

ALGEBRAS FROM SURFACES WITHOUT PUNCTURES

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ABSTRACT. We introduce a new class of finite dimensional gentle algebras, the *surface algebras*, which are constructed from an unpunctured Riemann surface with boundary and marked points by introducing cuts in internal triangles of an arbitrary triangulation of the surface. We show that surface algebras are endomorphism algebras of partial cluster-tilting objects in generalized cluster categories, we compute the invariant of Avella-Alaminos and Geiss for surface algebras and we provide a geometric model for the module category of surface algebras.

1. INTRODUCTION

We introduce a new class of finite dimensional gentle algebras, the *surface algebras*, which includes the hereditary, the tilted, and the cluster-tilted algebras of Dynkin type \mathbb{A} and Euclidean type $\tilde{\mathbb{A}}$. These algebras are constructed from an unpunctured Riemann surface with boundary and marked points by introducing cuts in internal triangles of an arbitrary triangulation of the surface.

To be more precise, let T be a triangulation of a bordered unpunctured Riemann surface S with a set of marked points M , and let (Q_T, I_T) be the bound quiver associated to T as in [16, 3]. The corresponding algebra $B_T = kQ_T/I_T$, over an algebraically closed field k , is a finite dimensional gentle algebra [3]. Moreover, B_T is the endomorphism algebra of the cluster-tilting object corresponding to T in the generalized cluster category associated to (S, M) , see [16, 13, 1, 12]. Each internal triangle in the triangulation T corresponds to an oriented 3-cycle in the quiver Q_T , and the relations for the algebra B_T state precisely that the composition of any two arrows in an oriented 3-cycle is zero in B_T .

If the surface is a disc or an annulus then the corresponding cluster algebra, as defined in [20], is acyclic, and, in this case, the algebra B_T is cluster-tilted of type \mathbb{A} , if S is a disc; and of type $\tilde{\mathbb{A}}$, if S is an annulus [16, 14]. It has been shown in [4] that every cluster-tilted algebra is the trivial extension of a tilted algebra C by the canonical C -bimodule $\text{Ext}_C^2(DC, C)$, where D denotes the standard duality. The quiver of the tilted algebra C contains no oriented cycle and can be obtained from the quiver of B_T by an admissible cut, that is, by deleting one arrow in every oriented 3-cycle. Moreover, it

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has been shown in [10] that an algebra is iterated tilted of Dynkin type \mathbb{A} of global dimension at most two if and only if it is the quotient of a cluster-tilted algebra of the same type by an admissible cut. It is then natural to ask, what kind of algebras we can get from admissible cuts of algebras B_T coming from other surfaces.

This motivates the definition of a *surface algebra*, which is constructed by cutting a triangulation T of a surface (S, M) at internal triangles. Cutting an internal triangle Δ means replacing the triangle Δ by a quadrilateral Δ^\dagger with one side on the boundary of the same surface S with an enlarged set of marked points M^\dagger , see Definition 3.1. Cutting as many internal triangles as we please, we obtain a partial triangulation T^\dagger of a surface with marked points (S, M^\dagger) , to which we can associate an algebra $B_{T^\dagger} = kQ_{T^\dagger}/I_{T^\dagger}$ in a very similar way to the construction of B_T from T , see Definition 3.3. This algebra B_{T^\dagger} is called a *surface algebra of type (S, M)* . A surface algebra is called an *admissible cut* if it is obtained by cutting every internal triangle exactly once.

Our first main results are the following.

Theorem 1. *Every surface algebra is isomorphic to the endomorphism algebra of a partial cluster-tilting object in a generalized cluster category. More precisely, if the surface algebra B_{T^\dagger} is given by the cut $(S, M^\dagger, T^\dagger)$ of the triangulated surface (S, M, T) , then*

$$B_{T^\dagger} \cong \text{End}_{\mathcal{C}_{(S, M^\dagger)}} T^\dagger,$$

where T^\dagger denotes the object in cluster category $\mathcal{C}_{(S, M^\dagger)}$ corresponding to T^\dagger .

Theorem 2. *If $(S, M^\dagger, T^\dagger)$ is an admissible cut of (S, M, T) then*

- (a) Q_{T^\dagger} is an admissible cut of Q_T .
- (b) B_{T^\dagger} is of global dimension at most two.
- (c) The tensor algebra of B_{T^\dagger} with respect to the B_{T^\dagger} -bimodule

$$\text{Ext}_{B_{T^\dagger}}^2(DB_{T^\dagger}, B_{T^\dagger})$$

is isomorphic to the algebra B_T .

Part (c) of Theorem 2 implies that the cluster category associated to B_{T^\dagger} is the same as the cluster category associated to the surface (S, M) . Therefore, the surface algebras B_{T^\dagger} which are admissible cuts are algebras of cluster type (S, M) in the sense of [2]. In [2], the authors study algebras of global dimension two whose cluster type is acyclic, which, in our setting, corresponds to admissible cuts from a disc or an annulus.

Applying the result of [10] mentioned above, we see that the admissible cut surface algebras of the disc with $n + 3$ marked points are precisely the iterated tilted algebras of type \mathbb{A}_n whose global dimension is at most two. For all surfaces other than the disc, we show that the surface algebras form several different classes under derived equivalence. For admissible cuts from the annulus, this recovers a result in [2].

To show that the surface algebras are not always derived equivalent, we use an invariant introduced by Avella-Alaminos and Geiss in [9], which we call the AG-invariant for short. For each surface algebra B_{T^\dagger} , we compute the AG-invariant in terms of the original triangulated surface (S, M, T) and the number of cuts on each boundary component, see Theorem 4.6. In particular, we show that already for the annulus there are different cuts coming from the same triangulation T such that the corresponding surface algebras have different AG invariants and hence are not derived equivalent.

We want to point out here that Ladkani has very recently also computed the AG invariant for the algebras B_T (without cuts) in the special case where each boundary component has exactly one marked point, and used it to classify the surfaces (S, M) such that any two triangulations T_1, T_2 of (S, M) give rise to derived equivalent algebras B_{T_1} and B_{T_2} , see [24]. Let us also mention that the AG-invariant has been used by Bobinski and Buan in [11] to classify algebras that are derived equivalent to cluster-tilted algebras of type \mathbb{A} and $\tilde{\mathbb{A}}$.

We then study the module categories of the surface algebras. Since surface algebras are gentle, their indecomposable modules are either string modules or band modules. In the special case where the surface is not cut, the indecomposable modules and the irreducible morphisms in the module category of the algebras B_T have been described by Brüstle and Zhang in [12] in terms of the of generalized arcs on the surface (S, M, T) . One of the main tools used in [12] is the description of irreducible morphisms between string modules by Butler and Ringel [15]. We generalize the results of Brüstle and Zhang to the case of arbitrary surface algebras B_{T^\dagger} , and we describe the indecomposable modules in terms of certain permissible generalized arcs in the surface $(S, M^\dagger, T^\dagger)$ and the irreducible morphisms in terms of pivots of these arcs in the surface. In this way, we construct a combinatorial category \mathcal{E}^\dagger of permissible generalized arcs in $(S, M^\dagger, T^\dagger)$ and define a functor

$$H: \mathcal{E}^\dagger \rightarrow \text{mod } B_{T^\dagger}.$$

This construction is inspired by the construction of the cluster category of type \mathbb{A} as a category of diagonals in a convex polygon by Caldero, Chapoton and the second author in [16]. We then show the following theorem.

Theorem 3.

- (a) The functor H is faithful and induces a dense, faithful functor from \mathcal{E}^\dagger to the category of string modules over B_{T^\dagger} . Moreover, H maps irreducible morphisms to irreducible morphisms and commutes with Auslander-Reiten translations.
- (b) If the surface S is a disc, then H is an equivalence of categories.
- (c) If the algebra B_{T^\dagger} is of finite representation type, then H is an equivalence of categories.

As an application of our results, we provide a geometric model for the module category of any iterated tilted algebra of type \mathbb{A} of global dimension two in terms of permissible diagonals in a partially triangulated polygon.

