = \begin{pmatrix} M_1^2 + |M_D|^2 & M_D\,\left(M_1\,e^{i\phi} + M_2\,e^{-i\phi}\right) \\ M_D\,\left(M_1\,e^{-i\phi} + M_2\,e^{i\phi}\right) & M_1^2 + |M_D|^2 \end{pmatrix}, \tag{6.59}$$

the eigenvalues

$$m_{1/2}^2 = \frac{1}{2}\left[M_1^2 + M_2^2 + 2|M_D|^2 \pm \sqrt{(M_1^2 - M_2^2)^2 + 4|M_D|^2\,\left(M_1^2 + M_2^2 + 2M_1M_2\,\cos(2\phi)\right)}\right], \tag{6.60}$$

and we are left with the case $M_D = 0$ of two decoupled (lefthanded) Majorana fields

6.3 Symmetries and conserved currents

For quantum fields (ϕ operator!) and in the case of the existence of symmetries variations at the surface become important. In the first case it is not possible to specify for instance ϕ and $\dot{\phi}$ on the surface σ_1. Also when the lagrangian is invariant under symmetries one can consider variations at the initial or final surface that do not affect the dynamics. Returning to δS the surface term is rewritten to

$$\begin{aligned}\delta S &= \int_R d^4x \ \ldots + \int_{\partial R} d\sigma_\mu \left\{ \frac{\delta\,\mathcal{L}}{\delta(\partial_\mu\phi)}\,\Delta\phi - \left[\frac{\delta\,\mathcal{L}}{\delta(\partial_\mu\phi)}\,\partial^\nu\phi - \mathcal{L}\,g^{\mu\nu}\right]\delta x_\nu \right\} \\ &\equiv \int_R d^4x \ \ldots + \int_{\partial R} d\sigma_\mu \left\{ \frac{\delta\,\mathcal{L}}{\delta(\partial_\mu\phi)}\,\Delta\phi - \theta^{\mu\nu}(x)\delta x_\nu \right\}, \end{aligned} \tag{6.61}$$

where

$$\theta^{\mu\nu} = \frac{\delta\,\mathcal{L}}{\delta(\partial_\mu\phi)}\,\partial^\nu\phi - \mathcal{L}\,g^{\mu\nu}. \tag{6.62}$$

The variation δS thus can be expressed as

$$\delta S \;=\; F(\sigma_1) - F(\sigma_2) = \int_{\partial R} d\sigma_\mu\ J^\mu(x) = \int_R d^4x\ \partial_\mu J^\mu(x), \tag{6.63}$$

with

$$J^\mu(x) = -\frac{\delta\,\mathcal{L}}{\delta(\partial_\mu\phi)}\,\Delta\phi + \theta^{\mu\nu}(x)\delta x_\nu. \tag{6.64}$$

Thus considering $\partial R = \sigma_2$ - σ_1 with $\sigma = (R^3, t)$, the presence of a symmetry that leaves the lagrangian invariant, requires the presence of a conserved quantity, $F(t_1) = F(t_2)$, which is the space-integral over the zero-component of a conserved current, $\partial_\mu J^\mu(x) = 0$,

$$F(t) = \int d^3x\ J^0(\boldsymbol{x}, t), \tag{6.65}$$

In the case of quantum fields, discussed in the next chapter, these conserved quantities become operators, which in a consistent picture precisely generate the symmetries.

As an example consider $U(1)$ transformations of the Dirac field proportional to the charge e,

$$\psi(x) \rightarrow e^{i\,e\Lambda}\psi(x) \quad \text{or} \quad \Delta\psi(x) = i\,e\Lambda\,\psi(x). \tag{6.66}$$

From the lagrangian for the Dirac field, we obtain (since $\delta x^\mu = 0$), omitting the small parameter Λ

$$\begin{aligned} j^\mu &= -\frac{\delta\,\mathcal{L}}{\delta(\partial_\mu\overline{\psi}^i)}\cdot(-i\,e\,\overline{\psi}^i) - \frac{\delta\,\mathcal{L}}{\delta(\partial_\mu\psi^i)}\cdot(i\,e\,\psi^i) \\ &= e\,\overline{\psi}\gamma^\mu\psi. \end{aligned} \tag{6.67}$$

For the complex scalar field the $U(1)$ transformations leave the lagrangian invariant and lead to the conserved current

$$j^\mu = i\,e\,\phi^* \overleftrightarrow{\partial_\mu}\,\phi. \tag{6.68}$$

These currents are conserved as discussed already in chapter 2. The integral over the zero-component, $Q = \int d^3x\, j^0(\boldsymbol{x}, t)$ is the conserved charge ($\dot{Q} = 0$). In the next section we will see the quantity show up as the charge operator.

6.4 Space-time symmetries

One kind of symmetries that leave the lagrangian invariant are the Poincaré transformations, the space-time translations and the Lorentz transformations.

6.4.1 Translations

Under translations (generated by P^{μ}_{op}) we have

$$\begin{aligned} \delta x^{\mu} &= \epsilon^{\mu}, & (6.69)\\ \Delta\phi(x) &= 0, & (6.70)\\ \delta\phi(x) &= -(\partial_{\mu}\phi)\delta x^{\mu} = -\epsilon^{\mu}\partial_{\mu}\phi(x). & (6.71) \end{aligned}$$

The behavior of the field under translations ($\delta\phi$) is governed by the translational behavior of the argument in such a way that $\Delta\phi = 0$. From Noether's theorem one sees that the current $\theta^{\mu\nu}\epsilon_{\nu}$ is conserved. Therefore there are four conserved currents $\theta^{\mu\nu}$ (ν labeling the currents!) and four conserved quantities

$$P^{\nu} = \int d\sigma_{\mu}\,\theta^{\mu\nu} = \int d^3x\,\theta^{0\nu}. \tag{6.72}$$

These are the energy and momentum. For quantized fields it will become the expressions of the hamiltonian and the momentum operators in terms of the fields, e.g. $H = \int d^3x\,\theta^{00}(x)$.

6.4.2 Lorentz transformations

In this case the transformation of the coordinates and fields are written as

$$\begin{aligned} \delta x^{\mu} &= \omega^{\mu\nu}x_{\nu}, \qquad (\omega^{\mu\nu}\text{antisymmetric}) & (6.73)\\ \Delta\phi^i(x) &= \frac{1}{2}\,\omega_{\rho\sigma}(i\,S^{\rho\sigma})^i_{\;j}\,\phi^j(x). & (6.74) \end{aligned}$$

Note that the coordinate transformations can be written in a form similar to that for the fields,

$$\Delta x^{\mu} = \frac{1}{2}\,\omega_{\rho\sigma}(a^{\rho\sigma})^{\mu}_{\;\nu}\,x^{\nu} \tag{6.75}$$

with

$$(a_{\rho\sigma})^{\mu}{}_{\nu} = g^{\rho\mu}g^{\sigma}{}_{\nu} - g^{\sigma\mu}g^{\rho}{}_{\nu}. \tag{6.76}$$

For the fields the behavior under Lorentz transformations has been the subject of chapter 3. Summarizing,

- **Scalar field** ϕ: $i\,S_{\rho\sigma} = 0$.
- **Dirac field** ψ: $i\,S_{\rho\sigma} = (1/4)[\gamma_{\rho},\gamma_{\sigma}]$
  This result for the Dirac field is the same as discussed in chapter 3, but stated as the general result. Any specific behavior under Lorentz transformations may be found by substituting a specific representation of the Dirac matrices. The general behavior (expressed in Dirac matrices) can be obtained by requiring them to behave as a four-vector, i.e. when $\psi \rightarrow L\psi$ with $L = 1 + (i/2)\omega_{\rho\sigma}S^{\rho\sigma}$ one has

$$\begin{aligned} &L^{-1}\gamma^{\mu}L = \Lambda^{\mu}_{\nu}\gamma^{\nu} \quad \text{with} \quad \Lambda^{\mu}_{\nu} = g^{\mu}_{\nu} + \omega^{\mu}_{\nu}\\ &[\gamma^{\mu}, i\,S^{\rho\sigma}] = (a^{\rho\sigma})^{\mu}_{\nu}\gamma^{\nu} = g^{\rho\mu}\gamma^{\sigma} - g^{\sigma\mu}\gamma^{\rho},\\ &i\,S^{\rho\sigma} = \frac{1}{4}[\gamma^{\rho},\gamma^{\sigma}] = -\frac{i}{2}\sigma^{\rho\sigma}. \end{aligned}$$

- **Vector field** A_{μ}: $i\,S_{\rho\sigma} = a_{\rho\sigma}$.

The current following from Noether's theorem is

$$
\begin{aligned}
J^{\mu} &= -\frac{\delta \mathcal{L}}{\delta(\partial_{\mu}\phi^{i})}\,\Delta\phi^{i} + \theta^{\mu\nu}\delta x_{\nu} \\
&= \frac{1}{2}\omega_{\rho\sigma}\left\{-\frac{\delta \mathcal{L}}{\delta(\partial_{\mu}\phi^{i})}\,(i\,S^{\rho\sigma})^{i}_{\;j}\phi^{j} + \theta^{\mu\rho}x^{\sigma} - \theta^{\mu\sigma}x^{\rho}\right\} && (6.77) \\
&= \frac{1}{2}\omega_{\rho\sigma}\left\{H^{\mu\rho\sigma} + \theta^{\mu\rho}x^{\sigma} - \theta^{\mu\sigma}x^{\rho}\right\} && (6.78) \\
&= \frac{1}{2}\omega_{\rho\sigma}\mathcal{M}^{\mu\rho\sigma}. && (6.79)
\end{aligned}
$$

Therefore there are six conserved currents $\mathcal{M}^{\mu\rho\sigma}$ labeled by ρ and σ (antisymmetric) and corresponding to it there exist conserved quantities

$$M^{\rho\sigma} = \int d^3x\; \mathcal{M}^{0\rho\sigma}. \tag{6.80}$$

A final note concernes the symmetry properties of $\theta^{\mu\nu}$. In general this tensor is not symmetric. In some applications (specifically coupling to gravity) it is advantageous to have an equivalent current that is symmetric. Defining

$$T^{\mu\nu} = \theta^{\mu\nu} - \partial_{\rho}G^{\rho\mu\nu}, \tag{6.81}$$

with $G^{\rho\mu\nu} = (H^{\rho\mu\nu} + H^{\mu\nu\rho} + H^{\nu\mu\rho})/2$, where $H^{\rho\mu\nu}$ is the quantity appearing in $\mathcal{M}^{\rho\mu\nu}$, one has a tensor that satisfies

$$
\begin{aligned}
&T^{\mu\nu} = T^{\nu\mu}, && (6.82) \\
&\partial_{\mu}T^{\mu\nu} = \partial_{\mu}\theta^{\mu\nu}, && (6.83) \\
&\mathcal{M}^{\mu\rho\sigma} = T^{\mu\rho}x^{\sigma} - T^{\mu\sigma}x^{\rho}. && (6.84)
\end{aligned}
$$

6.5 (Abelian) gauge theories

In section 6.2 we have seen *global gauge transformations* or gauge transformations of the first kind, e.g.

$$\phi(x) \rightarrow e^{i\,e\Lambda}\phi(x), \tag{6.85}$$

in which the $U(1)$ transformation goes over an angle $e\,\Lambda$, independent of x.

Gauge transformations of the second kind or *local gauge transformations* are transformations of the type

$$\phi(x) \rightarrow e^{i\,e\Lambda(x)}\phi(x), \tag{6.86}$$

i.e. the angle of the transformation depends on the space-time point x. The lagrangians which we have considered sofar are invariant under global gauge transformations and corresponding to this there exist a conserved Noether current. Any lagrangian containing derivatives, however, is not invariant under local gauge transformations,

$$
\begin{aligned}
\phi(x) &\rightarrow e^{i\,e\Lambda(x)}\phi(x), && (6.87) \\
\phi^{*}(x) &\rightarrow e^{-i\,e\Lambda(x)}\phi^{*}(x), && (6.88) \\
\partial_{\mu}\phi(x) &\rightarrow e^{i\,e\Lambda(x)}\partial_{\mu}\phi(x) + i\,e\,\partial_{\mu}\Lambda(x)\,e^{i\,e\Lambda(x)}\phi(x), && (6.89)
\end{aligned}
$$

where the last term spoils gauge invariance.

A solution is the one known as *minimal substitution* in which the derivative ∂_{μ} is replaced by a *covariant derivative* D_{μ} which satisfies

$$D_{\mu}\phi(x) \rightarrow e^{i\,e\Lambda(x)}D_{\mu}\phi(x). \tag{6.90}$$

For this purpose it is necessary to introduce a vector field A_μ,

$$D_\mu \phi(x) \equiv (\partial_\mu + i\,eA_\mu(x))\phi(x), \tag{6.91}$$

The required transformation for D_μ then demands

$$\begin{aligned}
D_\mu \phi(x) &= (\partial_\mu + i\,eA_\mu(x))\phi(x) \\
&\to e^{i\,e\Lambda}\partial_\mu\phi + i\,e\,(\partial_\mu\Lambda)\,e^{i\,e\Lambda}\phi + i\,eA'_\mu\,e^{i\,e\Lambda}\phi \\
&= e^{i\,e\Lambda}\left(\partial_\mu + i\,e(A'_\mu + \partial_\mu\Lambda)\right)\phi \\
&\equiv e^{i\,e\Lambda}\left(\partial_\mu + i\,eA_\mu\right)\phi.
\end{aligned} \tag{6.92}$$

Thus the covariant derivative has the correct transformation behavior provided

$$A_\mu \to A_\mu - \partial_\mu\Lambda, \tag{6.93}$$

the behavior which we have encountered before as a gauge freedom for massless vector fields with the (free) lagrangian density $\mathcal{L} = -(1/4)F_{\mu\nu}F^{\mu\nu}$. Replacing derivatives by covariant derivatives and adding the (free) part for the vector fields to the original lagrangian therefore produces a gauge invariant lagrangian,

$$\mathcal{L}(\phi, \partial_\mu\phi) \implies \mathcal{L}(\phi, D_\mu\phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{6.94}$$

As an example consider the Dirac lagrangian,

$$\mathcal{L} = \frac{i}{2}\left(\overline{\psi}\gamma^\mu\partial_\mu\psi - (\partial_\mu\overline{\psi})\gamma^\mu\psi\right) - M\,\overline{\psi}\psi.$$

Minimal substitution $\partial_\mu\psi \to (\partial_\mu + i\,eA_\mu)\psi$ leads to the gauge invariant lagrangian

$$\mathcal{L} = \frac{i}{2}\overline{\psi}\overset{\leftrightarrow}{\not\partial}\psi - M\,\overline{\psi}\psi - e\,\overline{\psi}\gamma^\mu\psi\,A_\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{6.95}$$

We note first of all that the coupling of the Dirac field (electron) to the vector field (photon) can be written in the familiar interaction form

$$\mathcal{L}_{\,int} = -e\,\overline{\psi}\gamma^\mu\psi\,A_\mu = -e\,j^\mu A_\mu, \tag{6.96}$$

involving the interaction of the charge ($\rho = j^0$) and three-current density ($\boldsymbol{j}$) with the electric potential ($\phi = A^0$) and the vector potential ($\boldsymbol{A}$), $-e\,j^\mu A_\mu = -e\,\rho\phi + e\,\boldsymbol{j}\cdot\boldsymbol{A}$. The equation of motion for the fermion follow from

$$\begin{aligned}
\frac{\delta\,\mathcal{L}}{\delta(\partial_\mu\overline{\psi})} &= -\frac{i}{2}\gamma^\mu\psi, \\
\frac{\delta\,\mathcal{L}}{\delta\overline{\psi}} &= \frac{i}{2}\not\partial\psi - M\,\psi - e\not A\,\psi
\end{aligned}$$

giving the Dirac equation in an electromagnetic field,

$$(i\not D - M)\,\psi = (i\not\partial - e\not A - M)\,\psi = 0. \tag{6.97}$$

For the photon the equations of motion follow from

$$\begin{aligned}
\frac{\delta\,\mathcal{L}}{\delta(\partial_\mu A_\nu)} &= -F^{\mu\nu}, \\
\frac{\delta\,\mathcal{L}}{\delta A_\nu} &= -e\overline{\psi}\gamma^\nu\psi,
\end{aligned}$$

giving the Maxwell equation coupling to the electromagnetic current,

$$\partial_\mu F^{\mu\nu} = j^\nu, \tag{6.98}$$

where $j^\mu = e\,\overline{\psi}\gamma^\mu\psi$.

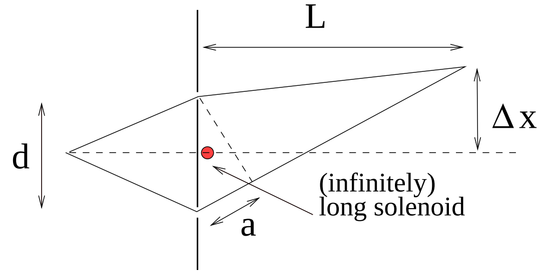


Figure 6.1: The Aharonov-Bohm experiment

6.6 The electromagnetic field and topology

The electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ as appearing for instance in the Lorentz force on a moving charge,

$$\boldsymbol{F} = e\,\boldsymbol{E} + e\boldsymbol{v} \times \boldsymbol{B} \tag{6.99}$$

are gauge invariant, while the electromagnetic field A_μ is not. Nevertheless the significance of A_μ is shown in the Aharonov-Bohm experiment. In this experiment an observable phase difference is measured for electrons moving through field-free space ($\boldsymbol{E} = \boldsymbol{B} = 0$ but $A_\mu \neq 0$), illustrated in fig. 6.1. The phase difference between the electrons travelling two different paths is given by

$$\delta = 2\pi\,\frac{a}{\lambda} = \frac{a}{\lambdabar} \approx \frac{\Delta x}{L}\,\frac{d}{\lambdabar},$$

or $\Delta x = (L\lambdabar/d)\delta$. This causes an interference pattern as a function of Δx. In the presence of an electromagnetic field A_μ an additional phase difference is observed. This occurs in the region outside the solenoid, where $A_\mu \neq 0$, but where $\boldsymbol{E} = \boldsymbol{B} = 0$. To be precise,

$$\begin{aligned}
\text{Inside solenoid:}\quad \boldsymbol{B} = B\,\hat{z} \qquad \boldsymbol{A} &= \frac{Br}{2}\,\hat{\phi} \\
&= \left(-\frac{By}{2}, \frac{Bx}{2}, 0\right), \\
\text{Outside solenoid:}\quad \boldsymbol{B} = 0 \qquad \boldsymbol{A} &= \frac{BR^2}{2r}\hat{\phi} \\
&= \left(-\frac{BR^2 y}{2r^2}, \frac{BR^2 x}{2r^2}, 0\right),
\end{aligned}$$

where $\hat{r} = (\cos\phi, \sin\phi, 0)$ and $\hat{\phi} = (-\sin\phi, \cos\phi, 0)$. The explanation of the significance of the vector potential $\boldsymbol{A}$ for the phase of the electron wave function is found in minimal substitution

$$\begin{aligned}
\psi \propto \exp\left(\frac{i}{\hbar}\,\boldsymbol{p}\cdot\boldsymbol{r}\right) \quad \longrightarrow \quad & \exp\left(\frac{i}{\hbar}\,(\boldsymbol{p} - e\boldsymbol{A})\cdot\boldsymbol{r}\right) \\
&= \exp\left(-i\,\frac{e}{\hbar}\,\boldsymbol{A}\cdot\boldsymbol{r}\right)\exp\left(\frac{i}{\hbar}\,\boldsymbol{p}\cdot\boldsymbol{r}\right).
\end{aligned}$$

The phase difference between the two paths of the electron then are given by

$$\begin{aligned}
\Delta\delta &= -\frac{e}{\hbar}\int_1 \boldsymbol{A}\cdot d\boldsymbol{r} + \frac{e}{\hbar}\int_2 \boldsymbol{A}\cdot d\boldsymbol{r} = \frac{e}{\hbar}\oint \boldsymbol{A}\cdot d\boldsymbol{r} \\
&= \frac{e}{\hbar}\int (\nabla\times\boldsymbol{A})\cdot d\boldsymbol{s} = \frac{e}{\hbar}\int \boldsymbol{B}\cdot d\boldsymbol{s} = \frac{e}{\hbar}\Phi,
\end{aligned} \tag{6.100}$$

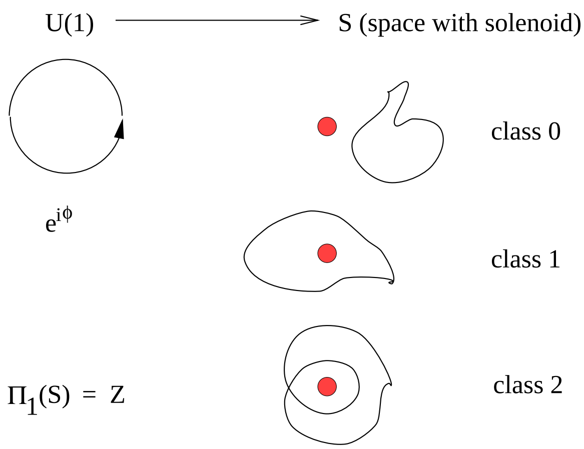


Figure 6.2: The topology of the space in the Aharonov-Bohm experiment, illustrating also that the group structure of the mapping of $U(1)$ into this space is that of the group of integers

where Φ is the flux through the solenoid. This phase difference is actually observed.

The situation, nevertheless, may look a bit akward. It is however, nothing more than a manisfestation of a nontrivial vacuum (a la the Dirac sea for fermions). The energy density $\frac{1}{2}(\boldsymbol{E}^2 + \boldsymbol{B}^2) = 0$ (hence a vacuum), but $\boldsymbol{A} \neq 0$ (hence there is structure in the vacuum).

Indeed when one considers the $\boldsymbol{A}$-field in the example of the Aharonov-Bohm effect outside the solenoid, it can be obtained from the situation $\boldsymbol{A} = 0$ by a gauge transformation,

$$\boldsymbol{A} = \nabla\chi \qquad \text{with} \qquad \chi = \frac{BR^2}{2}\,\phi \tag{6.101}$$

(ϕ being the azimuthal angle). This situation that $\boldsymbol{A}$ can be written as $\nabla\chi$ is always true when $\nabla \times \boldsymbol{A}$ = 0. The observable phase going around in general is

$$\Delta\delta = \frac{e}{\hbar}\oint \boldsymbol{A}\cdot d\boldsymbol{r} = \frac{e}{\hbar}\oint \nabla\chi\cdot d\boldsymbol{r} = \frac{e}{\hbar}\;\chi|_{\phi=0}^{\phi=2\pi}\,.$$

The function χ in Eq. 6.101, however, is multivalued as $\phi = 0$ and $\phi = 2\pi$ are the same point in space. If this would be be happening in empty space one would be in trouble. By going around an arbitrary loop a different number of times the electron wave function would acquire different phases or it would acquire an arbitrary phase in a point by contracting a loop continuously into that point. Therefore the gauge transformation must be uniquely defined in the space one is working in, i.e. being single-valued [3].

Now back to the Aharonov-Bohm experiment. The difference here is that we are working in a space with a 'defect' (the infinitely long solenoid). In such a space loops with different winding number (i.e. the number of times they go around the defect) cannot be continuously deformed into one another. Therefore space outside the solenoid doesn't care that χ is multi-valued. The Aharonov-Bohm experiment shows a realization of this possibility.

Another situation in which the topology of space can be used is in the case of superconductors. Consider a superconductor, forming a simple (connected) space. Below the critical temperature the magnetic field is squeezed out of the superconducting material, organizing itself in tiny flux tubes (Abrikosov strings), minimizing the space occupied by $\boldsymbol{B}$ fields. The only way for flux tubes to be

[3] The occurrence of nontrivial possibilities, i.e. nonobservable phases $\phi = 2\pi n$, has been employed by Dirac in constructing magnetic monopoles in electrodynamics

formed and move around without global consequences is when each flux tube contains a flux Φ such that $\Delta\delta = (e/\hbar)\Phi = n \cdot 2\pi$, giving no observable phase.

6.7 Exercises

Exercise 6.1

(a) Show that the Klein-Gordon equation for the real scalar field can be derived from the lagrangian density

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\,M^2\,\phi^2.$$

(b) Show that the Klein-Gordon equation for the complex scalar field (considering ϕ and ϕ^* as independent fields can be derived from the lagrangian density

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - M^2\,\phi^*\phi.$$

Exercise 6.2

(a) Show that the Maxwell equations can be derived from the lagrangian density

$$\mathcal{L} = -\frac{1}{4}\,F_{\mu\nu}F^{\mu\nu}.$$

(b) What is the form of the interaction term involving a current j_μ and the field A^μ that will give the inhomogeneous Maxwell equations, $\partial_\mu F^{\mu\nu} = j^\nu$.

(c) Show that the interaction term is invariant under gauge transformations only if the current j_μ is conserved, i.e. $\partial_\mu j^\mu = 0$ (Note that the addition of a total derivative to the lagrangian density does not modify the equations of motion).

Exercise 6.3
Show that the current $j_\mu = i\,\phi^*\,\overleftrightarrow{\partial}_\mu\,\phi$ for a scalar field is connected to a $U(1)$ transformation on the fields, $\phi \rightarrow e^{i\Lambda}\,\phi$.

Exercise 6.4
Given the Dirac equation for a spin-1/2 particle in an external electromagnetic field, $[i\gamma^\mu(\partial_\mu - i\,e\,A_\mu) - M]\psi = 0$, give the equation which is satisfied by $\psi^c = C\overline{\psi}^T$, and deduce what is the charge of the antiparticle as compared to a particle.

Exercise 6.5
Show that the Dirac equation for an electron with charge $-e$ in an electromagnetic field $A^\mu = (A^0, \boldsymbol{A})$ (see also exercise 6.4) satisfies in the non-relativistic limit de following equation for the 'upper component' ψ_u (two components),

$$\left(\frac{1}{2M}(\boldsymbol{p} + e\,\boldsymbol{A})^2 + \frac{e}{2M}\,\sigma\cdot\boldsymbol{B} - eA^0\right)\psi_u = E_{n.r.}\psi_u,$$

where $\boldsymbol{p} = -i\nabla$ and $E_{n.r.} = E - M$.

Exercise 6.6 (optional)
Proof the properties of the tensor $T^{\mu\nu}$ in section 6.3.2.
Hints: it may be useful to realize that $G^{\mu\rho\sigma}$ is totally antisymmetric; use $\partial_\mu T^{\mu\nu} = 0$ and $\partial_\mu\mathcal{M}^{\mu\rho\sigma} = 0$, in order to show that $\theta^{\rho\sigma} - \theta^{\sigma\rho} = \partial_\mu H^{\mu\rho\sigma}$; finally note that it is sufficient to show the last equation for $\int d\sigma_\mu\mathcal{M}^{\mu\rho\sigma}$.

Chapter 7

Quantization of fields

7.1 Canonical quantization

We will first recall the example of classical mechanics for one coordinate $q(t)$, starting from the lagrangian $L(q,\dot{q})$ also considered in the previous chapter,

$$L(q,\dot{q}) = \frac{1}{2}\,m\dot{q}^2 - V(q). \tag{7.1}$$

The Hamiltonian (also corresponding to a conserved quantity because of time translation invariance) is given by

$$\begin{aligned} H(p,q) &= p\,\dot{q} - L(q,\dot{q}(q,p)) &(7.2)\\ &= \frac{p^2}{2m} + V(q), &(7.3) \end{aligned}$$

where the (canonical) momentum $p = \partial L/\partial\dot{q}$, in our example $p = m\dot{q}$. Quantizing the system, canonical commutation relations between q and p are imposed,

$$[q,p] = i, \tag{7.4}$$

with a possible realization as operators in the Hilbert space of (coordinate space) wave functions through $q_{op}\psi(q) = q\psi(q)$ and $p_{op}\psi(q) = -id\psi/dq$.

An immediate generalization for fields can be obtained by considering them as coordinates, labeled by the position,

$$q_{\mathbf{x}}(t) = \frac{1}{\Delta^3 x}\int_{\Delta^3 x} d^3x\ \phi(\boldsymbol{x},t), \tag{7.5}$$

etc. The lagrangian is given by

$$\begin{aligned} L &= \int d^3x\ \mathcal{L}(\phi(x),\partial_\mu\phi(x)) = \int d^3x\ \mathcal{L}(\phi,\dot{\phi},\nabla\phi)\\ &= \sum_{\mathbf{x}} \Delta^3 x\ \mathcal{L}_{\ \mathbf{x}}\left(q_{\mathbf{x}},\dot{q}_{\mathbf{x}},q_{\mathbf{x}+\Delta\mathbf{x}}\right), &(7.6) \end{aligned}$$

In order to construct the hamiltonian it is necessary to find the canonical momenta,

$$p_{\mathbf{x}}(t) = \frac{\partial L}{\partial\dot{q}_{\mathbf{x}}} = \Delta^3 x\,\frac{\partial\,\mathcal{L}_{\mathbf{x}}}{\partial\dot{q}_{\mathbf{x}}} = \Delta^3 x\,\Pi_{\mathbf{x}}(t), \tag{7.7}$$

where $\Pi_{\mathbf{x}}(t)$ is obtained from the continuous field $\Pi(x) \equiv \delta\,\mathcal{L}\,/\delta(\partial_0\phi)$. The hamiltonian then is

$$H = \sum_{\mathbf{x}} \boldsymbol{p}_{\mathbf{x}} \dot{q}_{\mathbf{x}} - L \quad = \quad \sum_{\mathbf{x}} \Delta^3 x\, [\Pi_{\mathbf{x}} \dot{q}_{\mathbf{x}} - \mathcal{L}_{\,\mathbf{x}}] \tag{7.8}$$

$$= \quad \sum_{\mathbf{x}} \Delta^3 x\, \mathcal{H}_{\mathbf{x}} \tag{7.9}$$

where

$$\mathcal{H}(x) = \frac{\delta\,\mathcal{L}}{\delta\dot{\phi}(x)}\,\dot{\phi}(x) - \mathcal{L}(x). \tag{7.10}$$

Note that this indeed corresponds to zero component (Θ^{00}) of the conserved energy-momentum stress tensor $\Theta^{\mu\nu}$, discussed in the section 6.2.

As an example, for the scalar field theory, we have

$$\begin{aligned}
\mathcal{L}(x) &= \frac{1}{2}\,\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\,M^2\,\phi^2 \\
&= \frac{1}{2}\,(\partial_0\phi)^2 - \frac{1}{2}\,(\nabla\phi)^2 - \frac{1}{2}\,M^2\,\phi^2 && (7.11) \\
\Pi(x) &= \frac{\delta\,\mathcal{L}}{\delta\partial_0\phi)} = \partial_0\phi, && (7.12) \\
\mathcal{H}(x) &= \Theta^{00}(x) = \frac{1}{2}\,(\partial_0\phi)^2 + \frac{1}{2}\,(\nabla\phi)^2 + \frac{1}{2}\,M^2\,\phi^2. && (7.13)
\end{aligned}$$

For the quantization procedure we can formulate a number of basic axioms of quantum field theory. Sometimes it is useful to keep in mind that, formally, the fields $\phi(x)$ can be considered as regular operators in the Hilbert space after smearing with a test function f,

$$\Phi(f) = \int d^4x\;\phi(x)\,f(x). \tag{7.14}$$

In fact, the discretization procedure above is an explicit example, albeit with 'sharp' functions.

The following items are essential for quantization of a theory.

- **Canonical commutation relations**
  Quantization of the theory is achieved by imposing the canonical quantization condition $[q_{\mathbf{x}}(t), p_{\mathbf{y}}(t)] = i\,\delta_{\mathbf{xy}}$ or for the fields $\phi(x)$ and $\Pi(x)$ the socalled *equal time commutation relations*

$$[\phi(\boldsymbol{x},t), \Pi(\boldsymbol{y},t)] = i\,\delta^3(\boldsymbol{x}-\boldsymbol{y}), \tag{7.15}$$

  with furthermore the relations $[\phi(\boldsymbol{x},t), \phi(\boldsymbol{y},t)] = [\Pi(\boldsymbol{x},t), \Pi(\boldsymbol{y},t)] = 0$.

