THE COMPLEX INVERSION FORMULA REVISITED

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ABSTRACT. We give a simplified proof of the complex inversion formula for semigroups and — more generally — solution families for scalar-type Volterra equations, including the stronger versions on UMD spaces. Our approach is based on (elementary) Fourier analysis.

1. INTRODUCTION

In this paper we are concerned with the following question: Let X, Y be Banach spaces and let $S : [0, \infty) \longrightarrow \mathcal{L}(X, Y)$ be a strongly continuous mapping of finite exponential type $\omega_0(S)$. In what sense and under what conditions does the complex inversion inversion formula

$$\lim_{N\to\infty} \frac{1}{2\pi i} \int_{\omega-iN}^{\omega+iN} e^{tz}(\mathcal{L}S)(z)\, dz = S(t) \qquad (t > 0) \tag{1.1}$$

hold true? (Here $\omega > \omega_0(S)$ is fixed and $\mathcal{L}S$ denotes the Laplace transform of S).

Actually we are interested in the case that S is a solution family to a scalar-type Volterra equation (see Section 4 below), in particular that S is a C_0-semigroup. However, as in [1, Theorem 2.3.4] we do not confine to these applications but start very generally. (Of course, one could drop continuity from the assumptions on S towards weaker assumptions, but this is of secondary importance.)

Theorem 2.3.4 from [1] states that (1.1) holds in an "integrated form". From this one can then derive the standard result on semigroups (strong convergence on the domain of the generator). In the paper [3] DRIOUICH and EL-MENNAOUI showed that in case that X has the UMD property the convergence is strong on all of X. This was subsequently generalised from semigroups to solution families for scalar-type Volterra equations by CIORANESCU and LIZAMA in [2].

The aim of the present paper is to present new and much shorter proofs of these results, eventually even generalising them. Our approach uses some elementary Fourier analysis and has the advantage that the recent "UMD-results" become at least as simple as the classical ones, if not simpler. The results obtained are also more specific than the existing ones with respect to what happens with the approximation for small times (compare, e.g., Theorem 3.4 below with [4, Corollary III.5.15]).

One last remark before we go *medias in res*. All our results on the complex inversion formula remain true when we let the lower and the upper bound of the integral in (1.1) tend to infinity independently. One has to replace the Dirichlet kernel in our discussion by a somewhat more complicated expression, but it won't change the proofs apart from notation.

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Preliminary remarks and notation
Here and in the following, X, Y, Z always denote Banach spaces. The symbol $\mathbf{1}$ is used to denote characteristic function of the positive real axis, i.e. $\mathbf{1} = \chi_{[0,\infty)}$. So $\mathbf{1}' = \delta_0$ in the distributional sense, where δ_0 is the Dirac measure at 0. We write simply t to denote the real coordinate $(t \longmapsto t)$. All functions that live on $[0, \infty)$ are tacitly extended to $\mathbb{R}$ by 0 on $(-\infty, 0)$. For a mapping $S : [0, \infty) \longrightarrow \mathcal{L}(X, Y)$ and $\omega \in \mathbb{R}$ we define its *exponential shift* S_ω by

$$S_\omega(t) := e^{-\omega t} S(t) \qquad (t \geq 0).$$

The *exponential type* of S is

$$\omega_0(S) := \inf\{\omega \in \mathbb{R} \mid \exists\, M \geq 0 : \|S(t)\| \leq M e^{\omega t} \ (t \geq 0)\}.$$

If S is strongly measurable and of finite exponential type, we denote by

$$(\mathcal{L}S)(z) := \text{strong} - \int_0^\infty e^{-zt}\, S(t)\, dt \qquad (\operatorname{Re} z > \omega_0(f))$$

its *Laplace transform.* If $S : [0, \infty) \longrightarrow \mathcal{L}(X, Y)$ and $T : [0, \infty) \longrightarrow \mathcal{L}(Y, Z)$ are both strongly measurable and of finite exponential type, then the *convolution* $S*T : [0, \infty) \longrightarrow \mathcal{L}(X, Y)$ given by

$$(T * S)(t)x := \int_0^t T(t - s)S(s)x\, ds \qquad (x \in X)$$

is well-defined, strongly continuous and of finite exponential type; furthermore, one has

$$(T * S)_\omega = T_\omega * S_\omega \quad \text{and} \quad \mathcal{L}(T * S) = (\mathcal{L}T)(\mathcal{L}S). \tag{1.2}$$