The paper is organized as follows. In section 2, we recall definitions and results that we need at a later stage. Section 3 is devoted to the definition of surface algebras and their fundamental properties. The computation of the AG-invariant is contained in section 4 and, in section 5, we study the module categories of surface algebras in terms of arcs in the surface. The application to iterated tilted algebras of type \mathbb{A} is presented in section 6.

2. PRELIMINARIES AND NOTATION

In this section, we recall concepts that we need and fix notations.

2.1. Gentle algebras, string modules and band modules. Recall from [7] that a finite-dimensional algebra B is *gentle* if it admits a presentation $B = kQ/I$ satisfying the following conditions:

- (G1) At each point of Q start at most two arrows and stop at most two arrows.
- (G2) The ideal I is generated by paths of length 2.
- (G3) For each arrow b there is at most one arrow a and at most one arrow c such that $ab \in I$ and $bc \in I$.
- (G4) For each arrow b there is at most one arrow a and at most one arrow c such that $ab \notin I$ and $bc \notin I$.

An algebra $B = kQ/I$ where I is generated by paths and (Q, I) satisfies the two conditions (G1) and (G4) is called a *string algebra* (see [15]), thus every gentle algebra is a string algebra. Butler and Ringel have shown in [15] that, for a finite dimensional string algebra, there are two types of indecomposable modules, the string modules and the band modules. We recall the definitions here.

For any arrow $b \in Q_1$, we denote by b^{-1} a *formal inverse* for b , with $s(b^{-1}) = t(b)$, $t(b^{-1}) = s(b)$, and we set $(b^{-1})^{-1} = b$.

A *walk* of length $n \geq 1$ in Q is a sequence $w = a_1 \cdots a_n$ where each a_i is an arrow or a formal inverse of an arrow and such that $t(a_i) = s(a_{i+1})$, for any $i \in \{1, \dots, n-1\}$. The *source of the walk* w is $s(w) = s(a_1)$ and the *target of the walk* w is $t(w) = t(a_n)$. We define a *walk e_i of length zero* for any point $i \in Q_0$ such that $s(e_i) = t(e_i) = i$.

If (Q, I) is a bound quiver, a *string* in (Q, I) is either a walk of length zero or a walk $w = a_1 \cdots a_n$ of length $n \geq 1$ such that $a_i \neq a_{i+1}^{-1}$ for any $i \in \{1, \dots, n-1\}$ and such that no walk of the form $a_i a_{i+1} \cdots a_t$ nor its inverse belongs to I for $1 \leq i$ and $t \leq n$. A *band* is a string $b = a_1 \cdots a_n$ such that $s(b) = t(b)$, and any power of b is a string, but b itself is not a proper power of a string.

Any string w gives rise to a string module $M(w)$ over B , whose underlying vector space consists of the direct sum of one copy of the field k for each vertex in the string w , and the action of an arrow a on $M(w)$ is induced by

the identity morphism $1: k \rightarrow k$ between the copies of k at the endpoints of a , if a or a^{-1} is in w , and zero otherwise. Each band $b = a_1 a_2 \cdots a_n$ defines a family of band modules $M(b, \ell, \phi)$, where ℓ is a positive integer and ϕ is an automorphism of k^ℓ . The underlying vector space of $M(b, \ell, \phi)$ is the direct sum of one copy of k^ℓ for each vertex in b , and the action of an arrow a is induced by the identity morphism $1: k^\ell \rightarrow k^\ell$, if $a = a_1, a_2, \dots, a_{n-1}$, by the automorphism ϕ if $a = a_n$ and the action is zero if a is not in w .

2.2. The AG-invariant. First we recall from [9] the combinatorial definition of the derived invariant of Avella-Alaminos and Geiss. We will refer to this as the AG-invariant. Throughout let A be a gentle k -algebra with bound quiver (Q, I) , $Q = (Q_0, Q_1, s, t)$ where $s, t: Q_1 \rightarrow Q_0$ are the source and target functions on the arrows.

Definition 2.1. A *permitted path* of A is a path $C = a_1 a_2 \cdots a_n$ which is not in I . We say a permitted path is a *non-trivial permitted thread* of A if for all arrows $b \in Q_1$, neither bC or Cb is not a permitted path. These are the ‘maximal’ permitted paths of A . Dual to this, we define the *forbidden paths* of A to be a sequence $F = a_1 a_2 \cdots a_n$ such that $a_i \neq a_j$ unless $i = j$, and $a_i a_{i+1} \in I$, for $i = 1, \dots, n-1$. A forbidden path F is a *non-trivial forbidden thread* if for all $b \in Q_1$, neither bF or Fb is a forbidden path. We also require *trivial permitted* and *trivial forbidden threads*. Let $v \in Q_0$ such that there is at most one arrow starting at v and at most one arrow ending at v . Then the constant path e_v is a trivial permitted thread if when there are arrows $b, c \in Q_1$ such that $s(c) = v = t(b)$ then $bc \notin I$. Similarly, e_v is a trivial forbidden thread if when there are arrows $b, c \in Q_1$ such that $s(c) = v = t(b)$ then $bc \in I$.

Let \mathcal{H} denote the set of all permitted threads and \mathcal{F} denote the set of all forbidden threads.

Notice that each arrow in Q_1 , is both a permitted and a forbidden path. Moreover, the constant path at each sink and at each source will simultaneously satisfy the definition for a permitted and a forbidden thread because there are no paths going through v .

We fix a choice of functions $\sigma, \varepsilon: Q_1 \rightarrow \{-1, 1\}$ characterized by the following conditions.

- (1) If $b_1 \neq b_2$ are arrows with $s(b_1) = s(b_2)$, then $\sigma(b_1) = -\sigma(b_2)$.
- (2) If $b_1 \neq b_2$ are arrows with $t(b_1) = t(b_2)$, then $\varepsilon(b_1) = -\varepsilon(b_2)$.
- (3) If b, c are arrows with $s(c) = t(b)$ and $bc \notin I$, then $\sigma(c) = -\varepsilon(b)$.

Note that the functions need not be unique. Given a pair σ and ε , we can define another pair $\sigma' := -1\sigma$ and $\varepsilon' := -1\varepsilon$.

These functions naturally extend to paths in Q . Let $C = a_1 a_2 \cdots a_{n-1} a_n$ be a path. Then $\sigma(C) = \sigma(a_1)$ and $\varepsilon(C) = \varepsilon(a_n)$. We can also extend these functions to trivial threads. Let x, y be vertices in Q_0 , h_x the trivial permitted thread at x , and p_y the trivial forbidden thread at y . Then we

set

$$\begin{aligned}\sigma(h_x) &= -\varepsilon(h_x) = -\sigma(a), & \text{if } s(a) = x, \text{ or} \\ \sigma(h_x) &= -\varepsilon(h_x) = -\varepsilon(b), & \text{if } t(b) = x\end{aligned}$$

and

$$\begin{aligned}\sigma(p_y) &= \varepsilon(p_y) = -\sigma(c), & \text{if } s(c) = y, \text{ or} \\ \sigma(p_y) &= \varepsilon(p_y) = -\varepsilon(d), & \text{if } t(d) = y,\end{aligned}$$

where $a, b, c, d \in Q_1$. Recall that these arrows are unique if they exist.

Definition 2.2. The AG-invariant $\text{AG}(A)$ is defined to be a function depending on the ordered pairs generated by the following algorithm.

- (1) (a) Begin with a permitted thread of A , call it H_0 .
 (b) To H_i we associate F_i , the forbidden thread which ends at $t(H_i)$ and such that $\varepsilon(H_i) = -\varepsilon(F_i)$. Define $\varphi(H_i) := F_i$.
 (c) To F_i we associate H_{i+1} , the permitted thread which starts at $s(F_i)$ and such that $\sigma(F_i) = -\sigma(H_{i+1})$. Define $\psi(F_i) := H_{i+1}$.
 (d) Stop when $H_n = H_0$ for some natural number n . Define $m = \sum_{i=1}^n \ell(F_i)$, where $\ell(C)$ is the length (number of arrows) of a path C . In this way we obtain the pair (n, m) .
- (2) Repeat (1) until all permitted threads of A have occurred.
- (3) For each oriented cycle in which each pair of consecutive arrows form a relation, we associate the ordered pair $(0, n)$, where n is the length of the cycle.