- **Poincaré invariance**
  The fields must satisfy the following transformation properties

$$U(\Lambda,a)\,\phi^i(x)U^{-1}(\Lambda,a) = R^i_j(\Lambda^{-1})\,\phi^j(\Lambda x + a), \tag{7.16}$$

  or if

$$\begin{aligned}
U(\Lambda,a) &= 1 - i\epsilon_\mu P^\mu - \frac{i}{2}\,\omega_{\mu\nu}M^{\mu\nu}, \\
R^i_j(\Lambda^{-1}) &= 1 - \frac{i}{2}\,\omega_{\mu\nu}(S^{\mu\nu})^i_j,
\end{aligned}$$

  one has

$$\begin{aligned}
[P_\mu, \phi^i(x)] &= i\partial_\mu\phi^i(x), && (7.17) \\
[M_{\mu\nu}, \phi^i(x)] &= i(x_\mu\partial_\nu - x_\nu\partial_\mu)\phi^i(x) + (S_{\mu\nu})^i_j\phi^j(x). && (7.18)
\end{aligned}$$

- **Causality**
  Operators $\Phi(f)$ and $\Phi(g)$ for which the test functions are space-like separated, i.e. $(x-y)^2 < 0$ for x and y belonging to the support of the functions f and g respectively can be measured simultaneously (*macroscopic causality*). The measurements cannot influence each other or $[\Phi(f), \Phi(g)] = 0$. *Microscopic causality* implies local commutativity,

$$[\phi(x), \phi(y)] = 0 \quad \text{if} \quad (x-y)^2 < 0. \tag{7.19}$$

7.2 Field operators

Before discussing (real and complex) scalar fields and Dirac fields we recall the analogy with the well-known harmonic oscillator as an example of quantization using creation and annihilation operators, sometimes referred to as second quantization. In simplified form the hamiltonian is given by

$$H = \frac{1}{2}\,P^2 + \frac{1}{2}\,\omega^2\,Q^2, \tag{7.20}$$

where the coordinate Q and the momentum P satisfy the canonical commutation relations

$$[Q, P] = i, \quad [Q, Q] = [P, P] = 0 \tag{7.21}$$

Writing Q and P in terms of creation and annihilation operators, $a^\dagger$ and a,

$$Q = \frac{i}{\sqrt{2\omega}}(a - a^\dagger) \quad \text{and} \quad P = \sqrt{\frac{\omega}{2}}(a + a^\dagger) \tag{7.22}$$

it is straightforward to check that the commutation relations between Q and P are equivalent with the commutation relations

$$[a, a^\dagger] = 1, \quad [a, a] = [a^\dagger, a^\dagger] = 0. \tag{7.23}$$

The hamiltonian in this case can be expressed in the number operator $N = a^\dagger a$,

$$\begin{aligned} H &= \omega\left\{\left(\frac{P}{\sqrt{2\omega}} + i\sqrt{\frac{\omega}{2}}Q\right)\left(\frac{P}{\sqrt{2\omega}} - i\sqrt{\frac{\omega}{2}}Q\right) - \frac{i}{2}[Q, P]\right\} \\ &= \omega\left\{a^\dagger a + \frac{1}{2}\right\} = \omega\left\{N + \frac{1}{2}\right\}. \end{aligned} \tag{7.24}$$

It is straightforward to find the commutation relations between N and a and $a^\dagger$,

$$[N, a^\dagger] = a^\dagger, \quad \text{and} \quad [N, a] = -a. \tag{7.25}$$

Defining states $|n\rangle$ as eigenstates of N with eigenvalue n, $N|n\rangle = n|n\rangle$ one finds

$$\begin{aligned} N\,a^\dagger|n\rangle &= (n+1)\,a^\dagger|n\rangle \quad \longrightarrow \quad a^\dagger|n\rangle \propto |n+1\rangle, \\ N\,a|n\rangle &= (n-1)\,a|n\rangle \quad \longrightarrow \quad a|n\rangle \propto |n-1\rangle, \end{aligned}$$

i.e. $a^\dagger$ and a act as raising and lowering operators. Since

$$\langle n-1|n-1\rangle = \langle n|a^\dagger a|n\rangle = n\langle n|n\rangle,$$

and the norm of a state must be positive we see that a state $|0\rangle$ must exist for which $N|0\rangle = a|0\rangle = 0$. Therefore one has found for the harmonic oscillator the spectrum of eigenstates $|n\rangle$ (with n a non-negative integer) with $E_n = (n+1/2)\omega$.

7.2.1 The real scalar field

We have expanded the (classical) field in plane wave solutions, which we have split into positive and negative energy pieces with (complex) coefficients $a(\boldsymbol{k})$ and $a^*(\boldsymbol{k})$ multiplying them. The quantization of the field is achieved by quantizing the coefficients in the Fourier expansion, e.g. the real scalar field $\phi(x)$ becomes

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3\, 2E_k}\ \left[a(k)\, e^{-i\,k\cdot x} + a^\dagger(k)\, e^{i\,k\cdot x}\right], \tag{7.26}$$

where the Fourier coefficients $a(k)$ and $a^\dagger(k)$ are now operators. The canonical momentum becomes

$$\Pi(x) = \dot{\phi}(x) = \frac{-i}{2}\int \frac{d^3k}{(2\pi)^3}\ \left[a(k)\, e^{-i\,k\cdot x} - a^\dagger(k)\, e^{i\,k\cdot x}\right]. \tag{7.27}$$

It is easy to check that these equations can be inverted

$$a(k) \;=\; \int d^3x\ e^{i\,k\cdot x}\, i\, \overset{\leftrightarrow}{\partial}_0\, \phi(x), \tag{7.28}$$

$$a^\dagger(k') \;=\; \int d^3x\ \phi(x)\, i\, \overset{\leftrightarrow}{\partial}_0\, e^{i\,k'\cdot x} \tag{7.29}$$

It is straightforward to prove that the equal time commutation relations between $\phi(x)$ and $\Pi(x')$ are equivalent with 'harmonic oscillator - like' commutation relations between $a(k)$ and $a^\dagger(k')$, i.e.

$$\begin{aligned}&[\phi(x),\Pi(x')]_{x^0=x'^0} = i\,\delta^3(\boldsymbol{x}-\boldsymbol{x}') \quad \text{and}\\ &[\phi(x),\phi(x')] = [\Pi(x),\Pi(x')] = 0,\end{aligned} \tag{7.30}$$

is equivalent with

$$\begin{aligned}&[a(k),a^\dagger(k')] = (2\pi)^3\, 2E_k\, \delta^3(\boldsymbol{k}-\boldsymbol{k}') \quad \text{and}\\ &[a(k),a(k')] = [a^\dagger(k),a^\dagger(k')] = 0.\end{aligned} \tag{7.31}$$

The hamiltonian can be rewritten in terms of a number operator $N(k) = a^\dagger(k)a(k)$, which represents the 'number of particles' with momentum k.

$$\begin{aligned} H &= \int d^3x\ \left[\frac{1}{2}(\partial_0\phi)^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}M^2\,\phi^2\right]\\ &= \int \frac{d^3k}{(2\pi)^3\, 2k^0}\,\frac{k^0}{2}\left(a^\dagger(k)a(k) + a(k)a^\dagger(k)\right) \qquad (7.32)\\ &= \int \frac{d^3k}{(2\pi)^3\, 2k^0}\, k^0 \left(N(k) + \frac{1}{2}\right), \qquad (7.33)\end{aligned}$$

and similarly

$$\begin{aligned} P^i &= \int d^3x\ \Theta^{0i}(x) = \int d^3x\ \partial^0\phi\, \partial^i\phi \qquad (7.34)\\ &= \int \frac{d^3k}{(2\pi)^3\, 2k^0}\, k^i \left(N(k) + \frac{1}{2}\right). \qquad (7.35)\end{aligned}$$

Just as in the case of the harmonic oscillator it is essential (axiom) that there exists a ground state $|0\rangle$ that is annihilated by $a(k)$, $a(k)|0\rangle = 0$. The rest of the states are then obtained from the groundstate via the creation operator, defining particle states $|p\rangle = |p^0 = \sqrt{\boldsymbol{p}^2 + M^2}, \boldsymbol{p}\rangle$

$$|p\rangle = a^\dagger(p)|0\rangle, \tag{7.36}$$

and multiparticle states

$$|(p_1)^{n_1}\,(p_2)^{n_2}\,\ldots\rangle = \frac{(a^\dagger(p_1))^{n_1}}{\sqrt{n_1!}}\,\frac{(a^\dagger(p_2))^{n_2}}{\sqrt{n_2!}}\,\ldots|0\rangle, \tag{7.37}$$

normalized as

$$\langle p|p'\rangle = (2\pi)^3\, 2p^0\, \delta^3(\boldsymbol{p}-\boldsymbol{p}'), \tag{7.38}$$

and satisfying the completeness condition

$$\begin{aligned} 1 &= \int \frac{d^4p}{(2\pi)^4}\, 2\pi\, \delta(p^2 - M^2)\, \theta(p^0)\, |p\rangle\langle p| \\ &= \int \frac{d^3p}{(2\pi)^3\, 2E} |p\rangle\langle p|, \end{aligned} \tag{7.39}$$

where $E = \sqrt{\boldsymbol{p}^2 + M^2}$.

A problem is the zero-point energy because in this case there are an infinite number of oscillators. This will be subtracted as an (infinite) constant zero-point energy,

$$E_{z.p.} = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3\, 2k^0}\, k^0, \tag{7.40}$$

which amounts to redefining H as

$$H = \int d^3x \;\; :\mathcal{H}(x):= \int \frac{d^3k}{(2\pi)^3\, 2k^0}\, k^0\, N(k). \tag{7.41}$$

This procedure is known as *normal ordering*, i.e. writing all annihilation operators to the right of the creation operators (in this way any operator containing annihilation operators will have eigenvalue 0 for the vacuum!).

For the purpose of normal ordering it is convenient to decompose the field in positive and negative frequency parts,

$$\phi(x) = \phi_+(x) + \phi_-(x), \tag{7.42}$$

$$\phi_+(x) = \int \frac{d^3k}{(2\pi)^3\, 2E_k}\, a(k)\, e^{-i\,k\cdot x}, \tag{7.43}$$

$$\phi_-(x) = \int \frac{d^3k}{(2\pi)^3\, 2E_k}\, a^\dagger(k)\, e^{i\,k\cdot x}. \tag{7.44}$$

The normal ordered product can be expressed as

$$:\phi(x)\phi(y):= \phi_+(x)\phi_+(y) + \phi_-(x)\phi_+(y) + \phi_-(y)\phi_+(x) + \phi_-(x)\phi_-(y). \tag{7.45}$$

The 1-particle wave function is obtained as

$$\langle 0|\phi(x)|p\rangle = \langle 0|\phi_+(x)|p = \int \frac{d^3k}{(2\pi)^3\, 2E_k}\langle 0|a(k)a^\dagger(p)|0\rangle\, e^{-i\,k\cdot x} = e^{-i\,p\cdot x} \tag{7.46}$$

$$\langle p|\phi(x)|0\rangle = \langle p|\phi_-(x)|0\rangle = e^{i\,p\cdot x}. \tag{7.47}$$

In order to ensure the consistency of the theory it is necessary to check that the operators P^ν and $M^{\mu\nu}$ obtained from the conserved currents $\theta^{\rho\sigma}$ and $\mathcal{M}^{\mu\rho\sigma}$ are indeed the generators of the Poincaré group, i.e. that they satisfy the required commutation relations in Eqs (7.17) and (7.18).

The last item to be checked for the scalar field are the causality condition. In order to calculate $[\phi(x), \phi(y)]$ consider

$$
\begin{aligned}
[\phi_+(x), \phi_-(y)] &\equiv i\Delta_+(x-y), &(7.48)\\
&= \int \frac{d^3k}{(2\pi)^3\, 2E_k} \int \frac{d^3k'}{(2\pi)^3\, 2E'_k}\, e^{-i\,k\cdot x + i\,k'\cdot y}\, [a(k), a^\dagger(k')] \\
&= \int \frac{d^3k}{(2\pi)^3\, 2E_k}\, e^{-i\,k\cdot(x-y)}, &(7.49)\\
[\phi_-(x), \phi_+(y)] &\equiv i\Delta_-(x-y), &(7.50)\\
&= -\int \frac{d^3k}{(2\pi)^3\, 2E_k}\, e^{i\,k\cdot(x-y)} = -i\Delta_+(y-x), &(7.51)
\end{aligned}
$$

or as integrals over d^4k,

$$
\begin{aligned}
i\Delta_+(x) &= \int \frac{d^4k}{(2\pi)^4}\, \theta(k^0)\, \delta(k^2 - M^2)\, e^{-i\,k\cdot x}, &(7.52)\\
i\Delta_-(x) &= \int \frac{d^4k}{(2\pi)^4}\, \theta(-k^0)\, \delta(k^2 - M^2)\, e^{-i\,k\cdot x} &(7.53)\\
&= -\int \frac{d^4k}{(2\pi)^4}\, \theta(k^0)\, \delta(k^2 - M^2)\, e^{i\,k\cdot x} = -i\Delta_+(-x)
\end{aligned}
$$

The result for the *invariant commutator function* is

$$
\begin{aligned}
[\phi(x), \phi(y)] &= i\Delta(x-y)\\
&= i\left(\Delta_+(x-y) + \Delta_-(x-y)\right), &(7.54)
\end{aligned}
$$

which has the following properties

(i) $i\Delta(x) = i\Delta_+(x) + i\Delta_-(x)$ can be expressed as

$$
i\Delta(x) = \int \frac{d^3k}{(2\pi)^3\, 2k^0}\, \epsilon(k^0) e^{-i\,k\cdot x}, \tag{7.55}
$$

where $\epsilon(k^0) = \theta(k^0)$ - $\theta(-k^0)$.

(ii) $\Delta(x)$ is a solution of the homogeneous Klein-Gordon equation.

(iii) $\Delta(0, \boldsymbol{x}) = 0$ and hence $\Delta(x) = 0$ for $x^2 < 0$.

(iv) The equal time commutation relations follow from

$$
\left.\frac{\partial}{\partial t}\Delta(x)\right|_{t=0} = \delta^3(x). \tag{7.56}
$$

(v) For M = 0,

$$
\Delta(x) = -\frac{\epsilon(x^0)}{2\pi}\, \delta(x^2). \tag{7.57}
$$

7.2.2 The complex scalar field

In spite of the similarity with the case of the real field, we will consider it as a repetition of the quantization procedure, extending it with the charge operator and the introduction of particle and antiparticle operators. The field satisfies the Klein-Gordon equation and the density current ($U(1)$ transformations) and the energy-momentum tensor are

$$\begin{aligned} j_\mu &= i\,\phi^* \overleftrightarrow{\partial}_\mu \phi, &(7.58)\\ \Theta_{\mu\nu} &= \partial_\mu \phi^* \partial_\nu \phi - \mathcal{L}\, g_{\mu\nu}. &(7.59)\end{aligned}$$

The quantized fields are written as

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3\, 2E_k} \left[a(k)\, e^{-i\,k\cdot x} + b^\dagger(k)\, e^{i\,k\cdot x}\right], \tag{7.60}$$

$$\phi^\dagger(x) = \int \frac{d^3k}{(2\pi)^3\, 2E_k} \left[b(k)\, e^{-i\,k\cdot x} + a^\dagger(k)\, e^{i\,k\cdot x}\right], \tag{7.61}$$

$$\tag{7.62}$$

and satisfy the equal time commutation relation

$$[\phi(x), \partial_0 \phi^\dagger(y)]_{x^0=y^0} = i\,\delta^3(x-y), \tag{7.63}$$

which is equivalent to the relations

$$[a(k), a^\dagger(k')] = [b(k), b^\dagger(k')] = (2\pi)^3\, 2E_k\, \delta^3(\boldsymbol{k}-\boldsymbol{k}'). \tag{7.64}$$

The hamiltonian is as before given by the normal ordered expression

$$\begin{aligned} H &= \int d^3x \;:\Theta^{00}(x): \\ &= \int \frac{d^3k}{(2\pi)^3\, 2E_k}\; E_k \;:\left[a^\dagger(k)a(k) + b(k)b^\dagger(k)\right]: \\ &= \int \frac{d^3k}{(2\pi)^3\, 2E_k}\; E_k \left[a^\dagger(k)a(k) + b^\dagger(k)b(k)\right], \end{aligned} \tag{7.65}$$

i.e. particles (created by $a^\dagger$) and antiparticles (created by $b^\dagger$) with the same momentum contribute equally to the energy. Also the charge operator involves normal ordering (in order to give the vacuum eigenvalue zero),

$$\begin{aligned} Q &= i\int d^3x \;:\left[\phi^\dagger\, \partial_0 \phi - \partial_0 \phi^\dagger\, \phi(x)\right]: \\ &= \int \frac{d^3k}{(2\pi)^3\, 2E_k} \;:\left[a^\dagger(k)a(k) - b(k)b^\dagger(k)\right]: \\ &= \int \frac{d^3k}{(2\pi)^3\, 2E_k} \left[a^\dagger(k)a(k) - b^\dagger(k)b(k)\right]. \end{aligned} \tag{7.66}$$

The commutator of ϕ and $\phi^\dagger$ is as for the real field given by

$$[\phi(x), \phi^\dagger(y)] = i\Delta(x-y). \tag{7.67}$$

7.2.3 The Dirac field

From the lagrangian density

$$\mathcal{L} = \frac{i}{2}\, \bar\psi \gamma^\mu \overleftrightarrow{\partial}_\mu \psi - M\, \bar\psi \psi, \tag{7.68}$$

the conserved density and energy-momentum currents are easily obtained,

$$j_\mu = \bar{\psi}\gamma_\mu\psi, \tag{7.69}$$

$$\Theta_{\mu\nu} = \frac{i}{2}\,\bar{\psi}\gamma_\mu \overleftrightarrow{\partial}_\nu\,\psi - \left(\frac{i}{2}\,\bar{\psi}\,\overleftrightarrow{\not\partial}\,\psi - M\,\bar{\psi}\psi\right) g_{\mu\nu}. \tag{7.70}$$

The canonical momentum and the hamiltonian are given by

$$\Pi(x) = \frac{\delta\,\mathcal{L}}{\delta\dot{\psi}(x)} = i\,\psi^\dagger(x), \tag{7.71}$$

$$\begin{aligned}\mathcal{H}(x) &= \Theta^{00}(x) = -\frac{i}{2}\bar{\psi}\gamma^i\partial_i\psi + M\,\bar{\psi}\psi \\ &= i\bar{\psi}\gamma^0\partial_0\psi = i\psi^\dagger\partial_0\psi,\end{aligned} \tag{7.72}$$

where the last line is obtained by using the Dirac equation.

The quantized fields are written

$$\psi(x) = \sum_s \int \frac{d^3k}{(2\pi)^3\,2E_k}\,\left[b(k,s)u(k,s)e^{-i\,k\cdot x} + d^\dagger(k,s)v(k,s)e^{i\,k\cdot x}\right], \tag{7.73}$$

$$\bar{\psi}(x) = \sum_s \int \frac{d^3k}{(2\pi)^3\,2E_k}\,\left[b^\dagger(k,s)\bar{u}(k,s)e^{i\,k\cdot x} + d(k,s)\bar{v}(k,s)e^{-i\,k\cdot x}\right]. \tag{7.74}$$

In terms of the operators for the b and d quanta the hamiltonian and charge operators are (omitting mostly the spin summation in the rest of this section)

$$H = \int d^3x\ :\psi^\dagger(x)\,i\partial_0\psi(x): \tag{7.75}$$

$$= \int \frac{d^3k}{(2\pi)^3\,2E_k}\,E_k\ :\left[b^\dagger(k)b(k) - d(k)d^\dagger(k)\right]: \tag{7.76}$$

$$Q = \int d^3x\ :\psi^\dagger\,\psi: \tag{7.77}$$

$$= \int \frac{d^3k}{(2\pi)^3\,2E_k}\ :\left[b^\dagger(k)b(k) + d(k)d^\dagger(k)\right]:, \tag{7.78}$$

which seems to cause problems as the antiparticles (d-quanta) contribute negatively to the energy and the charges of particles (b-quanta) and antiparticles (d-quanta) are the same.

The solution is the introduction of anticommutation relations,

$$\{b(k,s), b^\dagger(k',s')\} = \{d(k,s), d^\dagger(k',s')\} = (2\pi)^3\,2E_k\,\delta^3(\boldsymbol{k}-\boldsymbol{k}')\,\delta_{ss'}. \tag{7.79}$$

Note that achieving normal ordering, i.e. interchanging creation and annihilation operators, then leads to additional minus signs and

$$H = \int \frac{d^3k}{(2\pi)^3\,2E_k}\,E_k\,\left[b^\dagger(k)b(k) + d^\dagger(k)d(k)\right] \tag{7.80}$$

$$Q = \int \frac{d^3k}{(2\pi)^3\,2E_k}\,\left[b^\dagger(k)b(k) - d^\dagger(k)d(k)\right]. \tag{7.81}$$

Also for the field and the canonical conjugate momentum anticommutation relations are considered,

$$\begin{aligned}\{\psi_i(x), \psi_j^\dagger(y)\}_{x^0=y^0} &= \int \frac{d^3k}{(2\pi)^3\,2k^0}\left[\sum_s u_i(k,s)u_j^\dagger(k,s)\,e^{-i\,k\cdot(x-y)}\right. \\ &\qquad \left. + v_i(k,s)v_j^\dagger(k,s)\,e^{i\,k\cdot(x-y)}\right]_{x^0=y^0}.\end{aligned}$$

Using the positive and negative energy projection operators discussed in section 4, one has

$$
\begin{aligned}
\{\psi_i(x), \psi_j^\dagger(y)\}_{x^0=y^0} &= \int \frac{d^3k}{(2\pi)^3\, 2k^0} \Big[(\not k + M)(\gamma_0)_{ij}\, e^{-i\,k\cdot(x-y)} \\
&\qquad + (\not k - M)(\gamma_0)_{ij}\, e^{i\,k\cdot(x-y)}\Big]_{x^0=y^0} \\
&= \int \frac{d^3k}{(2\pi)^3\, 2k^0}\, 2k^0\, (\gamma_0\gamma_0)_{ij}\, e^{i\,\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} \\
&= \delta^3(\boldsymbol{x}-\boldsymbol{y})\,\delta_{ij}. \qquad (7.82)
\end{aligned}
$$

For the scalar combination $\{\psi(x), \bar\psi(y)\}$ at arbitrary times one has

$$
\begin{aligned}
\{\psi(x), \bar\psi(y)\} &= \int \frac{d^3k}{(2\pi)^3\, 2k^0} \Big[(\not k + M)\, e^{-i\,k\cdot(x-y)} + (\not k - M)\, e^{i\,k\cdot(x-y)}\Big] \\
&= (i\not\partial_x + M) \int \frac{d^3k}{(2\pi)^3\, 2k^0} \Big[e^{-i\,k\cdot(x-y)} + -e^{i\,k\cdot(x-y)}\Big] \\
&= (i\not\partial_x + M)\, i\Delta(x-y), \qquad (7.83)
\end{aligned}
$$

where $i\Delta = i\Delta_+ + i\Delta_-$ is the same invariant commutator function as encountered before. When we would have started with commutation relations for the field ψ and the canonical momentum, we would have obtained

$$
[\psi(x), \bar\psi(y)] = (i\not\partial_x + M)\, i\Delta^1(x-y), \qquad (7.84)
$$

where $i\Delta^1 = i\Delta_+ - i\Delta_-$, which however has wrong causality properties! Therefore the relation between spin and statistics is required to get micro-causality (which is also known as the *spin statistics theorem*).

7.2.4 The electromagnetic field

From the lagrangian density

$$
\mathcal{L} = -\frac{1}{4}\, F_{\mu\nu} F^{\mu\nu}, \qquad (7.85)
$$

the canonical momenta are

$$
\begin{aligned}
\Pi^0 &= \frac{\delta\mathcal{L}}{\delta\dot A_0} = 0, \qquad (7.86) \\
\Pi^i &= \frac{\delta\mathcal{L}}{\delta\dot A_i} = F^{i0} = E^i, \qquad (7.87)
\end{aligned}
$$

which reflects the gauge freedom, but has the problem of being noncovariant, as the vanishing of Π^0 induces a constraint. It is possible to continue in a covariant way with the lagrangian

$$
\mathcal{L} = -\frac{1}{4}\, F_{\mu\nu} F^{\mu\nu} - \frac{\lambda}{2}(\partial_\mu A^\mu)^2,
$$

This gives the equations of motion discussed before, it implies the Lorentz constraint and leads to the canonical momenta

$$
\begin{aligned}
\Pi^0 &= -\lambda(\partial_\rho A^\rho), \qquad (7.88) \\
\Pi^i &= E^i. \qquad (7.89)
\end{aligned}
$$

If one wants to impose canonical commutation relations $\partial_\mu A^\mu = 0$ cannot hold as an operator identity, but we must restrict ourselves to the weaker condition

$$
\langle B|\partial_\mu A^\mu|A\rangle = 0, \qquad (7.90)
$$

for physical states $|A\rangle$ and $|B\rangle$.

The quantized field is expanded as

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3\, 2E_k} \sum_{\lambda=0}^{3} \epsilon_\mu^{(\lambda)}(k) \left[c(k,\lambda)e^{-i\,k\cdot x} + c^\dagger(k,\lambda)e^{i\,k\cdot x}\right], \tag{7.91}$$

with four independent vectors $\epsilon_\mu^{(\lambda)}$, containing a time-like photon, a longitudinal photon and two transverse photons. The canonical equal time commutation relations are

$$[A_\mu(x), \Pi_\nu(y)]_{x^0=y^0} = i\, g_{\mu\nu}\, \delta^3(\boldsymbol{x}-\boldsymbol{y}), \tag{7.92}$$

where $\Pi^\nu = F^{\nu 0} - \lambda\, g^{\nu 0}\,(\partial_\rho A^\rho)$ and we have furthermore $[A_\mu(x), A_\nu(y)] = [\Pi_\mu(x), \Pi_\nu(y)] = 0$. In fact the commutation relations imply

$$[\dot{A}_\mu(x), A_\nu(y)]_{x^0=y^0} = i\, g_{\mu\nu}\, \delta^3(\boldsymbol{x}-\boldsymbol{y}), \tag{7.93}$$

and are equivalent with

$$[c(k,\lambda), c^\dagger(k'), \lambda')] = -g^{\lambda\lambda'}\, 2E_k\, (2\pi)^3\, \delta^3(\boldsymbol{k}-\boldsymbol{k}'). \tag{7.94}$$

Note that for the transverse states there are no problems with the normalization and the statistics of the states as $-g^{ij} = \delta^{ij}$ for $i,j = 1,2$. The hamiltonian in terms of the creation and annihilation operators is (after normal ordering) given by

$$H = \int \frac{d^3k}{(2\pi)^3\, 2k^0}\, k^0 \left[\sum_{\lambda=1}^{3} c^\dagger(k,\lambda)c(k,\lambda) - c^\dagger(k,0)c(k,0)\right], \tag{7.95}$$

which does exhibit problems with the time-like photon. These problems are solved by the Lorentz constraint between physical states given above, for which it is sufficient that $\partial_\mu A_+^\mu|A\rangle = 0$, where A_+^μ is the part of the vector field containing the annihilation operators. It gives

$$\sum_{\lambda=0}^{3} k^\mu \epsilon_\mu^{(\lambda)}(k) c(k,\lambda)|A\rangle = 0. \tag{7.96}$$

Choosing $k^\mu = (|k^3|, 0, 0, k^3)$ this reads

$$\begin{aligned} &\left(|k^3|a^{(0)}(k) - k^3 a^{(3)}(k)\right)|A\rangle = 0 \\ &\left(a^{(0)}(k) \mp a^{(3)}(k)\right)|A\rangle = 0, \end{aligned} \tag{7.97}$$

i.e. *one* time-like photon by itself is not allowed! This solves the problems with the normalization and the negative energies.

7.3 Exercises

Exercise 7.1

Proof that the equal time commutation relations between $\phi(x)$ and $\Pi(x')$ are equivalent with the commutation relations between $a(k)$ and $a^\dagger(k')$

Exercise 7.2

(a) Show that

$$e^{-i\,a^\mu P_\mu}\, \phi(x)\, e^{i\,a^\mu P_\mu} = \phi(x+a)$$

requires $[P_\mu, \phi(x)] = i\partial_\mu \phi(x)$.

(b) Check the above commutation relation $[P_\mu, \phi(x)]$ for the (real) scalar field using the expressions for the fields and momentum operator in terms of creation and annihilation operators. Is the subtraction of 'zero point' contributions essential in this check?

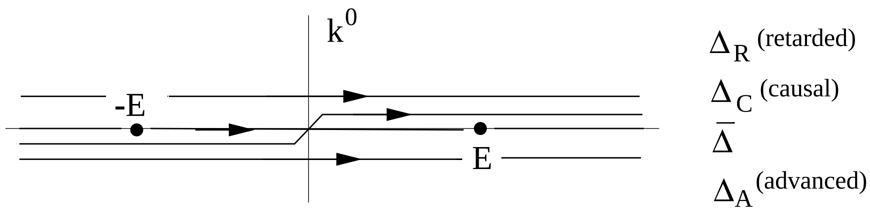
Exercise 7.3 (Greens functions)

(a) Show that
$$D(x) = \frac{1}{(2\pi)^4}\int d^4k\; e^{-ikx}\delta(k^2 - M^2)f(k)$$
is a solution of the homogeneous Klein-Gordon equation, $(\Box + M^2)\,D(x) = 0$.

(b) Show that
$$D(x) = \frac{1}{(2\pi)^4}\int d^4k\; e^{-ikx}\frac{1}{k^2 - M^2}$$
is a solution of the inhomogeneous Klein-Gordon equation, $(\Box + M^2)\,D(x) = -\delta^4(x)$.

(c) What are the poles in the integral under (b) in the (complex) k_0 plane?

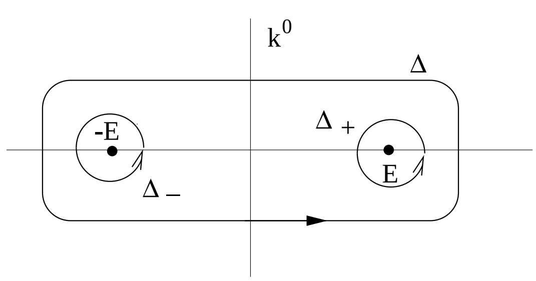
(d) Depending on the paths in the k_0 plane going from $k_0 = -\infty$ to $k_0 = +\infty$ one can distinguish four different Greens functions:



Show that for Δ_R we can write
$$\Delta_R = \frac{1}{(2\pi)^4}\int d^4k\; e^{-ikx}\frac{1}{k^2 - M^2 + i\epsilon k^0}$$
which amounts to shifting the poles into the lower complex plane. Give also the expression for Δ_C.

(e) Show that $\Delta_R(x) = 0$ if $x^0 < 0$. (Similarly one can show that for the advanced Greens function $\Delta_A(x) = 0$ if $x^0 > 0$).

(f) By performing the k_0 integration over a closed path C going around the poles show that
$$\Delta(x) = -\frac{1}{(2\pi)^4}\int_C d^4k\; e^{-ikx}\frac{1}{k^2 - M^2}$$
are homogeneous solutions (of the form in a) in the following cases:



(Note that one needs step functions $f(k) \propto \theta(k^0)$ or $\epsilon(k^0) = \theta(k^0) - \theta(-k^0)$).

(g) Show that the homogeneous solution Δ in (f) satisfies $\Delta(0, \boldsymbol{x}) = 0$ and hence $\Delta(x) = 0$ for $x^2 < 0$.

(h) Show that

$$\Delta_C(x) = \theta(x^0)\, \Delta_+(x) - \theta(-x^0)\, \Delta_-(x).$$

Chapter 8

Discrete symmetries

In this chapter we discuss the discrete symmetries, parity (P), time reversal (T) and charge conjugation (C). The consequences of P, T and C for classical quantities is shown in the table 1. The time reversal operator, which transforms an incoming state into an outgoing state must be an anti-unitary operator, i.e. $T_{op}|\ \rangle = \langle\ |$ implies that $T_{op}\,c|\psi\rangle = c^* T_{op}|\psi\rangle$.