Besides this type of convolution we will encounter (in Section 4) $\mu*S$, where $S : [0, \infty) \longrightarrow \mathcal{L}(X, Y)$ is strongly continuous and μ is a locally finite complex Borel measure on $[0, \infty)$. The convolution $\mu * S : [0, \infty) \longrightarrow \mathcal{L}(X, Y)$ is then given as

$$(\mu * S)(t)x := \int_0^t S(t - s)x\, \mu(ds) \qquad (x \in X)$$

and is again strongly continuous. With the obvious definition of μ_ω we have $(\mu * S)_\omega = \mu_\omega * S_\omega$; if μ_ω happens to be a bounded measure, $\mathcal{L}\mu$ is defined in the obvious way, and one has $\mathcal{L}(\mu * S) = (\mathcal{L}\mu)(\mathcal{L}S)$.

A third situation envolves functions on the whole real line, and is described in the following. A strongly measurable mapping $S : \mathbb{R} \longrightarrow \mathcal{L}(X, Y)$ is said to be *strongly-*$\mathbf{L^2}$, if $S(t)x \in \mathbf{L^2}(\mathbb{R}; Y)$ for every $x \in X$. By the closed graph theorem one then has

$$\|S\|_2 := \sup_{x \in X, \|x\| \leq 1} \|S(t)x\|_{\mathbf{L^2}(\mathbb{R};Y)} < \infty.$$

The mapping S is said to be *uniformly-*$\mathbf{L^2}$ if there exists a function $0 \leq g \in \mathbf{L^2}(\mathbb{R})$ such that

$$\|S(t)x\|_Y \leq g(t)\, \|x\|_X \qquad (t \in \mathbb{R},\, x \in X).$$

The function g is said to be a scalar majorant of S. We will have occasion to use the following form of Young's inequality. The proof is trivial.

Lemma 1.1. (Young's inequality)
Let X, Y be Banach spaces, let $S : \mathbb{R} \longrightarrow \mathcal{L}(X, Y)$ be strongly-$\mathbf{L^2}$, and let $T : \mathbb{R} \longrightarrow \mathcal{L}(Y, Z)$ be uniformly-$\mathbf{L^2}$ with scalar majorant g. Then the convolution $T * S$ defined by

$$(T * S)(t)x := \int_{\mathbb{R}} T(t - s)S(s)x\, ds \qquad (x \in X,\, t \in \mathbb{R})$$

exists and satisfies $(T * S) \in \mathbf{C_0}(\mathbb{R}; \mathcal{L}_s(X, Z))$ *with*

$$\sup_{t \geq 0} \|(T * S)(t)\|_{\mathcal{L}(X,Y)} \leq \|g\|_2 \|S\|_2 . \tag{1.3}$$

One may choose $Y = Z$ and $T(s) = f(s)I$, $g(t) = |f(t)|$ in the lemma, so the estimate (1.3) becomes

$$\sup_{t \geq 0} \|(f * S)(t)\|_{\mathcal{L}(X,Y)} \leq \|f\|_2 \|S\|_2 ,$$

and this shows that with S fixed the mapping

$$(f \longmapsto f * S) : \mathbf{L^2}(\mathbb{R}) \longrightarrow \mathbf{C_0}(\mathbb{R}; \mathcal{L}(X, Y))$$

is continuous. On the other hand, if we choose $X = \mathbb{C}$, then (1.3) shows that with fixed T the mapping

$$(f \longmapsto T * f) : \mathbf{L^2}(\mathbb{R}; Y) \longrightarrow \mathbf{C_0}(\mathbb{R}; Z)$$

is continuous.

For $N > 0$ we denote by D_N the *Dirichlet kernel*, i.e.

$$D_N(t) := \frac{\sin(Nt)}{\pi t} \qquad (t \in \mathbb{R}).$$

Then, as is well known (or by a short computation)

$$D_N * f = \frac{1}{2\pi} \int_{-N}^{N} e^{ist} \widehat{f}(s)\, ds$$

where f is integrable and $\widehat{f}$ denotes its Fourier transform. (The function f may be vector- or operator-valued, of course.) The following is an easy consequence of Plancherel's theorem.