We define $\text{AG}(A): \mathbb{N}^2 \rightarrow \mathbb{N}$ where $\text{AG}(A)(n, m)$ is the number of times the ordered pair (n, m) is formed by the above algorithm.

The algorithm defining $\text{AG}(A)$ can be thought of as dictating a walk in the quiver Q , where we move forward on permitted threads and backward on forbidden threads.

Remark 2.3. Note that the steps (1b) and (1c) of this algorithm give two different bijections φ and ψ between the set of permitted threads \mathcal{H} and the set of forbidden threads which do not start and end in the same vertex. We will often refer to the permitted (respectively forbidden) thread “corresponding” to a given forbidden (respectively permitted) thread. This correspondence is referring to the bijection φ (respectively ψ).

Theorem 2.4. (a) *Any two derived equivalent gentle algebras have the same AG-invariant.*
 (b) *Gentle algebras which have at most one (possibly non-oriented) cycle in their quiver are derived equivalent if and only if they have the same AG-invariant.*

Proof. See [9, Theorems A and C]. ■

2.3. Surfaces and triangulations. In this section, we recall a construction of [20] in the case of surfaces without punctures.

Let S be a connected oriented unpunctured Riemann surface with boundary ∂S and let M be a non-empty finite subset of the boundary ∂S . The elements of M are called *marked points*. We will refer to the pair (S, M) simply by *unpunctured surface*. The orientation of the surface will play a crucial role.

We say that two curves in S *do not cross* if they do not intersect each other except that endpoints may coincide.

Definition 2.5. An *arc* γ in (S, M) is a curve in S such that

- (a) the endpoints are in M ,
- (b) γ does not cross itself,
- (c) the relative interior of γ is disjoint from M and from the boundary of S ,
- (d) γ does not cut out a monogon or a digon.

Definition 2.6. A *generalized arc* is a curve in S which satisfies the conditions (a), (c) and (d) of Definition 2.5.

Curves that connect two marked points and lie entirely on the boundary of S without passing through a third marked point are called *boundary segments*. Hence an arc is a curve between two marked points, which does not intersect itself nor the boundary except possibly at its endpoints and which is not homotopic to a point or a boundary segment.

Each generalized arc is considered up to isotopy inside the class of such curves. Moreover, each generalized arc is considered up to orientation, so if a generalized arc has endpoints $a, b \in M$ then it can be represented by a curve that runs from a to b , as well as by a curve that runs from b to a .

For any two arcs γ, γ' in S , let $e(\gamma, \gamma')$ be the minimal number of crossings of γ and γ' , that is, $e(\gamma, \gamma')$ is the minimum of the numbers of crossings of curves α and α' , where α is isotopic to γ and α' is isotopic to γ' . Two arcs γ, γ' are called *non-crossing* if $e(\gamma, \gamma') = 0$. A *triangulation* is a maximal collection of non-crossing arcs. The arcs of a triangulation cut the surface into *triangles*. Since (S, M) is an unpunctured surface, the three sides of each triangle are distinct (in contrast to the case of surfaces with punctures). A triangle in T is called an *internal triangle* if none of its sides is a boundary segment. We often refer to the triple (S, M, T) as a *triangulated surface*.

Any two triangulations of (S, M) have the same number of elements, namely

$$n = 6g + 3b + |M| - 6,$$

where g is the genus of S , b is the number of boundary components and $|M|$ is the number of marked points. The number n is called the *rank* of (S, M) . Moreover, any two triangulations of (S, M) have the same number of triangles, namely

$$n - 2(g - 1) - b.$$

b	g	$ M $	surface
1	0	$n + 3$	disc
1	1	$n - 3$	torus with disc removed
1	2	$n - 9$	genus 2 surface with disc removed
2	0	n	annulus
2	1	$n - 6$	torus with 2 discs removed
2	2	$n - 12$	genus 2 surface with 2 discs removed
3	0	$n - 3$	pair of pants

TABLE 2.1. Examples of unpunctured surfaces

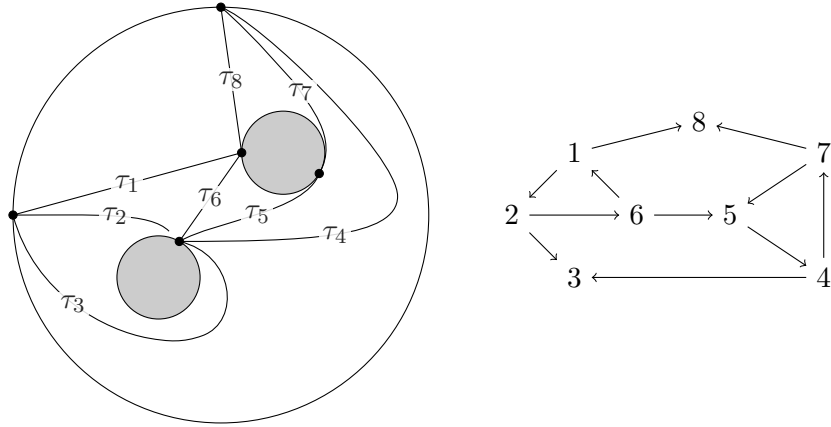


FIGURE 2.1. A triangulation and its quiver

Note that $b > 0$ since the set M is not empty. We do not allow n to be negative or zero, so we have to exclude the cases where (S, M) is a disc with one, two or three marked points. Table 2.1 gives some examples of unpunctured surfaces.

2.4. Jacobian algebras from surfaces. If $T = \{\tau_1, \tau_2, \dots, \tau_n\}$ is a triangulation of an unpunctured surface (S, M) , we define a quiver Q_T as follows. Q_T has n vertices, one for each arc in T . We will denote the vertex corresponding to τ_i simply by i . The number of arrows from i to j is the number of triangles Δ in T such that the arcs τ_i, τ_j form two sides of Δ , with τ_j following τ_i when going around the triangle Δ in the counter-clockwise orientation, see Figure 2.1 for an example. Note that the interior triangles in T correspond to oriented 3-cycles in Q_T .

In [20], the authors associate a cluster algebra $\mathcal{A}(Q_T)$ to this quiver; the cluster algebras obtained in this way are called cluster algebras from (unpunctured) surfaces and have been studied in [20, 21, 26, 25, 18], and the corresponding cluster categories in [16, 12].

Following [3, 23], let W be the sum of all oriented 3-cycles in Q_T . Then W is a potential, in the sense of [17], which gives rise to a Jacobian algebra $B_T = \text{Jac}(Q_T, W)$, which is defined as the quotient of the path algebra of the quiver Q_T by the two-sided ideal generated by the subpaths of length two of each oriented 3-cycle in Q_T .

Proposition 2.7. *B_T is a gentle algebra.*

Proof. This is shown in [3]. ■

2.5. Cuts and admissible cuts. Let Q be a quiver. An oriented cycle C in Q is called *chordless* if C is a full subquiver of Q and has the property that for every vertex v in C , there is exactly one arrow in C starting and one arrow in C ending at v . We define a *cut* of Q to be a subset of the set of arrows of Q with the property that each arrow in the cut lies in an oriented chordless cycle in Q . Following [19], such a cut is called *admissible* if it contains exactly one arrow of each oriented chordless cycle of Q .

Now let $C = kQ/I$ be a quotient of the path algebra of Q by an admissible ideal I . Then an algebra is said to be obtained from C by a cut if it is isomorphic to a quotient $kQ/\langle I \cup \Gamma \rangle$, where Γ is a cut of Q .

Proposition 2.8. *Any algebra obtained by a cut from a gentle algebra is gentle.*

Proof. If a bound quiver satisfies the conditions (G1)–(G4) of the definition of a gentle algebra, then deleting an arrow will produce a bound quiver that still satisfies these conditions. ■

3. ALGEBRAS FROM SURFACES WITHOUT PUNCTURES

In this section, we introduce the surface algebras and develop their fundamental properties.