8.1 Parity

The parity operator transforms

$$x^\mu = (t, \boldsymbol{r}) \longrightarrow \tilde{x}^\mu \equiv x_\mu = (t, -\boldsymbol{r}). \tag{8.1}$$

We will consider the transformation properties for a fermion field $\psi(x)$, writing

$$\psi(x) \longrightarrow P_{op}\psi(x)P_{op}^{-1} = \eta_P A \psi(\tilde{x}) \equiv \psi^p(\tilde{x}), \tag{8.2}$$

where η_P is the intrinsic parity of the field and A is a 4×4 matrix acting in the spinor space. Both ψ^p and ψ satisfy the Dirac equation. We can determine A, starting with the Dirac equation for $\psi(x)$,

$$(i\gamma^\mu \partial_\mu - M)\,\psi(x) = 0.$$

After parity transforming x to $\tilde{x}$ the Dirac equation becomes after some manipulations

$$\left(i\gamma^\mu \tilde{\partial}_\mu - M\right)\psi(\tilde{x}) = 0,$$

Table 8.1: *The behavior of classical quantities under P, T, and C*

quantity	P	T	C
t	t	-t	t
$\boldsymbol{r}$	-$\boldsymbol{r}$	$\boldsymbol{r}$	$\boldsymbol{r}$
x^μ	$\tilde{x}^\mu \equiv x_\mu$	$-\tilde{x}^\mu$	x^μ
E	E	E	E
$\boldsymbol{p}$	-$\boldsymbol{p}$	-$\boldsymbol{p}$	$\boldsymbol{p}$
p^μ	$\tilde{p}^\mu$	$\tilde{p}^\mu$	p^μ
$\boldsymbol{L}$	$\boldsymbol{L}$	-$\boldsymbol{L}$	$\boldsymbol{L}$
$\boldsymbol{s}$	$\boldsymbol{s}$	-$\boldsymbol{s}$	$\boldsymbol{s}$
$\lambda = \boldsymbol{s}\cdot\hat{\boldsymbol{p}}$	$-\lambda$	λ	λ

$$
\begin{aligned}
(i\tilde{\gamma}^{\mu}\partial_{\mu} - M)\,\psi(\tilde{x}) &= 0,\\
(i\gamma^{\mu\dagger}\partial_{\mu} - M)\,\psi(\tilde{x}) &= 0,\\
(i\gamma^{\mu}\partial_{\mu} - M)\,\gamma_0\;\psi(\tilde{x}) &= 0.
\end{aligned} \tag{8.3}
$$

Therefore $\gamma_0\psi(\tilde{x})$ is again a solution of the Dirac equation and we have

$$\psi^p(x) \;= \gamma_0\;\psi(\tilde{x}). \tag{8.4}$$

It is straightforward to apply this to the explicit field operator $\psi(x)$ using

$$\gamma_0\; u(k,m) \;=\; u(\tilde{k},m), \tag{8.5}$$

$$\gamma_0\; v(k,m) \;=\; -v(\tilde{k},m), \tag{8.6}$$

(check this for the standard representation; if helicity λ is used instead of the z-component of the spin m, the above operation reverses the sign of λ). The result is

$$\psi^p(x) \;=\; P_{op}\;\psi(x)\;P_{op}^{-1} = \eta_P\gamma_0\psi(\tilde{x}) \tag{8.7}$$

$$
\begin{aligned}
&= \sum_\lambda\int\frac{d^3k}{(2\pi)^3\;2E_k}\eta_P\left[b(k,\lambda)\gamma_0 u(k,\lambda)\,e^{-ik\cdot\tilde{x}} + d^\dagger(k,\lambda)\gamma_0 v(k,\lambda)\,e^{ik\cdot\tilde{x}}\right]\\
&= \sum_\lambda\int\frac{d^3k}{(2\pi)^3\;2E_k}\eta_P\left[b(k,\lambda)u(\tilde{k},-\lambda)\,e^{-i\tilde{k}\cdot x} - d^\dagger(k,\lambda)v(\tilde{k},-\lambda)\,e^{i\tilde{k}\cdot x}\right]\\
&= \sum_\lambda\int\frac{d^3\tilde{k}}{(2\pi)^3\;2E_{\tilde{k}}}\eta_P\left[b(k,\lambda)u(\tilde{k},-\lambda)\,e^{-i\tilde{k}\cdot x} - d^\dagger(k,\lambda)v(\tilde{k},-\lambda)\,e^{i\tilde{k}\cdot x}\right]\\
&= \sum_\lambda\int\frac{d^3k}{(2\pi)^3\;2E_k}\eta_P\left[b(\tilde{k},-\lambda)u(k,\lambda)\,e^{-ik\cdot x} - d^\dagger(\tilde{k},-\lambda)v(k,\lambda)\,e^{ik\cdot x}\right].
\end{aligned} \tag{8.8}
$$

From this one sees immediately that

$$P_{op}\;b(k,\lambda)\;P_{op}^{-1} \;=\; \eta_P\;b(\tilde{k},-\lambda), \tag{8.9}$$

$$P_{op}\;d(k,\lambda)\;P_{op}^{-1} \;=\; -\eta_P^*\;d(\tilde{k},-\lambda), \tag{8.10}$$

i.e. choosing η_P is real ($\eta_P = \pm 1$) particle and antiparticle have opposite parity.

In the same way as the Fermion field, one can also consider the scalar field and vector fields. For the scalar field

$$\phi(x) \;\longrightarrow\; P_{op}\;\phi(x)\;P_{op}^{-1} \;=\; \eta_P\;\phi(\tilde{x}), \tag{8.11}$$

and for the vector field

$$A^\mu(x) \;\longrightarrow\; P_{op}\;A^\mu(x)\;P_{op}^{-1} \;=\; -A_\mu(\tilde{x}). \tag{8.12}$$

The latter behavior of the vector field will be discussed further below.

8.2 Charge conjugation

We have already seen the particle-antiparticle symmetry with under what we will call charge conjugation the behavior

$$\psi(x) \;\longrightarrow\; \psi^c(x) = \eta_C\;C\;\bar{\psi}^T(x), \tag{8.13}$$

the latter being also a solution of the Dirac equation. The action on the spinors (using $C = i\gamma^2\gamma^0 = -i\rho^1\sigma^2$ in standard representation) gives

$$C\;\bar{u}^T(k,m) \;=\; v(k,m), \tag{8.14}$$

$$C\;\bar{v}^T(k,m) \;=\; u(k,m), \tag{8.15}$$

where one must be aware of the fact that when σ_i is the spin 1/2 representation acting on spinors χ, then $-\sigma_i^*$ is the (equivalent) conjugate representation acting on spinors χ^*. The transformation between these representations is given by $\bar{\chi} \propto \epsilon\chi^*$ and $-\boldsymbol{\sigma}^* = \sigma^2\boldsymbol{\sigma}\sigma^2$. Therefore

$$\begin{aligned}
\psi^c(x) &= C_{op}\ \psi(x)\ C_{op}^{-1} = \eta_C C\bar{\psi}^T(x) &(8.16)\\
&= \sum_\lambda \int \frac{d^3k}{(2\pi)^3\ 2E_k}\eta_C\left[d(k,\lambda)C\bar{v}^T(k,\bar{\lambda})\,e^{-ik\cdot x} + b^\dagger(k,\lambda)C\bar{u}^T(k,\bar{\lambda})\,e^{ik\cdot x}\right]\\
&= \sum_\lambda \int \frac{d^3k}{(2\pi)^3\ 2E_k}\eta_C\left[d(k,\bar{\lambda})u(k,\lambda)\,e^{-ik\cdot x} + b^\dagger(k,\bar{\lambda})v(k,\lambda)\,e^{ik\cdot x}\right]. &(8.17)
\end{aligned}$$

This shows that

$$C_{op}\ b(k,\lambda)\ C_{op}^{-1} = \eta_C\ d(k,\bar{\lambda}), \qquad (8.18)$$

$$C_{op}\ d(k,\lambda)\ C_{op}^{-1} = \eta_C^*\ b(k,\bar{\lambda}). \qquad (8.19)$$

8.3 Time reversal

The time reversal operator transforms

$$x^\mu = (t,\boldsymbol{r}) \longrightarrow -\tilde{x}^\mu \equiv -x_\mu = (-t,\boldsymbol{r}). \qquad (8.20)$$

We will again consider the transformation properties for a fermion field $\psi(x)$, writing

$$\psi(x) \longrightarrow T_{op}\psi(x)T_{op}^{-1} = \eta_T A\psi(-\tilde{x}) \equiv \psi^t(-\tilde{x}), \qquad (8.21)$$

where A is a 4×4 matrix acting in the spinor space. As time reversal will transform 'bra' into 'ket', $T_{op}|\phi\rangle = \langle\phi^t| = (|\phi^t\rangle)^*$, it is antilinear[1]. Norm conservation requires T_{op} to be anti-unitary[2]. For a quantized field one has

$$T_{op}\psi_k(x)b_kT_{op}^{-1} = \psi_k^*(x)T_{op}b_kT_{op}^{-1},$$

i.e. to find $\psi^t(-\tilde{x})$ that is a solution of the Dirac equation, we start with the complex conjugated Dirac equation for ψ,

$$((i\gamma^\mu)^*\partial_\mu - M)\,\psi(x) = 0.$$

The (time-reversed) Dirac equation becomes,

$$\begin{aligned}
&\left(-(i\gamma^\mu)^*\tilde{\partial}_\mu - M\right)\psi(-\tilde{x}) = 0,\\
&(i\tilde{\gamma}^{\mu *}\partial_\mu - M)\,\psi(-\tilde{x}) = 0,\\
&\left(i\gamma^{\mu T}\partial_\mu - M\right)\psi(-\tilde{x}) = 0,\\
&\left(-iC^{-1}\gamma^\mu C\partial_\mu - M\right)\psi(-\tilde{x}) = 0.\\
&\left(i(\gamma_5 C)^{-1}\gamma^\mu\gamma_5 C\partial_\mu - M\right)\psi(-\tilde{x}) = 0.\\
&(i\gamma^\mu\partial_\mu - M)\,\gamma_5 C\psi(-\tilde{x}) = 0. \qquad (8.22)
\end{aligned}$$

Therefore $\gamma_5 C\psi(-\tilde{x})$ is again a solution of the (ordinary) Dirac equation and we can choose (phase is convention)

$$\psi^t(x) = i\,\gamma_5\,C\,\psi(-\tilde{x}). \qquad (8.23)$$

[1] A is antilinear if $A(\lambda|\phi\rangle + \mu|\psi\rangle) = \lambda^* A|\phi\rangle + \mu^* A|\psi\rangle$.

[2] An antilinear operator is anti-unitary if $A^\dagger = A^{-1}$. One has $\langle A\phi|A\psi\rangle = \langle\phi|\psi\rangle^* = \langle A\psi|A\phi\rangle^* = \langle\psi|A^\dagger A\phi\rangle = \langle\psi|\phi\rangle$.

Table 8.2: *The transformation properties of physical states for particles (a) and antiparticles ($\bar{a}$).*

state	P	T	C
$\vert a;\boldsymbol{p},\lambda\rangle$	$\vert a;-\boldsymbol{p},-\lambda\rangle$	$\langle a;-\boldsymbol{p},\lambda\vert$	$\vert\bar{a};\boldsymbol{p},\lambda\rangle$
$\vert\bar{a};\boldsymbol{p},\lambda\rangle$	$\vert\bar{a};-\boldsymbol{p},-\lambda\rangle$	$\langle\bar{a};-\boldsymbol{p},\lambda\vert$	$\vert a;\boldsymbol{p},\lambda\rangle$

In the standard representation $i\gamma_5 C = \sigma_2$ and it is straightforward to apply this to the explicit field operator $\psi(x)$ using

$$i\,\gamma_5 C\; u(k,\lambda) = u^*(\tilde{k},\lambda), \tag{8.24}$$
$$i\,\gamma_5 C\; v(k,\lambda) = v^*(\tilde{k},\lambda), \tag{8.25}$$

(check this for the standard representation). The result is

$$\psi^t(x) = T_{op}\;\psi(x)\;T_{op}^{-1} = i\,\eta_T\gamma_5 C\psi(-\tilde{x}) \tag{8.26}$$
$$\begin{aligned}
&= \sum_\lambda \int \frac{d^3k}{(2\pi)^3\;2E_k}\eta_T\left[b(k,\lambda)i\gamma_5 C u(k,\lambda)\,e^{ik\cdot\tilde{x}} + d^\dagger(k,\lambda)i\gamma_5 C v(k,\lambda)\,e^{-ik\cdot\tilde{x}}\right]\\
&= \sum_\lambda \int \frac{d^3k}{(2\pi)^3\;2E_k}\eta_T\left[b(k,\lambda)u^*(\tilde{k},\lambda)\,e^{i\tilde{k}\cdot x} + d^\dagger(k,\lambda)v^*(\tilde{k},\lambda)\,e^{-i\tilde{k}\cdot x}\right]\\
&= \sum_\lambda \int \frac{d^3\tilde{k}}{(2\pi)^3\;2E_k}\eta_T\left[b(k,\lambda)u^*(\tilde{k},\lambda)\,e^{i\tilde{k}\cdot x} + d^\dagger(k,\lambda)v^*(\tilde{k},\lambda)\,e^{-i\tilde{k}\cdot x}\right]
\end{aligned}$$
$$= \sum_\lambda \int \frac{d^3k}{(2\pi)^3\;2E_k}\eta_T\left[b(\tilde{k},\lambda)u^*(k,\lambda)\,e^{ik\cdot x} + d^\dagger(\tilde{k},\lambda)v^*(k,\lambda)\,e^{-ik\cdot x}\right] \tag{8.27}$$

From this one obtains

$$T_{op}\;b(k,\lambda)\;T_{op}^{-1} = \eta_T\;b(\tilde{k},\lambda), \tag{8.28}$$
$$T_{op}\;d(k,\lambda)\;T_{op}^{-1} = \eta_T^*\;d(\tilde{k},\lambda). \tag{8.29}$$

In table 2 the behavior of particle states under the various transformations has been summarized. Note that applying an anti-unitary transformation such as T_{op} one must take for the matrix element the complex conjugate. Therefore one has $\langle A|k\rangle = \langle A|b^\dagger(k)|0\rangle = \langle A|T^\dagger T\,b^\dagger(k)\,T^\dagger T|0\rangle^* = \langle A^t|b^\dagger(\tilde{k})|0\rangle^*$ $= \langle 0|b(\tilde{k})|A^t\rangle = \langle\tilde{k}|A^t\rangle$.

8.4 Bilinear combinations

In quantities such as currents and lagrangians often bilinear combinations of spinor fields are encountered. Since there are 16 independent 4×4 matrices, there are 16 independent of these bilinear combinations. They are the following

$$S(x) = \bar{\psi}(x)\psi(x) \qquad \text{(scalar)} \tag{8.30}$$
$$V^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x) \qquad \text{(vector)} \tag{8.31}$$
$$T^{\mu\nu}(x) = \bar{\psi}(x)\sigma^{\mu\nu}\psi(x) \qquad \text{(tensor)} \tag{8.32}$$
$$A^\mu(x) = \bar{\psi}(x)\gamma_5\gamma^\mu\psi(x) \qquad \text{(axial vector)} \tag{8.33}$$
$$P(x) = i\bar{\psi}(x)\gamma_5\psi(x) \qquad \text{(pseudoscalar)} \tag{8.34}$$

Table 8.3: *The behavior of the independent bilinear combinations of fermi fields under P, C, and T*

	P	C	T	Θ = PCT
$S(x)$	$S(\tilde{x})$	$S(x)$	$S(-\tilde{x})$	$S(-x)$
$V^\mu(x)$	$V_\mu(\tilde{x})$	$-V^\mu(x)$	$V_\mu(-\tilde{x})$	$-V^\mu(-x)$
$T^{\mu\nu}(x)$	$T_{\mu\nu}(\tilde{x})$	$-T^{\mu\nu}(x)$	$-T_{\mu\nu}(-\tilde{x})$	$T^{\mu\nu}(-x)$
$A^\mu(x)$	$-A_\mu(\tilde{x})$	$A^\mu(x)$	$A_\mu(-\tilde{x})$	$-A^\mu(-x)$
$P(x)$	$-P(\tilde{x})$	$P(x)$	$-P(-\tilde{x})$	$P(-x)$

The matrix $\sigma^{\mu\nu} \equiv (i/2)[\gamma^\mu, \gamma^\nu]$. The 16 combinations of Dirac matrices appearing above are linearly independent. Applying the results from the previous sections it is straightforward to determine the behavior of the combinations under P, C, and T, as well as under the combined operation Θ = PCT (see Table 3). As the coupling of the photon field to fermions is given by an interaction term in the lagrangian of the form $:\bar{\psi}(x)\gamma^\mu\psi(x): A_\mu(x)$ and behaves as a scalar one sees immediately that the photon field $A^\mu(x)$ behaves in the same way as the vector combination $\bar{\psi}(x)\gamma^\mu\psi(x)$. Note that the lagrangian density $\mathcal{L}(x) \to \mathcal{L}(-x)$ under Θ.

8.5 Form factors

Currents play an important role in field theory. In many applications the expectation values of currents are needed, e.g. for the vector current $V^\mu(x)$,

$$\langle p', s'|V^\mu(x)|p, s\rangle. \tag{8.35}$$

The x-dependence can be accounted for straightforwardly using translation invariance, $V^\mu(x) = e^{i\,P_{op}\cdot x}V^\mu(0)e^{-i\,P_{op}\cdot x}$. This implies

$$\langle p'|V^\mu(x)|p\rangle = e^{-i\,(p-p')\cdot x}\langle p'|V^\mu(0)|p\rangle. \tag{8.36}$$

As an example consider the vector current for a point fermion,

$$V^\mu(x) =: \bar{\psi}(x)\gamma^\mu\psi(x):, \tag{8.37}$$

of which the expectation value between momentum eigenstates can be simply found,

$$\langle p'|V^\mu(0)|p\rangle = \bar{u}(p')\gamma^\mu u(p). \tag{8.38}$$

In general the expectation value between momentum states can be more complicated,

$$\langle p'|V^\mu(x)|p\rangle = \bar{u}(p')?^\mu(p', p)u(p)\,e^{-i\,(p-p')\cdot x}, \tag{8.39}$$

where $?^\mu(p', p)$ can be built from any combination of Dirac matrices (1, γ^μ, $\sigma^{\mu\nu}$, $\gamma_5\gamma^\mu$ or γ_5), momenta (p^μ or p'^μ) or constant tensors ($\epsilon_{\mu\nu\rho\sigma}$ or $g_{\mu\nu}$). For instance for nucleons one has

$$?_\mu(p', p) = \gamma_\mu\,F_1(q^2) + \frac{i\sigma_{\mu\nu}q^\nu}{2M}\,F_2(q^2), \tag{8.40}$$

where the coefficients are functions of invariants constructed out of the momenta, in this case only $q^2 = (p' - p)^2$, called *form factors*. There are several other terms that also have the correct tensorial behavior such as

$$q^\mu\,F_3(q^2), \qquad \frac{(p^\mu + p'^\mu)}{2M}\,F_{..}(q^2), \qquad \frac{\gamma_5\sigma^{\mu\nu}q_\nu}{2M}\,F_{..}(q^2), \tag{8.41}$$

but that are eliminated because of relations between Dirac matrices, e.g. $\gamma_5\sigma_{\mu\nu} = (1/2)\epsilon_{\mu\nu\rho\sigma}\gamma^\rho\gamma^\sigma$, or relations that follow from the equations of motion, e.g. the Gordon decomposition

$$\bar{u}(p\prime)\gamma^\mu u(p) = \frac{1}{2M}\bar{u}(p')\left[(p'+p)^\mu + i\sigma^{\mu\nu}q_\nu\right]u(p), \tag{8.42}$$

where $q = p' - p$, or relations based on hermiticity of the operator or P, C and T invariance.

- **Hermiticity**

$$\begin{aligned} \langle p'|V^\mu(0)|p\rangle &= \bar{u}(p')?^\mu(p',p)u(p) \\ = \langle p'|V^{\mu\dagger}(0)|p\rangle &= \langle p|V^\mu(0)|p'\rangle^* \\ &= (\bar{u}(p)?^\mu(p,p')u(p'))^* \\ &= \left(u^\dagger(p)\gamma_0?^\mu(p,p')u(p')\right)^\dagger \\ &= u^\dagger(p')?^{\mu\dagger}(p,p')\gamma_0 u(p). \end{aligned}$$

Therefore hermiticity implies for the structure of $?^\mu(p',p)$ that

$$\gamma_0?^{\mu\dagger}(p,p')\gamma_0 = ?^\mu(p',p). \tag{8.43}$$

- **Parity**

$$\begin{aligned} &\langle p'|V^\mu(x)|p\rangle = e^{i\,q\cdot x}\,\bar{u}(p')?^\mu(p',p)u(p) \\ &= \langle p'|P_{op}^\dagger P_{op}V^\mu(x)P_{op}^\dagger P_{op}|p\rangle \\ &= \langle\tilde{p}'|V_\mu(\tilde{x})|\tilde{p}\rangle = e^{i\,\tilde{q}\cdot\tilde{x}}\,\bar{u}(\tilde{p}')?_\mu(\tilde{p}',\tilde{p})u(\tilde{p}) \\ &\qquad = e^{i\,q\cdot x}\,\bar{u}(p')\gamma_0?_\mu(\tilde{p}',\tilde{p})\gamma_0 u(p). \end{aligned}$$

Therefore parity invariance implies for the structure of $?^\mu(p',p)$ that

$$?^\mu(p',p) = \gamma_0?_\mu(\tilde{p}',\tilde{p})\gamma_0, \tag{8.44}$$

- **Time reversal**

$$\begin{aligned} &\langle p'|V^\mu(x)|p\rangle = e^{i\,q\cdot x}\,\bar{u}(p')?^\mu(p',p)u(p) \\ &= \langle p'|T_{op}^\dagger T_{op}V^\mu(x)T_{op}^\dagger T_{op}|p\rangle \\ &= \langle\tilde{p}'|V_\mu(-\tilde{x})|\tilde{p}\rangle = e^{i\,\tilde{q}\cdot\tilde{x}}\,[\bar{u}(\tilde{p}')]^*?^*_\mu(\tilde{p}',\tilde{p})u^*(\tilde{p}) \\ &\qquad = e^{i\,q\cdot x}\,\bar{u}(p')\,(i\gamma_5 C)?^*_\mu(\tilde{p}',\tilde{p})(i\gamma_5 C)u(p). \end{aligned}$$

Therefore time reversal invariance implies for the structure of $?^\mu(p',p)$ that

$$?^\mu(p',p) = (i\,\gamma_5 C)?^*_\mu(\tilde{p}',\tilde{p})(i\,\gamma_5 C). \tag{8.45}$$

8.6 Exercises

Exercise 8.1
The matrix element of the electromagnetic current between nucleon states is written as

$$< p'|J_\mu(x)|p >= e^{-i(p-p')\cdot x}\;\;\bar{u}(p')?_\mu u(p)$$

Here

$$\begin{aligned} ?_\mu &= \gamma_\mu F_1(q^2) + \frac{i\sigma_{\mu\nu}q^\nu}{2M}F_2(q^2) \\ &= \gamma_\mu H_1(q^2) - \frac{p_\mu + p_\mu\prime}{2M}H_2(q^2) \end{aligned}$$

(a) Use the Dirac equation to prove the Gordon-decomposition

$$\bar{u}(p\prime)\gamma^\mu u(p) = \frac{1}{2M}\bar{u}(p')[(p'+p)^\mu + i\sigma^{\mu\nu}(p'-p)_\nu]u(p)$$

(b) Give the relation between H_i en F_i.

(c) Show that hermiticity of the current requires that F_1 en F_2 are real.

Exercise 8.2

(a) Calculate the current matrix element (using the explicit fermion spinors) in the Breit-frame in which $q = (0, 0, 0, |\boldsymbol{q}|)$, $p = (E, 0, 0, -|\boldsymbol{q}|/2)$, $p' = (E, 0, 0, |\boldsymbol{q}|/2)$ with $E^2 = M^2 + |\boldsymbol{q}|^2/4$ and express them in terms of the Sachs form factors depending on $Q^2 = -q^2$,

$$\begin{aligned} G_E &= F_1 - \frac{Q^2}{4M^2}F_2, \\ G_M &= F_1 + F_2. \end{aligned}$$

(b) Show that in the interaction with the electromagnetic field,

$$\mathcal{L}_{\,int} = e j_\mu A^\mu$$

the charge is given by $e\,G_E(0)$ and the magnetic moment by $e\,G_M(0)/2M$.

Exercise 8.3

(a) Show that because of parity conservation no term of the form

$$\gamma^5 \frac{\sigma_{\mu\nu}q^\nu}{2M} F_3(q^2)$$

can appear in the current matrix element.

(b) Show that such a term is also not allowed by time-reversal symmetry.

(c) Show that if a term of this form would exist, it would correspond to an electric dipole moment $d = -e\,F_3(0)/2M$ (Interaction term $d\,\sigma \cdot \boldsymbol{E}$).

Chapter 9

Path integrals and quantum mechanics

9.1 Time evolution as path integral

The time evolution from $t_0 \to t$ of a quantum mechanical system is generated by the Hamiltonian,

$$U(t, t_0) = e^{-i(t-t_0)H} \tag{9.1}$$

or

$$i\frac{\partial}{\partial t}U(t, t_0) = H\, U(t, t_0) \tag{9.2}$$

Two situations can be distinguished:

(i) *Schrödinger picture*, in which the operators are time-independent, $A_S(t) = A_S$ and the states are time dependent, $|\psi_S(t)\rangle = U(t, t_0)|\psi_S(t_0)\rangle$,

$$i\frac{\partial}{\partial t}|\psi_S\rangle = H\,|\psi_S\rangle, \tag{9.3}$$

$$i\frac{\partial}{\partial t}A_S \equiv 0. \tag{9.4}$$

(ii) *Heisenberg picture*, in which the states are time-independent, $\psi_H(t)\rangle = |\psi_H\rangle$, and the operators are time-dependent, $A_H(t) = U^{-1}(t, t_0)\, A_H(t_0)\, U(t, t_0)$,

$$i\frac{\partial}{\partial t}|\psi_H\rangle \equiv 0, \tag{9.5}$$

$$i\frac{\partial}{\partial t}A_H = [A_H, H]. \tag{9.6}$$

Of these the Heisenberg picture is most appropriate for quantum field theory since the field operators do depend on the position and one would like to have position and time on the same footing.

Consider the two (time-independent) Heisenberg states:

$$\begin{array}{ll} |q, t\rangle & Q_H(t)|q, t\rangle \equiv q|q, t\rangle, \\ |q', t'\rangle & Q_H(t')|q', t'\rangle \equiv q'|q', t'\rangle, \end{array} \tag{9.7}$$

and Schrödinger states

$$\begin{aligned} |q\rangle &= |q(t)\rangle \qquad Q_S|q\rangle \equiv q|q\rangle, \\ |q'\rangle &= |q'(t)\rangle \qquad Q_S|q'\rangle \equiv q'|q'\rangle. \end{aligned} \tag{9.8}$$

Choose t as the starting point with $|q\rangle = |q,t\rangle$ and $Q_H(t) = Q_S$ and study the evolution of the system by calculating the quantum mechanical overlap amplitude

$$\langle q', t'|q, t\rangle = \langle q'|e^{-i\,H(t'-t)}|q\rangle. \tag{9.9}$$

Dividing the interval from $t \equiv t_0$ to $t' \equiv t_n$ into n pieces of length $\Delta\tau$ and using completeness (at each time t_i) one writes

$$\begin{aligned} \langle q', t'|q, t\rangle &= \langle q'|e^{-i\,n\,H\Delta\tau}|q\rangle = \langle q'|\left(e^{-i\,H\Delta\tau}\right)^n|q\rangle \\ &= \int dq_1 \int \dots \int dq_{n-1}\,\langle q'|e^{-i\,H\Delta\tau}|q_{n-1}\rangle\,\langle q_{n-1}|\dots|q_1\rangle\,\langle q_1|e^{-i\,H\Delta\tau}|q,t\rangle. \end{aligned} \tag{9.10}$$

The purpose of this is to calculate the evolution for an infinitesimal time interval. The hamiltonian is an operator $H = H(P,Q)$ expressed in terms of the operators P and Q. These can be written in coordinate or momentum representation as

$$Q = \int dq\;|q\rangle\,q\,\langle q| \tag{9.11}$$

$$P = \int dq\;|q\rangle\left(-i\frac{\partial}{\partial q}\right)\langle q| = \int \frac{dp}{2\pi}\;|p\rangle\,p\,\langle p|, \tag{9.12}$$

where the transformation between coordinate and momentum space involves

$$\langle q|p\rangle = e^{i\,p\cdot q}. \tag{9.13}$$

At least for a simple hamiltonian such as consisting of a kinetic energy term and a local potential, $H(P,Q) = K(P) + V(Q) = (P^2/2M) + V(Q)$ one can split $e^{-i\,H\Delta\tau} \approx e^{-i\,K\Delta\tau}\,e^{-i\,V\Delta\tau}$, with the correction[1] being of order $(\Delta\tau)^2$. Then

$$\begin{aligned} \langle q_{j+1}, t_{j+1}|q_j, t_j\rangle &\approx \langle q_{j+1}|e^{-i\,K\Delta\tau}\,e^{-i\,V\Delta\tau}|q_j\rangle \\ &= \int \frac{dp_j}{2\pi}\,\langle q_{j+1}|p_j\rangle\,\langle p_j|e^{-i\,K\Delta\tau}\,e^{-i\,V\Delta\tau}|q_j\rangle. \end{aligned} \tag{9.15}$$

By letting the kinetic and potential parts act to left and right respectively one can express the expectation values in terms of integrals containing $H(p_j, q_j)$, which is a hamiltonian in which the operators are replaced by real-numbered variables. Combining the two terms gives

$$\begin{aligned} \langle q_{j+1}, t_{j+1}|q_j, t_j\rangle &= \int \frac{dp}{2\pi}\;e^{i\,p(q_{j+1}-q_j)}\;e^{-i\,H(p_j,q_j)\Delta\tau} \\ &= \int \frac{dp_j}{2\pi}\;e^{i\,p(q_{j+1}-q_j)-i\,\Delta\tau H(p_j,q_j)} \\ &= \int \frac{dp_j}{2\pi}\;\exp\left(i\,\Delta\tau\left[p_j\dot{q}_j - H(p_j,q_j)\right]\right), \end{aligned} \tag{9.16}$$

[1]For this, use the Campbell-Baker-Hausdorff formula,

$$e^A\,e^B = e^C \quad \text{with} \quad C = A + B + \frac{1}{2}[A,B] + \frac{1}{12}[A,[A,B]] + \frac{1}{12}[[A,B],B] + \dots \tag{9.14}$$

or for the full interval

$$
\begin{aligned}
\langle q', t'|q, t\rangle &= \int \prod_{\tau} \frac{dq(\tau)\,dp(\tau)}{2\pi}\ \exp\left(i\sum_{t}^{t'} \Delta\tau\,[p\dot{q} - H(p,q)]\right) \\
&= \int \frac{\mathcal{D}q\,\mathcal{D}p}{2\pi}\ \exp\left(i\int_t^{t'} d\tau\,[p\dot{q} - H(p,q)]\right), \qquad (9.17)
\end{aligned}
$$

where $\mathcal{D}q$ and $\mathcal{D}p$ indicate functional integrals. The importance of this expression is that it expresses a quantum mechanical amplitude as a *path integral* with in the integrand a classical hamiltonian function.

Before proceeding we also give the straightforward extension to more than one degree of freedom,

$$
\begin{aligned}
&\langle q_1', \ldots, q_N', t'|q_1, \ldots, q_N, t\rangle \qquad (9.18) \\
&= \int \prod_{n=1}^{N} \frac{\mathcal{D}q_n\,\mathcal{D}p_n}{2\pi}\ \exp\left(i\int_t^{t'} d\tau\left[\sum_n p_n\dot{q}_n - H(p, q_1, \ldots, q_N)\right]\right).
\end{aligned}
$$

9.2 Functional integrals

In this section I want to give a fairly heuristic discussion of functional integrals. What one is after is the meaning of

$$
\int \mathcal{D}\alpha\ F[\alpha], \qquad (9.19)
$$

where $F[\alpha]$ is a functional that represents a mapping from a function space $\mathcal{F}$ of (real) functions α $(R \to R)$ into the real numbers R, i.e. $F[\alpha] \in R$. Examples are

$$
\begin{aligned}
F[\alpha] &= \alpha \equiv \int dx\,\alpha(x), \\
F[\alpha] &= \alpha^2 \equiv \int dx\,\alpha^2(x), \\
F[\alpha, \beta] &= \alpha\, G\, \beta \equiv \int dx\,dy\ \alpha(x)\,G(x,y)\,\beta(y), \\
F[\alpha] &= \exp\left(-\frac{1}{2}\alpha^2\right) \equiv \exp\left(-\frac{1}{2}\int dx\,\alpha^2(x)\right), \\
F[\alpha, \alpha^*] &= \exp\left(-\alpha^*\,K\,\alpha\right) \equiv \exp\left(-\int dx\,\alpha^*(x)\,K(x,y)\,\alpha(y)\right).
\end{aligned}
$$

Two working approaches are the following:

(i) Divide space on which α acts into cells, i.e. $\alpha(x) \to \alpha_x$ and $F[\alpha] = F(\alpha_x)$ changes into a multivariable function, while

$$
\int \mathcal{D}\alpha\ F[\alpha] = \int \left(\prod_x \frac{d\alpha_x}{N}\right)\ F(\alpha_x) \qquad (9.20)
$$

becomes a multidimensional integral.

(ii) Write α as the sum of basis functions, $\alpha = \sum_n \alpha_n\, f_n$ and consider $F[\alpha] = F(\alpha_n)$ as a multivariable function, with

$$
\int \mathcal{D}\alpha\ F[\alpha] = \int \left(\prod_n \frac{d\alpha_n}{N}\right)\ F(\alpha_n) \qquad (9.21)
$$

again a multidimensional integral. In both cases N is an appropriately choosen normalization constant in such a way that the integral is finite. Note that procedure (i) is an example of the more general procedure under (ii).