Lemma 1.2. *Let* $f \in \mathbf{L^2}(\mathbb{R})$. *Then* $D_N * f \to f$ *in* $\mathbf{L^2}(\mathbb{R})$ *as* $N \to \infty$.

2. General Laplace transforms

Let X, Y be Banach spaces and let $S : [0, \infty) \longrightarrow \mathcal{L}(X, Y)$ be a strongly continuous mapping of finite exponential type $\omega_0(S)$. Note that

$$\begin{aligned} K_N(t) &:= \frac{1}{2\pi i} \int_{\omega - iN}^{\omega + iN} e^{tz} (\mathcal{L}S)(z)\, dz \\ &= e^{\omega t} \frac{1}{2\pi} \int_{-N}^{N} e^{ist} (\mathcal{L}S)(\omega + is)\, ds = e^{\omega t} (D_N * S_\omega)(t) \end{aligned} \tag{2.1}$$

whence $(K_N)_\omega = D_N * S_\omega$. If we replace S by $a * S$ with a scalar function a we arrive at our first result.

Proposition 2.1. *Let* X, Y *be Banach spaces, let* $S : [0, \infty) \longrightarrow \mathcal{L}(X, Y)$ *be strongly continuous, and let* $a \in \mathbf{L^1_{loc}}[0, \infty)$ *be a scalar functions, both* a *and* S *of finite exponential type. Then for every* $\omega > \omega_0(S), \omega_0(a)$ *one has*

$$\lim_{N \to \infty} \frac{1}{2\pi i} \int_{\omega - iN}^{\omega + iN} e^{tz} \mathcal{L}(a * S)(z)\, dz = (a * S)(t)$$

in $\mathcal{L}(X, Y)$, *uniformly in* t *from compact subsets of* $[0, \infty)$.

Proof. Replace S by $a * S$ in (2.1) to obtain

$$e^{-\omega t} K_N(t) = D_N * (a * S)_\omega = D_N * a_\omega * S_\omega$$

by (1.2). Now $D_N * a_\omega \to a_\omega$ in $\mathbf{L^2}(\mathbb{R})$ and hence, by Young's inequality, $D_N * a_\omega * S_\omega \to a_\omega * S_\omega$ in $\mathcal{L}(X, Y)$, uniformly in $t \geq 0$. Multiplying everything by $e^{\omega t}$ concludes the proof. □

Proposition 2.1 does not quite cover [1, Theorem 2.3.4]; however, with some more terminological effort the same proof would work. In any case, Proposition 2.1 will suffice for the applications we have in mind, and it is certainly more general than [2, Lemma 5], where the authors need $a \in \mathbf{C^1}$ and assert only strong convergence and uniformity only in t from compact subsets of $(0, \infty)$.

We would like to point out that we do not claim Proposition 2.1 to be new, although it might be (as we do not know of a reference). Our emphasis is on the idea of the proof, which can be put as follows. The complex inversion formula is nothing else but the convergence of the partial inverse Fourier transforms. In a first step one establishes $\mathbf{L^2}$-convergence; then a convolution with another $\mathbf{L^2}$-term yields uniform convergence to something which — with some luck — is just a weighted form of what one is interested in.

This idea will in the following be applied to the case of the complex inversion formula in its bare (i.e., non-integral) form. The following observation will also be helpful.

Lemma 2.2. *Let X, Y be a Banach space, let $S : [0, \infty) \longrightarrow \mathcal{L}(X, Y)$ be strongly continuous of finite exponential type, and let $\omega > \omega_0(S)$. Then*

$$t(D_N * S_\omega) \sim (D_N * [tS(t)]_\omega)$$

by which is meant that for each $x \in X$

$$\lim_{N\to\infty} \Big[t(D_N * S_\omega) - (D_N * [tS(t)]_\omega) \Big] x = 0$$

uniformly in $t \geq 0$.