Let (S, M) be a surface without punctures, T a triangulation, Q_T the corresponding quiver, and B_T the Jacobian algebra. Throughout this section, we assume that, if S is a disc, then M has at least 5 marked points, thus we exclude the disc with 4 marked points. Recall that the oriented 3-cycles in the quiver Q_T are in bijection with the interior triangles in the triangulation T .

3.1. Cuts of triangulated surfaces. We want to define a geometric object which corresponds to cuts of the quiver Q_T , and, to that purpose, we modify the set M to a new set of marked points M^\dagger , and we modify the triangulation T to a partial triangulation T^\dagger of the surface (S, M^\dagger) .¹

We need some terminology. Each arc has two ends defined by trisecting the arc and deleting the middle piece. If Δ is a triangle with sides α, β, γ , then the six ends of the three sides can be matched into three pairs such that each pair forms one of the angles of the triangle Δ .

¹We use the dagger symbol \dagger to indicate the cut.

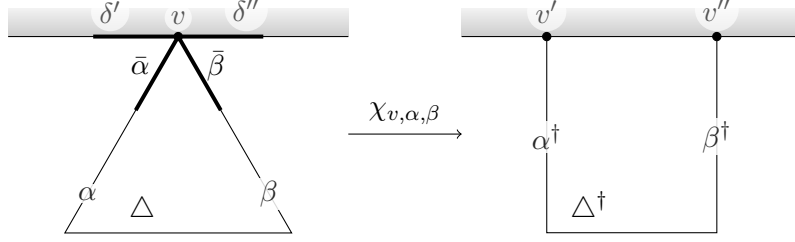


FIGURE 3.1. A local cut at v relative to α, β . The internal triangle Δ in T becomes a quasi-triangle Δ^\dagger in T^\dagger .

Let Δ be an internal triangle of T and let $v \in M$ be one of its vertices. Let δ' and δ'' be two curves on the boundary, both starting at v but going in opposite directions, and denote by v' and v'' their respective endpoints. Moreover, choose δ', δ'' short enough such that v' and v'' are not in M , and no point of M other than v lies on the curves δ', δ'' . We can think of v', v'' being obtained by moving the point v a small amount in either direction along the boundary, see Figure 3.1 for an example. Define

$$\chi_{v, \Delta}(M) = (M \setminus \{v\} \cup \{v', v''\}).$$

Let $\bar{\alpha}$ and $\bar{\beta}$ be ends of two sides α, β of Δ such that $\bar{\alpha}, \bar{\beta}$ form an angle of Δ at v . If $\bar{\gamma}$ is an end of an arc $\gamma \in T$ such that $\bar{\gamma}$ is incident to v , let $\bar{\gamma}'$ be a curve in the interior of S homotopic to

$$\begin{cases} \text{the concatenation of } \bar{\gamma} \text{ and } \delta', \text{ if } \bar{\alpha} \text{ lies between } \bar{\gamma} \text{ and } \bar{\beta}, \text{ or } \bar{\gamma} = \bar{\alpha}; \\ \text{the concatenation of } \bar{\gamma} \text{ and } \delta'', \text{ if } \bar{\beta} \text{ lies between } \bar{\gamma} \text{ and } \bar{\alpha}, \text{ or } \bar{\gamma} = \bar{\beta}. \end{cases}$$

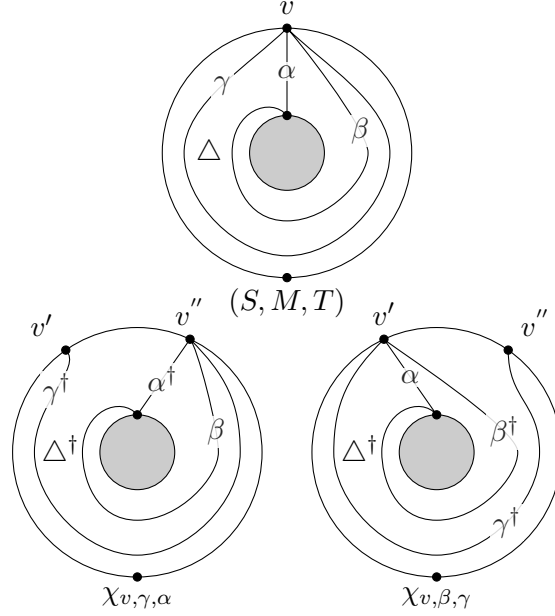
Then let $\chi_{v, \alpha, \beta}(\gamma)$ be the arc obtained from γ by replacing the end $\bar{\gamma}$ by $\bar{\gamma}'$. If both ends of γ are incident to v then $\chi_{v, \alpha, \beta}(\gamma)$ is obtained from γ by replacing both ends with the appropriate new end; this is the case in the example in Figure 3.2. If $\gamma \neq \alpha, \beta$ then, abusing notation, we will denote the arc $\chi_{v, \alpha, \beta}(\gamma)$ again by γ . The arcs obtained from α and β will be denoted by $\alpha^\dagger = \chi_{v, \alpha, \beta}(\alpha)$ and $\beta^\dagger = \chi_{v, \alpha, \beta}(\beta)$, see Figure 3.1. Define

$$\chi_{v, \alpha, \beta}(T) = (T \setminus \{\gamma \in T \mid \gamma \text{ incident to } v\}) \cup \{\chi_{v, \alpha, \beta}(\gamma) \mid \gamma \text{ incident to } v\}.$$

Finally, let $\chi_{v, \alpha, \beta}(S, M, T) = (S, \chi_{v, \Delta}(M), \chi_{v, \alpha, \beta}(T))$. Let us point out that $(S, \chi_{v, \Delta}(M))$ is a surface which has exactly one marked point more than (S, M) , and that $\chi_{v, \alpha, \beta}(T)$ has the same number of arcs as T . Therefore $\chi_{v, \alpha, \beta}(T)$ is a partial triangulation of the surface $(S, \chi_{v, \Delta}(M))$. We denote by Δ^\dagger the quadrilateral with sides $\alpha^\dagger, \beta^\dagger, \gamma$ and the new boundary segment between v' and v'' .

Definition 3.1.

- (1) The partially triangulated surface $\chi_{v, \alpha, \beta}(S, M, T)$ is called the *local cut* of (S, M, T) at v relative to α, β .

FIGURE 3.2. All of the possible cuts at vertex v

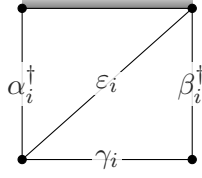
- (2) A *cut* of the triangulated surface (S, M, T) is a partially triangulated surface $(S, M^\dagger, T^\dagger)$ obtained by applying a sequence of local cuts $\chi_{v_1, \alpha_1, \beta_1}, \dots, \chi_{v_t, \alpha_t, \beta_t}$ to (S, M, T) , subject to the condition that the triangle Δ_i in the i -th step is still an internal triangle after $i - 1$ steps.

Thus we are allowed to cut each internal triangle of T at most once. The quadrilaterals Δ_i^\dagger in T^\dagger corresponding to Δ_i in T are called *quasi-triangles*. Note that a quasi-triangle is a quadrilateral that has exactly one side on the boundary.

Definition 3.2. A cut of (S, M, T) is called an *admissible cut* if every internal triangle of T is cut exactly once.

3.2. Definition of surface algebras. Let $(S, M^\dagger, T^\dagger)$ be a cut of (S, M, T) given by the sequence $(\chi_{v_i, \alpha_i, \beta_i})_{i=1,2,\dots,t}$. Note that each of the pairs (α_i, β_i) corresponds to a pair of vertices (α_i, β_i) in the quiver Q_T and each triangle Δ_i to an arrow $\alpha_i \rightarrow \beta_i$ or $\beta_i \rightarrow \alpha_i$.