Consider the gaussian functional as an example.

$$\begin{aligned} \int \mathcal{D}\alpha\ \exp\left(-\frac{1}{2}\alpha^2\right) &= \int \left(\prod_x \frac{d\alpha_x}{N}\right) \exp\left(-\frac{1}{2}\sum_x \Delta x\, \alpha_x^2\right) \\ &= \prod_x \int \frac{d\alpha_x}{N} \exp\left(-\frac{1}{2}\Delta x\, \alpha_x^2\right) \\ &= \prod_x \left(\frac{1}{N}\sqrt{\frac{2\pi}{\Delta x}}\right) \equiv 1. \end{aligned} \tag{9.22}$$

The last equality is obtained by defining the right measure (normalization N) in the integration. Physical answers will usually come out as the ratio of two functional integrals and are thus independent of the chosen measure. Having defind the Gaussian integral, the following integrals can be derived for a symmetric or hermitean kernel K,

$$\int \mathcal{D}\alpha\ \exp\left(-\frac{1}{2}\,\alpha\, K\, \alpha\right) = \frac{1}{\sqrt{\det K}}, \tag{9.23}$$

$$\int \mathcal{D}\alpha\, \mathcal{D}\alpha^*\ \exp\left(-\alpha^*\, K\, \alpha\right) = \frac{1}{\det K}. \tag{9.24}$$

A useful property of functional integration is the translation invariance,

$$\int \mathcal{D}\alpha\ F[\alpha] = \int \mathcal{D}\alpha\ F[\alpha+\beta]. \tag{9.25}$$

As an important application of translation invariance we mention the identity

$$\int \mathcal{D}\alpha\ \exp\left(-\omega\alpha - \frac{1}{2}\alpha^2\right) = \exp\left(\frac{1}{2}\omega^2\right), \tag{9.26}$$

or with $\omega \to \pm i\,\omega$,

$$\int \mathcal{D}\alpha\ \exp\left(\pm i\,\omega\alpha - \frac{1}{2}\alpha^2\right) = \exp\left(-\frac{1}{2}\omega^2\right). \tag{9.27}$$

Functional differentiation is defined as

$$\left[\frac{\delta}{\delta\alpha(x)}, \alpha(y)\right] = \delta(x-y), \tag{9.28}$$

or in discretized form

$$\frac{\delta}{\delta\alpha(x)} \longrightarrow \frac{1}{\Delta x}\,\frac{\partial}{\partial\alpha_x}. \tag{9.29}$$

Examples are

$$\frac{\delta}{\delta\alpha(x)}\,\alpha = \frac{\delta}{\delta\alpha(x)}\int dy\,\alpha(y) = \int dy\,\delta(x-y) = 1, \tag{9.30}$$

$$\frac{\delta}{\delta\alpha(x)}\,\alpha G\beta = \int dy\, G(x,y)\beta(y) \tag{9.31}$$

$$\frac{\delta}{\delta\alpha(x)}\ \exp\left(-\frac{1}{2}\alpha^2\right) = -\alpha(x)\exp\left(-\frac{1}{2}\alpha^2\right), \tag{9.32}$$

$$\frac{\delta}{\delta\alpha(x)}\ \exp\left(-\alpha\,\beta\right) = -\beta(x)\exp\left(-\alpha\,\beta\right). \tag{9.33}$$

For applications to fermion fields, we need to consider anticommuting Grassmann variables, i.e. $\theta\eta = -\eta\theta$, $\theta^2 = 0$. In principle the definitions of functionals is the same, e.g. the Gaussian-type functionals,

$$F[\theta,\theta^*] \;=\; \exp\left(-\theta^*\,\theta\right) \;\equiv\; \exp\left(-\int dx\,\theta^*(x)\,\theta(x)\right),$$

$$F[\theta,\theta^*] \;=\; \exp\left(-\theta^*\,K\,\theta\right) \;\equiv\; \exp\left(-\int dx\,\theta^*(x)\,K(x,y)\,\theta(y)\right).$$

Integration for Grassmann variables is defined as

$$\int d\theta\; 1 \;=\; 0 \qquad \text{and} \qquad \int d\theta\;\theta \;=\; 1. \tag{9.34}$$

This gives for the Gaussian integral

$$\begin{aligned}\int \mathcal{D}\theta^*\,\mathcal{D}\theta\;\exp\left(-\theta^*\,\theta\right) \;&=\; \int \left(\prod_x \frac{d\theta^*_x\,d\theta_x}{N}\right)\;\exp\left(-\sum_x \Delta x\,\theta^*_x\,\theta_x\right)\\ &=\; \prod_x \int \frac{d\theta^*_x\,d\theta_x}{N}\;\exp\left(-\Delta x\,\theta^*_x\,\theta_x\right)\\ &=\; \prod_x \int \frac{d\theta^*_x\,d\theta_x}{N}\;\left(1-\Delta x\,\theta^*_x\,\theta_x\right) = 1.\end{aligned} \tag{9.35}$$

We note that from the first to the second line one needs to realize that a pairs of different Grassmann variables behaves as ordinary complex variables. In the expansion of the exponential (from second to third line), however, products of the same pair appear, which vanish. the same pairs It is now easy to check that

$$\int \mathcal{D}\theta^*\,\mathcal{D}\theta\;\exp\left(-\theta^*\,K\,\theta\right) = \det K. \tag{9.36}$$

Functional derivation of Grassmann-valued functions is given by

$$\left\{\frac{\delta}{\delta\theta(x)},\theta(y)\right\} = \delta(x-y), \tag{9.37}$$

leading to e.g.

$$\frac{\delta}{\delta\eta(x)}\,e^{-\bar\theta\eta} = \bar\theta(x) = \bar\theta(x)\,e^{-\bar\theta\eta}, \tag{9.38}$$

$$\frac{\delta}{\delta\bar\eta(x)}\,e^{-\bar\eta\theta} = -\theta(x) = -\theta(x)\,e^{-\bar\eta\theta}. \tag{9.39}$$

The translation invariance property and the application to the 'Fourier transform' identity of Gaussian integrals, remains valid for Grassmann variables.

9.3 Time ordered products

As an example of working with functional integrals consider the expression for $K(q',t';q,t)$ for a lagrangian $L(q,\dot q) = \frac{1}{2}\dot q^2 - V(q)$ and the corresponding hamiltonian $H(p,q) = \frac{1}{2}p^2 + V(q)$. The expression

$$\langle q',t'|q,t\rangle = \int \frac{\mathcal{D}q\,\mathcal{D}p}{2\pi}\;\exp\left(i\int_t^{t'} d\tau\left[p\dot q - \frac{1}{2}p^2 - V(q)\right]\right), \tag{9.40}$$

can be rewritten after rewriting the integrand as $-\frac{1}{2}(p-\dot{q})^2 + \frac{1}{2}\dot{q}^2 - V(q) = -\frac{1}{2}(p-\dot{q})^2 + L(q,\dot{q})$. The result is

$$\langle q',t'|q,t\rangle = \int \mathcal{D}q \quad \exp\left(i\int_t^{t'} d\tau\ L(q,\dot{q})\right), \tag{9.41}$$

which was considered the starting point for path integral quantization by Feynman.

Note, however, that not always the $\mathcal{D}p$ integration can be removed that easily. A counter example is the lagrangian $L(q,\dot{q}) = \frac{1}{2}\dot{q}^2 f(q)$ for which $H(p,q) = p^2/[2f(q)]$. As discussed for instance in Ryder the $\mathcal{D}p$ integration can still be removed but one ends up with an effective lagrangian in the path integral

$$\langle q',t'|q,t\rangle = \int \mathcal{D}q \quad \exp\left(i\int_t^{t'} d\tau\ L_{eff}(q,\dot{q})\right), \tag{9.42}$$

which is of the form $L_{eff}(q,\dot{q}) = L(q,\dot{q}) - \frac{i}{2}\delta(0)\ln f(q)$.

Making use of path integrals it is straightforward to calculate the expectation value $\langle q',t'|Q(s)|q,t\rangle$ of an operator $Q(s)$ if $t \le s \le t'$. By sandwiching the time s in one of the infinitesimal intervals, $t_j \le s \le t_{j+1}$, we have

$$\begin{aligned}\langle q',t'|Q(s)|q,t\rangle &= \int \prod_i dq_i\ \langle q',t'|q_n,t_n\rangle\ \ldots \\ &\qquad \times \langle q_{j+1},t_{j+1}|Q(s)|q_j,t_j\rangle\ \ldots\ \langle q_1,t_1|q,t\rangle.\end{aligned} \tag{9.43}$$

Using $Q(s)|q_j,t_j\rangle = q(s)|q_j,t_j\rangle$, one gets

$$\langle q',t'|Q(s)|q,t\rangle = \int \frac{\mathcal{D}q\,\mathcal{D}p}{2\pi}\ q(s)\ \exp\left(i\int_t^{t'} d\tau\,[p\dot{q} - H(p,q)]\right) \tag{9.44}$$

$$= \int \mathcal{D}q \quad q(s)\ \exp\left(i\int_t^{t'} d\tau\ L(q,\dot{q})\right). \tag{9.45}$$

Defining the time ordered product of operators

$$\mathcal{T}\,Q(t_1)\ldots Q(t_n) \equiv Q(t_{i_1})\ldots Q(t_{i_n}), \tag{9.46}$$

where $t_{i_1} \ge \ldots \ge t_{i_n}$ is a permutation of $\{t_1,\ldots,t_n\}$, one has

$$\begin{aligned}&\langle q',t'|\mathcal{T}\,Q(t_1)\ldots Q(t_n)|q,t\rangle \\ &\quad = \int \mathcal{D}q \quad q(t_{i_1})\ldots q(t_{i_n})\ \exp\left(i\int_t^{t'} d\tau\ L(q,\dot{q})\right).\end{aligned} \tag{9.47}$$

9.4 The interaction picture

In the interaction picture, a separation is made of the hamiltonian $H = H_0 + H_I$. The fast time evolution is described in H_0 while H_I is considered as a perturbation. The interaction picture is defined as

$$U_I(t',t) \equiv e^{-i\,H_0(t'-t)}, \tag{9.48}$$

$$\psi_I(t) \equiv e^{i\,H_0 t}\psi_S(t) = e^{i\,H_0 t}\,e^{-i\,Ht}\,\psi_S(0) = e^{-iH_I t}\psi_S(0), \tag{9.49}$$

$$A_I(t) \equiv e^{i\,H_0 t}A_S(t)e^{-i\,H_0 t} = e^{i\,H_0 t}A_S e^{-i\,H_0 t}, \tag{9.50}$$

i.e. if $H_I = 0$ it is the Heisenberg picture and the evolution $\langle q', t'|q, t\rangle$ is described through the operators by H_0. The evolution of the (interaction) states is described only by H_I,

$$\begin{aligned} i\frac{\partial}{\partial t}|\psi_I\rangle &= H_I(t)\,\psi_I(t), &(9.51)\\ i\frac{\partial}{\partial t}A_I &= [A_I, H_0], &(9.52)\end{aligned}$$

and

$$\psi_I(t') = U(t', t)\,\psi_I(t), \tag{9.53}$$

$$i\frac{\partial}{\partial t}U(t', t) = H_I\,U(t', t) \tag{9.54}$$

with $U(t, t) = 1$ or

$$U(t', t) = 1 - i\int_t^{t'} d\tau\; H_I(\tau)\,U(\tau, t), \tag{9.55}$$

which can be solved by iteration, i.e. writing

$$U(t', t) = \sum_{n=0}^{\infty} U^{(n)}(t', t), \tag{9.56}$$

one has

$$\begin{aligned} U^{(0)}(t', t) &= 1\\ U^{(1)}(t', t) &= 1 - i\int_t^{t'} d\tau\; H_I(\tau)\,U^{(0)}(\tau, t) - U^{(0)}(t', t)\\ &= (-i)\int_t^{t'} d\tau_1\; H_I(\tau_1)\\ &\;\vdots\\ U^{(n)}(t', t) &= 1 - i\int_t^{t'} d\tau\; H_I(\tau)\,U^{(n-1)}(\tau, t) - U^{(n-1)}(t', t)\\ &= (-i)^n\int_t^{t'} d\tau_1\int_t^{\tau_1} d\tau_2\ldots\int_t^{\tau_{n-1}} d\tau_n\; H_I(\tau_1)\,\ldots\,H_I(\tau_n)\\ &= \frac{(-i)^n}{n!}\int_t^{t'} d\tau_1\int_t^{t'} d\tau_2\ldots\int_t^{t'} d\tau_n\;\mathcal{T}\,H_I(\tau_1)\,\ldots\,H_I(\tau_n). \end{aligned} \tag{9.57}$$

The last equality is illustrated for the second term $U^{(2)}$ in the following. The integration

$$\int_t^{t'} d\tau_1\int_t^{\tau_1} d\tau_2\; H_I(\tau_1)\,H_I(\tau_2)$$

can also be performed by first integrating over τ_2 but changing the integration limits (check!). Thus we can write the integration as the sum of the two expressions (multiplying with 1/2),

$$\begin{aligned} = \;&\frac{1}{2}\int_t^{t'} d\tau_1\int_t^{\tau_1} d\tau_2\; H_I(\tau_1)\,H_I(\tau_2)\\ &+\frac{1}{2}\int_t^{t'} d\tau_2\int_{\tau_2}^{t'} d\tau_1\; H_I(\tau_1)\,H_I(\tau_2), \end{aligned}$$

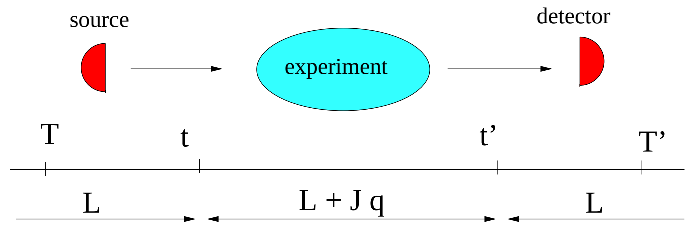


Figure 9.1: Physical picture of vacuum to vacuum amplitude in presence of a source term

Now the integration can be extended by adding theta functions,

$$\begin{aligned} = \; & \frac{1}{2}\int_t^{t'} d\tau_1 \int_t^{t'} d\tau_2 \; H_I(\tau_1)\, H_I(\tau_2)\theta(\tau_1-\tau_2) \\ & \quad + \frac{1}{2}\int_t^{t'} d\tau_2 \int_t^{t'} d\tau_1 \; H_I(\tau_1)\, H_I(\tau_2)\theta(\tau_1-\tau_2), \end{aligned}$$

which can be rewritten (by interchanging in the second term the names of the integration variables)

$$= \frac{1}{2}\int_t^{t'} d\tau_1 \int_t^{t'} d\tau_2 \; \left[H_I(\tau_1)\, H_I(\tau_2)\theta(\tau_1-\tau_2) + H_I(\tau_2)\, H_I(\tau_1)\theta(\tau_2-\tau_1)\right],$$

the desired result.

The time evolution operator, therefore, can be written (symbolically) as

$$U(t',t) = \mathcal{T}\,\exp\left(-i\int_t^{t'} d\tau\, H_I(\tau)\right), \tag{9.58}$$

and we have as expected

$$\begin{aligned} \langle q', t'|q,t\rangle^V &= \langle q',t'|\mathcal{T}\,\exp\left(-i\int_t^{t'} d\tau\, H_I(Q(\tau))\right)|q,t\rangle^{V=0} \\ &= \int \frac{\mathcal{D}q\,\mathcal{D}p}{2\pi}\,\exp\left(-i\int_t^{t'} d\tau\, H_I(q)\right)\exp\left(i\int_t^{t'} d\tau\,\left[p\dot{q} - H_0(p,q)\right]\right) \\ &= \int \frac{\mathcal{D}q\,\mathcal{D}p}{2\pi}\,\exp\left(i\int_t^{t'} d\tau\,\left[p\dot{q} - H(p,q)\right]\right). \end{aligned} \tag{9.59}$$

9.5 The ground-state to ground-state amplitude

By introducing a source-term, $L(q,\dot{q}) \rightarrow L(q,\dot{q}) + J(t)\cdot q$, it is possible to switch on an interaction, physically pictured as, say, the creation of an electron (think of a radio-tube, making the electron) and the absorption of an electron (think of a detector). Before and after these processes there is only the vacuum or ground-state $|0\rangle$. Consider, furthermore, a set $|n\rangle$ of physical eigenstates. The Heisenberg state $|q,t\rangle$ is related to the Schrödinger state $|q\rangle$ by $|q,t\rangle = e^{i\,Ht}|q\rangle$, i.e. for the physical states

$$\langle q,t|n\rangle = \langle q|n\rangle\, e^{-iE_n t} \tag{9.60}$$

with for example

$$\begin{aligned} \langle x|n\rangle &= \phi_n(x) = e^{i\,k_n x} \quad \text{for plane waves,} \\ \langle x|n\rangle &= e^{-\omega^2 x^2/2} \quad \text{for g.s. harmonic oscillator.} \end{aligned}$$

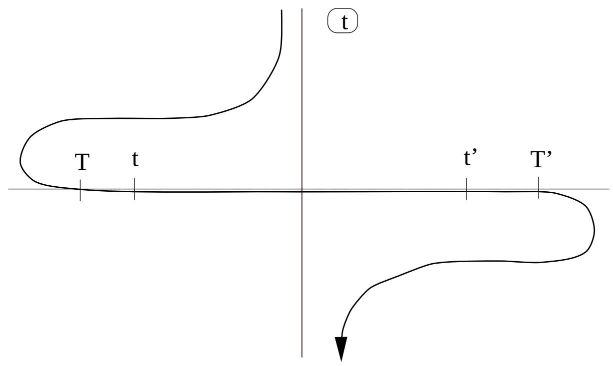


Figure 9.2: Analytic continuation of boundaries T and T'

Considering the source to be present between times t and t', which in turn are embedded between an early time T and a future time T', i.e. $T < t < t' < T'$, one has

$$\langle q', t' | q, t \rangle^J = \int \mathcal{D}q \; \exp\left(i \int_t^{t'} d\tau \; [L(q,\dot{q}) + J(\tau) q] \right), \tag{9.61}$$

$$\langle Q', T' | Q, T \rangle^J = \int dq' \, dq \langle Q', T' | q', t' \rangle \, \langle q', t' | q, t \rangle^J \, \langle q, t | Q, T \rangle, \tag{9.62}$$

with

$$\langle q, t | Q, T \rangle = \sum_n \langle q, t | n \rangle \, \langle n | Q, T \rangle = \sum_n \phi_n(q) \, \phi_n^*(Q) \, e^{-i\,E_n (t-T)}. \tag{9.63}$$

We can project out the groundstate by an analytic continuation in the time, $T \to i\,\infty$ and $T' \to -i\,\infty$, in which case

$$\lim_{T \to i\infty} \langle q, t | Q, T \rangle = \phi_0(q) \, \phi_0^*(Q) \, e^{-i\,E_0 (t-T)} \propto e^{-E_0 \cdot \infty}.$$

We define the generating functional $W[J]$ as

$$W[J] = \lim_{\substack{T' \to -i\infty \\ T \to i\infty}} \frac{\langle Q', T' | Q, T \rangle^J}{e^{-i\,E_0 (T'-T)} \phi_0(Q') \, \phi_0^*(Q)} \tag{9.64}$$

$$= \int dq' \, dq \phi_0^*(q', t') \langle q', t' | q, t \rangle^J \, \phi_0(q, t) \tag{9.65}$$

$$= \langle 0_{out} | 0_{in} \rangle. \tag{9.66}$$

The factor that has been divided out in the first line of this equation is a numerical factor depending on the boundaries of the space-time volume (T and T'). The generating functional precisely represents the vacuum to vacuum amplitude from initial ('in') to final('out') situation in the presence of a source. The time ordered product of operators can be expressed as

$$\left. \frac{\delta^n W[J]}{\delta J(t_1) \ldots \delta J(t_n)} \right|_{J=0} = (i)^n \, \langle 0 | \mathcal{T} \, Q(t_1) \ldots Q(t_n) | 0 \rangle, \tag{9.67}$$

an equality that easily can be proven by inserting the expression for the $\langle q', t' | q, t \rangle^J$ as the time ordered product involving $H_I = -J(\tau) Q(\tau)$ and using the definition in section 9.4.

9.6 Euclidean formulation

Neglecting multiplicative factors we have the generating functional

$$W[J] = \lim_{\substack{T' \to -i\infty \\ T \to i\infty}} \int \mathcal{D}q \;\exp\left(i\int_T^{T'} d\tau\,\left[L\left(q,\frac{dq}{d\tau}\right) + J(\tau)q(\tau)\right]\right), \tag{9.68}$$

which is, however, ill-defined. Better is the use of imaginary time $t = -i\tilde{t}$, such that

$$\begin{array}{rcl} t \to i\infty & \leftrightarrow & \tilde{t} \to -\infty, \\ t' \to -i\infty & \leftrightarrow & \tilde{t}' \to \infty. \end{array}$$

In terms of the imaginary time we can write the Euclidean generating functional,

$$W_E[J] = \int \mathcal{D}q \;\exp\left(\int_{-\infty}^{\infty} d\tilde{\tau}\,\left[L\left(q, i\frac{dq}{d\tilde{\tau}}\right) + J(\tilde{\tau})q(\tilde{\tau})\right]\right) \tag{9.69}$$

$$= \int \mathcal{D}q \;\exp\left(-\int_{-\infty}^{\infty} d\tilde{\tau}\,\left[L_E\left(q, \frac{dq}{d\tilde{\tau}}\right) - J(\tilde{\tau})q(\tilde{\tau})\right]\right), \tag{9.70}$$

where

$$L_E\left(q,\frac{dq}{d\tilde{t}}\right) \equiv -L\left(q, i\frac{dq}{d\tilde{t}}\right), \tag{9.71}$$

e.g. when

$$L\left(q,\frac{dq}{dt}\right) = \frac{1}{2}\left(\frac{dq}{dt}\right)^2 - V(q) \tag{9.72}$$

$$= -\frac{1}{2}\left(\frac{dq}{d\tilde{t}}\right)^2 - V(q),$$

$$L_E\left(q,\frac{dq}{d\tilde{t}}\right) = \frac{1}{2}\left(\frac{dq}{d\tilde{t}}\right)^2 + V(q), \tag{9.73}$$

which is a positive definite quantity, ensuring convergence for the functional integral $W_E[J]$. As discussed in the previous section the interesting quantities are obtained from functional differentiation with respect to the sources. The differentiations in $W[J]$ and $W_E[J]$ are related,

$$\frac{1}{W[J]}\,\frac{\delta^n W[J]}{\delta J(t_1)\ldots\delta J(t_n)}\bigg|_{J=0} = (i)^n\,\frac{1}{W_E[J]}\,\frac{\delta^n W_E[J]}{\delta J(\tilde{t}_1)\ldots\delta J(\tilde{t}_n)}\bigg|_{\substack{J=0 \\ \tilde{t}=it}}, \tag{9.74}$$

where the expressions have been divided by $W[J]$ and $W_E[J]$ in order to get rid of dependence on multiplicative factors.

9.7 Exercises

Exercise 9.1
Convince yourself by discretizing the space that

$$\int \mathcal{D}\alpha\;\exp\left(-\frac{1}{2}\,\alpha\,K\,\alpha\right) = \frac{1}{\sqrt{\det K}},$$

$$\int \mathcal{D}\alpha\,\mathcal{D}\alpha^*\exp\left(-\alpha^*\,K\,\alpha\right) = \frac{1}{\det K}.$$

for complex valued functions and

$$\int \mathcal{D}\theta^* \, \mathcal{D}\theta \;\; \exp\left(-\theta^* \, K \, \theta\right) = \det K.$$

for Grassmann valued functions.

Exercise 9.2
Check the examples of functional derivation and the "Fourier transform" property of functional integrals in section 2 for complex and Grassmann valued functions.

Exercise 9.3
Proof the relation

$$\left.\frac{\delta^n W[J]}{\delta J(t_1) \ldots \delta J(t_n)}\right|_{J=0} = (i)^n \, \langle 0|\mathcal{T} \, Q(t_1) \ldots Q(t_n)|0\rangle.$$

(see remarks at the end of section 9.5).

Chapter 10

Feynman diagrams for scattering amplitudes

10.1 Generating functional for scalar fields

10.1.1 Generalization to quantum fields

The connection for quantum fields with the foregoing is made by generalizing to a system with more degrees of freedom, i.e. $\phi(\mathbf{x},t)$ is considered as a set of quantum operators $\phi_{\mathbf{x}}(t)$ in the Heisenberg picture and

$$\begin{aligned}\langle \phi'; x', t' | \phi; x, t\rangle &= \int \mathcal{D}\phi\,\mathcal{D}\Pi\ \exp\left(i\int_t^{t'} d\tau\, d^3x\,\left[\Pi(x)\dot{\phi}(x) - \mathcal{H}(\phi,\Pi)\right]\right) \\ &= \int \mathcal{D}\phi\ \exp\left(i\int_{\sigma}^{\sigma'} d^4x\ \mathcal{L}(x)\right) \qquad (10.1)\end{aligned}$$

and

$$\begin{aligned}W[J] &= \lim_{\substack{T\to i\infty \\ T'\to -i\infty}} \int \mathcal{D}\phi\ \exp\left(i\int_T^{T'} d^4x\ [\mathcal{L}(\phi,\partial_\mu\phi) + J\phi]\right) \qquad (10.2) \\ &= \int \mathcal{D}\phi\ \exp\left(-\int_{-\infty}^{\infty} d^4x_E\ [\mathcal{L}_{\ E}(\phi,\partial_\mu\phi) - J\phi]\right) \qquad (10.3)\end{aligned}$$

The Euclidean formulation is as before, implying at the level of four vectors in coordinate and momentum space (when $k\cdot x \equiv k_E\cdot x_E$) for instance

$$x^4 = i\,x^0 = i\,t \qquad\qquad k^4 = -i\,k^0$$

$$dx^2 = (dx^0)^2 - \textstyle\sum_{i=1}^3 (dx^i)^2 = -\sum_{\mu=1}^4 (dx^\mu)^2 \qquad\qquad k^2 = (k^0)^2 - \sum_{i=1}^3 (k^i)^2 = -\sum_{\mu=1}^4 (k^\mu)^2$$

$$d^4x_E = i\,d^4x \qquad\qquad d^4k_E = -i\,d^4k$$

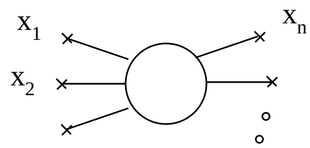
$$\mathcal{L} = \tfrac{1}{2}\,\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}\,M^2\,\phi^2 \qquad\qquad \mathcal{L}_{\ E} = \tfrac{1}{2}\,(\partial_E^\mu\phi)^2 + \tfrac{1}{2}\,M^2\,\phi^2$$

Furthermore we have

$$\langle 0|\mathcal{T}\,\phi(x_1)\ldots\phi(x_n)|0\rangle = (-i)^n \left.\frac{\delta^n W[J]}{\delta J(x_1)\ldots\delta J(x_n)}\right|_{J=0}$$

$$\equiv \quad G^{(n)}(x_1, \ldots, x_n), \tag{10.4}$$

or pictorially

$$G^{(n)}(x_1, \ldots, x_n) =$$ 

10.1.2 $W[J]$ for the free scalar field

For the (free) lagrangian

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\,M^2\,\phi^2, \tag{10.5}$$

the generating functional is (using $\overleftarrow{\partial}_\mu = -\,\overrightarrow{\partial}_\mu$) given by

$$\begin{aligned} W_0[J] &= \int \mathcal{D}\phi\ \exp\left(-i\int d^4x\,\left[\frac{1}{2}\,\phi(\partial_\mu\partial^\mu + M^2)\phi - J\phi\right]\right) \qquad (10.6)\\ &= \int \prod_x d\phi_x\ \exp\left(-i\sum_x \Delta^4 x\,\left[\frac{1}{2}\,\sum_y \Delta^4 y\phi_x\,K_{xy}\,\phi_y - J_x\phi_x\right]\right) \end{aligned}$$

where K_{xy} is the discretized form of $(\partial_\mu\partial^\mu + M^2)\,\delta^4(x-y)$. Rewriting the exponential

$$\begin{aligned} &-\frac{1}{2}\,i\sum_x \Delta^4 x\left[\sum_y \Delta^4 y\ \phi_x\,K_{xy}\,\phi_y - 2\,J_x\phi_x\right]\\ &= -\frac{1}{2}\,i\sum_x \Delta^4 x\sum_y \Delta^4 y\left(\phi_x - \sum_z \Delta^4 z\,J_z\,K^{-1}_{zx}\right)K_{xy}\left(\phi_y - \sum_{z'} \Delta^4 z'\,K^{-1}_{yz'}\,J_{z'}\right)\\ &\qquad + \frac{1}{2}\,i\sum_z \Delta^4 z\,\sum_{z'} \Delta^4 z'\,J_z\,K^{-1}_{zz'}\,J_{z'}, \end{aligned}$$

and using

$$\int \prod_\alpha d\phi_\alpha\ \exp\left(-\frac{1}{2}\sum_{\beta,\gamma} \phi_\beta\,K_{\beta\gamma}\,\phi_\gamma\right) = \frac{1}{\sqrt{\det K}} \tag{10.7}$$

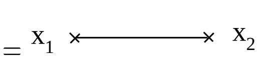
one finds

$$W_0[J] \propto \exp\left(-\frac{1}{2}\,i\int d^4x\,d^4y\ J(x)\,\Delta_F(x-y)\,J(y)\right), \tag{10.8}$$

where

$$\left(\partial_\mu\partial^\mu + M^2\right)\Delta_F(x-y) = -\delta^4(x-y), \tag{10.9}$$

i.e. Δ_F (the socalled Feynman propagator) is the Green's function of the Klein-Gordon equation. Furthermore,

$$i\Delta_F(x_1 - x_2) = \frac{(-i)^2}{W[0]}\ \frac{\delta^2 W[J]}{\delta J(x_1)\,\delta J(x_2)}\bigg|_{J=0} = \tag{10.10}$$ 

$$= \langle 0|\mathcal{T}\,\phi(x_1)\,\phi(x_2)|0\rangle \qquad \text{provided } W[0] = 1. \tag{10.11}$$

In order to determine Δ_F, consider the general solution of

$$\left(\partial_\mu\partial^\mu + M^2\right)\Delta_F(x) = -\delta^4(x),$$

which can be written as

$$\Delta(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{-i\,k\cdot x}\,\tilde{\Delta}(k) \tag{10.12}$$

with $(k^2 - M^2)\tilde{\Delta}(k) = 1$ or

$$\tilde{\Delta}(k) = \frac{1}{k^2 - M^2} = \frac{1}{(k^0)^2 - \mathbf{k}^2 - M^2} = \frac{1}{(k^0)^2 - E^2}. \tag{10.13}$$

Depending on the path choosen in the complex k^0 plane (see exercise 7.1) one distinguishes the *retarded* Green's function

$$\Delta_R(x) = \int \frac{d^4k}{(2\pi)^4}\,\frac{e^{-i\,k\cdot x}}{(k^0 + i\epsilon)^2 - E^2}, \tag{10.14}$$

satisfying $\Delta_R(\mathbf{x}, t) = 0$ for $t < 0$,
the *advanced* Green's function

$$\Delta_A(x) = \int \frac{d^4k}{(2\pi)^4}\,\frac{e^{-i\,k\cdot x}}{(k^0 - i\epsilon)^2 - E^2}, \tag{10.15}$$

satisfying $\Delta_A(\mathbf{x}, t) = 0$ for $t > 0$,
the *causal* Green's function

$$\Delta_C(x) = \int \frac{d^4k}{(2\pi)^4}\,\frac{e^{-i\,k\cdot x}}{(k^0)^2 - E^2 + i\epsilon} = \int \frac{d^4k}{(2\pi)^4}\,\frac{e^{-i\,k\cdot x}}{k^2 - M^2 + i\epsilon}, \tag{10.16}$$

and finally the Green's function

$$\bar{\Delta}(x) = \mathcal{P}\int \frac{d^4k}{(2\pi)^4}\,\frac{e^{-i\,k\cdot x}}{k^2 - M^2}. \tag{10.17}$$

They are solutions of the inhomogeneous equation. The solution of the homogeneous equation $\left(\partial_\mu\partial^\mu + M^2\right)\Delta(x) = 0$ can also be written as an integral in k-space,

$$\Delta(x) = -\int_C \frac{d^4k}{(2\pi)^4}\,\frac{e^{-i\,k\cdot x}}{k^2 - M^2}, \tag{10.18}$$

where C is a closed contour in the complex k^0-plane. The contours for

$$\begin{aligned} i\,\Delta(x) &= [\phi(x), \phi(0)] = i\Delta_+(x) + i\Delta_-(x) \\ i\,\Delta_+(x) &= [\phi_+(x), \phi_-(0)] = \int \frac{d^3k}{(2\pi)^3\,2E}\,e^{-i\,k\cdot x} \\ i\,\Delta_-(x) &= [\phi_-(x), \phi_+(0)] = \int \frac{d^3k}{(2\pi)^3\,2E}\,e^{-i\,k\cdot x} \end{aligned}$$

are also shown in exercise (7.1). It is straightforward (closing contours in the appropriate half of the complex plane) to prove that

$$i\,\Delta_C(x) = \theta(x^0)\,i\,\Delta_+(x) - \theta(-x^0)\,i\,\Delta_-(x). \tag{10.19}$$

In order to see which is the appropriate Green's function to be used in the generating functional $W_0[J]$ one can consider the well-defined Euclidean formulation or explicitly consider $\langle 0|\mathcal{T}\,\phi(x_1)\,\phi(x_2)|0\rangle$.