Proof. We perform integration by parts to obtain

$$\frac{1}{2\pi} \int_{-N}^{N} t e^{ist} \widehat{S_\omega}(s)\, ds = \frac{1}{2\pi i} e^{its} \widehat{S_\omega} \Big|_{s=-N}^{s=N} - \frac{1}{2\pi i} \int_{-N}^{N} e^{ist} \widehat{S_\omega}'(s)\, ds,$$

and clearly $\widehat{S_\omega}' = -i\widehat{[tS(t)]_\omega}$. The statement now follows from the Riemann-Lebesgue Lemma. □

Remark 2.3. Here is a curious remark. The proof of [1, Theorem 2.3.4] relies on [1, Lemma 2.3.5] which states that

$$\frac{1}{2\pi i} \int_{\omega - iN}^{\omega + iN} \frac{e^{zt}}{z} e_{-z}\, dz \to \chi_{[0,t]} \qquad (N \to \infty, t > 0, \omega > 0)$$

in the Banach space $X = \mathbf{L^1}[0, \infty)$ where $e_{-z} = e^{-z\cdot}$. This in fact follows from Proposition 2.1: Let S be the right shift semigroup on X and $a = \mathbf{1}$. Then $(\mathbf{1} * S)(t)$ is convolution with $\chi_{[0,t]}$ and $(\mathcal{L}S)(z)$ is convolution with e_{-z}, as is easily seen. But the $\mathcal{L}(X)$-norm of the operator "convolution with f" equals the $\mathbf{L^1}$-norm of f (see [5] for the easy proof).

3. Semigroups

In this section we apply the results of the previous section to C_0-semigroups. Although it is a special case of the situation considered in Section 4, it is worthwhile to deal with the semigroup case first. We begin with the complex inversion formula in its integral form, cp. [4, Theorem III.5.14].

Theorem 3.1. *Let A generate a bounded C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X. Then for every $\epsilon > 0$,*

$$\int_0^t T(s)\,ds = \lim_{N\to\infty} \frac{1}{2\pi i}\int_{\epsilon-iN}^{\epsilon+iN} \frac{e^{zt}}{z} R(z,A)\,dz \tag{3.1}$$

in norm, the convergence being uniform in t from bounded intervals of $[0,\infty)$.

Proof. This follows from Proposition 2.1 with $X = Y$, $S = T$ and $a = \mathbf{1}$. □

Now we pass to the plain form of the inversion formula. Recall that for any C_0-semigroup T on a Banach space X one has

$$(T * T)(t) = \int_0^t T(t-s)T(s)\,ds = tT(t) \qquad (t \geq 0) \tag{3.2}$$

The definition of K_N from (2.1) in this context reads

$$K_N(t) := \frac{1}{2\pi i}\int_{\omega-iN}^{\omega+iN} e^{tz} R(z,A)\,dz \tag{3.3}$$

where A is the generator of T. Here is the result on strong convergence.

Theorem 3.2. *Let A be the generator of a C_0-semigroup T on the Banach space X, let $\omega > \omega_0(T)$, and define K_N by (3.3). Suppose that X is a UMD space. Then for every $x \in X$*

$$tK_N(t)x \to tT(t)x \qquad \text{as } N \to \infty$$

uniformly in t from bounded subintervals of $[0,\infty)$.

Proof. By (2.1) one has $K_N = e^{\omega t}(D_N * T_\omega)$. Now Lemma 2.2 and (3.2) yield

$$e^{-\omega t} tK_N = t(D_N * T_\omega) \sim D_N * [tT(t)]_\omega = D_N * T_\omega * T_\omega = T_\omega * (D_N * T_\omega).$$

Hence by Young's inequality it suffices to show that for every $x \in X$ one has $D_N * T_\omega x \to T_\omega x$ in $\mathbf{L^2}(\mathbb{R}; X)$. Now, consider the operator family $L_N := (f \mapsto D_N * f)$, $N > 1$ on $\mathbf{L^2}(\mathbb{R}; X)$. Since D_N is essentially the difference of two shifted Hilbert transforms, the UMD property yields that $\sup_N \|L_N\|_{\mathcal{L}(\mathbf{L^2}(\mathbb{R};X))} < \infty$. The space $\mathbf{L^2}(\mathbb{R}) \otimes X$ is dense in $\mathbf{L^2}(\mathbb{R}; X)$, whence $L_N \to I$ strongly. Hence indeed $D_N * T_\omega x = L_N(T_\omega x) \to T_\omega x$ in $\mathbf{L^2}(\mathbb{R}; X)$, and this is what we needed. □

Applying Theorem 3.2 with $A = 0$ and $X = \mathbb{C}$ yields the following.