The collection T^\dagger is a partial triangulation of (S, M^\dagger) ; at each local cut $\chi_{v_i, \alpha_i, \beta_i}$ the arcs $\alpha_i^\dagger, \beta_i^\dagger, \gamma_i$ together with the boundary segment between v_i' and v_i'' form a quasi-triangle Δ_i^\dagger in T^\dagger . Choose a diagonal ε_i for each of these quasi-triangles, as in Figure 3.3, and let $\overline{T}^\dagger = T^\dagger \cup \{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_t\}$. Then \overline{T}^\dagger is a triangulation of (S, M^\dagger) . Let $Q_{\overline{T}^\dagger}$ be the associated quiver. Note that each quasi-triangle Δ_i^\dagger gives rise to a subquiver with four vertices $\alpha_i^\dagger, \beta_i^\dagger, \gamma_i,$

FIGURE 3.3. A choice for ε_i in Δ_i^\dagger

and ε_i , consisting of a 3-cycle with vertices ε_i , γ_i and either α_i^\dagger or β_i^\dagger , and one arrow connecting the fourth vertex (either β_i^\dagger or α_i^\dagger) to the vertex ε_i . We may suppose without loss of generality that these subquivers have the form

$$\begin{array}{ccccc} \alpha_i^\dagger & \rightarrow & \varepsilon_i & \longrightarrow & \gamma_i \\ & & \swarrow & & \searrow \\ & & & & \beta_i^\dagger \end{array}$$

Now we define a new quiver Q_{T^\dagger} corresponding to the partial triangulation T^\dagger by deleting the vertices ε_i and replacing each of the paths of length two given by $\alpha_i^\dagger \rightarrow \varepsilon_i \rightarrow \gamma_i$ by an arrow $\alpha_i^\dagger \rightarrow \gamma_i$. Thus each quasi-triangle Δ_i^\dagger in T^\dagger gives rise to a subquiver of Q_{T^\dagger} of the form $\alpha_i^\dagger \rightarrow \gamma_i \rightarrow \beta_i^\dagger$.

Next, we define relations on the quiver Q_{T^\dagger} . First note that the quiver $Q_{\overline{T}^\dagger}$ comes with the potential $W_{\overline{T}^\dagger}$ and the Jacobian algebra $B_{\overline{T}^\dagger} = kQ_{\overline{T}^\dagger}/I_{\overline{T}^\dagger}$, where $I_{\overline{T}^\dagger}$ is generated by the set \mathcal{R} consisting of all subpaths of length two of all oriented 3-cycles. In particular, for each quasi-triangle Δ_i^\dagger , we have the three relations $\varepsilon_i \rightarrow \gamma_i \rightarrow \beta_i^\dagger$, $\gamma_i \rightarrow \beta_i^\dagger \rightarrow \varepsilon_i$, and $\beta_i^\dagger \rightarrow \varepsilon_i \rightarrow \delta_i$. Denote by \mathcal{R}_i the set of these three relations. Define I_{T^\dagger} to be the two-sided ideal generated by all relations in

$$\left(\mathcal{R} \setminus \left(\bigcup_{i=1}^t \mathcal{R}_i \right) \right) \cup \left(\bigcup_{i=1}^t \{ \alpha_i^\dagger \rightarrow \gamma_i \rightarrow \beta_i^\dagger \} \right).$$

Thus each Δ_i^\dagger corresponds to a subquiver $\alpha_i^\dagger \rightarrow \gamma_i \rightarrow \beta_i^\dagger$ with a zero relation.

Definition 3.3. A *surface algebra of type* (S, M) is a bound quiver algebra $B_{T^\dagger} = kQ_{T^\dagger}/I_{T^\dagger}$ where $(S, M^\dagger, T^\dagger)$ is a cut of a triangulated unpunctured surface (S, M, T) .

3.3. Properties of surface algebras.

Lemma 3.4. *If $(S, M^\dagger, T^\dagger)$ is obtained from (S, M, T) by the sequence of local cuts $(\chi_{v_i, \alpha_i, \beta_i})_{i=1, \dots, t}$, then Q_{T^\dagger} is isomorphic to the quiver obtained from Q_T by deleting the arrows $\beta_i \rightarrow \alpha_i$ for $i = 1, 2, \dots, t$.*

Proof. This follows immediately from the construction. ■

Theorem 3.5. *If $(S, M^\dagger, T^\dagger)$ is an admissible cut of (S, M, T) then*

- (a) Q_{T^\dagger} is an admissible cut of Q_T ;
- (b) B_{T^\dagger} is of global dimension at most two, and B_{T^\dagger} is of global dimension one if and only if B_T is a hereditary algebra of type \mathbb{A} or $\tilde{\mathbb{A}}$;
- (c) The tensor algebra of B_{T^\dagger} with respect to the B_{T^\dagger} -bimodule

$$\text{Ext}_{B_{T^\dagger}}^2(DB_{T^\dagger}, B_{T^\dagger})$$

is isomorphic to the Jacobian algebra B_T .

Proof. Part (a). The oriented 3-cycles in Q_T are precisely the chordless cycles in Q_T and, by Lemma 3.4, Q_{T^\dagger} is obtained from Q_T by deleting exactly one arrow in each chordless cycles; this shows (a).

Part (b). Since Q_{T^\dagger} does not contain any oriented cycles, the ideal I_{T^\dagger} is generated by monomial relations which do not overlap. This immediately implies $\text{gl. dim } B_{T^\dagger} \leq 2$.

If B_{T^\dagger} is of global dimension at most one, then the ideal I_{T^\dagger} is trivial and hence the ideal I_T is trivial too. It follows that Q_T has no oriented cycles, $T^\dagger = T$, $B_{T^\dagger} = B_T$, and that T is a triangulation without internal triangles. The only unpunctured surfaces that admit such a triangulation are the disc and the annulus, corresponding to the case where B_T is hereditary algebra of type \mathbb{A} or $\tilde{\mathbb{A}}$, respectively.

Part (c). Let $\widetilde{B_{T^\dagger}}$ denote the tensor algebra. It follows from [4] that its quiver is obtained from Q_{T^\dagger} by adding on arrow $\beta_i^\dagger \rightarrow \alpha_i^\dagger$ for each relation $\alpha_i^\dagger \rightarrow \gamma_i \rightarrow \beta_i^\dagger$; thus the quiver of $\widetilde{B_{T^\dagger}}$ is isomorphic to the quiver Q_T . Moreover, it follows from [22, Theorem 6.12] that $\widetilde{B_{T^\dagger}}$ is a Jacobian algebra with potential \widetilde{W} given by the sum of all 3-cycles $\alpha_i^\dagger \rightarrow \gamma_i \rightarrow \beta_i^\dagger \rightarrow \alpha_i^\dagger$; thus $\widetilde{B_{T^\dagger}} \cong B_T$. ■

Corollary 3.6. *The admissible cut surface algebras of the disc with $n + 3$ marked points are precisely the iterated tilted algebras of type \mathbb{A}_n whose global dimension is at most two.*

Proof. In [10], the authors have shown that the quotients by an admissible cut of a cluster-tilted algebra of type \mathbb{A}_n are precisely the iterated tilted algebras of type \mathbb{A}_n of global dimension at most two. By [16], the cluster-tilted algebras of type \mathbb{A}_n are precisely the algebras B_T , where T is a triangulation of the disc with $n + 3$ marked points. The result now follows from Theorem 3.5. ■

Theorem 3.7. *Every surface algebra is isomorphic to the endomorphism algebra of a partial cluster tilting object in a generalized cluster category. More precisely, if the surface algebra B_{T^\dagger} is given by the cut $(S, M^\dagger, T^\dagger)$ of the triangulated surface (S, M, T) , then*

$$B_{T^\dagger} \cong \text{End}_{\mathcal{C}(S, M^\dagger)} T^\dagger,$$

where T^\dagger denotes the object in the generalized cluster category $\mathcal{C}(S, M^\dagger)$ corresponding to T^\dagger .

Proof. Let \overline{T}^\dagger be the completion of T^\dagger as in the construction of the quiver Q_{T^\dagger} . By [20], the triangulation \overline{T}^\dagger corresponds to a cluster in the cluster algebra of (S, M^\dagger) , hence \overline{T}^\dagger also corresponds to a cluster-tilting object in the generalized cluster category $\mathcal{C}(S, M^\dagger)$, see [1, 12]. Thus T^\dagger is a partial cluster-tilting object in $\mathcal{C}(S, M^\dagger)$. The endomorphism algebra $\text{End}_{\mathcal{C}(S, M^\dagger)} \overline{T}^\dagger$ of the cluster-tilting object \overline{T}^\dagger is isomorphic to the Jacobian algebra $B_{\overline{T}^\dagger}$.