Firstly, in the (well-defined) Euclidean formulation starting with $\mathcal{L}_E = \frac{1}{2}(\partial_E^\mu\phi)^2 + \frac{1}{2}M^2\phi^2$ - $J\phi$ with the equation of motion $\left((\partial_E^\mu)^2 - M^2\right)\phi(x) = -J(x)$, the generating functional can be written

$$\begin{aligned} W_0[J] &= \int \mathcal{D}\phi\;\exp\left(\int d^4x_E\left[\frac{1}{2}\phi\left(\partial_E^\mu\partial_E^\mu - M^2\right)\phi + J\phi\right]\right) && (10.20) \\ &= \exp\left(-\frac{1}{2}\int d^4x_E\,d^4y_E\;J(x)\left(-i\,\Delta_F(x-y)\right)J(y)\right), && (10.21) \end{aligned}$$

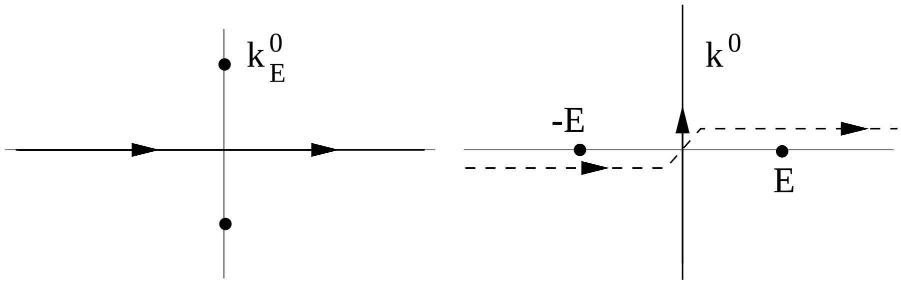


Figure 10.1: Contour in the k^0 plane for the Feynman propagator in Euclidean and Minkowski space

where

$$\left((\partial_E^\mu)^2 - M^2\right)(i\,\Delta_F(x-y)) = -\delta^4(x-y) \tag{10.22}$$

or (see fig. 6.1 for countours)

$$\Delta_F(x) = -i\int_{k_E^0=-\infty}^{k_E^0=\infty}\frac{d^4k_E}{(2\pi)^4}\,\frac{e^{-i\,k_E\cdot x_E}}{k_E^2+M^2} \tag{10.23}$$

$$= \int_{k^0=-i\infty}^{k^0=i\infty}\frac{d^4k}{(2\pi)^4}\,\frac{e^{-i\,k\cdot x}}{k^2-M^2}, \tag{10.24}$$

where the latter contour can be deformed to the contour for Δ_C, i.e. $\Delta_F = \Delta_C$.

Also calculating the time-ordered product explicitly we find

$$\begin{aligned} i\,\Delta_F(x-y) \\ &= \langle 0|\mathcal{T}\,\phi(x)\,\phi(y)|0\rangle \\ &= \theta(x^0-y^0)\langle 0|\phi(x)\,\phi(y)|0\rangle + \theta(y^0-x^0)\langle 0|\phi(y)\,\phi(x)|0\rangle \\ &= \theta(x^0-y^0)\langle 0|\phi_+(x)\,\phi_-(y)|0\rangle + \theta(y^0-x^0)\langle 0|\phi_+(y)\,\phi_-(x)|0\rangle \\ &= \theta(x^0-y^0)\langle 0|[\phi_+(x),\phi_-(y)]|0\rangle - \theta(y^0-x^0)\langle 0|[\phi_-(x),\phi_+(y)]|0\rangle \\ &= \theta(x^0-y^0)\,i\,\Delta_+(x-y) - \theta(y^0-x^0)\,i\,\Delta_-(x-y) \\ = i\,\Delta_C(x-y). \end{aligned} \tag{10.25}$$

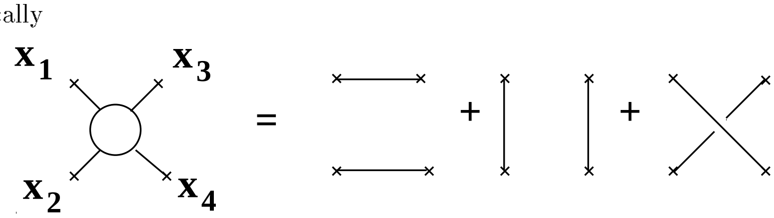
Knowing the explicit form of $W_0[J]$ it is straightforward to calculate the 4-points Green's function $G^{(4)}$ in terms of $G^{(2)} = i\,\Delta_F$. Neglecting multiplicative factors or equivalently assuming that $W[0] = 1$, we have

$$G^{(4)}(x_1,x_2,x_3,x_4) = \langle 0|\mathcal{T}\,\phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)|0\rangle \tag{10.26}$$

$$= (-i)^4\,\frac{\delta^4}{\delta J(x_1)\ldots\delta J(x_4)}\,\exp\left(-\frac{1}{2}i\int J\,\Delta_F\,J\right)\Bigg|_{J=0} \tag{10.27}$$

$$\begin{aligned} &= -\left[\Delta_F(x_1-x_2)\,\Delta_F(x_3-x_4) + \Delta_F(x_1-x_3)\,\Delta_F(x_2-x_4)\right. \\ &\qquad \left. + \Delta_F(x_1-x_4)\,\Delta_F(x_2-x_3)\right], \end{aligned} \tag{10.28}$$

or diagrammatically



10.1.3 Connected Green's functions

The Green's function appear in the expansion of $W[J]$, for which we can write

$$W[J] = \sum_n \frac{i^n}{n!} \int d^4x_1 \ldots d^4x_n \; G^{(n)}(x_1, \ldots, x_n)\, J(x_1) \ldots J(x_n). \tag{10.29}$$

Writing

$$W[J] = \exp\left(i\, Z[J]\right), \tag{10.30}$$

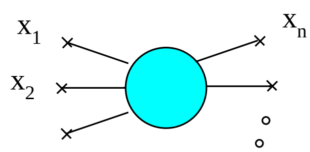
with

$$Z[J] = \sum_n \frac{i^{n-1}}{n!} \int d^4x_1 \ldots d^4x_n \; G_c^{(n)}(x_1, \ldots, x_n)\, J(x_1) \ldots J(x_n). \tag{10.31}$$

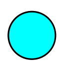
we obtain the connected Green's functions,

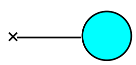
$$G_c^{(n)}(x_1, \ldots, x_n) = (-i)^{n-1} \left. \frac{\delta^n Z[J]}{\delta J(x_1) \ldots \delta J(x_n)} \right|_{J=0}, \tag{10.32}$$

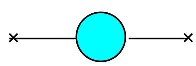
which are denoted

$G_c^{(n)}(x_1, \ldots, x_n) =$ 

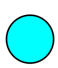
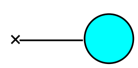
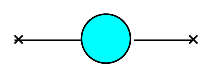
Specific examples are

$G_c^{(0)} =$  (vacuum bubble)

$G_c^{(1)}(x_1) =$  (tadpole)

$G_c^{(2)}(x_1, x_2) =$  (connected propagator)

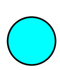
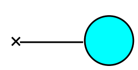
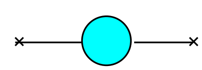
The proof that $W[J] = \exp(i\, Z[J])$ can be performed inductively. I will just write down the first few terms, i.e. if

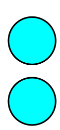
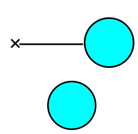
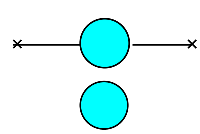
$i\, Z[J] =$  $+ i \int d^4x_1 \{$  $\} + \frac{i^2}{2!} \int d^4x_1\, d^4x_2 \{$  $\}$,

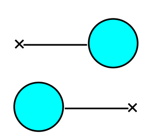
then

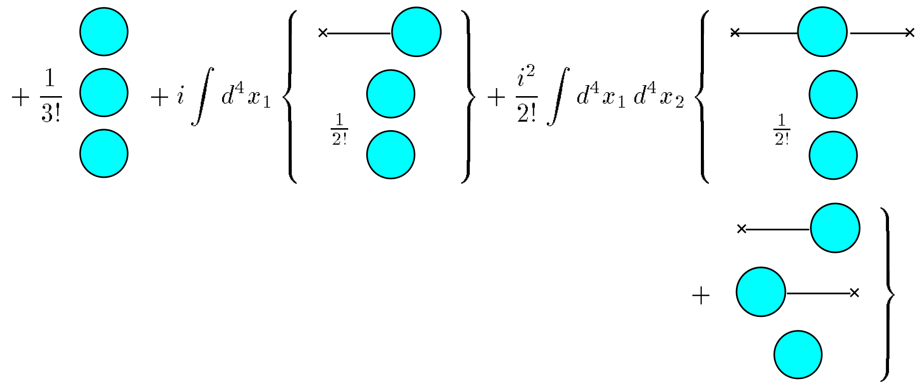
$$\exp(i\, Z[J]) = 1 + i\, Z[J] + \frac{(i\, Z[J])^2}{2!} + \frac{(i\, Z[J])^3}{3!} + \ldots$$

$= 1$

$+$  $+ i \int d^4x_1 \{$  $\} + \frac{i^2}{2!} \int d^4x_1\, d^4x_2 \{$  $\}$

$+ \frac{1}{2!}$  $+ i \int d^4x_1 \{$  $\} + \frac{i^2}{2!} \int d^4x_1\, d^4x_2 \{$ 

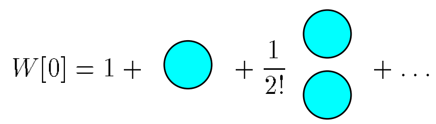
$+$  $\}$



$+ \ldots$

Notes:

(i) In $G^{(n)}$ all appropriate $G_c^{(n)}$ appear with correct combinatorial factors (this will be discussed later).
(ii) Note that $W[J]$ can be rewritten as an expansion of *source connected* Green's functions by dividing out $W[0]$,



i.e. the vacuum blobs appear as a multiplicative factor.
(iii) $G_{sc}^{(1)} = G_c^{(1)}$.
(iv) If

$$\left.\frac{\delta W[J]}{\delta J(x)}\right|_{J=0} = i \, \langle 0|\phi(x)|0\rangle = 0,$$

i.e. the vacuum expectation value of the field $\phi(x)$ is zero, implying the absence of tadpoles, then $G_{sc}^{(n)} = G_c^{(n)}$ for $n \leq 3$.

For the free scalar theory considered we have

$$Z_0[J] = -\frac{1}{2}\int d^4x \, d^4y \; J(x)\, \Delta_F(x-y)\, J(y), \tag{10.33}$$

implying as expected

$$G_c^{(2)}(x_1 - x_2) = (-i)\left.\frac{\delta^2 Z[J]}{\delta J(x_1)\,\delta J(x_2)}\right|_{J=0} \tag{10.34}$$

$$= G^{(2)}(x_1 - x_2) = i\,\Delta_F(x_1 - x_2), \tag{10.35}$$

$$G_c^{(n)}(x_1, \ldots, x_n) = (-i)^n \left.\frac{\delta^n Z[J]}{\delta J(x_1)\ldots\delta J(x_n)}\right|_{J=0} = 0, \tag{10.36}$$

for $n \geq 3$.

10.1.4 $W[J]$ in the interacting case

When interactions are present, i.e. $\mathcal{L}(\phi) = \mathcal{L}_0(\phi) + \mathcal{L}_I(\phi)$, the generating functional can be written,

$$W[J] = \int \mathcal{D}\phi \; \exp\left(i\int d^4x \, [\mathcal{L}_{\,0}(\phi) + \mathcal{L}_{\,I}(\phi) + J\phi]\right) \tag{10.37}$$

$$
\begin{aligned}
&= \exp\left(i\int d^4z\,\mathcal{L}_I\left(\frac{1}{i}\,\frac{\delta}{\delta J(z)}\right)\right) \\
&\qquad \int \mathcal{D}\phi\ \exp\left(i\int d^4x\,[\mathcal{L}_0(\phi)+J\phi]\right) \qquad (10.38)\\
&= \exp\left(i\int d^4z\,\mathcal{L}_I\left(\frac{1}{i}\,\frac{\delta}{\delta J(z)}\right)\right) \\
&\qquad \exp\left(-\frac{1}{2}\,i\int d^4x\,d^4y\,J(x)\,\Delta_F(x-y)\,J(y)\right) \qquad (10.39)\\
&= \exp\left(\frac{1}{2}\int d^4x\,d^4y\,\frac{\delta}{\delta\phi(x)}\,i\,\Delta_F(x-y)\,\frac{\delta}{\delta\phi(y)}\right) \\
&\qquad \exp\left(i\int d^4z\,[\mathcal{L}_I(\phi)+J(z)\phi(z)]\right)\Bigg|_{\phi=0}. \qquad (10.40)
\end{aligned}
$$

This expression will be the one from which Feynman rules will be derived, with propagators ($i\,\Delta_F$) being connected to vertices ($i\,\mathcal{L}_I$) according to the above expression for the functional $W[J]$.

Consider as an example the interaction

$$\mathcal{L}_I(\phi) = -\frac{g}{4!}\,\phi^4$$

in the scalar field theory discussed sofar. To zeroth order in the coupling one has

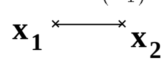
$$W^{(0)}[J] = \exp\left(-\frac{1}{2}\,i\int J\,\Delta_F\,J\right), \qquad (10.41)$$

and

$$W^{(0)}[0] = 1, \qquad (10.42)$$

$$\langle 0|\mathcal{T}\,\phi(x_1)\,\phi(x_2)|0\rangle^{(0)} = (-i)^2\,\frac{\delta^2 W^{(0)}[J]}{\delta J(x_1)\,\delta J(x_2)}\Bigg|_{J=0}$$

$$= i\,\Delta_F(x_1-x_2) = \qquad (10.43)$$

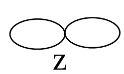


To first order in g one has (see Ryder, p. 206/207)

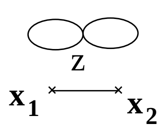
$$
\begin{aligned}
W^{(1)}[J] &= i\int d^4z\,\mathcal{L}_I\left(\frac{1}{i}\,\frac{\delta}{\delta J(z)}\right)\ \exp\left(-\frac{1}{2}\,i\int J\,\Delta_F\,J\right) \qquad (10.44)\\
&= -i\,\frac{g}{4!}\int d^4z\left\{-3\,\Delta_F^2(0)+6i\,\Delta_F(0)\left[\int d^4x\,\Delta_F(z-x)\,J(x)\right]^2\right.\\
&\qquad \left.+\left[\int d^4x\,\Delta_F(z-x)\,J(x)\right]^4\right\}\exp\left(-\frac{1}{2}\,i\int J\,\Delta_F\,J\right),
\end{aligned}
$$

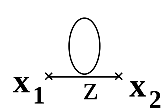
and

$$W^{(1)}[0] = -i\,\frac{g}{8}\int d^4z \qquad , \qquad (10.45)$$

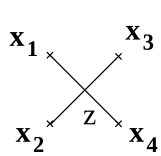


$$\langle 0|\mathcal{T}\,\phi(x_1)\,\phi(x_2)|0\rangle^{(1)} = -i\,\frac{g}{8}\int d^4z \qquad - i\,\frac{g}{2}\int d^4z \qquad , \qquad (10.46)$$

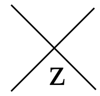




$$\langle 0|\mathcal{T}\,\phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)|0\rangle_c^{(1)} = -ig\int d^4z \qquad . \qquad (10.47)$$



The presence of the vertex thus has diagrammatically the following effect in configuration space,

$$-i\,g \int d^4z$$ 

.

introducing a vertex point to which four propagators are connected.

10.2 Interactions and the S-matrix

10.2.1 The S-matrix

The S-matrix transforms inial state free particle states (*in-states*) $|\alpha; in\rangle = |p_1, \ldots p_n; in\rangle$ into final state particle states (*out-states*) $|\beta; out\rangle = |p'_1, \ldots p'_m; out\rangle$ (suppressing except momenta all other quantum numbers),

$$\begin{aligned} S_{\beta\alpha} \equiv \langle\beta; out|\alpha; in\rangle \quad &\Leftrightarrow \quad \langle\beta; in|S = \langle\beta; out| \\ &\Leftrightarrow \quad |\alpha; in\rangle = S|\alpha; out\rangle. \end{aligned} \tag{10.48}$$

The properties of the S-matrix are
(1) The vacuum is invariant or $|S_{00}| = 1$.
Proof: $\langle 0; in|S = \langle 0; out| = e^{i\varphi_0}\langle 0; in|$ (choose $\varphi_0 = 0$).
(2) The one-particle state is invariant (conservation of energy and momentum; translation invariance),
$\langle p; out|S|p'; in\rangle = \langle p; out|p'; in\rangle = \langle p; in|p'; in\rangle = \langle p; out|p'; out\rangle = \langle p|p'\rangle$.
(3) S is unitary (it conserves the scalar product from initial to final state).
Proof: $\langle\alpha; in|S = \langle\alpha; out|$ and $S^\dagger|\alpha; in\rangle = |\alpha; out\rangle$,
thus $\langle\beta; in|SS^\dagger|\alpha; in\rangle = \langle\beta; out|\alpha; out\rangle = \delta_{\alpha\beta} \leftrightarrow SS^\dagger = 1$.

Next, this will be translated to the action on fields. In line with the consideration of the generating functional representing the vacuum to vacuum amplitude we consider fields ϕ_{in}, ϕ_{out} and the interpolating field $\phi(x)$,

$t = -\infty$		$t = +\infty$
$\phi_{in}(x)$	$\phi(x)$	$\phi_{out}(x)$

where ϕ_{in} and ϕ_{out} transform under the Poincaré group as scalar fields and satisfy the homogeneous Klein-Gordon equation with the physical mass M,

$$\left(\partial_\mu\partial^\mu + M^2\right)\phi_{in}(x) = 0, \tag{10.49}$$

while ϕ satisfies the inhomogeneous Klein-Gordon equation with the bare mass M_0 (this is the mass appearing in the lagrangian $\mathcal{L}_0$),

$$\left(\partial_\mu\partial^\mu + M_0^2\right)\phi(x) = J(x). \tag{10.50}$$

The fact that ϕ_{in} and ϕ_{out} satisfy the homogeneous Klein-Gordon equation with the physical mass M implies that they create particles and antiparticles as discussed, e.g.

$$\phi_{in}(x) = \int \frac{d^3k}{(2\pi)^3\, 2E}\left[a(k)\, f_k(x) + a^\dagger(k)\, f_k^*(x)\right]. \tag{10.51}$$

The field ϕ can be expressed in $\phi_{in/out}$ using retarded or advanced Green's functions,

$$\left(\partial_\mu\partial^\mu + M^2\right)\phi(x) = J(x) + (M^2 - M_0^2)\phi(x) = \tilde{J}(x) \tag{10.52}$$

$$\phi(x) = \sqrt{Z}\phi_{in}(x) - \int d^4y\, \Delta_R(x-y)\, \tilde{J}(y)$$
$$= \sqrt{Z}\phi_{in}(x) - \int d^4y\, d^4z\, \Delta_R(x-y)\, K(y,z)\, \phi(z) \quad (10.53)$$
$$\phi(x) = \sqrt{Z}\phi_{out}(x) - \int d^4y\, d^4z\, \Delta_A(x-y)\, K(y,z)\, \phi(z). \quad (10.54)$$

Although the above, as it stands, implies the strong (operator) convergence $\phi(x) \to \sqrt{Z}\phi_{in}(x)$, this can actually not be used as it would imply $[\phi(x), \phi(y)] = Z\, [\phi_{in}(x), \phi_{in}(y)] = iZ\, \Delta(x-y)$, a causality condition that can be proven to imply the absence of interactions. The convergence therefore must be weakened to

$$\langle\alpha|\phi^f(t)\beta\rangle \overset{t\to-\infty}{\longrightarrow} \sqrt{Z}\langle\alpha|\phi^f_{in}(t)\beta\rangle \quad (10.55)$$

for normalizable states $|\alpha\rangle$ and $|\beta\rangle$ and $\phi^f(t) \equiv \int d^3(x)\, f^*(x)\, i\, \overleftrightarrow{\partial}_0\, \phi(x)$ with f a normalizable solution of the Klein-Gordon equation (wave packet). Considering plane waves one sees from

$$\begin{aligned}\langle 0|\phi(x)|p\rangle &= \lim_{t\to\infty} \sqrt{Z}\,\langle 0|\phi_{out}(x)|p\rangle = \sqrt{Z}\, e^{-i\,p\cdot x} \\ &= \lim_{t\to-\infty} \sqrt{Z}\,\langle 0|\phi_{in}(x)|p\rangle = \sqrt{Z}\, e^{-i\,p\cdot x}\end{aligned}$$

that identical normalization of (single-particle) plane waves in initial and final state implies the same wave function normalization Z for in and out fields.

The relation between S-matrix and in- and out-fields is: (4) In and out fields transform as $\phi_{in}(x) = S\,\phi_{out}(x)S^{-1}$.
Proof:

$$\langle\beta; out|\phi_{out} = \left\{ \begin{array}{l} \langle\beta; in|\phi_{in}S \\ \langle\beta; in|S\phi_{out} \end{array} \right\} \to \phi_{in}\, S = S\,\phi_{out}(x).$$

Finally we check that as expected S does not spoil Poincaré invariance (5) S is invariant under Poincaré transformations: $U(\Lambda, a)SU^{-1}(\Lambda, a) = S$.
Proof: $\phi_{in}(\Lambda x + a) = U\phi_{in}(x)U^{-1} = US\phi_{out}(x)S^{-1}U^{-1}$
$= US\,U^{-1}U\,\phi_{out}(x)\,U^{-1}U\,S^{-1}U^{-1}$
$= USU^{-1}\,\phi_{out}(\Lambda x + a)\,US^{-1}U^{-1} \Leftrightarrow USU^{-1} = S.$

10.2.2 The relation between S and $W[J]$

For this purpose the source term $J\phi$ in the lagrangian density is treated in the interaction picture, giving a time evolution

$$U_T^{T'}[J] = \mathcal{T}\exp\left(i\int_T^{T'} d^4x\, J(x)\phi(x)\right), \quad (10.56)$$

which satisfies the property

$$\frac{\delta U_T^{T'}[J]}{\delta J(x)} = i\, U_t^{T'}[J]\,\phi(x)U_T^t[J], \quad (10.57)$$

or for $U[J] \equiv U_{-\infty}^{\infty}[J]$

$$\begin{aligned}\frac{\delta U[J]}{\delta J(x)} &= i\, U_t^{\infty}[J]\,\phi(x)U_{-\infty}^t[J] \quad (10.58)\\ &\overset{t\to\infty}{\longrightarrow} i\,\phi(x)\,U[J] \to i\sqrt{Z}\,\phi_{out}(x)\,U[J] \\ &\overset{t\to-\infty}{\longrightarrow} i\,U[J]\,\phi(x) \to i\sqrt{Z}\,U[J]\,\phi_{in}(x).\end{aligned}$$

Since $\delta U[J]/\delta J(x)$ satisfies the same equations as $\phi(x)$ and we know the limits we can as with $\phi(x)$ express it in terms of advanced and retarded Green's functions,

$$\begin{aligned}\frac{\delta U[J]}{\delta J(x)} &= i\sqrt{Z}\,U[J]\,\phi_{in}(x) - \int d^4y\,d^4z\,\Delta_R(x-y)\,K(y,z)\,\frac{\delta U[J]}{\delta J(z)} \\ &= i\sqrt{Z}\,\phi_{out}(x)\,U[J] - \int d^4y\,d^4z\,\Delta_A(x-y)\,K(y,z)\,\frac{\delta U[J]}{\delta J(z)}.\end{aligned}$$

Taking the difference between the two expressions

$$\begin{aligned}&\sqrt{Z}\,(\phi_{out}(x)\,U[J] - U[J]\,\phi_{in}(x)) \\ &\quad = i\int d^4y\,d^4z\,(\Delta_R(x-y) - \Delta_A(x-y))\,K(y,z)\,\frac{\delta U[J]}{\delta J(z)},\end{aligned} \tag{10.59}$$

or

$$[\phi_{in}(x), S\,U[J]] = \frac{i}{\sqrt{Z}}\int d^4y\,d^4z\,\Delta(x-y)K(y,z)\,\frac{\delta}{\delta J(z)}\,S\,U[J]. \tag{10.60}$$

In order to find a solution to this equation note that, with the use of the Baker-Campbell-Hausdorff formula $e^{-B}\,A\,e^{+B} = A + [A,B]$ (for the case that $[A,B]$ is a c-number), one has

$$[A, e^B] = [A,B]e^B, \tag{10.61}$$

$$[A, e^B\,e^C] = [A, B+C]e^B\,e^C, \tag{10.62}$$

provided $[A,B]$ and $[A,C]$ are c-numbers. Thus applied to the field $\phi(x) = \phi_+(x) + \phi_-(x)$,

$$\begin{aligned}&\left[\phi(x), e^{\int d^4x\,\phi_-(x)\,f(x)}\;e^{\int d^4y\,\phi_+(y)\,f(y)}\right] \\ &\quad = \int d^4y\,[\phi(x),\phi(y)]\,f(y)\;e^{\int d^4z\,\phi_-(z)\,f(z)}\;e^{\int d^4z\,\phi_+(z)\,f(z)},\end{aligned}$$

i.e.

$$\left[\phi(x), :e^{\int d^4z\,\phi(z)\,f(z)}:\right] = i\int d^4y\,\Delta(x-y)\,f(y)\;\;:e^{\int d^4z\,\phi(z)\,f(z)}:, \tag{10.63}$$

where the normal ordered expression $:e^{\int \phi f}:$ is used, which is equal to the expression $e^{\int \phi_- f}\;e^{\int \phi_+ f}$ with creation operators placed left of annihilation operators. Thus

$$S\,U[J] =: \exp\left(\frac{1}{\sqrt{Z}}\int d^4x\,d^4y\,\phi_{in}(x)\,K(x,y)\,\frac{\delta}{\delta J(y)}\right):\;F[J], \tag{10.64}$$

where $F[J]$ is some (arbitrary) functional. Noting that $\langle 0|:e^A:|0\rangle = 1$ it follows that

$$F[J] = \langle 0|S\,U[J]|0\rangle = \langle 0|U[J]|0\rangle = \langle 0_{out}|0_{in}\rangle \propto W[J], \tag{10.65}$$

while for $J = 0$ one has $U[0] = 1$, i.e.

$$S =: \exp\left(\frac{1}{\sqrt{Z}}\int d^4x\,d^4y\,\phi_{in}(x)\,K(x,y)\,\frac{\delta}{\delta J(y)}\right):\;\left.\frac{W[J]}{W[0]}\right|_{J=0}. \tag{10.66}$$

Therefore, an S-matrix element between momentum eigenstates in initial and final states is found by considering those source-connected Green's functions (action of $\delta/\delta J$ on $W[J]/W[0]$) where the external sources $J(x_i)$ are replaced by the particle wave functions (which are the result of acting with $\phi_{in}(x_i)$ on momentum eigenstates). Note that the Green's function connecting the external point x_i with the bubble is annihilated by $K(x,y)$.

Usually we are interested in the part of the S-matrix describing the scattering,

$$S_{fi} = \delta_{fi} - i\,(2\pi)^4\,\delta^4(P_i - P_f)\,\mathcal{M}_{fi}, \tag{10.67}$$

which is obtained considering only connected diagrams.

Explicitly, using that

$$\phi_{in}(x) = \int \frac{d^3k}{(2\pi)^3\, 2E} \left[a(k)\, f_k(x) + a^\dagger(k)\, f_k^*(x)\right].$$

we get

$$\langle p', \ldots |S| p, \ldots \rangle = \int d^4x' \ldots d^4x \ldots f_{p'}^*(x') \ldots i\overrightarrow{K}_{x'} \ldots$$
$$\times\, G_c(x', \ldots, x, \ldots)\; i\overleftarrow{K}_x \ldots f_p(x) \ldots, \tag{10.68}$$

where G_c is the connected Green's function, $iK_{x'} = (\Box_{x'} + M^2)$ precisely annihilating an external propagator $i\Delta_F$ in the Green's function.

Next we introduce the Fourier transform (after extracting a momentum conserving delta function coming from translation invariance, see exercise 10.2),

$$(2\pi)^4\, \delta^4(p_1 + \ldots + p_n)\, G^{(n)}(p_1, \ldots, p_n) =$$
$$\int \prod_{i=1}^{n} d^4x_i\; e^{i\, p_i \cdot x_i}\, G^{(n)}(x_1, \ldots, x_n), \tag{10.69}$$

and the (amputated) 1PI Green's functions

$$?^{(n)}(p_1, \ldots, p_n) = \left(\prod_{j=1}^{n} \frac{-i}{\Delta(p_j)}\right) G^{(n)}(p_1, \ldots, p_n), \tag{10.70}$$

where $\Delta(p)$ is the Fourier transform of $\Delta_F(x)$. It is straightforward to check that the S-matrix element now precisely is given by the amputated Green's function multiplied with the momentum space wave functions of the particles in initial and final state (by which we refer to the quantities multiplying the plane wave $e^{\pm i\, p\cdot x}$ in the field expansion, i.e. 1 for scalar case, $u(p)$, $v(p)$, $\bar{u}(p)$ and $\bar{v}(p)$ for fermions and $\epsilon_\mu^{(\lambda)}(p)$ for vector fields).

10.3 Feynman rules

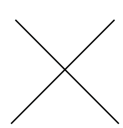
10.3.1 The real scalar field

The procedure to obtain the matrix element is commonly summarized by a set of rules known as *Feynman rules*. They start with the *propagator* ($i\Delta_F(k)$) in momentum space, which is determined by the inverse of the operator found in the quadratic term in the lagrangian i.e.

$$= \frac{i}{k^2 - M^2 + i\epsilon}$$

(in fact the inverse of the operator found in the quadratic term is for real scalar fields also multiplied by a factor 2, which cancels the factor 1/2 in the quadratic piece; the factor 2 corresponds to the two-points Green's function having two identical ends).

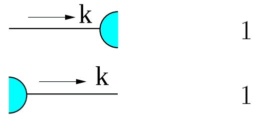
For the interaction terms in the lagrangian, to be precise $i\,\mathcal{L}_I$ *vertices* in momentum space are introduced,



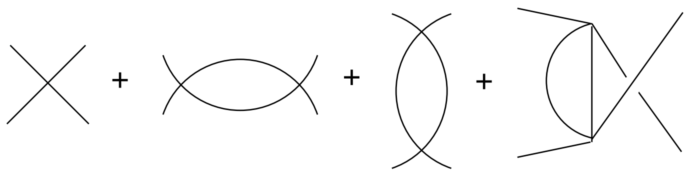
$$= -i\, g$$

(multiplied with 4! corresponding to the allowed number of permutations of identical particles). At these vertices each line can be assigned a momentum, but overall momentum conservation at a vertex is understood.

Corresponding to external particles wave functions are introduced



In order to calculate the connected amplitude $-i\mathcal{M}_{fi}$ appearing in the S-matrix element these ingredients are combined using Eqs 10.68 and 10.38, which is summarized in the following rules:
(**Rule 1**) Start with external legs (incoming particles/outgoing particles) and draw all possible topologically different connected diagrams, for example up to order g^2 the scattering of two neutral spin 0 particles (real scalar field) is described by

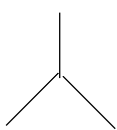
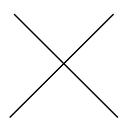


(**Rule 2**) The contribution of each diagram is obtained by multiplying the contributions from propagators, vertices and external particle wave functions in that diagram. Note that in calculating amputated Green's functions external lines are neglected, or calculating full Green's functions external lines are treated as propagators.
(**Rule 3**) Carry out the integration over all internal momenta (keeping track of momentum conservation at all vertices!)
(**Rule 4**) Add a symmetry factor $1/S$ corresponding to permutation of internal lines and vertices (keeping external lines fixed). If problems arise go back to the defining expression for the generating function in 10.38.