Corollary 3.3. *Let $\omega > 0$. Then $\lim_{N\to\infty} t(D_N * \mathbf{1}_\omega) = t\mathbf{1}_\omega$ uniformly in $t \geq 0$.*

Without X being a UMD space, the theorem cannot be true, the canonical counterexample being the shift semigroup on $\mathbf{L^1}(\mathbb{R})$, see [1, Example 3.12.3]. The classical result ([1, Proposition 3.12.1], [4, Corollary III.5.15]) is that one always has strong convergence if $x \in \mathcal{D}(A)$. We aim at showing that one has actually convergence in the norm of $\mathcal{L}(X, X_{-1})$, where X_{-1} is the first extrapolation space (cf. [4, Section II.5])

Theorem 3.4. *Let A be the generator of a C_0-semigroup T on the Banach space X. Take $\omega > \omega_0(T)$ and define K_N by (3.3). Let $\lambda \in \varrho(A)$ be arbitrary. Then*

$$\lim_{N\to\infty} tK_N(t)R(\lambda, A) \;=\; tT(t)R(\lambda, A)$$

in norm, uniformly in t from bounded subintervals of $[0, \infty)$.

Proof. By shifting the generator, we can assume that $\omega_0(T) \geq 0$, so $\omega > 0$. We abbreviate $R := R(\lambda, A), C := AR(\lambda, A)$. The fundamental formula for semigroups reads $T - I = A(\mathbf{1} * T)$. Multiplying this by R from the right yields

$$TR = R + (\mathbf{1} * T)C.$$

So

$$e^{-\omega t}K_N R = D_N * T_\omega R = (D_N * \mathbf{1}_\omega)R + (D_N * \mathbf{1}_\omega * T_\omega)C.$$

By the now well-known arguments, the second summand tends to $\mathbf{1}_\omega * T_\omega C = [\mathbf{1} * TC]_\omega$ uniformly in $t \geq 0$. By Corollary 3.3 we know that $t(D_N * \mathbf{1}_\omega) \to t\mathbf{1}_\omega$ uniformly in $t \geq 0$. To sum up, we obtain

$$\lim_{N\to\infty} tK_N R \;=\; e^{\omega t}\big(t\mathbf{1}_\omega R + t[\mathbf{1} * T]_\omega C\big) \;=\; t\big(R + (\mathbf{1} * T)C\big) \;=\; tT(t)R$$

in norm, and the convergence is uniform on bounded subintervals of $[0, \infty)$. □

Remark 3.5. (Integrated semigroups)
One may ask for analogues of Theorems 3.2 and 3.4 for integrated semigroups. If A generates an α-times integrated semigroup S, say, one has the relation

$$S = \frac{t^\alpha}{\Gamma(\alpha+1)}\mathbf{1} + A(\mathbf{1} * S).$$

But $t^\alpha/\Gamma(\alpha+1) = \mathbf{1} * [t^{\alpha-1}/\Gamma(\alpha)]$, and so the proof of Theorem 3.4 takes over to give

$$\lim_{N\to\infty} tK_N(t)R(\lambda, A) \;= tS(t)R(\lambda, A)$$

in norm, uniformly in t from bounded subintervals of $[0, \infty)$. However, apart from trivial cases we do not see how we could prove the analogue of Theorem 3.2.

4. Volterra equations

The previous results on semigroups are only special cases of a more general theorem on (scalar-type) Volterra equations. In this case one is given a function $a \in \mathbf{L}^1_{\mathbf{loc}}[0, \infty)$ and one considers the abstract Volterra equation

$$u = x + A(a * u) \qquad (x \in X). \tag{4.1}$$

The well-posedness of this equation corresponds to the existence of a strongly continuous *solution family* $(S(t))_{t\geq 0}$ satisfying

$$\int_0^t a(t-s)S(s)x\,ds \in \mathcal{D}(A) \quad \text{and} \quad S(t)x - x = A\int_0^t a(t-s)S(s)x\,ds$$

for every $x \in X, t \geq 0$. In short notation, this means just

$$S - I = A(a * S).$$

In case $a \equiv 1$, S is a semigroup. It is convenient (and generally done so) to assume that a and S are of finite exponential type $\omega_0 \geq 0$. In that case S and a are Laplace transformable and

$$H(z) := \mathcal{L}(S)(z) = R(z, z(\mathcal{L}a)(z)A) \qquad (\operatorname{Re} z > \omega_0).$$