Let $\chi_{v_i, \alpha_i, \beta_i}$ be one of the local cuts that defines the cut $(S, M^\dagger, T^\dagger)$. In the quiver $Q_{\overline{T}^\dagger}$, we have the corresponding subquiver

$$\begin{array}{ccc} \alpha_i^\dagger & \xrightarrow{\quad} & \varepsilon_i & \xrightarrow{\quad} & \gamma_i \\ & & \swarrow & & \searrow \\ & & \beta_i^\dagger & & \end{array}$$

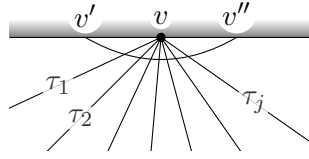
and there are no other arrows in $Q_{\overline{T}^\dagger}$ starting or ending at ε_i . Each vertex in this subquiver corresponds to an indecomposable summand of the cluster-tilting object \overline{T}^\dagger , and each non-zero path corresponds to a non-zero morphism between the indecomposable summands associated to the endpoints of the path. Thus in $\mathcal{C}(S, M^\dagger)$, there are non-zero morphisms $f: \beta_i^\dagger \rightarrow \gamma_i$, $g: \gamma_i \rightarrow \varepsilon_i$, $h_i: \varepsilon_i \rightarrow \beta_i^\dagger$, $u_i: \varepsilon_i \rightarrow \alpha_i^\dagger$, and the compositions $g_i f_i$, $h_i g_i$, $f_i h_i$ are zero, but the composition $u_i g_i$ is non-zero.

Removing the summand ε_i from the cluster-tilting object \overline{T}^\dagger and considering $\text{End}_{\mathcal{C}(S, M^\dagger)}(\overline{T}^\dagger \setminus \varepsilon_i)$, the only non-zero morphisms between the summands α_i^\dagger , β_i^\dagger , and γ_i are f_i and $u_i g_i$, and the composition $(u_i g_i) f_i$ is zero, since $g_i f_i$ is zero. Thus, locally, the quiver of $\text{End}_{\mathcal{C}(S, M^\dagger)}(\overline{T}^\dagger \setminus \varepsilon_i)$ is $\alpha_i^\dagger \rightarrow \gamma_i \rightarrow \beta_i^\dagger$ and the composition of the two arrows is zero in $\text{End}_{\mathcal{C}(S, M^\dagger)}(\overline{T}^\dagger \setminus \varepsilon_i)$.

The endomorphism algebra $\text{End}_{\mathcal{C}(S, M^\dagger)}(T^\dagger)$ of the partial cluster tilting object T^\dagger is obtained by a finite number of such local transformations, each of which is corresponding to a local cut $\chi_{v_i, \alpha_i, \beta_i}$. Thus $\text{End}_{\mathcal{C}(S, M)}(T^\dagger) = kQ_{T^\dagger}/I_{T^\dagger} = B_{T^\dagger}$. \blacksquare

Theorem 3.8. *Every surface algebra is gentle.*

Proof. Let B_{T^\dagger} be the surface algebra obtained by a cut $(S, M^\dagger, T^\dagger)$ of a triangulated surface (S, M, T) . The algebra B_T is gentle by Proposition 2.7, and the result then follows from Lemma 3.4 and Proposition 2.8. \blacksquare

FIGURE 4.1. The fan at v

4. COMPUTING THE AG-INVARIANT OF SURFACE ALGEBRAS

In this section, we give an explicit formula for the AG-invariant of an arbitrary surface algebra in terms of the surface. The key idea is to interpret the permitted threads as complete fans and the forbidden threads as triangles or quasi-triangles in the partial triangulation T^\dagger .

4.1. Permitted threads and complete fans. Let $(S, M^\dagger, T^\dagger)$ be a cut of a triangulated unpunctured surface (S, M, T) , and let v be a point in M^\dagger . Let v', v'' be two points on the boundary, but not in M^\dagger , such that there is a curve δ on the boundary from v' to v'' passing through v , but not passing through any other point of M^\dagger . Let γ be a curve from v' to v'' which is homotopic to δ but disjoint from the boundary except for its endpoints, such that the number of crossings between γ and T^\dagger is minimal. The sequence of arcs $\tau_1, \tau_2, \dots, \tau_j$ in T^\dagger that γ crosses in order is called the *complete fan at v in T^\dagger* . The arcs $\tau_2, \dots, \tau_{j-1}$ are called the *interior arcs of the fan* and j is called the *width of the fan*. See Figure 4.1

Note that the sequence is of width zero if no arc in T^\dagger is incident to v . Fans of width zero are called *empty fans*, and fans of width one are called *trivial fans*. Every arc in T^\dagger is contained in exactly two non-empty complete fans, namely the two fans at the two endpoints of the arc.

Lemma 4.1. *The construction of Q_{T^\dagger} induces a bijection between the set \mathcal{H} of permitted threads in Q_{T^\dagger} and the non-empty complete fans of $(S, M^\dagger, T^\dagger)$. Moreover, under this bijection, the trivial permitted threads correspond to the trivial fans.*

Proof. From the construction of Q_{T^\dagger} it follows that the non-trivial permitted threads of Q_{T^\dagger} correspond to the complete fans in $(S, M^\dagger, T^\dagger)$ of width at least 2.

Now consider a trivial permitted thread; this corresponds to a vertex $x \in Q_{T^\dagger}$ with at most one arrow starting at x and at most one arrow ending at x . By the construction of Q_{T^\dagger} , the vertex x corresponds to an arc in T^\dagger , and this arc is contained in exactly two complete fans, one at each endpoint of the arc. If both fans are non-trivial, then this configuration generates either two arrows ending at the vertex x in Q_{T^\dagger} or two arrows starting at the vertex x , which contradicts our assumption that x corresponds to a trivial permitted thread. Hence one of the two fans must be trivial.

Since we are excluding the case where (S, M) is a disc with four marked points, it follows from the construction of T^\dagger , that it is impossible that both fans at the endpoint of an arc are trivial. This shows that the trivial permitted threads are in bijection with the trivial complete fans. ■

4.2. Forbidden threads, triangles and quasi-triangles.

Lemma 4.2. *The construction of Q_{T^\dagger} induces a bijection between the set of forbidden threads of length at most two of Q_{T^\dagger} and the set of non-internal triangles and quasi-triangles in T^\dagger . Moreover, under this bijection*

- (a) *forbidden threads of length two correspond to quasi-triangles,*
- (b) *forbidden threads of length one correspond to triangles with exactly one side on the boundary,*
- (c) *forbidden threads of length zero correspond to triangles with exactly two sides on the boundary.*

Remark 4.3. (a) Each internal triangle in T^\dagger gives rise to three forbidden threads of length three.
 (b) There are no forbidden threads in Q_{T^\dagger} of length greater than three.
 (c) If $T^\dagger = T$ then there are no forbidden threads of length two.

Proof. Forbidden threads of length two can only occur at the local cuts, since all relations in B_T occur in oriented 3-cycles, and hence give forbidden threads of length three. Thus the forbidden threads of length two are precisely the paths $\alpha_i^\dagger \rightarrow \gamma_i \rightarrow \beta_i^\dagger$ in Q_{T^\dagger} which were obtained from the 3-cycles $\alpha_i \rightarrow \gamma_i \rightarrow \beta_i \rightarrow \alpha_i$ by the cuts $\chi_{v_i, \alpha_i, \beta_i}$. These paths correspond to the quasi-triangle Δ_i^\dagger . This shows (a).

A forbidden thread of length one in Q_{T^\dagger} is an arrow $x \rightarrow y$ which does not appear in any relation, which means that the arcs corresponding to x and y in the triangulation T^\dagger bound a triangle which is not an internal triangle and also not a quasi-triangle and which has two interior sides (corresponding to x and y). This shows (b).