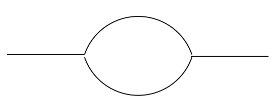
For the symmetry factor consider the examples (given in G. 't Hooft and M. Veltman, *Diagrammar*) in the case of the interaction terms

$$\mathcal{L}_I(\phi) = -\frac{f}{3!}\,\phi^3 - \frac{g}{4!}\,\phi^4. \tag{10.71}$$

The vertices are:

 $= -i\,f$  $= -i\,g$

Consider first the lowest order self-energy diagram,

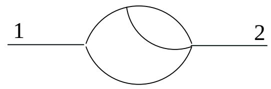


Draw two points corresponding to the two vertices and draw in each of these points the lines coming out of the vertices:

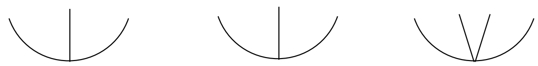
Now count in how many ways the lines can be connected with the same topological result. External line 1 can be attached in six, after that line 2 can in three ways. Then there are two ways to connect the remaining lines such that the desired diagram results. Divide by the permutational factors of the vertices, which have been included in the definition of vertices (here 3! for each vertex). Finally divide by the number of permutation of the points that have identical vertices (here 2!). The total result is

$$\frac{1}{S} = \frac{6 \times 3 \times 2}{3!\, 3!\, 2!} = \frac{1}{2}.$$

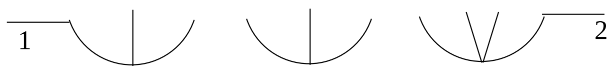
As a second example consider the diagram



There are three vertices,



After connecting line 1 (6 ways) and line 2 (4 ways) we have



leaving $6 \times 3 \times 2$ ways to connect the rest as to get the desired topology. Dividing by vertex factors and permutations of identical vertices, the result is

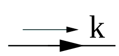
$$\frac{1}{S} = \frac{6 \times 4 \times 6 \times 3 \times 2}{3!\, 3!\, 4!\, 2!} = \frac{1}{2}.$$

10.3.2 Complex scalar fields

The case of complex scalar fields can be considered as two independent fields, or equivalently as independent fields ϕ and ϕ^*. The generating functional in the interacting case can be written as

$$\begin{aligned} &W[J,J^*] \\ &\quad = \int \mathcal{D}\phi\ \exp\left(i\int d^4x\ \left[\phi^*\left(-\partial_\mu\partial^\mu - M^2\right)\phi + \mathcal{L}_I(\phi) + J^*\phi + J\phi^*\right]\right) \\ &\quad = \exp\left(i\int d^4z\ \mathcal{L}_I\left(\frac{1}{i}\frac{\delta}{\delta J(z)},\frac{1}{i}\frac{\delta}{\delta J^*(z)}\right)\right) \\ &\qquad\quad \exp\left(-\int d^4x\, d^4y\, J^*(x)\, i\Delta_F(x-y)\, J(y)\right). \end{aligned} \tag{10.72}$$

In Feynman diagrams the propagator is still given by $i\Delta_F(k)$, but it connects a source with its complex conjugate and therefore is oriented, denoted



$$= \frac{i}{k^2 - M^2 + i\epsilon}$$

Note that in this case the propagator does not have identical ends, i.e. there is no combinatorial factor like in the scalar case.

10.3.3 Dirac fields

For fermions the generating functional is given by

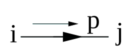
$$W[\eta,\bar\eta] \quad = \quad \int \mathcal{D}\overline{\psi}\,\mathcal{D}\psi\ \exp\left(i\int d^4x\ \left[\bar\psi\left(-i\partial\!\!\!/ - M\right)\psi + \mathcal{L}_I(\psi) + \bar\eta\psi + \bar\psi\eta\right]\right)$$

$$= \exp\left(i \int d^4z\ \mathcal{L}_I\left(\frac{1}{i}\frac{\delta}{\delta\eta(z)}, -\frac{1}{i}\frac{\delta}{\delta\bar{\eta}(z)}\right)\right)$$
$$\exp\left(-\int d^4x\, d^4y\, \bar{\eta}(x)\, iS_F(x-y)\, \eta(y)\right), \tag{10.73}$$

where iS_F is the Feynman propagator for fermions, which is the solution of $(i\not{\partial} + M)S_F(x) = \delta^4(x)$ (i.e. minus the inverse of the operator appearing in the quadratic piece) and is given by

$$iS_F(x-y) = \langle 0|\mathcal{T}\,\bar{\psi}(x)\psi(y)|0\rangle =$$
$$= (i\not{\partial}_x + M)\, i\Delta_F(x-y) \tag{10.74}$$
$$= \int \frac{d^4p}{(2\pi)^4}\, e^{-i\,p\cdot x}\, \frac{\not{p} + M}{p^2 - M^2 + i\epsilon}, \tag{10.75}$$

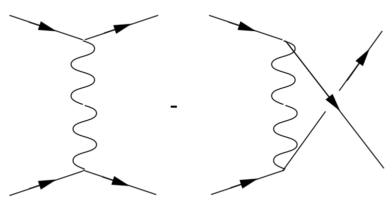
and the (oriented) propagator in Feynman diagrams involving fermions is



$$= \left(\frac{i}{\not{p} - M + i\epsilon}\right)_{ji} = \frac{i(\not{p} + M)_{ji}}{p^2 - M^2 + i\epsilon}$$

The time ordered functions are obtained by functional derivatives from $W(\eta, \bar{\eta})$, but the anticommutating properties of Grassmann variables impy some additional minus sign in Feynman diagrams, namely

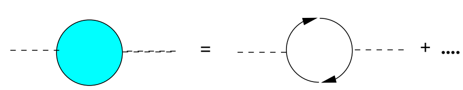
(**Rule 5**) Feynman diagrams which only differ by exchanging identical fermions in initial or final state have a relative minus sign, e.g. in $e^-e^- \to e^-e^-$ scattering (Møller scattering) the lowest order contribution is



(see next section for e-e-γ vertex).

(**Rule 6**) Each closed fermion loop gets a sign -1.

The latter rule is illustrated in the example of an interaction term $\mathcal{L}_I = :g\bar{\psi}(x)\psi(x)\phi(x):$ in an interacting theory with fermionic and scalar fields. The two-points Green's function $\langle 0|\mathcal{T}\,\phi(x)\,\phi(y)|0\rangle$ contains a fermionic loop contribution,



which arises from the quadratic term in $\exp\left(i\int d^4z\ \mathcal{L}_I\right)$,

$$-\frac{g^2}{2}\int dz\, dz'\, \frac{\delta^2}{\delta\bar{\eta}(z)\,\delta\eta(z)}\, \frac{\delta^2}{\delta\bar{\eta}(z')\,\delta\eta(z')}$$

and the quadratic term in $W_0[\eta, \bar{\eta}]$,

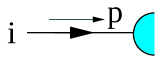
$$-\frac{1}{2}\int dx\, dy\, dx'\, dy'\ \bar{\eta}(x)S(x-y)\eta(y)\ \bar{\eta}(x')S(x'-y')\eta(y').$$

The result is

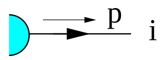
$$\propto g^2\, S(z-z')\, S(z'-z) = -g^2\, iS(z-z')\, iS(z'-z),$$

which contains an extra minus sign as compared to a bosonic loop.

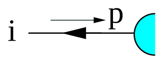
The wave functions for fermions are given by



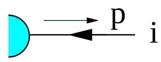
$u_i(p)$ incoming fermion



$\bar{u}_i(p)$ outgoing fermion



$\bar{v}_i(p)$ incoming antifermion



$v_i(p)$ outgoing antifermion

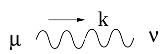
In writing down the expressions for Feynman diagrams one has to be aware that the wave functions and the propagators have more than one component. It is necessary to start at the end of a fermion line (with the above arrow convention an outgoing fermion wave function $\bar{u}$ or an incoming antifermion wave function $\bar{v}$) and keep on following that line, writing down the propagators till the beginning of the line (incoming fermion u or outgoing antifermion v) has been reached. As an example some scattering processes in quantum electrodynamics will be discussed in the next section after the introduction of the Feynman rules for vector fields.

10.3.4 Vector fields and Quantum Electrodynamics

As the most important example of vector fields consider the lagrangian density for quantum electrodynamics (QED),

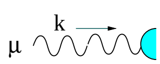
$$\mathcal{L} = -\frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{2}(\partial\cdot A)^2 + \bar{\psi}(i\partial\!\!\!/ - M)\psi - e\,\bar{\psi}\gamma^\mu\psi\,A_\mu. \qquad (10.76)$$

In addition to the fermion propagator and fermion wave functions discussed in the previous section we have the photon propagator[1]

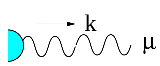


$$= \quad -i\left[\frac{g_{\mu\nu}}{k^2+i\epsilon} - \left(1-\frac{1}{\lambda}\right)\frac{k_\mu k_\nu}{(k^2+i\epsilon)^2}\right],$$

where the $k_\mu k_\nu$ terms in the case where the photon couples to a conserved current (such as $\bar{\psi}\gamma^\mu\psi$) will not contribute. Particular choices of λ are $\lambda = 1$ (Feynman propagator or Feynman gauge) and $\lambda = \infty$ (Landau gauge). The wave functions are given by

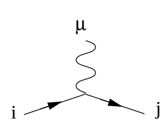


$\epsilon_\mu(k)$ incoming photon



$\epsilon^*_\mu(k)$ outgoing photon

The vertex for the coupling of photon to the electron is given by



$$= \quad -i\,e(\gamma_\mu)_{ji}.$$

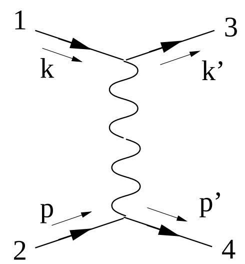
[1]For a massive vector boson inversion of the quadratic term including the Lorentz constraint $\propto \frac{\lambda}{2}(\partial_\mu A^\mu)^2$ leads to the propagator

$$i\,D_{\mu\nu}(k) = i\left[\frac{-g_{\mu\nu}}{k^2-M^2+i\epsilon} + \left(1-\frac{1}{\lambda}\right)\frac{k_\mu k_\nu}{(k^2-M^2+i\epsilon)(k^2-\frac{1}{\lambda}M^2+i\epsilon)}\right].$$

10.4 Some examples

10.4.1 eμ scattering

The first example is the electromagnetic scattering of an electron and a muon. To lowest order ($\propto \alpha = e^2/4\pi$) only one diagram contributes. The diagram and momenta and the commonly used invariants (Mandelstam variables) for a $2 \to 2$ scattering process are



$$\begin{aligned} s &= (k+p)^2 = m^2 + M^2 + 2\,k\cdot p \\ &= (k'+p')^2 = m^2 + M^2 + 2\,k'\cdot p' \\ t &= (k-k')^2 = q^2 = 2\,m^2 - 2\,k\cdot k' \\ &= (p-p')^2 = 2\,M^2 - 2\,p\cdot p' \\ u &= (k-p')^2 = m^2 + M^2 - 2\,k\cdot p' \\ &= (k'-p)^2 = m^2 + M^2 - 2\,k'\cdot p \\ s &+ t + u = \sum m_i^2 = 2\,m^2 + 2\,M^2 \end{aligned}$$

The scattering amplitude is given by

$$-i\mathcal{M} = \bar{u}(k', s_3)(-ie)\gamma^\mu u(k, s_1)\,\frac{-ig_{\mu\nu}}{q^2}\,\bar{u}(p', s_4)(-ie)\gamma^\nu u(p, s_2), \tag{10.77}$$

Note that the $q^\mu q^\nu$ term in the photon propagator are irrelevant because the photon couples to a conserved current. If we are interested in the scattering process of an unpolarized initial state and we are not interested in the spins in the final state we need $|\mathcal{M}|^2$ summed over spins in the final state ($\sum_{s_3,s_4}$) and averaged over spins in the initial state ($1/2 \times 1/2 \times \sum_{s_1,s_2}$) which can be written as

$$\begin{aligned} |\mathcal{M}|^2 &= \frac{1}{4}\sum_{s_1,s_2,s_3,s_4} |\bar{u}(k', s_3)\gamma^\mu u(k, s_1)\,\frac{e^2}{q^2}\,\bar{u}(p', s_4)\gamma_\mu u(p, s_2)|^2 \\ &= \frac{e^4}{q^4}\,L^{(m)}_{\mu\nu}\,L^{\mu\nu\,(M)}, \end{aligned} \tag{10.78}$$

[using that $(\bar{u}(k')\gamma_\mu u(k))^* = \bar{u}(k)\gamma_\mu u(k')$], where

$$\begin{aligned} L^{(m)}_{\mu\nu} &= \frac{1}{2}\sum_{s,s'} \bar{u}(k', s')\gamma_\mu u(k, s)\bar{u}(k, s)\gamma_\nu u(k', s') \\ &= \frac{1}{2}\,Tr\,[(\not{k}' + m)\gamma_\mu(\not{k} + m)\gamma_\nu] \\ &= 2\left[k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu}(k\cdot k' - m^2)\right] \\ &= 2\,k_\mu k'_\nu + 2\,k_\nu k'_\mu + q^2\,g_{\mu\nu} \end{aligned} \tag{10.79}$$

Combining $L^{(m)}_{\mu\nu}$ and $L^{\mu\nu\,(M)}$ one obtains

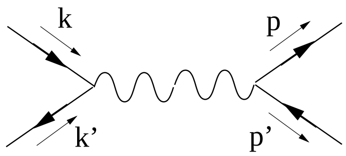
$$L^{(m)}_{\mu\nu}\,L^{\mu\nu\,(M)} = 2\left[s^2 + u^2 + 4t(M^2 + m^2) - 2(M^2 + m^2)^2\right] \tag{10.80}$$

and

$$\left|\frac{\mathcal{M}}{4\pi}\right|^2 = \frac{2\,\alpha^2}{t^2}\left[s^2 + u^2 + 4t(M^2 + m^2) - 2(M^2 + m^2)^2\right]. \tag{10.81}$$

10.4.2 $e^-e^+ \to \mu^-\mu^+$ scattering

The second example is the annihilation of an electron pair and creation of a muon pair. To lowest order ($\propto \alpha = e^2/4\pi$) only one diagram contributes. The diagram, the masses, momenta and invariants are



$$
\begin{aligned}
s &= (k+k')^2 = (p+p')^2 \\
t &= (k-p)^2 = (k'-p')^2 \\
u &= (k-p')^2 = (k'-p)^2
\end{aligned}
$$

The scattering amplitude squared (spins summed and averaged) is given by

$$
\begin{aligned}
|\mathcal{M}|^2 &= \frac{1}{4}\sum_{s_1,s_2,s_3,s_4} |\bar{u}(p,s_3)\gamma^\mu v(p',s_4)\,\frac{e^2}{s}\,\bar{v}(k,s_1)\gamma_\mu u(k',s_2)|^2 \\
&= \frac{e^4}{s^2}\left(\frac{1}{2}\sum_{s_1,s_2}\bar{v}(k,s_1)\gamma_\mu u(k',s_2)\,\bar{u}(k',s_2)\gamma_\nu v(k,s_1)\right) \\
&\quad\times\left(\frac{1}{2}\sum_{s_3,s_4}\bar{u}(p,s_3)\gamma^\mu v(p',s_4)\,\bar{v}(p',s_4)\gamma^\nu u(p,s_3)\right) \\
&= \frac{e^4}{s^2}\left(\frac{1}{2}Tr(\not{k}-m)\gamma_\mu(\not{k}'+m)\gamma_\nu\right)\left(\frac{1}{2}Tr(\not{p}+M)\gamma^\mu(\not{p}'-M)\gamma^\nu\right) \\
&= 4\,\frac{e^4}{s^2}\left(k_\mu k'_\nu + k_\nu k'_\mu - \frac{1}{2}g_{\mu\nu}s\right)\left(p^\mu p'^\nu + p_\nu p'^\mu - \frac{1}{2}g^{\mu\nu}s\right)
\end{aligned}
$$

and

$$
\left|\frac{\mathcal{M}}{4\pi}\right|^2 = \frac{2\,\alpha^2}{s^2}\left[t^2+u^2+4s(M^2+m^2)-2(M^2+m^2)^2\right]. \tag{10.82}
$$

Note the similarity in the amplitudes for eμ scattering and $e^-e^+ \to \mu^-\mu^+$. Basically the same diagram is calculated and the result are the same after the interchange of $s \leftrightarrow t$. This is known as crossing symmetry. Similarly, for instance Møller scattering ($e^-e^- \to e^-e^-$) and Bhabha scattering ($e^-e^+ \to e^-e^+$) are related using crossing symmetry.

10.5 Exercises

Exercise 10.1
Give the full Green's functions $G^{(4)}(x_1,\ldots x_4)$, the source connected Green's function $G^{(4)}_{sc}$ and the connected Green's function $G^{(4)}_c$ for the interacting case to first order in the coupling constant g as obtained from the full expression for $W[J]$ in section 1.4

Exercise 10.2

(a) Show that the translation properties of the fields and the vacuum imply

$$
G^{(n)}(x_1+a,\ldots,x_n+a) = G^{(n)}(x_1,\ldots,x_n).
$$

(b) Show next that this implies that

$$
\int\prod_{i=1}^{n} d^4x_i\; e^{i\,p_i\cdot x_i}\,G^{(n)}(x_1,\ldots,x_n) \;\propto\; (2\pi)^4\,\delta^4(p_1+\ldots+p_n),
$$

hence we can write

$$(2\pi)^4\,\delta^4(p_1+\ldots+p_n)\,G^{(n)}(p_1,\ldots,p_n)=\int\prod_{i=1}^{n}d^4x_i\;e^{i\,p_i\cdot x_i}\,G^{(n)}(x_1,\ldots,x_n).$$

Exercise 10.3
Check the combinatorial factors in 10.45 and 10.46 for the diagrams

using the rules given in section 10.3.

Exercise 10.4

(a) Write down the Feynman diagrams contributing to electron-electron scattering, e(p_1) + e(p_2) $\rightarrow$ e(p_1') + e(p_2') in lowest order in α.

(b) Calculate the quadratic pieces and interference terms,

$$|M|^2=T_{11}+T_{22}-T_{12}-T_{21},$$

in the amplitude. Express the contributions in invariants s, t and u. Show that the amplitude is symmetric under the interchange of $t\leftrightarrow u$.

Chapter 11

Scattering theory

11.1 kinematics in scattering processes

11.1.1 Phase space

The 1-particle state is denoted $|p\rangle$. It is determined by the energy-momentum four vector $p = (E, \boldsymbol{p})$ which satisfies $p^2 = E^2 - \boldsymbol{p}^2 = m^2$. A physical state has positive energy. The phase space is determined by the weight factors assigned to each state in the summation or integration over states, i.e. the 1-particle phase space is

$$\int \frac{d^3\boldsymbol{p}}{(2\pi)^3\,2E} = \int \frac{d^4\boldsymbol{p}}{(2\pi)^4}\,\theta(p^0)\,(2\pi)\delta(p^2 - m^2), \tag{11.1}$$

(proof this!). This is generalized to the multi-particle phase space

$$d\mathcal{R}(p_1, \ldots, p_n) = \prod_{i=1}^{n} \frac{d^3p_i}{(2\pi)^3\,2E_i}, \tag{11.2}$$

and the *reduced phase space element* by

$$d\mathcal{R}(s, p_1, \ldots, p_n) = (2\pi)^4\,\delta^4(P - \sum_i p_i)\,d\mathcal{R}(p_1, \ldots, p_n), \tag{11.3}$$

which is useful because the total 4-momentum of the final state usually is fixed by overall momentum conservation. Here s is the invariant mass of the n-particle system, $s = (p_1 + \ldots + p_n)^2$. It is a useful quantity, for instance for determining the threshold energy for the production of a final state $1 + 2 + \ldots + n$. In the CM frame the threshold value for s obviously is

$$s_{threshold} = \left(\sum_{i=1}^{n} m_i\right)^2. \tag{11.4}$$

For two particle states $|p_a, p_b\rangle$ we start with the four vectors $p_a = (E_a, \boldsymbol{p}_a)$ and $p_b = (E_b, \boldsymbol{p}_b)$ satisfying $p_a^2 = m_a^2$ and $p_b^2 = m_b^2$, and the total momentum four-vector $P = p_a + p_b$. The quantity

$$s = P^2 = (p_a + p_b)^2, \tag{11.5}$$

is referred to as the invariant mass squared. Its square root, $\sqrt{s}$ is for obvious reasons known as the center of mass (CM) energy.

To be specific let us consider two frequently used frames. The first is the CM system. In that case

$$p_a = (E_a^{cm}, \boldsymbol{q}), \tag{11.6}$$

$$p_b = (E_b^{cm}, -\boldsymbol{q}), \tag{11.7}$$

It is straightforward to proof that the unknowns in the particular system can be expressed in the invariants (m_a, m_b and s). Proof that

$$|\mathbf{q}| = \sqrt{\frac{(s-m_a^2-m_b^2)^2 - 4\,m_a^2 m_b^2}{4s}} = \sqrt{\frac{\lambda(s,m_a^2,m_b^2)}{4s}}, \tag{11.8}$$

$$E_a^{cm} = \frac{s+m_a^2-m_b^2}{2\sqrt{s}}, \tag{11.9}$$

$$E_b^{cm} = \frac{s-m_a^2+m_b^2}{2\sqrt{s}}. \tag{11.10}$$

The function $\lambda(s,m_a^2,m_b^2)$ is a function symmetric in its three arguments, which in the specific case also can be expressed as $\lambda(s,m_a^2,m_b^2) = 4(p_a\cdot p_b)^2 - 4p_a^2\,p_b^2$.

The second frame considered explicitly is the socalled target rest frame in which one of the particles (called the target) is at rest. In that case

$$p_a = (E_a^{trf}, \boldsymbol{p}_a^{trf}), \tag{11.11}$$

$$p_b = (m_b, \mathbf{0}), \tag{11.12}$$

Also in this case one can express the energy and momentum in the invariants. Proof that

$$E_a^{trf} = \frac{s-m_a^2-m_b^2}{2m_b}, \tag{11.13}$$

$$|\boldsymbol{p}_a^{trf}| = \frac{\sqrt{\lambda(s,m_a^2,m_b^2)}}{2m_b}. \tag{11.14}$$

One can, for instance, use the first relation and the abovementioned threshold value for s to calculate the threshold for a specific n-particle final state in the target rest frame,

$$E_a^{lab}(threshold) = \frac{1}{2\,m_b}\left((\sum_i m_i)^2 - m_a^2 - m_b^2\right). \tag{11.15}$$

Explicit calculation of the reduced two-body phase space element gives

$$\begin{aligned} d\mathcal{R}(s,p_1,p_2) &= \frac{1}{(2\pi)^2}\frac{d^3p_1}{2E_1}\frac{d^3p_2}{2E_2}\delta^4(P-p_1-p_2) \\ &\overset{CM}{=} \frac{1}{(2\pi)^2}\frac{d^3q}{4\,E_1\,E_2}\delta(\sqrt{s}-E_1-E_2) \\ &= \frac{1}{(2\pi)^2}\,d\Omega(\hat{q})\frac{\boldsymbol{q}^2\,d|\boldsymbol{q}|}{4\,E_1\,E_2}\delta(\sqrt{s}-E_1-E_2) \end{aligned}$$

which using $|\boldsymbol{q}|\,d|\boldsymbol{q}| = E_1\,dE_1 = E_2\,dE_2$ gives

$$\begin{aligned} d\mathcal{R}(s,p_1,p_2) &= \frac{|\boldsymbol{q}|}{(2\pi)^2}\,d\Omega(\hat{q})\frac{d(E_1+E_2)}{4(E_1+E_2)}\delta(\sqrt{s}-E_1-E_2) \\ &= \frac{|\boldsymbol{q}|}{4\pi\sqrt{s}}\,\frac{d\Omega(\hat{q})}{4\pi} = \frac{\sqrt{\lambda_{12}}}{8\pi\,s}\,\frac{d\Omega(\hat{q})}{4\pi}, \end{aligned} \tag{11.16}$$

where λ_{12} denotes $\lambda(s,m_1^2,m_2^2)$.

11.1.2 Kinematics of 2 $\rightarrow$ 2 scattering processes

The simplest scattering process is 2 particles in and 2 particles out. Examples appear in

$$\begin{aligned} \pi^- + p &\rightarrow \pi^- + p && (11.17)\\ &\rightarrow \pi^0 + n && (11.18)\\ &\rightarrow \pi^+ + \pi^- + n && (11.19)\\ &\rightarrow \ldots . && (11.20) \end{aligned}$$

The various possibilities are referred to as different reaction channels, where the first is referred to as elastic channel and the set of all other channels as the inelastic channels. Of course there are not only 2-particle channels. The initial state, however, usually is a 2-particle state, while the final state often arises from a series of 2-particle processes combined with the decay of an intermediate particle (resonance).

Consider the process $a + b \rightarrow c + d$. An often used set of invariants are the Mandelstam variables,

$$\begin{aligned} s &= (p_a + p_b)^2 = (p_c + p_d)^2 && (11.21)\\ t &= (p_a - p_c)^2 = (p_b - p_d)^2 && (11.22)\\ u &= (p_a - p_d)^2 = (p_b - p_c)^2 && (11.23) \end{aligned}$$

which are not independent as $s + t + u = m_a^2 + m_b^2 + m_c^2 + m_d^2$. The variable s is always larger than the minimal value $(m_a + m_b)^2$. A specific reaction channel starts contributing at the threshold value $s^{thr} = (\sum_i m_i)^2$. Instead of the scattering angle, which for the above 2 $\rightarrow$ 2 process in the case of azimuthal symmetry is defined as $\hat{\boldsymbol{p}}_a \cdot \hat{\boldsymbol{p}}_c = \cos\theta$ one can use in the CM the invariant

$$t \equiv (p_a - p_c)^2 \stackrel{CM}{=} m_a^2 + m_c^2 - 2\,E_a E_c + 2\,q q' \cos\theta^{cm},$$

with $q = \sqrt{\lambda_{ab}}/2\sqrt{s}$ and $q' = \sqrt{\lambda_{cd}}/2\sqrt{s}$. The minimum and maximum values for t correspond to θ^{cm} being 0 or 180 degrees,

$$\begin{aligned} t_{min}^{max} &= m_a^2 + m_c^2 - 2\,E_a E_c \pm 2\,q q' \\ &= m_a^2 + m_c^2 - \frac{(s + m_a^2 - m_b^2)(s + m_c^2 - m_d^2)}{2\,s} \pm \frac{\sqrt{\lambda\lambda'}}{2\,s}. && (11.24) \end{aligned}$$

Using the relation between t and $\cos\theta^{cm}$ it is straightforward to express $d\Omega^{cm}$ in dt, $dt = 2\,qq'\,d\cos\theta^{cm}$ and obtain for the two-body phase space element

$$\begin{aligned} d\mathcal{R}(s, p_c, p_d) &= \frac{q'}{4\pi\sqrt{s}}\frac{d\Omega^{cm}}{4\pi} = \frac{\sqrt{\lambda_{cd}}}{8\pi\,s}\frac{d\Omega^{cm}}{4\pi} && (11.25)\\ &= \frac{dt}{8\pi\,\sqrt{\lambda_{ab}}} = \frac{dt}{16\pi\,q\,\sqrt{s}}. && (11.26) \end{aligned}$$

11.2 Cross sections and lifetimes

11.2.1 Scattering process

For a scattering process $a + b \rightarrow c + \ldots$ (consider for convenience the target rest frame) the cross section $\sigma(a + b \rightarrow c + \ldots)$ is defined as the proportionality factor in

$$\frac{N_c}{T} = \sigma(a + b \rightarrow c + \ldots) \cdot N_b \cdot \text{flux}(a),$$

where V and T indicate the volume and the time in which the experiment is performed, N_c/T indicates the number of particles c detected in the scattering process, N_b indicates the number of (target) particles b, which for a density ρ_b is given by $N_b = \rho_b \cdot V$, while the flux of the beam particles a is $\text{flux}(a) = \rho_a \cdot v_a^{trf}$. The proportionality factor has the dimension of area and is called the cross section, i.e.

$$\sigma = \frac{N}{T \cdot V}\,\frac{1}{\rho_a \rho_b\, v_a^{trf}}. \tag{11.27}$$

Although this at first sight does not look covariant, it is. N and $T \cdot V$ are covariant. Using $\rho_a^{trf} = \rho_a^{(0)} \cdot \gamma_a = \rho_a^{(0)} \cdot E_a^{lab}/m_a$ (where $\rho_a^{(0)}$ is the rest frame density) and $v_a^{lab} = p_a^{lab}/E_a^{lab}$ we have

$$\rho_a \rho_b\, v_a^{trf} = \frac{\rho_a^{(0)}\rho_b^{(0)}}{4\, m_a m_b}\, 2\sqrt{\lambda_{ab}}$$

or with $\rho_a^{(0)} = 2\, m_a$,

$$\sigma = \frac{1}{2\sqrt{\lambda_{ab}}}\,\frac{N}{T \cdot V}. \tag{11.28}$$

11.2.2 Decay of particles

For the decay of particle a one has macroscopically

$$\frac{dN}{dt} = -?\, N, \tag{11.29}$$

i.e. the amount of decaying particles is proportional to the number of particles with proportionality factor the em decay width ?. From the solution

$$N(t) = N(0)\, e^{-\Gamma t} \tag{11.30}$$

one knows that the decay time $\tau = 1/?$. Microscopically one has

$$\frac{N_{decay}}{T} = N_a \cdot ?$$

or

$$? = \frac{N}{T \cdot V}\,\frac{1}{\rho_a}. \tag{11.31}$$

This quantity is not covariant, as expected. The decay time for moving particles τ is related to the decay time in the rest frame of that particle (the proper decay time τ_0) by $\tau = \gamma\, \tau_0$. For the (proper) decay width one thus has

$$?_0 = \frac{1}{2\, m_a}\,\frac{N}{T \cdot V}. \tag{11.32}$$

In both the scattering cross section and the decay constant the quantity N/TV appears. For this we employ in essence *Fermi's Golden rule* stating that when the S-matrix element is written as

$$S_{fi} = \delta_{fi} - (2\pi)^4\, \delta^4(P_i - P_f)\, i\mathcal{M}_{fi} \tag{11.33}$$

(in which we can calculate $-i\mathcal{M}_{fi}$ using Feynman diagrams), the number of scattered or decayed particles is given by

$$N = \left|(2\pi)^4\, \delta^4(P_i - P_f)\, i\mathcal{M}_{fi}\right|^2\, d\mathcal{R}(p_1, \ldots, p_n). \tag{11.34}$$

One of the δ functions can be rewritten as $T \cdot V$ (remember the normalization of plane waves),

$$\begin{aligned}
&\left|(2\pi)^4\, \delta^4(P_i - P_f)\right|^2 \\
&\quad = (2\pi)^4\, \delta^4(P_i - P_f) \int_{V,T} d^4x\, e^{i(P_i - P_f)\cdot x} \\
&\quad = (2\pi)^4\, \delta^4(P_i - P_f) \int_{V,T} d^4x = V \cdot T\, (2\pi)^4\, \delta^4(P_i - P_f).
\end{aligned}$$

(Using normalized wave packets these somewhat ill-defined manipulations can be made more rigorous). The result is

$$\frac{N}{T \cdot V} = |\mathcal{M}_{fi}|^2 \, d\mathcal{R}(s, p_1, \ldots, p_n). \tag{11.35}$$

Combining this with the expressions for the width or the cross section one obtains for the decay width

$$? \qquad = \frac{1}{2m} \int d\mathcal{R}(m^2, p_1, \ldots, p_n) \, |\mathcal{M}|^2 \tag{11.36}$$

$$\stackrel{2-body\ decay}{=} \frac{q}{32\pi^2 \, m^2} \int d\Omega \, |M|^2. \tag{11.37}$$

The *differential cross section* (final state not integrated over) is given by

$$d\sigma = \frac{1}{2\sqrt{\lambda_{ab}}} \, |\mathcal{M}_{fi}|^2 \, d\mathcal{R}(s, p_1, \ldots, p_n), \tag{11.38}$$

and for instance for two particles

$$d\sigma = \frac{q'}{q} \left| \frac{\mathcal{M}(s, \theta_{cm})}{8\pi \sqrt{s}} \right|^2 d\Omega_{cm} = \frac{\pi}{\lambda_{ab}} \left| \frac{\mathcal{M}(s,t)}{4\pi} \right|^2 dt. \tag{11.39}$$

This can be used to get the full expression for $d\sigma/dt$ for $e\mu$ and e^+e^- scattering, for which the amplitudes squared have been calculated in the previous chapter. The amplitude $-\mathcal{M}/8\pi\sqrt{s}$ is the one to be compared with the quantum mechanical scattering amplitude $f(E,\theta)$, for which one has $d\sigma/d\Omega = |f(E,\theta)|^2$. The sign difference comes from the (conventional) sign in relation between S and quantummechanical and relativistic scattering amplitude, respectively.