As in the case of semigroups, one can ask under what conditions and in what sense the inversion of the Laplace transform converges to S. The definition of K_N now reads

$$K_N(t) := \frac{1}{2\pi i}\int_{\omega-iN}^{\omega+iN} e^{tz}H(z)\,dz, \tag{4.2}$$

where as usual $\omega > \omega_0$ is fixed. The following result is the exact generalisation of the corresponding result for semigroups.

Theorem 4.1. *Let a, S, A, H, K_N as above, and let $\lambda \in \varrho(A)$ be arbitrary. Then*

$$\lim_{N\to\infty} tK_N(t)R(\lambda, A) \;=\; tS(t)R(\lambda, A)$$

in norm, uniformly in t from bounded subintervals of $[0,\infty)$.

Proof. By definition of a solution family, one has $S = I + A(a*S)$, so $SR = R + (a*S)C$, where $R := R(\lambda, A)$ and $C := AR(\lambda, A)$. Therefore, the proof of Theorem 3.4 carries over almost literally. □

The analogue of Theorem 3.2 is not so easy to obtain, and in general will not hold without additional assumptions on a. The assumptions we make are of a technical kind, chosen to make our proof work. However, they are weaker and easier to verify than in the reference paper [2].

Theorem 4.2. *Let S and a as before. Suppose there exists $b \in \mathbf{L^1_{loc}}[0,\infty)$ such that the following holds:*

1) $(a*b)(t) = ta(t), \quad t \geq 0$;
2) $b_\omega \in \mathbf{L^2}(0,\infty)$;
3) $(b' * S)_\omega$ *is uniformly-*$\mathbf{L^2}$.

(Here, b' is the distributional derivative of b on $\mathbb{R}$.) Define

$$K_N(t) := \frac{1}{2\pi}\int_{-N}^{N} e^{t(is+\omega)}H(is+\omega)\,ds.$$

Then if X is a UMD space, $\lim_{N\to\infty} tK_N(t)x = tS(t)x$ uniformly in t from compact subsets of $[0,\infty)$, for each $x \in X$.

Before we give the proof, which is more or less along the lines of the semigroup case, let us comment on the theorem. Condition (1) says that $\mathcal{L}(b) = -\mathcal{L}(a)'/\mathcal{L}(a)$. The crucial point in applying the theorem is therefore to be able to recognize $\mathcal{L}(a)'/\mathcal{L}(a)$ as a Laplace transform. The second is no essential condition, as we might always choose ω large enough. The third is more delicate, as it imposes regularity on b. We need that $b' * S$ is meaningful, hence b' should be a Radon measure, i.e. $b \in \mathbf{BV_{loc}}[0,\infty)$. A feasible condition that implies (2) and (3) is that b_{ω_0} is bounded and $(b')_\omega$ is a bounded measure.

Example 4.3. Let $\alpha = 0$ or $\operatorname{Re}\alpha > 0$, $a(t) = t^\alpha$, $\omega > 0$. Then $b(t) \equiv \alpha + 1$ clearly satisfies (1) and (2). Moreover, $b' = (\alpha+1)\delta_0$ and so also (3) is satisfied. The case $\alpha = 0$ recovers the semigroup case; the case $\alpha = 1$ corresponds to S being a cosine function.

Example 4.3 also appears in [2, p.191]; however, in our case it is much easier to verify.