Finally, a trivial forbidden thread in Q_{T^\dagger} is a vertex x that is either a sink with only one arrow ending at x , or a source with only one arrow starting at x , or the middle vertex of a zero relation of length two and the two arrows from this zero relation are the only two arrows at x . If x is such a source or sink, then it follows from the construction of Q_{T^\dagger} that the triangle on one side of the arc corresponding to x in T^\dagger must not generate an arrow in Q_{T^\dagger} . Thus x is a side of a triangle that has two sides on the boundary. On the other hand, if x is such a middle vertex of a zero relation, then the triangle on one of side of the arc corresponding to x in T^\dagger must be an internal triangle or a quasi-triangle. On the other side, we must have a triangle with two sides on the boundary, since there are no further arrows at x .

Conversely, given any triangle with two sides on the boundary and the third side γ corresponding to a vertex $x \in Q_{T^\dagger}$, we can deduce that x is a trivial forbidden thread, because, on the other side of γ , we have either a

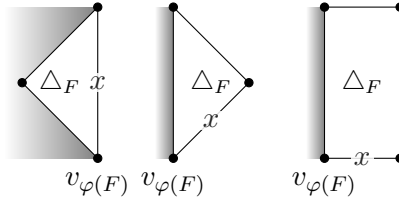


FIGURE 4.2. The relative position of the marked point $v_{\varphi(F)}$; on the left, F is the trivial forbidden thread; in the middle, F is of length one; and, on the right, F is of length two.

triangle with one side on the boundary or an internal triangle, respectively quasi-triangle, thus x is either a vertex with only one arrow incident to it or the middle vertex of a zero relation with no further arrows incident to it. ■

If $F \in \mathcal{F}$ is a forbidden thread, we denote by Δ_F the corresponding triangle or quasi-triangle in Q_{T^\dagger} , and if $H \in \mathcal{H}$ is a permitted thread, we denote by v_H the marked point in H such that H corresponds to the complete fan at v_H .

4.3. AG-invariant. We now want to compute the AG-invariant of B_{T^\dagger} in terms of the partially triangulated surface $(S, M^\dagger, T^\dagger)$. To do so, we need to describe how to go from a forbidden thread to the following permitted thread and from a permitted thread to the following forbidden thread as in the AG-algorithm.

Lemma 4.4. *Let $F \in \mathcal{F}$ be a forbidden thread in Q_{T^\dagger} , and let $x = s(F)$ be its starting vertex in Q_{T^\dagger} . Then there exists a unique permitted thread $\varphi(F) \in \mathcal{H}$ such that $s(\varphi(F)) = x$ and $\sigma(\varphi(F)) = -\sigma(F)$. Moreover, the marked point $v_{\varphi(F)}$ of the complete fan of $\varphi(F)$ is the unique vertex of Δ_F that is the starting point of the arc corresponding to x in T^\dagger , with respect to the counterclockwise orientation of Δ_F , see Figure 4.2.*

Proof. The existence and uniqueness of $\varphi(F)$ follows already from [9], but we include a proof here, for convenience. If F is a trivial forbidden thread, then there is at most one arrow start at x , hence there is a unique permitted thread $\varphi(F)$ starting at x . By Lemma 4.1, $\varphi(F)$ corresponds to a complete fan in which the first arc corresponds to x . It follows that the vertex of this fan is the one described in the Lemma.

Now suppose that F is not trivial, and let α be its initial arrow. Then there are two permitted threads starting at x , one of which has also α as initial arrow. However, since $\sigma(F) = \sigma(\alpha)$, the condition $\sigma(\varphi(F)) = -\sigma(F)$ excludes the possibility that $\varphi(F)$ starts with α ; thus $\varphi(F)$ is the other permitted thread starting at x . Again using Lemma 4.1, we see that $\varphi(F)$ corresponds to the complete fan whose vertex is the marked point $v_{\varphi(F)}$

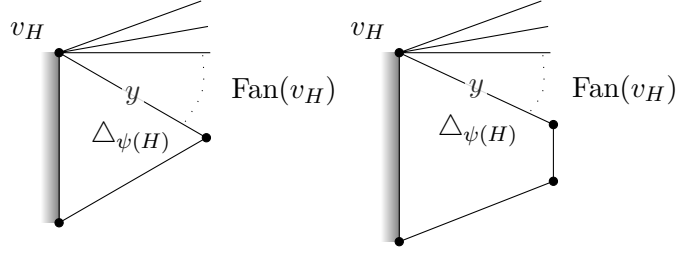


FIGURE 4.3.

described in the Lemma. Note that if α is the only arrow starting at x , then $\varphi(F)$ trivial. \blacksquare

Lemma 4.5. *Let $H \in \mathcal{H}$ be a permitted thread in Q_{T^\dagger} and let $y = t(H)$ be its terminal vertex in Q_{T^\dagger} . Then there exists a unique forbidden thread $\psi(H) \in \mathcal{F}$ such that $t(\psi(H)) = y$ and $\varepsilon(\psi(H)) = -\varepsilon(H)$. Moreover, the triangle or quasi-triangle $\Delta_{\psi(H)}$ of $\psi(H)$ is the unique triangle or quasi-triangle which is adjacent to the arc corresponding to y and incident to v_H , see Figure 4.3.*

Proof. Again, the existence and uniqueness already follows from [9], but we include a proof here too. It follows from Lemma 4.1 that the side of the triangle $\Delta_{\psi(H)}$ which is following y in the counterclockwise order must be a boundary segment, since the fan at v_H is a complete fan and $y = t(H)$. It follows from Lemma 4.2 that the forbidden thread corresponding to $\Delta_{\psi(H)}$ ends in y . Let b be the terminal arrow in H . If $\Delta_{\psi(H)}$ has two sides on the boundary, then $\psi(H)$ is a trivial forbidden thread and $\varepsilon(\psi(H)) = -\varepsilon(b) = -\varepsilon(H)$. Note that in this case, b is the only arrow in Q_{T^\dagger} ending at y , and, since $\varepsilon(b) = \varepsilon(H)$, it follows that $\psi(H)$ is unique.

Suppose now that $\Delta_{\psi(H)}$ has only one side on the boundary. Since this boundary segment follows y in the counterclockwise orientation, $\Delta_{\psi(H)}$ induces an arrow b' in Q_{T^\dagger} , with $t(b') = y$, and this arrow is the terminal arrow of $\psi(H)$. Since b and b' end at the same vertex, we have $\varepsilon(\psi(H)) = \varepsilon(b') = -\varepsilon(b) = -\varepsilon(H)$. This shows the existence of $\psi(H)$, and its uniqueness follows as above. \blacksquare

We are now able to prove the main result of this section, the computation of the AG-invariant for an arbitrary surface algebra.

For any triangulated surface (S, M, T) and any boundary component C of S , denote by $M_{C,T}$ the set of marked points on C that are incident to at least one arc in T . Let $n(C, T)$ be the cardinality of $M_{C,T}$, and let $m(C, T)$ be the number of boundary segments on C that have both endpoints in $M_{C,T}$.

Theorem 4.6. *Let $B = B_{T^\dagger}$ be a surface algebra of type (S, M, T) given by a cut $(S, M^\dagger, T^\dagger)$. Then the AG-invariant $AG(B)$ can be computed as follows:*

- (a) *The ordered pairs $(0, 3)$ in $AG(B)$ are in bijection with the internal triangles in T^\dagger , and there are no ordered pairs $(0, m)$ with $m \neq 3$.*
- (b) *The ordered pairs (n, m) in $AG(B)$ with $n \neq 0$ are in bijection with the boundary components of S . Moreover, if C is a boundary component, then the corresponding ordered pair (n, m) is given by*

$$n = n(C, T) + \ell, \quad m = m(C, T) + 2\ell,$$

where ℓ is the number of local cuts $\chi_{v, \alpha, \beta}$ in $(S, M^\dagger, T^\dagger)$ such that v is a point on C .