11.3 Unitarity condition

The unitarity of the S-matrix, i.e.

$$(S^\dagger)_{fn} \, S_{ni} = \delta_{fi}$$

implies for the scattering matrix $\mathcal{M}$,

$$\left[\delta_{fn} + i(2\pi)^4 \, \delta^4(P_f - P_n) \, (\mathcal{M}^\dagger)_{fn}\right] \left[\delta_{ni} - i(2\pi)^4 \, \delta^4(P_i - P_n) \, \mathcal{M}_{ni}\right] = \delta_{fi},$$

or

$$-i \left[\mathcal{M}_{fi} - (\mathcal{M}^\dagger)_{fi}\right] = -\sum_n (\mathcal{M}^\dagger)_{fn} \, (2\pi)^4 \, \delta^4(P_i - P_n) \, \mathcal{M}_{ni}. \tag{11.40}$$

Since the amplitudes also depend on all momenta the full result for two-particle intermediate states is (in CM, see 11.25)

$$-i \left[\mathcal{M}_{fi} - (\mathcal{M}^\dagger)_{fi}\right] = -\sum_n \int d\Omega(\hat{q}_n) \mathcal{M}^*_{nf}(\mathbf{q}_f, \mathbf{q}_n) \, \frac{q_n}{16\pi^2 \sqrt{s}} \mathcal{M}_{ni}(\mathbf{q}_i, \mathbf{q}_n). \tag{11.41}$$

Partial wave expansion

Often it is useful to make a partial wave expansion for the amplitude $\mathcal{M}(s,\theta)$ or $\mathcal{M}(\mathbf{q}_i, \mathbf{q}_f)$,

$$\mathcal{M}(s,\theta) = -8\pi \sqrt{s} \sum_\ell (2\ell + 1) M_\ell(s) \, P_\ell(\cos\theta), \tag{11.42}$$

(in analogy with the expansion for $f(E,\theta)$ in quantum mechanics; note the sign and $\cos\theta = \hat{q}_i \cdot \hat{q}_f$). Inserted in the unitarity condition for $\mathcal{M}$,

$$i \left[\frac{\mathcal{M}}{8\pi \sqrt{s}} - \frac{\mathcal{M}^\dagger}{8\pi \sqrt{s}}\right]_{fi} = \sum_n \int d\Omega_n \, \frac{\mathcal{M}^*_{nf}}{8\pi \sqrt{s}} \, \frac{q_n}{2\pi} \, \frac{\mathcal{M}_{ni}}{8\pi \sqrt{s}},$$

we obtain

$$\text{LHS} = -i \sum_\ell (2\ell+1)\, P_\ell(\hat{q}_i \cdot \hat{q}_f)\, \left((M_\ell)_{fi} - (M_\ell^\dagger)_{fi}\right),$$

while for the RHS use is made of

$$P_\ell(\hat{q}\cdot\hat{q}') = \sum_m \frac{4\pi}{2\ell+1}\, Y_m^{(\ell)}(\hat{q})\, Y_m^{(\ell)*}(\hat{q}')$$

and the orthogonality of the $Y_m^{(\ell)}$ functions to prove that

$$\text{RHS} = 2\sum_n \sum_\ell (2\ell+1) P_\ell(\hat{q}_i\cdot\hat{q}_f)\,(M_\ell^\dagger)_{fn}\, q_n\, (M_\ell)_{ni},$$

i.e.

$$-i\left((M_\ell)_{fi} - (M_\ell^\dagger)_{fi}\right) = 2\sum_n (M_\ell^\dagger)_{fn}\, q_n\, (M_\ell)_{ni}. \tag{11.43}$$

If only one channel is present this simplifies to

$$-i\,(M_\ell - M_\ell^*) = 2\, q M_\ell^*\, M_\ell, \tag{11.44}$$

or $Im\, M_\ell = q\,|M_\ell|^2$, which allows writing

$$M_\ell(s) = \frac{S_\ell(s)-1}{2i\,q} = \frac{e^{2i\,\delta_\ell(s)}-1}{2i\,q}, \tag{11.45}$$

where $S_\ell(s)$ satisfies $|S_\ell(s)| = 1$ and $\delta_\ell(s)$ is the phase shift.

In general a given channel has $|S_\ell(s)| \le 1$, parametrized as $S_\ell(s) = \eta_\ell(s)\,\exp(2i\,\delta_\ell(s))$. Using

$$\sigma = \int d\Omega\, \frac{q'}{q} \left|\frac{\mathcal{M}(s,\theta^{cm})}{8\pi\sqrt{s}}\right|^2,$$

in combination with the partial wave expansion for the amplitudes $\mathcal{M}$ and the orthogonality of the Legendre polynomials immediately gives for the elastic channel,

$$\begin{aligned}\sigma_{el} &= 4\pi\sum_\ell (2\ell+1)|M^{(\ell)}(s)|^2 \\ &= \frac{4\pi}{q^2}\sum_\ell (2\ell+1)\left|\frac{\eta_\ell\, e^{\,2i\,\delta_\ell}-1}{2i}\right|^2,\end{aligned} \tag{11.46}$$

and for the case that this is the only channel (purely elastic scattering, $\eta = 1$) the result

$$\sigma_{el} = \frac{4\pi}{q^2}\sum_\ell (2\ell+1)\,\sin^2\delta_\ell. \tag{11.47}$$

From the imaginary part of $\mathcal{M}(s,0)$, the total cross section can be determined. Show that

$$\begin{aligned}\sigma_T &= \frac{4\pi}{q}\sum_\ell (2\ell+1)\, Im\, M^{(\ell)}(s) \\ &= \frac{2\pi}{q^2}\sum_\ell (2\ell+1)\,(1-\eta_\ell\,\cos 2\delta_\ell).\end{aligned} \tag{11.48}$$

The difference is the inelastic cross section,

$$\sigma_{inel} = \frac{\pi}{q^2}\sum_\ell (2\ell+1)\,(1-\eta_\ell^2). \tag{11.49}$$

Note that the total cross section is maximal in the case of full absorption, $\eta = 0$, in which case, however, $\sigma_{el} = \sigma_{inel}$.

11.4 Unstable particles

For a stable particle the propagator is

$$i\Delta(k) = \frac{i}{k^2 - M^2 + i\epsilon} \tag{11.50}$$

(note that we have disregarded spin). The prescription for the pole structure, i.e. one has poles at $k^0 = \pm(E - i\epsilon)$ where $E = +\sqrt{\boldsymbol{k}^2 + M^2}$ guarantees the correct behavior, specifically one has for $t > 0$ that the Fourier transform is

$$\int dk^0\; e^{-ik^0 t}\,\Delta(k) \propto \int dk^0\; \frac{e^{-ik^0 t}}{(k^0 - E + i\epsilon)(k^0 + E - i\epsilon)} \quad \overset{(t>0)}{\propto} \quad e^{-iEt},$$

i.e. $\propto U(t,0)$, the time-evolution operator. For an unstable particle one expects that

$$U(t,0) \propto e^{-i(E - i\Gamma/2)t},$$

such that $|U(t,0)|^2 = e^{-\Gamma t}$. This is achieved with a propagator

$$i\Delta_R(k) = \frac{i}{k^2 - M^2 + iM?} \tag{11.51}$$

(again disregarding spin). The quantity ? is precisely the width for unstable particles. This is (somewhat sloppy!) seen by considering the (truncated) 1PI two-points vertex

$$?^{(2)} = \frac{-i}{\Delta}$$

as the amplitude $-i\mathcal{M}$ for scattering a particle into itself through the decay channels as intermediate states. The unitarity condition then states

$$\begin{aligned} 2\,Im\,\Delta_R^{-1}(k) &= \sum_n \int d\mathcal{R}(p_1,\ldots,p_n)\mathcal{M}^\dagger_{Rn}\,(2\pi)^4\,\delta^4(k - P_n)\,\mathcal{M}_{nR} \\ &= 2M\sum_n ?_n = 2M\,?. \end{aligned} \tag{11.52}$$

This shows that ? is the width of the resonance, which is given by a sum of the partial widths into the different channels. It is important to note that the physical width of a particle is the imaginary part of the two-points vertex at $s = M^2$.

For the amplitude in a scattering process going through a resonance, it is straightforward to write down the partial wave amplitude,

$$(M_\ell)_{ij}(s) = \frac{-M\sqrt{?_i ?_j}}{s - M^2 + iM?}. \tag{11.53}$$

(Proof this using the unitarity condition for partial waves). From this one sees that a resonance has the same shape in all channels but different strength. Limiting ourselves to a resonance in one channel, it is furthermore easy to proof that the cross section is given by

$$\sigma_{el} = \frac{2\pi}{q^2}\,(2\ell + 1)\,\frac{M^2 ?^2}{(s - M^2)^2 + M^2 ?^2}, \tag{11.54}$$

reaching the unitarity limit for $s = M^2$, where furthermore $\sigma_{inel} = 0$. This characteristic shape of a resonance is called the Breit-Wigner shape. The half-width of the resonance is $M?$. The phase shift in the resonating channel near the resonance is given by

$$\tan\delta_\ell(s) = \frac{M?}{M^2 - s}, \tag{11.55}$$

showing that the phase shift at resonance rises through $\delta = \pi/2$ with a 'velocity' $\partial\delta/\partial s = 1/M?$, i.e. a fast change in the phase shift for a narrow resonance. Note that because of the presence of a background the phase shift at resonance may actually be shifted.
Three famous resonances are:

- The Δ-resonance seen in pion-nucleon scattering. Its mass is $M = 1232$ MeV, its width $? =$ 120 MeV. At resonance the cross section $\sigma_T(\pi^+ p)$ is about 210 mb. The cross section $\sigma_T(\pi^- p)$ also shows a resonance with the same width with a value of about 70 mb. This implies that the resonance has spin $J = 3/2$ (decaying in a P-wave ($\ell = 1$) pion-nucleon state) and isospin $I = 3/2$ (the latter under the assumption that isospin is conserved for the strong interactions).
- The J/ψ resonance in e^+e^- scattering. This is a narrow resonance discovered in 1974. Its mass is $M = 3096.88$ MeV, the full width is $? = 88$ keV, the partial width into e^+e^- is $?_{ee} = 5.26$ keV.
- The Z^0 resonance in e^+e^- scattering with $M = 91.2$ GeV, $? = 2.49$ GeV. Essentially this resonance can decay into quark-antiquark pairs or into pairs of charged leptons. All these decays can be seen and leave an 'invisible' width of 498 MeV, which is attributed to neutrinos. Knowing that each neutrino contributes about 160 MeV (see next chapter), one can reconstruct the resonance shape for different numbers of neutrino species. Three neutrinos explain the resonance shape. The cross section at resonance is about 30 nb.

11.5 Exercises

Exercise 11.1
Show that the cross section for electron-electron scattering (exercise 10.4) can be written as

$$\frac{d\sigma}{dt} = \frac{4\pi\alpha^2}{s(s-4m^2)}\left\{f(t,u) + g(t,u) + f(u,t) + g(u,t)\right\},$$

with

$$\begin{aligned} f(t,u) &= \frac{1}{t^2}\left[\frac{1}{2}(s^2+u^2) + 4m^2(t-m^2)\right] \\ g(t,u) &= \frac{2}{tu}\left[(\frac{1}{2}s - m^2)(\frac{1}{2}s - 3m^2)\right] \end{aligned}$$

Exercise 11.2
Show that unitarity fixes the the P-wave amplitude near the Δ-pole,

$$M_1(s) = \frac{-M_\Delta ?_\Delta/q}{s - M_\Delta^2 + iM_\Delta ?_\Delta},$$

under the assumption that ? is approximately constant near this pole.

Chapter 12

The standard model

12.1 Non-abelian gauge theories

In chapter 10 we have considered quantum electrodynamics as an example of a gauge theory. The photon field A_μ was introduced as to render the lagrangian invariant under local gauge transformations. The extension to non-abelian gauge theories is straightforward. The symmetry group is a Lie-group G generated by generators T_a, which satisfy commutation relations

$$[T_a, T_b] = i\, c_{abc}\, T_c, \tag{12.1}$$

with c_{abc} known as the *structure constants* of the group. For a compact Lie-group they are antisymmetric in the three indices. In an abelian group the structure constants would be zero (for instance the trivial example of $U(1)$). Consider a field transforming under the group,

$$\phi(x) \longrightarrow e^{i\,\theta^a(x) L_a}\phi(x) \stackrel{inf.}{=} (1 + i\,\theta^a(x) L_a)\,\phi(x) \tag{12.2}$$

where L_a is a representation matrix for the representation to which ϕ belongs, i.e. for a three-component field $\vec{\phi}$ under an $SO(3)$ or $SU(2)$ symmetry transformation,

$$\vec{\phi} \longrightarrow e^{i\,\vec{\theta}\cdot\vec{L}}\vec{\phi} \approx \vec{\phi} - \vec{\theta}\times\vec{\phi}. \tag{12.3}$$

The complication arises (as in the abelian case) when one considers for a lagrangian density $\mathcal{L}(\phi, \partial_\mu\phi)$ the behavior of $\partial_\mu\phi$ under a local gauge transformation, $\underline{U}(\theta) = e^{i\,\theta^a(x) L_a}$,

$$\phi(x) \longrightarrow \underline{U}(\theta)\phi(x), \tag{12.4}$$

$$\partial_\mu\phi(x) \longrightarrow \underline{U}(\theta)\partial_\mu\phi(x) + (\partial_\mu\underline{U}(\theta))\,\phi(x). \tag{12.5}$$

Introducing as many gauge fields as there are generators in the group, which are conveniently combined in the matrix valued field $\underline{W}_\mu = W^a_\mu L_a$, one defines

$$\underline{D}_\mu\phi(x) \equiv \left(\partial_\mu - ig\,\underline{W}_\mu\right)\phi(x), \tag{12.6}$$

and one obtains after transformation

$$\underline{D}_\mu\phi(x) \longrightarrow \underline{U}(\theta)\partial_\mu\phi(x) + (\partial_\mu\underline{U}(\theta))\,\phi(x) - ig\,\underline{W}'_\mu\,\underline{U}(\theta)\phi(x).$$

Requiring that $\underline{D}_\mu\phi$ transforms as $\underline{D}_\mu\phi \to \underline{U}(\theta)\,\underline{D}_\mu\phi$, i.e.

$$\underline{D}_\mu\phi(x) \longrightarrow \underline{U}(\theta)\partial_\mu\phi(x) - ig\,\underline{U}(\theta)\,\underline{W}_\mu\phi(x),$$

one obtains

$$\underline{W}'_\mu = \underline{U}(\theta)\,\underline{W}_\mu\,\underline{U}^{-1}(\theta) - \frac{i}{g}\,\left(\partial_\mu \underline{U}(\theta)\right)\underline{U}^{-1}(\theta), \tag{12.7}$$

or infinitesimal

$$W'^a_\mu = W^a_\mu - c_{abc}\theta^b W^c_\mu + \frac{1}{g}\,\partial_\mu\theta^a = W^a_\mu + \frac{1}{g}\,D_\mu\theta^a.$$

It is necessary to introduce the free lagrangian density for the gauge fields just like the term $-(1/4)F_{\mu\nu}F^{\mu\nu}$ in QED. For abelian fields $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = (i/g)[D_\mu, D_\nu]$ is gauge invariant. In the nonabelian case $\partial_\mu \underline{W}^a_\nu - \partial_\nu \underline{W}^a_\mu$ does not provide a gauge invariant candidate for $\underline{G}_{\mu\nu} = G^a_{\mu\nu}L_a$, as can be checked easily. Expressing $\underline{G}_{\mu\nu}$ in terms of the covariant derivatives provides a gauge invariant definition for $\underline{G}_{\mu\nu}$ with

$$\underline{G}_{\mu\nu} = \frac{i}{g}\,[\underline{D}_\mu, \underline{D}_\nu] = \partial_\mu \underline{W}_\nu - \partial_\nu \underline{W}_\mu - ig\,[\underline{W}_\mu, \underline{W}_\nu], \tag{12.8}$$

with

$$G^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\,c_{abc}W^b_\mu W^c_\nu, \tag{12.9}$$

It transforms like

$$\underline{G}_{\mu\nu} \;\rightarrow\; \underline{U}(\theta)\,\underline{G}_{\mu\nu}\,\underline{U}^{-1}(\theta). \tag{12.10}$$

The gauge-invariant lagrangian density is now constructed as

$$\mathcal{L}(\phi, \partial_\mu\phi) \longrightarrow \mathcal{L}(\phi, D_\mu\phi) - \frac{1}{2}\mathrm{Tr}\,\underline{G}_{\mu\nu}\underline{G}^{\mu\nu} = \mathcal{L}(\phi, D_\mu\phi) - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu\,a} \tag{12.11}$$

with the standard normalization $\mathrm{Tr}(L_a L_b) = (1/2)\delta_{ab}$. Note that the gauge fields must be massless, as a mass term $\propto M_W^2 W^a_\mu W^{\mu\,a}$ would break gauge invariance.

12.1.1 QCD, an example of a nonabelian gauge theory

As an example of a nonabelian gauge theory consider *quantum chromodynamics* (QCD), the theory describing the interactions of the colored quarks. The existence of an extra degree of freedom for each species of quarks is evident for several reasons, e.g. the necessity to have an antisymmetric wave function for the Δ^{++} particle consisting of three *up* quarks (each with charge $+(2/3)e$). With the quarks belonging to the fundamental (three-dimensional) representation of $SU(3)_C$, i.e. having three components in color space

$$\psi = \begin{pmatrix} \psi_r \\ \psi_g \\ \psi_b \end{pmatrix},$$

the wave function of the baryons (such as nucleons and deltas) form a singlet under $SU(3)_C$,

$$|\mathrm{color}\rangle = \frac{1}{\sqrt{6}}\left(|rgb\rangle - |grb\rangle + |gbr\rangle - |bgr\rangle + |brg\rangle - |rbg\rangle\right). \tag{12.12}$$

The nonabelian gauge theory that is obtained by making the 'free' quark lagrangian, for one specific species (flavor) of quarks just the Dirac lagrangian for an elementary fermion,

$$\mathcal{L} = i\,\bar\psi \not{\partial}\psi - m\,\bar\psi\psi,$$

invariant under local $SU(3)_C$ transformations has proven to be a good candidate for the microscopic theory of the strong interactions. The representation matrices for the quarks and antiquarks in the fundamental representation are given by

$$\begin{aligned} F_a &= \frac{\lambda_a}{2} \quad \text{for quarks,} \\ F_a &= -\frac{\lambda^*_a}{2} \quad \text{for antiquarks,} \end{aligned}$$

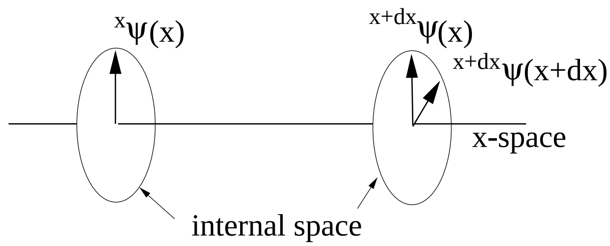


Figure 12.1: The vectors belonging to internal space located at each point in (one-dimensional) space

which satisfy commutation relations $[F_a, F_b] = i\, f_{abc} F_c$ in which f_{abc} are the (completely antisymmetric) structure constants of $SU(3)$ and where the matrices λ_a are the eight Gell-Mann matrices [1]. The (locally) gauge invariant lagrangian density is

$$\mathcal{L} = -\frac{1}{4}\, F^a_{\mu\nu} F^{\mu\nu\, a} + i\,\bar{\psi} D\!\!\!/ \psi - m\,\bar{\psi}\psi, \tag{12.13}$$

with

$$\begin{aligned} D_\mu \psi &= \partial_\mu \psi - ig\, A^a_\mu F_a \psi, \\ F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\, c_{abc} A^b_\mu A^c_\nu . \end{aligned}$$

Note that the term $i\,\bar{\psi} D\!\!\!/ \psi = i\,\bar{\psi}\partial\!\!\!/\psi + g\,\bar{\psi} A\!\!\!/^a F_a \psi = i\,\bar{\psi}\partial\!\!\!/\psi + j^{\mu\, a} A^a_\mu$ with $j^{\mu\, a} = \bar{\psi}\gamma^\mu F_a \psi$ describes the interactions of the gauge bosons A^a_μ (gluons) with the color current of the quarks (this is again precisely the Noether current corresponding to color symmetry transformations). Note furthermore that the lagrangian terms for the gluons contain interaction terms corresponding to vertices with three gluons and four gluons due to the nonabelian character of the theory. For writing down the complete set of Feynman rules it is necessary to account for the gauge symmetry in the quantization procedure. This will lead (depending on the choice of gauge conditions) to the presence of ghost fields. (For more details see e.g. Ryder, chapter 7.)

12.1.2 A geometric picture of gauge theories

A geometric picture of gauge theories is useful for comparison with general relativity and topological considerations (such as we have seen in the Aharonov-Bohm experiment). Consider the space $\prod_x {}^x G$ (called a fibre bundle). At each space-time point x there is considered to be a copy of an internal space G (say spin or isospin). In each of these spaces a reference frame is defined. ${}^x\psi(x)$ denotes a field vector $\psi(x)$ which belongs to a representation of G, i.e. forms a vector in the internal space (see fig. 12.1). The superscript x denotes that it is expressed with respect to the frame at point x, i.e. the

[1] The Gell-Mann matrices are the eight traceless hermitean matrices generating $SU(3)$ transformations,

$$\lambda_1 = \begin{pmatrix} & 1 & \\ 1 & & \\ & & \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} & -i & \\ i & & \\ & & \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} & & 1 \\ & & \\ 1 & & \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} & & -i \\ & & \\ i & & \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} & & \\ & & 1 \\ & 1 & \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} & & \\ & & -i \\ & i & \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$$

basis of ${}^xG$. Let fields $A^a_\mu(x)$ determine the 'parallel displacements' in the internal space, i.e. connect the basis for ${}^xG$ and ${}^{x+dx}G$,

$$
\begin{aligned}
{}^{x+dx}\psi(x) &= \left(1 + ig\,dx^\mu\,A^a_\mu(x)T_a\right)^x\,\psi(x) &(12.14)\\
&= \left(1 + ig\,dx^\mu\,\underline{A}_\mu(x)\right)^x\,\psi(x), &(12.15)
\end{aligned}
$$

which connects two identical vectors, but expresses them with respect to different bases.

If there is a 'true' difference in the vector $\psi(x)$ and $\psi(x+dx)$ it is denoted with the covariant derivative connecting the vectors expressed with respect to the same basis, i.e.

$$
{}^{x+dx}\psi(x+dx) = \left(1 + dx^\mu\,\underline{D}_\mu\right)^{x+dx}\,\psi(x), \tag{12.16}
$$

which because of the presence of the 'connection' $\underline{A}_\mu$ differs from the total change between ${}^x\psi(x)$ and ${}^{x+dx}\psi(x+dx)$,

$$
{}^{x+dx}\psi(x+dx) = (1 + dx^\mu\,\partial_\mu)^x\,\psi(x). \tag{12.17}
$$

The three equations given so far immediately give

$$
\underline{D}_\mu = \partial_\mu - ig\,\underline{A}_\mu(x). \tag{12.18}
$$

We note that local gauge invariance requires that we can modify all local systems with a (local) unitary transformation $S(x)$. The relation in Eq. 12.16, should be independent of such transformations, requiring that the 'connection' $\underline{A}_\mu(x)$ is such that $\underline{D}_\mu \to S(x)\underline{D}_\mu S^{-1}(x)$.

A 'constant' vector that only rotates because of the arbitrary definitions of local frames satisfies $\underline{D}_\mu\,\psi(x) = 0$, i.e.

$$
\left[\partial_\mu - ig\,\underline{A}_\mu(x)\right]\psi(x) = 0
$$

or considering a path $x^\mu(s)$ from a fixed origin (0) to point x,

$$
\frac{dx^\mu}{ds}\left[\partial_\mu - ig\,\underline{A}_\mu(x(s))\right]\psi(x) = 0,
$$

which is solved by

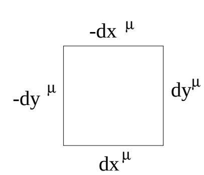
$$
\begin{aligned}
\frac{d\psi(s)}{ds} &= ig\,\underline{A}_\mu(s)\,\psi(s)\,\frac{dx^\mu}{ds},\\
\psi(s) &= \exp\left(ig\int_0^s ds'\,\frac{dx^\mu}{ds'}\,\underline{A}_\mu(s')\right)\psi(0),
\end{aligned}
$$

which is the path-ordered integral denoted

$$
\psi(x) = P\,e^{ig\int_P dx^\mu\,\underline{A}_\mu(x)}\psi(0). \tag{12.19}
$$

This gives rise to a (path dependent) phase in each point.

In principle such a phase in a given point is not observable. However, if two different paths to the same point give different phases the effects can be observed. What is this physical effect by which the 'connection' $\underline{A}_\mu$ can be observed? For this consider the phase around a closed loop,



For the constant vector, it is given by

$$\begin{aligned}\psi(x) &= (1 - i\,dy^\rho \underline{D}_\rho)(1 - i\,dx^\sigma \underline{D}_\sigma)(1 + i\,dy^\nu \underline{D}_\nu)(1 + i\,dx^\mu \underline{D}_\mu)\psi(x) \\ &= \left(1 + dx^\mu dy^\nu\,[\underline{D}_\mu, \underline{D}_\nu]\right)\psi(x) \\ &= \left(1 - ig\,d\sigma^{\mu\nu}\,\underline{G}_{\mu\nu}\right)\psi(x),\end{aligned}$$

where $\underline{G}_{\mu\nu} = (i/g)[\underline{D}_\mu, \underline{D}_\nu]$. Similarly as the definition of the covariant derivative the effect is thus frame-independent and we have the transformation law $\underline{G}_{\mu\nu} \rightarrow S(x)\underline{G}_{\mu\nu}S^{-1}(x)$. In geometric language the effect on parallel transport of a vector depends on the 'curvature' $\underline{G}_{\mu\nu}$. Only if this quantity is nonzero a physically observable effect of $\underline{A}_\mu$ exists. If it is zero one has $\oint dx^\mu \underline{A}_\mu(x) = 0$, and equivalently $\underline{A}_\mu$ can be considered as a pure gauge effect, which means that by an appropriate transformation $S(x)$ it can be gauged away (see example for Aharonov-Bohm effect).

12.2 Spontaneous symmetry breaking

In this section we consider the situation that the groundstate of a physical system is degenerate. Consider as an example a ferromagnet with an interaction hamiltonian of the form

$$H = -\sum_{i>j} J_{ij}\,\mathbf{S}_i \cdot \mathbf{S}_j,$$

which is rotationally invariant. If the temperature is high enough the spins are oriented randomly and the (macroscopic) ground state is spherically symmetric. If the temperature is below a certain critical temperature ($T < T_c$) the kinetic energy is no longer dominant and the above hamiltonian prefers a lowest energy configuration in which all spins are parallel. In this case there are many possible groundstates (determined by a fixed direction in space). This characterizes spontaneous symmetry breaking, the groundstate itself appears degenerate. As there can be one and only one groundstate, this means that there is more than one possibility for the groundstate. Nature will choose one, usually being (slightly) prejudiced by impurities, external magnetic fields, i.e. in reality a not perfectly symmetric situation.

Nevertheless, we can disregard those 'perturbations' and look at the ideal situation, e.g. a theory for a scalar degree of freedom (a scalar field) having three (real) components,

$$\vec{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix},$$

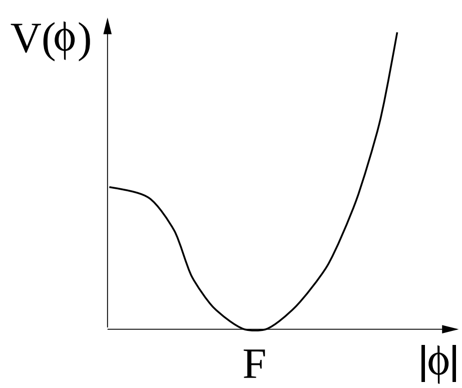


Figure 12.2: The symmetry-breaking 'potential' in the lagrangian for the case that $m^2 < 0$.

with a lagrangian density of the form

$$\mathcal{L} = \frac{1}{2}\partial_\mu\vec{\phi}\partial^\mu\vec{\phi} \underbrace{-\frac{1}{2}\,m^2\,\vec{\phi}\cdot\vec{\phi} - \frac{1}{4}\,\lambda(\vec{\phi}\cdot\vec{\phi})^2}_{-V(\vec{\phi})}. \tag{12.20}$$

The potential $V(\vec{\phi})$ is shown in fig. 12.2. Classically the (time-independent) ground state is found for a constant field ($\nabla\vec{\phi} = 0$) and the condition

$$\frac{\partial V}{\partial\vec{\phi}} = 0 \qquad \longrightarrow \qquad \vec{\phi}\cdot\vec{\phi} = 0 \text{ or } -\frac{m^2}{\lambda} \equiv F^2,$$

the latter only forming a minimum for $m^2 < 0$. In this situation one speaks of *spontaneous symmetry breaking*. The classical groundstate appears degenerate. Any constant field ϕ with 'length' $|\vec{\phi}| = F$ is a possible groundstate. The presence of a nonzero value for the classical groundstate value of the field will have an effect when the field is quantized. As nature, a quantum field theory has only *one* nondegenerate groundstate $|0\rangle$. Writing the field $\vec{\phi}$ as a sum of a classical and a quantum field, $\phi = \vec{\varphi}_c + \vec{\phi}_{\text{quantum}}$ where for the (operator-valued) coefficients in the quantum field one wants $\langle 0|c^\dagger = c|0\rangle = 0$ one has

$$\langle 0|\vec{\phi}|0\rangle = \vec{\varphi}_c. \tag{12.21}$$

Stability of the action requires $\vec{\varphi}_c$ to be the classical groundstate. However, the fact that the groundstate is unique requires a well-defined value for the groundstate expectation value, which can be nonzero. Therefore a choice must be made, say

$$\langle 0|\vec{\phi}|0\rangle = \begin{pmatrix} 0 \\ 0 \\ F \end{pmatrix}. \tag{12.22}$$

The situation now is the following. The original lagrangian contained an SO(3) invariance under (length conserving) rotations among the three fields, while the lagrangian including the nonzero groundstate expectation value chosen by nature, has less symmetry. It is only invariant under rotations around the 3-axis.

It is appropriate to redefine the field as

$$\vec{\phi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ F+\eta \end{pmatrix}, \tag{12.23}$$

such that $\langle 0|\varphi_1|0\rangle = \langle 0|\varphi_2|0\rangle = \langle 0|\eta|0\rangle = 0$. The field along the third axis plays a special role because of the choice of the v.e.v. In order to see the consequences for the particle spectrum of the theory we construct the lagrangian in terms of the fields φ_1, φ_2 and η. It is sufficient to do this to second order in the fields as the higher (cubic, etc.) terms constitute interaction terms. The result is

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\,(\partial_\mu\varphi_1)^2 + \frac{1}{2}\,(\partial_\mu\varphi_2)^2 + \frac{1}{2}\,(\partial_\mu\eta)^2 - \frac{1}{2}\,m^2\,(\varphi_1^2+\varphi_2^2) \\ &\quad -\frac{1}{2}\,m^2\,(F+\eta)^2 - \frac{1}{4}\,\lambda\,(\varphi_1^2+\varphi_2^2+F^2+\eta^2+2\,F\eta)^2 \qquad (12.24) \\ &= \frac{1}{2}\,(\partial_\mu\varphi_1)^2 + \frac{1}{2}\,(\partial_\mu\varphi_2)^2 + \frac{1}{2}\,(\partial_\mu\eta)^2 + m^2\,\eta^2 + \ldots\,. \qquad (12.25) \end{aligned}$$

Therefore there are *2 massless scalar particles*, corresponding to the number of broken generators (in this case rotations around 1 and 2 axis) and *1 massive scalar particle* with mass $m_\eta^2 = -2m^2$. The massless particles are called *Goldstone bosons*.

12.2.1 Realization of symmetries

In this section we want to discuss a bit more formal the two possible ways that a symmetry can be implemented. They are known as the *Weyl mode* or the *Goldstone mode*:

Weyl mode. In this mode the lagrangian *and* the vacuum are both invariant under a set of symmetry transformations generated by Q^a, i.e. for the vacuum $Q^a|0\rangle = 0$. In this case the spectrum is described by degenerate representations of the symmetry group. Known examples are rotational symmetry and the fact that the the spectrum shows multiplets labeled by angular momentum ℓ (with members labeled by m). The generators Q^a (in that case the rotation operators L_z, L_x and L_y or instead of the latter two L_+ and L_-) are used to label the multiplet members or transform them into one another. A bit more formal, if the generators Q^a generate a symmetry, i.e. $[Q^a, H] = 0$, and $|a\rangle$ and $|a'\rangle$ belong to the same multiplet (there is a Q^a such that $|a'\rangle = Q^a|a\rangle$) then $H|a\rangle = E_a|a\rangle$ implies that $H|a'\rangle = E_a|a'\rangle$, i.e. a and a' are degenerate states.