We now head for the *proof* of Theorem 4.2. As in the semigroup case we apply Lemma 2.2 and obtain

$$e^{-\omega t}tK_N = t(D_N * S_\omega) \;\sim\; D_N * [tS(t)]_\omega,$$

so we have to analyse $\mathcal{L}(tS(t)) = -H'$ further. Observe that $\mathbf{1} * b' = (\mathbf{1} * b)' = \mathbf{1}' * b = \delta_0 * b = b$ and hence

$$z(\mathcal{L}b)(z) = z\mathcal{L}(\mathbf{1} * b')(z) = z\frac{1}{z}(\mathcal{L}b')(z) = (\mathcal{L}b')(z) \qquad (\operatorname{Re} z > \omega_0).$$

Since $H(z) = R(z, z(\mathcal{L}a)(z)A)$, a little computation finally reveals that

$$\begin{aligned} H'(z) &= \Big(-\Big(\frac{1}{z} + \frac{(\mathcal{L}a)'(z)}{(\mathcal{L}a)(z)}\Big) + z\frac{(\mathcal{L}a)'(z)}{(\mathcal{L}a)(z)}H(z)\Big)H(z) \\ &= \Big(-(\mathcal{L}\mathbf{1})(z) + (\mathcal{L}b)(z) - z(\mathcal{L}b)(z)H(z)\Big)H(z), \end{aligned}$$

cp. [2, Lemma 1]. (We used that $(\mathcal{L}b)(\mathcal{L}a) = \mathcal{L}(ta(t)) = -(\mathcal{L}a)'$.) Hence

$$\begin{aligned} \mathcal{L}(tS(t)) = -H' &= \big(\mathcal{L}(\mathbf{1} - b) + \mathcal{L}(b' * S)\big)(\mathcal{L}S) \\ &= \mathcal{L}\big([\mathbf{1} - b + (b' * S)] * S\big), \end{aligned}$$

and by uniqueness of Laplace transforms, we obtain

$$tS(t) = [\mathbf{1} - b + (b' * S)] * S = C * S$$

with $C := [\mathbf{1} - b + (b' * S)]$. Thanks to hypotheses (2) and (3), C_ω is uniformly-$\mathbf{L^2}$, and so the same arguments as in the proof of Theorem 3.2 yield what we wanted to prove. □

Let us point out that Theorem 4.2 is an improvement in comparison to [2, Theorem 1]. There it was required that

1) $a(t)$ is 3-regular;
2) $\mathcal{L}a = O(|z|^{-1})$ for $|z| > 1$;
3) $z(\mathcal{L}a)'(z)/(\mathcal{L}a)(z)$ is locally analytic.

The authors do not specify the region of local analyticity, but from the proof it is clear that they mean locally analytic on $\mathbb{C}_+^\infty$. Since $(\mathcal{L}a)(z)$ does only exist for $\operatorname{Re} z > \omega_0$ the whole set of hypotheses seems a little strange; e.g., in their definition of 3-regular (taken from [6, Definition 7.3]) one considers functions living on $\mathbb{C}_+$. Anyway, with the help of our Theorem 4.2 we can relax hypotheses in the following way.

Corollary 4.4. *Let S be an exponentially bounded solution family for the scalar-type Volterra equation (4.1), and let $\omega_0 := \max\{\omega_0(a), \omega_0(S)\}$. Suppose that the function $z(\mathcal{L}a)'/(\mathcal{L}a)$ is locally analytic on $(\operatorname{Re} z > \omega_0) \cup \{\infty\}$. Then the hypotheses — and hence also the conclusion — of Theorem 4.2 hold true.*

Proof. Let $F(z) := z(\mathcal{L}a)'/(\mathcal{L}a)$. Then $F(z + \omega_0)$ is locally analytic on $\mathbb{C}_+^\infty$. So we may apply [6, Lemma 10.1] to conclude that there is a constant c and a function $g \in \mathbf{L^1}(0, \infty)$ such that $c + \mathcal{L}g = F(z + \omega_0)$, i.e., $F = c + \mathcal{L}g_{-\omega_0}$. Let $b := \mathbf{1} * g_{-\omega_0} + c\mathbf{1}$. Then $b_{\omega_0} = \mathbf{1}_{\omega_0} * g + c\mathbf{1}_{\omega_0}$ is a bounded function and $(b')_\omega = (g_{-\omega_0})_\omega + c\delta_0 = g_{\omega-\omega_0} + c\delta_0$ is a bounded measure. Hence conditions (2) and (3) of Theorem 4.2 are satisfied. Furthermore, $\mathcal{L}b = z^{-1}(\mathcal{L}g_{-\omega_0} + c) = z^{-1}F(z) = (\mathcal{L}a)'/(\mathcal{L}a)$, and this is condition (1). □

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