Goldstone mode. In this mode the lagrangian is invariant but $Q^a|0\rangle \neq 0$ for a number of generators. This means that they are operators that create states from the vacuum, denoted $|\pi^a(k)\rangle$. As the generators for a symmetry are precisely the zero-components of a conserved current $J^a_\mu(x)$ integrated over space, there must be a nonzero expectation value $\langle 0|J^a_\mu(x)|\pi^a(k)\rangle$. Using translation invariance and as k_μ is the only four vector on which this matrix element could depend one may write

$$\langle 0|J^a_\mu(x)|\pi^a(k)\rangle = f_\pi\, k_\mu\, e^{i\,k\cdot x} \quad (f_\pi \neq 0) \tag{12.26}$$

for all the states labeled by a corresponding to 'broken' generators. Taking the derivative,

$$\langle 0|\partial^\mu J^a_\mu(x)|\pi^a(k)\rangle = f_\pi\, k^2\, e^{i\,k\cdot x} = f_\pi\, m^2_{\pi^a}\, e^{i\,k\cdot x}. \tag{12.27}$$

If the transformations in the lagrangian give rise to a symmetry the Noether currents are conserved, $\partial^\mu J^a_\mu = 0$, irrespective of the fact if they annihilate the vacuum, and one must have $m_{\pi^a} = 0$, i.e. a massless Goldstone boson for each 'broken' generator. Note that for the fields $\pi^a(x)$ one would have the relation $\langle 0|\pi^a(x)|\pi^a(k)\rangle = e^{i\,k\cdot x}$, suggesting the stronger relation $\partial^\mu J^a_\mu(x) = f_\pi\, m^2_{\pi^a}\, \pi^a(x)$.

12.2.2 Chiral symmetry

An example of spontaneous symmetry breaking is chiral symmetry breaking in QCD. Neglecting at this point the local color symmetry, the lagrangian for the quarks consists of the free Dirac lagrangian for each of the types of quarks, called flavors. Including a sum over the different flavors (up, down, strange, etc.) one can write

$$\mathcal{L} = \overline{\psi}(i\partial\!\!\!/ - M)\psi, \tag{12.28}$$

where ψ is extended to a vector in *flavor space* and M is a diagonal matrix,

$$\psi = \begin{pmatrix} \psi_u \\ \psi_d \\ \vdots \end{pmatrix}, \qquad M = \begin{pmatrix} m_u & & \\ & m_d & \\ & & \ddots \end{pmatrix} \tag{12.29}$$

(Note that each of the entries in the vector for ψ is a 4-component Dirac spinor). This lagrangian density then is invariant under unitary (vector) transformations in the flavor space,

$$\psi \longrightarrow e^{i\,\vec{\alpha}\cdot\vec{T}}\,\psi, \tag{12.30}$$

which for instance including only two flavors form an $SU(2)_V$ symmetry (isospin symmetry) generated by the Pauli matrices, $\vec{T} = \vec{\tau}/2$. The conserved currents corresponding to this symmetry transformation are found directly using Noether's theorem (see chapter 6),

$$\vec{V^\mu} = \overline{\psi}\gamma^\mu\vec{T}\psi. \tag{12.31}$$

Using the Dirac equation, it is easy to see that one gets

$$\partial_\mu \vec{V}^\mu = i\,\overline{\psi}\,[M,\vec{T}]\,\psi. \tag{12.32}$$

Furthermore $\partial_\mu \vec{V}^\mu = 0 \Longleftrightarrow [M,\vec{T}] = 0$. From group theory (Schur's theorem) one knows that the latter can only be true, if in flavor space M is proportional to the unit matrix, $M = m \cdot 1$. I.e. $SU(2)_V$ (isospin) symmetry is good if the up and down quark masses are identical. This situation, both are very small, is what happens in the real world. This symmetry is realized in the Weyl mode with the spectrum of QCD showing an almost perfect isospin symmetry, e.g. a doublet (isospin 1/2) of nucleons, proton and neutron, with almost degenerate masses (M_p = 938.3 MeV/c^2 and M_n =939.6 MeV/c^2), but also a triplet (isospin 1) of pions, etc.

There exists another set of symmetry transformations, socalled axial vector transformations,

$$\psi \longrightarrow e^{i\,\vec{\alpha}\cdot\vec{T}\gamma_5}\psi, \tag{12.33}$$

which for instance including only two flavors form $SU(2)_A$ transformations generated by the Pauli matrices, $\vec{T}\gamma_5 = \vec{\tau}\gamma_5/2$. Note that these transformations also work on the spinor indices. The currents corresponding to this symmetry transformation are again found using Noether's theorem,

$$\vec{A}^\mu = \overline{\psi}\gamma^\mu \vec{T}\gamma_5\psi. \tag{12.34}$$

Using the Dirac equation, it is easy to see that one gets

$$\partial_\mu \vec{A}^\mu = i\,\overline{\psi}\,\{M,\vec{T}\}\,\gamma_5\,\psi. \tag{12.35}$$

In this case $\partial_\mu \vec{A}^\mu = 0$ will be true if the quarks have zero mass, which is approximately true for the up and down quarks. Therefore the world of up and down quarks describing pions, nucleons and atomic nuclei has not only an isospin or vector symmetry $SU(2)_V$ but also an axial vector symmetry $SU(2)_A$. This combined symmetry is what one calls *chiral symmetry.*

That the massless theory has this symmetry can also be seen by writing it down for the socalled lefthanded and righthanded fermions, $\psi_{R/L} = \frac{1}{2}(1 \pm \gamma_5)\psi$, in terms of which the Dirac lagrangian density looks like

$$\mathcal{L} = i\,\overline{\psi}_L \partial\!\!\!/ \psi_L + i\,\overline{\psi}_R \partial\!\!\!/ \psi_R - \overline{\psi}_R M \psi_L - \overline{\psi}_L M \psi_R. \tag{12.36}$$

If the mass is zero the lagrangian is split into two disjunct parts for L and R showing that there is a direct product $SU(2)_L \otimes SU(2)_R$ symmetry, generated by $\vec{T}_{R/L} = \frac{1}{2}(1 \pm \gamma_5)\vec{T}$, which is equivalent to the V-A symmetry. This symmetry, however, is by nature not realized in the Weyl mode. How can we see this. The chiral fields ψ_R and ψ_L are transformed into each other under parity. Therefore realization in the Weyl mode would require that all particles come double with positive and negative parity, or, stated equivalently, parity would not play a role in the world. We know that mesons and baryons (such as the nucleons) have a well-defined parity that is conserved.

The conclusion is that the original symmetry of the lagrangian is spontaneously broken and as the vector part of the symmetry is the well-known isospin symmetry, nature has choosen the path

$$SU(2)_L \otimes SU(2)_R \quad \Longrightarrow \quad SU(2)_V\,,$$

i.e. the lagrangian density is invariant under left (L) and right (R) rotations independently, while the groundstate is only invariant under isospin rotations ($R = L$). From the number of broken generators it is clear that one expects three massless Goldstone bosons, for which the field (according to the discussion in the previous subsection) has the same behavior under parity, etc. as the quantity $\partial_\mu A^\mu(x)$, i.e. (leaving out the flavor structure) the same as $\bar{\psi}\gamma_5\psi$, i.e. behaves as a pseudoscalar particle (spin zero, parity minus). In the real world, where the quark masses are not completely zero, chiral symmetry is not perfect. Still the basic fact that the generators acting on the vacuum give a nonzero result (i.e. $f_\pi \neq 0$ remains, but the fact that the symmetry is not perfect and the right hand side of Eq. 12.35 is nonzero, gives also rise to a nonzero mass for the Goldstone bosons according to Eq. 12.27. The Goldstone bosons of QCD are the pions for which f_π = 93 MeV and which have a mass of $m_\pi \approx$ 138 MeV/c^2, much smaller than any of the other mesons or baryons.

12.3 The Higgs mechanism

The Higgs mechanism occurs when spontaneous symmetry breaking happens in a gauge theory where gauge bosons have been introduced in order to assure the local symmetry. Considering the same example with rotational symmetry ($SO(3)$) as for spontaneous symmetry breaking of a scalar field (Higgs field) with three components, made into a gauge theory,

$$\mathcal{L} = -\frac{1}{4}\,\vec{G}_{\mu\nu}\cdot\vec{G}^{\mu\nu} + \frac{1}{2}\,D_\mu\vec{\phi}\cdot D^\mu\vec{\phi} - V(\vec{\phi}), \tag{12.37}$$

where

$$\begin{aligned} D_\mu\vec{\phi} &= \partial_\mu\vec{\phi} - ig\,\vec{W}^a_\mu L_a\vec{\phi} \\ &= \partial_\mu\vec{\phi} + g\,\vec{W}_\mu \times vec\phi, \qquad (12.38) \\ \vec{G}_{\mu\mu} &= \partial_\mu\vec{W}_\nu - \partial_\nu\vec{W}_\mu + g\vec{W}_\mu \times \vec{W}_\nu \end{aligned}$$

(12.39)

where the explicit (conjugate, in this case three-dimensional) representation $(L_a)_{ij} = -i\,\epsilon_{aij}$ has been used and where the quantities $\vec{W}_\mu$ and $\vec{G}_{\mu\nu}$ are also three-component vectors. The symmetry is broken in the same way as before and the same choice for the vacuum,

$$\langle 0|\vec{\phi}|0\rangle = \begin{pmatrix} 0 \\ 0 \\ F \end{pmatrix}.$$

is made. The difference comes when we reparametrize the field $\vec{\phi}$. We have the possibility to perform local gauge transformations. Therefore we can always rotate the field ϕ into the z-direction, i.e.

$$\vec{\phi} = \begin{pmatrix} 0 \\ 0 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ F+\eta \end{pmatrix}. \tag{12.40}$$

Explicitly one then has

$$D_\mu\vec{\phi} = \partial_\mu\vec{\phi} + g\,\vec{W}_\mu \times \vec{\phi} = \begin{pmatrix} gF\,W^2_\mu + g\,W^2_\mu\,\eta \\ -gF\,W^1_\mu - g\,W^1_\mu\,\eta \\ 0 \end{pmatrix},$$

which gives for the lagrangian density up to quadratic terms

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}\,\vec{G}_{\mu\nu}\cdot\vec{G}^{\mu\nu} + \frac{1}{2}\,D_\mu\vec{\phi}\cdot D^\mu\vec{\phi} - \frac{1}{2}\,m^2\vec{\phi}\cdot\vec{\phi} - \frac{\lambda}{4}\,(\vec{\phi}\cdot\vec{\phi})^2 \\ &= -\frac{1}{4}\,(\partial_\mu\vec{W}_\nu - \partial_\nu\vec{W}_\mu)\cdot(\partial^\mu\vec{W}^\nu - \partial^\nu\vec{W}^\mu) - \frac{1}{2}\,g^2F^2\,(W^1_\mu W^{\mu\,1} + W^2_\mu W^{\mu\,2}) \\ &\quad + \frac{1}{2}\,(\partial_\mu\eta)^2 + m^2\,\eta^2 + \ldots, \end{aligned} \tag{12.41}$$

from which one reads off that the particle content of the theory consists of one massless gauge boson (W^3_μ), two massive bosons (W^1_μ and W^2_μ with $M_W = gF$) and a massive scalar particle (η with $m^2_\eta = -2\,m^2$. The latter is a spin 0 particle (real scalar field) called a Higgs particle. Note that the number of massless gauge bosons (in this case one) coincides with the number of generators corresponding to the remaining symmetry (in this case rotations around the 3-axis), while the number of massive gauge bosons coincides with the number of 'broken' generators.

One may wonder about the degrees of freedom, as in this case there are no massless Goldstone bosons. Initially there are 3 massless gauge fields (each, like a photon, having two independent spin components) and three scalar fields (one degree of freedom each), thus 9 independent degrees of freedom. After symmetry breaking the same number (as expected) comes out, but one has 1 massless gauge field (2), 2 massive vector fields or spin 1 bosons (2×3) and one scalar field (1), again 9 degrees of freedom.

12.4 The standard model $SU(2)_W \otimes U(1)_Y$

The symmetry ideas discussed before play an essential role in the standard model that describes the elementary particles, the quarks (up, down, etc.), the leptons (elektrons, muons, neutrinos, etc.) and the gauge bosons responsible for the strong, electromagnetic and weak forces. In the standard model one starts with a very simple basic lagrangian for (massless) fermions which exhibits more symmetry than observed in nature. By introducing gauge fields and breaking the symmetry a more complex lagrangian is obtained, that gives a good description of the physical world. The procedure, however, implies certain nontrivial relations between masses and mixing angles that can be tested experimentally and sofar are in excellent agreement with experiment.

The lagrangian for the leptons consists of three families each containing an elementary fermion (electron e^-, muon μ^- or tau τ^-), its corresponding neutrino (ν_e, ν_μ and ν_τ) and their antiparticles. As they are massless, left- and righthanded particles, $\psi_{R/L} = \frac{1}{2}(1 \pm \gamma_5)\psi$ decouple. For the neutrino only a lefthanded particle (and righthanded antiparticle) exist. Thus

$$\mathcal{L}^{(f)} = i\,\bar{e}_R \partial\!\!\!/ e_R + i\,\bar{e}_L \partial\!\!\!/ e_L + i\,\bar{\nu}_{eL} \partial\!\!\!/ \nu_{eL} + (\mu, \tau). \tag{12.42}$$

One introduces a (weak) $SU(2)_W$ symmetry under which e_R forms a singlet, while the lefthanded particles form a doublet, i.e.

$$L = \begin{pmatrix} \nu_e \\ e_L \end{pmatrix} \qquad \text{with } I_W = \frac{1}{2} \text{ and } I_W^3 = \begin{cases} +1/2 \\ -1/2 \end{cases}$$

and

$$R = e_R \qquad \text{with } I_W = 0 \text{ and } I_W^3 = 0.$$

Thus the lagrangian density is

$$\mathcal{L}^{(f)} = i\,\bar{L} \partial\!\!\!/ L + i\,\bar{R} \partial\!\!\!/ R, \tag{12.43}$$

which has an $SU(2)_W$ symmetry under transformations $e^{i\vec{\alpha}\cdot\vec{T}}$, explicitly

$$L \stackrel{SU(2)_W}{\longrightarrow} e^{i\,\vec{\alpha}\cdot\vec{\tau}/2} L, \tag{12.44}$$

$$R \stackrel{SU(2)_W}{\longrightarrow} R. \tag{12.45}$$

One notes that the charges of the leptons can be obtained as $Q = I_W^3 - 1/2$ for lefthanded particles and $Q = I_W^3 - 1$ for righthanded particles. This is written as

$$Q = I_W^3 + \frac{Y}{2}, \tag{12.46}$$

and Y is considered as an operator that generates a $U(1)_Y$ symmetry, under which the lefthanded and righthanded particles transform with $e^{i\beta Y/2}$,

$$L \stackrel{U(1)_Y}{\longrightarrow} e^{-i\,\beta/2} L, \tag{12.47}$$

$$R \stackrel{U(1)_Y}{\longrightarrow} e^{-i\,\beta} R. \tag{12.48}$$

Next the $SU(2)_W \otimes U(1)_Y$ symmetry is made into a local symmetry introducing gauge fields $\vec{W}_\mu$ and B_μ,

$$D_\mu L = \partial_\mu L - \frac{i}{2}\, g\, \vec{W}_\mu \cdot \vec{\tau}\, L + \frac{i}{2}\, g'\, B_\mu\, L, \tag{12.49}$$

$$D_\mu R = \partial_\mu R + i\, g'\, B_\mu\, R, \tag{12.50}$$

where $\vec{W}_\mu$ is a triplet of gauge bosons with $I_W = 1$, $I_W^3 = \pm 1$ or 0 and $Y_W = 0$ (thus $Q = I_W^3$) and B_μ is a singlet under $SU(2)_W$ ($I_W = I_W^3 = 0$) and also has $Y_W = 0$. Putting this in leads to

$$\mathcal{L}^{(f)} = \mathcal{L}^{(f1)} + \mathcal{L}^{(f2)}, \tag{12.51}$$

$$\begin{aligned}
\mathcal{L}^{(f1)} &= i\,\bar{R}\gamma^\mu(\partial_\mu + ig' B_\mu)R + i\,\bar{L}\gamma^\mu(\partial_\mu + \frac{i}{2}g' B_\mu - \frac{i}{2}\,\vec{W}_\mu \cdot \vec{\tau})L \\
\mathcal{L}^{(f2)} &= -\frac{1}{4}(\partial_\mu \vec{W}_\nu - \partial_\nu \vec{W}_\mu + g\,\vec{W}_\mu \times \vec{W}_\nu)^2 - \frac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2.
\end{aligned}$$

In order to break the symmetry to the symmetry of the physical world, the $U(1)_Q$ symmetry (generated by the charge operator), a complex Higgs field

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(\theta_2 + i\theta_1) \\ \frac{1}{\sqrt{2}}(\theta_4 - i\theta_3) \end{pmatrix} \tag{12.52}$$

with $I_W = 1/2$ and $Y_W = 1$ is introduced, with the following lagrangian density consisting of a symmetry breaking piece and a coupling to the fermions,

$$\mathcal{L}^{(h)} = \mathcal{L}^{(h1)} + \mathcal{L}^{(h2)}, \tag{12.53}$$

where

$$\begin{aligned}
\mathcal{L}^{(h1)} &= (D_\mu\phi)^\dagger(D^\mu\phi)\underbrace{-m^2\,\phi^\dagger\phi - \lambda\,(\phi^\dagger\phi)^2}_{-V(\phi)}, \\
\mathcal{L}^{(h2)} &= -G_e(\bar{L}\phi R + \bar{R}\phi^\dagger L),
\end{aligned}$$

and

$$D_\mu\phi = (\partial_\mu - \frac{i}{2}\,g\vec{W}_\mu \cdot \vec{\tau} - \frac{i}{2}\,g'\,B_\mu)\phi. \tag{12.54}$$

The Higgs potential $V(\phi)$ is choosen such that it gives rise to spontaneous symmetry breaking with $\phi^\dagger\phi = -m^2/2\lambda \equiv v/\sqrt{2}$. For the classical field the choice $\theta_4 = v$ is made. Using *local* gauge invariance θ_i for $i = 1$, 2 and 3 may be eliminated (the necessary $SU(2)_W$ rotation is precisely $e^{-i\vec{\theta}(x)\cdot\tau}$), leading to the parametrization

$$\phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + h(x) \end{pmatrix} \tag{12.55}$$

and

$$D_\mu\phi = \begin{pmatrix} -\frac{ig}{2}\left(\frac{W_\mu^1 - iW_\mu^2}{\sqrt{2}}\right)(v+h) \\ \partial_\mu h + \frac{i}{2}\left(\frac{gW_\mu^3 - g'B_\mu}{\sqrt{2}}\right)(v+h) \end{pmatrix}. \tag{12.56}$$

Up to cubic terms, this leads to the lagrangian

$$\begin{aligned}
\mathcal{L}^{(h1)} &= \frac{1}{2}(\partial_\mu h)^2 + m^2\,h^2 + \frac{g^2\,v^2}{8}\left[(W_\mu^1)^2 + (W_\mu^2)^2\right] \\
&\quad + \frac{v^2}{8}\left(gW_\mu^3 - g'B_\mu\right)^2 + \ldots \qquad (12.57) \\
&= \frac{1}{2}(\partial_\mu h)^2 + m^2\,h^2 + \frac{g^2\,v^2}{8}\left[(W_\mu^+)^2 + (W_\mu^-)^2\right] \\
&\quad + \frac{(g^2 + g'^2)\,v^2}{8}(Z_\mu)^2 + \ldots, \qquad (12.58)
\end{aligned}$$

where the quadratically appearing gauge fields that are furthermore eigenstates of the charge operator are

$$W^{\pm}_\mu = \frac{1}{\sqrt{2}}\left(W^1_\mu \pm i\, W^2_\mu\right), \tag{12.59}$$

$$Z_\mu = \frac{g\, W^3_\mu - g'\, B_\mu}{\sqrt{g^2+g'^2}} \equiv \cos\theta_W\; W^3_\mu - \sin\theta_W\; B_\mu, \tag{12.60}$$

$$A_\mu = \frac{g'\, W^3_\mu + g\, B_\mu}{\sqrt{g^2+g'^2}} \equiv \sin\theta_W\; W^3_\mu + \cos\theta_W\; B_\mu, \tag{12.61}$$

and correspond to three massive particle fields ($W^{\pm}$ and Z^0) and one massless field (photon γ) with

$$M_W^2 = \frac{g^2\, v^2}{4}, \tag{12.62}$$

$$M_Z^2 = \frac{g^2\, v^2}{4\,\cos^2\theta_W} = \frac{M_W^2}{\cos^2\theta_W}, \tag{12.63}$$

$$M_\gamma^2 = 0. \tag{12.64}$$

The weak mixing angle is related to the ratio of coupling constants, $g'/g = \tan\theta_W$.

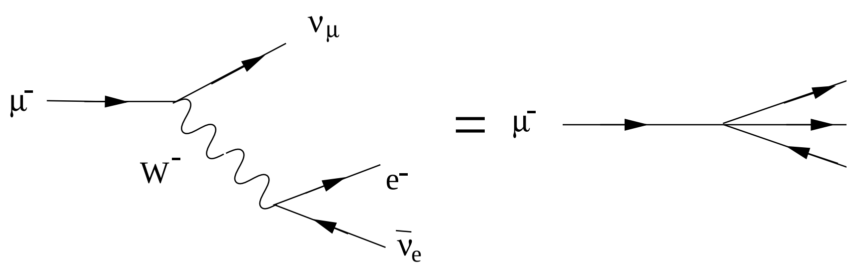
The coupling of the fermions to the physical gauge bosons are contained in $\mathcal{L}^{(f1)}$ giving

$$\begin{aligned}\mathcal{L}^{(f1)} = {} & i\,\bar{e}\gamma^\mu\partial_\mu e + i\,\bar{\nu}_e\gamma^\mu\partial_\mu\nu_e + g\sin\theta_W\;\bar{e}\gamma^\mu e\,A_\mu \\ & -\frac{g}{\cos\theta_W}\left(\sin^2\theta_W\;\bar{e}_R\gamma^\mu e_R - \frac{1}{2}\cos 2\theta_W\;\bar{e}_L\gamma^\mu e_L + \frac{1}{2}\,\bar{\nu}_e\gamma^\mu\nu_e\right)\,Z_\mu \\ & -\frac{g}{\sqrt{2}}\left(\bar{\nu}_e\gamma^\mu e_L\,W^+_\mu + \bar{e}_L\gamma^\mu\nu_e\,W^-_\mu\right).\end{aligned} \tag{12.65}$$

From the coupling to the photon, we can read off

$$e = g\,\sin\theta_W = g'\cos\theta_W. \tag{12.66}$$

The coupling of electrons or muons to their respective neutrinos, for instance in the amplitude for the decay of the muon



is given by

$$\begin{aligned}-i\,\mathcal{M} &= \frac{g^2}{2}(\bar{\nu}_\mu\gamma^\rho\mu_L)\frac{g_{\rho\sigma}+\dots}{k^2+M_W^2}(\bar{e}_L\gamma^\sigma\nu_e) \\ &\approx \frac{g^2}{8\,M_W^2}\underbrace{(\bar{\nu}_\mu\gamma^\rho(1-\gamma_5)\mu)}_{(j_L^{(\mu)\dagger})_\rho}\underbrace{(\bar{e}\gamma^\sigma(1-\gamma^5)\nu_e)}_{(j_L^{(e)})^\rho} \qquad (12.67) \\ &\equiv \frac{G_F}{\sqrt{2}}\,(j_L^{(\mu)\dagger})_\rho\,(j_L^{(e)})^\rho, \qquad (12.68)\end{aligned}$$

the well-known four-point interaction introduced by Fermi to explain the weak interactions, i.e. one has the relation

$$\frac{G_F}{\sqrt{2}} = \frac{e^2}{8\, M_W^2\, \sin^2\theta_W} \tag{12.69}$$

Note that the parameters g, g' and v determine a number of experimentally measurable quantities, such as

$$\begin{aligned} e^2/4\pi &\approx 1/137, & (12.70)\\ G_F &= 1.1664 \times 10^{-5}\ \mathrm{GeV}^2, & (12.71)\\ \sin^2\theta_W &= 0.2325 \pm 0.0008, & (12.72)\\ M_W &= 80.2\ \mathrm{GeV}, & (12.73)\\ M_Z &= 91.17\ \mathrm{GeV}. & (12.74) \end{aligned}$$

The coupling of the Z^0 to fermions is given by

$$I_W^3\, \frac{1}{2}(1-\gamma_5) - \sin^2\theta_W\, Q \equiv \frac{1}{2}\, C_V - \frac{1}{2}\, C_A\, \gamma_5, \tag{12.75}$$

with

$$\begin{aligned} C_V &= I_W^3 - 2\, \sin^2\theta_W\, Q, & (12.76)\\ C_A &= I_W^3. & (12.77) \end{aligned}$$

From this coupling it is straightforward to calculate the partial width for Z^0 into a fermion-antifermion pair,

$$?\,(Z^0 \rightarrow f\bar{f}) = \frac{M_Z}{48\pi}\, \frac{g^2}{\cos^2\theta_W}\, (C_V^2 + C_A^2). \tag{12.78}$$

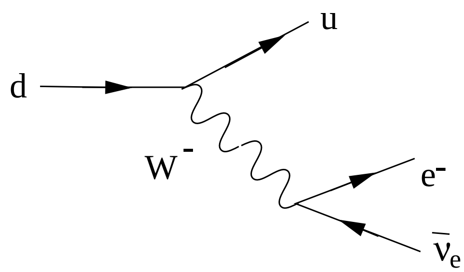
For the electron, muon or tau, leptons with $C_V = -1/2 + 2\sin^2\theta_W \approx -0.05$ and $C_A = -1/2$ we calculate $?\,(e^+e^-) \approx 78.5$ MeV (exp. $?_e \approx ?_\mu \approx ?_\tau \approx 83$ MeV). For each neutrino species (with $C_V = 1/2$ and $C_A = 1/2$ one expects $?\,(\bar{\nu}\nu) \approx 155$ MeV. Comparing this with the total width into (invisible!) channels, $?_{invisible} = 480$ MeV one sees that three families of (light) neutrinos are allowed. Actually including corrections corresponding to higher order diagrams the agreement for the decay width into electrons can be calculated much more accurately and the number of allowed (light) neutrinos turns to be even closer to three.

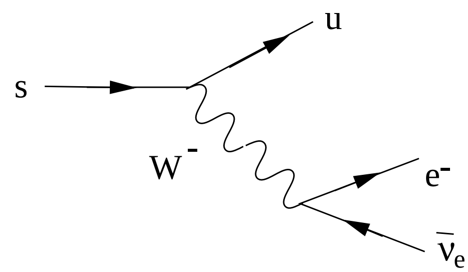
The masses of the fermions and the coupling to the Higgs particle are contained in $\mathcal{L}^{(h2)}$. With the choosen v.e.v. for the Higgs field, one obtains

$$\begin{aligned} \mathcal{L}^{(h2)} &= -\frac{G_e\, v}{\sqrt{2}}\, (\bar{e}_L e_R + \bar{e}_R e_L) - \frac{G_e}{\sqrt{2}}\, (\bar{e}_L e_R + \bar{e}_R e_L)\, h \\ &= -m_e\, \bar{e} e - \frac{m_e}{v}\, \bar{e} e h. & (12.79) \end{aligned}$$

First, the mass of the electron comes from the spontaneous symmetry breaking but is not predicted (it is in the coupling G_e). The coupling to the Higgs particle is weak as the value for v calculated e.g. from the M_W mass is about 250 GeV, i.e. m_e/v is extremely small.

Finally we want to say something about the weak properties of the quarks, as appear for instance in the decay of the neutron or the decay of the Λ (quark content uds),





$n \longrightarrow p e^- \bar{\nu}_e \quad \Longleftrightarrow \quad d \longrightarrow u e^- \bar{\nu}_e,$ $\qquad \Lambda \longrightarrow p e^- \bar{\nu}_e \quad \Longleftrightarrow \quad s \longrightarrow u e^- \bar{\nu}_e.$

The quarks also turn out to fit into doublets of $SU(2)_W$ for the lefthanded species and into singlets for the righthanded singlets. A complication arises as it are not the 'mass' eigenstates that appear in the weak isospin doublets but linear combinations of them,

$$\begin{pmatrix} u \\ d' \end{pmatrix} \quad \begin{pmatrix} c \\ s' \end{pmatrix} \quad \begin{pmatrix} t \\ b' \end{pmatrix},$$

where

$$\begin{aligned} \begin{bmatrix} d' \\ s' \\ b' \end{bmatrix} &= \begin{bmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{bmatrix} \begin{bmatrix} d \\ s \\ b \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta_C & \sin\theta_C & \sim 0.004 \\ -\sin\theta_C & \cos\theta_C & \sim 0.04 \\ \sim 0.01 & \sim 0.04 & \sim 0.999 \end{bmatrix} \begin{bmatrix} d \\ s \\ b \end{bmatrix}. \end{aligned}$$

This mixing allows all quarks with $I_W^3 = -1/2$ to decay into an up quark, but with different strength. Comparing neutron decay and Λ decay one can get an estimate of the mixing parameter V_{us} in the socalled Cabibbo-Kobayashi-Maskawa mixing matrix, $\sin\theta_C \approx 0.221$ (do not confuse with the value of $\sin^2\theta_W$). In principle one complex phase is allowed in the most general form of the CKM matrix, which can account for the (observed) CP violation of the weak interactions. This is only true if the mixing matrix is at least three-dimensional, i.e. CP violation requires three generations.

12.5 Exercises

Exercise 12.1
Consider the case of the Weyl mode for symmetries. Prove that if the generators Q^a generate a symmetry, i.e. $[Q^a, H] = 0$, and $|a\rangle$ and $|a'\rangle$ belong to the same multiplet (there is a Q^a such that $|a'\rangle = Q^a|a\rangle$) then $H|a\rangle = E_a|a\rangle$ implies that $H|a'\rangle = E_a|a'\rangle$, i.e. a and a' are degenerate states.

Exercise 12.2
Derive for the vector and axial vector currents, $\vec{V}^\mu = \bar{\psi}\gamma^\mu\vec{T}\psi$ and $\vec{A}^\mu = \bar{\psi}\gamma^\mu\gamma_5\vec{T}\psi$

$$\begin{aligned} \partial_\mu \vec{V}^\mu &= i\,\bar{\psi}\,[M, \vec{T}]\,\psi, \\ \partial_\mu \vec{A}^\mu &= i\,\bar{\psi}\,\{M, \vec{T}\}\,\gamma_5\,\psi. \end{aligned}$$

Exercise 12.3

(a) The coupling of the Z^0 particle to fermions is described by the vertex

$$-i\,\frac{g}{2\cos\theta_W}\,\left(c_V^f\,\gamma^\mu - c_A^f\,\gamma^\mu\gamma_5\right),$$

with

$$c_V = I_W^3 - 2Q\,\sin^2\theta_W\,,$$
$$c_A = I_W^3\,.$$

Write down the matrix element squared (averaged over initial spins and summed over final spins) for the decay of the Z^0. Neglect the masses of fermions and use the fact that the sum over polarizations is

$$\sum_{\lambda=1}^{3} \epsilon_\mu^{(\lambda)}(p)\epsilon_\nu^{(\lambda)*}(p) = -g_{\mu\nu} + \frac{p_\mu p_\nu}{M^2}).$$

to calculate the width $?(Z^0 \to f\bar{f})$,

$$?(Z^0 \to f\bar{f}) = \frac{M_Z}{48\pi}\,\frac{g^2}{\cos^2\theta_W}\,\left(c_V^{f\,2} + c_A^{f\,2}\right).$$

(b) Calculate the width to electron-positron pair, $?(Z^0 \to e^+e^-)$, and the width to a pair of neutrino's, $?(Z^0 \to \nu_e\bar{\nu}_e)$. The mass of the Z^0 is $M_Z = 91$ GeV, the weak mixing angle is given by $\sin^2\theta_W = 0.2325$.

Exercise 12.4
Calculate the lifetime $\tau = 1/?$ for the top quark (t) assuming that the dominant decay mode is

$$\mathrm{t} \to \mathrm{b} + W^+.$$

In the standard model this coupling is described by the vertex

$$\frac{-i\,g}{2\sqrt{2}}\,(\gamma^\mu - \gamma^\mu\gamma_5)\,.$$

The masses are $m_t = 175$ GeV, $m_b = 5$ GeV and $M_W = 80$ GeV.