Proof. Part (a) follows directly from the construction of B_{T^\dagger} . To show (b), let

$$(H_0, F_0, H_1, F_1, \dots, H_{n-1}, F_{n-1}, H_n = H_0)$$

be a sequence obtained by the AG-algorithm. Thus each H_i is a permitted thread in Q_{T^\dagger} , each F_i is a forbidden thread, $F_i = \psi(H_i)$, and $H_{i+1} = \varphi(F_i)$, where φ and ψ are the maps described in the Lemmas 4.4 and 4.5 respectively.

By Lemma 4.1, each permitted thread H_i corresponds to a non-empty complete fan, thus to a marked point v_{H_i} in M^\dagger . On the other hand, Lemma 4.2 implies that each forbidden thread F_i corresponds to a triangle or quasi-triangle Δ_{F_i} in T^\dagger .

Let C be the boundary component containing v_{H_0} . Lemma 4.5 implies that Δ_{F_0} contains a boundary segment incident to v_{H_0} , and it follows then from Lemma 4.4 that $v_{H_1} \in M^\dagger$ is a marked point on the same boundary component C . Note that if F_0 is a non-trivial forbidden thread then v_{H_0} and v_{H_1} are the two endpoints of the unique boundary segment in Δ_{F_0} , and, if F_0 is a trivial forbidden thread, the Δ_{F_0} is a triangle with two sides on the boundary and v_{H_0}, v_{H_1} are the two endpoints of the side of Δ_{F_0} that is not on the boundary.

Recursively, we see that each of the marked points $v_{H_i} \in M^\dagger$ lies on the boundary component C and that the set of points $\{v_{H_0}, v_{H_1}, \dots, v_{H_{n-1}}\}$ is precisely the set of marked points in M^\dagger that lie on C and that are incident to at least one arc in T^\dagger . In particular, $n = n(C, T) + \ell$.

Recall that the number m in the ordered pair (n, m) is equal to the sum of the number of arrows appearing in the forbidden threads F_0, F_1, \dots, F_{n-1} . For each i , the number of arrows in F_i is zero if Δ_{F_i} has two sides on the boundary component C ; it is one if Δ_{F_i} is a triangle with one side on C ; and it is two if Δ_{F_i} is a quasi-triangle with one side on C . Taking the sum, we see that m is the number of triangles in T^\dagger that have exactly one side on C plus twice the number of quasi-triangle in T^\dagger that have exactly one side on C . Thus m is equal to the number of boundary segments in (S, M, T)

on C that have both endpoints in $M_{C,T}$ plus twice the number of local cuts $\chi_{v,\alpha,\beta}$ with v a marked point in C . \blacksquare

Remark 4.7. The theorem holds for arbitrary cuts of (S, M, T) , thus, in particular, it computes the AG-invariant of the Jacobian algebra B_T corresponding to the uncut triangulated surface (S, M, T) .

Corollary 4.8. *Any admissible cut surface algebra of the disc with $n + 3$ marked points is derived equivalent to the path algebra A of the quiver*

$$1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 3 \xrightarrow{a_3} \dots \xrightarrow{a_{n-2}} n-1 \xrightarrow{a_{n-1}} n$$

Proof. This statement already follows from Corollary 3.6, since tilting induces an equivalence of derived categories. We give here an alternative proof using Theorem 4.6. The algebra A has AG-invariant $(n + 1, n - 1)$, since in the AG algorithm we get the sequence

$$\begin{array}{ll} H_0 = a_1 a_2 \cdots a_{n-1} & F_0 = e_n \\ H_1 = e_n & F_1 = a_{n-1} \\ H_2 = e_{n-1} & F_2 = a_{n-2} \\ \vdots & \vdots \\ H_n = e_1 & F_n = e_1 \\ H_{n+1} = H_0 & \end{array}$$

which contains $n + 1$ permissible threads and $n - 1$ arrows in the forbidden threads.

On the other hand, Theorem 4.6 implies that the AG-invariant of an admissible cut surface algebra B_{T^\dagger} of the disc is given by

$$(n(C, T) + \ell, m(C, T) + 2\ell).$$

It follows from an easy induction that the number of internal triangles in T is equal to $n + 1 - n(C, T)$. Since the cut is admissible, this is also the number of local cuts. Thus

$$n(C, T) + \ell = n + 1.$$

Moreover, $m(C, T)$ is equal to the number of boundary segments minus $2(n + 3 - n(C, T))$, thus $m(C, T) = 2n(C, T) - n - 3$, and it follows that

$$m(C, T) + 2\ell = n - 1.$$

Alternatively, since there is only one ordered pair (n, m) in the AG-invariant, the number m is the number of arrows in the quiver, which is equal to $n - 1$, since the quiver of an admissible cut from the disc with $n + 3$ marked points is a tree on n vertices.

Thus the AG-invariant of B_{T^\dagger} is equal to the AG-invariant of A . Since the quivers of both algebras have no cycles, now the result follows from Theorem 2.4. \blacksquare

Remark 4.9. It also follows from Theorem 4.6 that for surfaces other than the disc, the surface algebras obtained by admissible cuts of a fixed triangulation are in general not all derived equivalent.

4.4. Surface algebras of annulus type. We make some observations for surface algebras coming from admissible cuts of a triangulated annulus.

Corollary 4.10. *Let S be an annulus, and let B_1, B_2 be surface algebras of type (S, M, T) obtained by two admissible cuts of the same triangulation. Then*

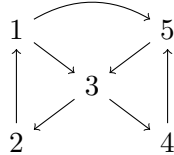
- (a) B_1 and B_2 are derived equivalent if and only if B_1 and B_2 have the same AG-invariant.
- (b) If on each boundary component the number of local cuts is the same in the two admissible cuts then B_1 and B_2 are derived equivalent.

Proof. (a) The quivers of B_1 and B_2 have exactly one cycle, since they are admissible cuts of the annulus. Thus Theorem 2.4 implies that $\text{AG}(B_1) = \text{AG}(B_2)$ if and only if B_1 and B_2 are derived equivalent.

(b) Theorem 4.6 shows that $\text{AG}(B_1) = \text{AG}(B_2)$ if the number of local cuts on any given boundary component is the same in both admissible cuts. ■

Remark 4.11. In [2], the authors associate a weight to each of the algebras appearing in Corollary 4.10 and show that two such algebras are derived equivalent if and only if they have the same weight (respectively the same absolute weight if the marked points are equally distributed over the two boundary components).

Example 4.12. We end this section with an example coming from an annulus. Let (S, M, T) be the triangulated surface given in Figure 4.4 and B the corresponding cluster-tilted algebra of type $\tilde{\mathbb{A}}_4$ with the following quiver



Applying Theorem 4.6, we see that $\text{AG}(B)$ is given by the ordered pairs $\{(0, 3), (0, 3), (1, 0), (2, 1)\}$ where $(1, 0)$ is associated to the inner boundary component and $(2, 1)$ the outer boundary component. If we consider only admissible cuts, then the AG-invariant of the resulting algebras will be given by only two ordered pairs, one per boundary component. In Figures 4.5–4.7 we list the bound quivers given by the different possible admissible cuts grouped by their AG-invariant. We use dashed lines to represent the zero relations induced by the cuts. There are two internal triangles, hence there are nine distinct admissible cuts. If we make sure to cut both boundary components, we will get five algebras having the AG-invariant given by $\{(2, 2), (3, 3)\}$. It follows from [9, Section 7] and Theorem 2.4, that these are iterated tilted algebras of type $\tilde{\mathbb{A}}_4$. The other four admissible cuts of (S, M, T) split into two AG-classes: $\{(1, 0), (4, 5)\}$, corresponding to cutting the outer boundary component twice, and $\{(3, 4), (2, 1)\}$ cutting the inner boundary component twice. See Figure 4.6 and Figure 4.7, respectively.

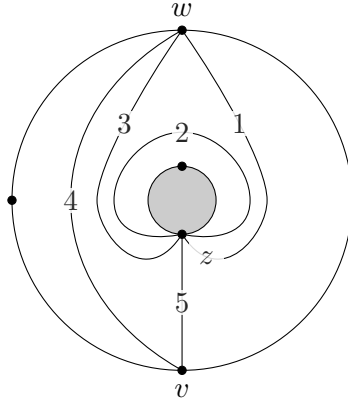


FIGURE 4.4. Triangulated annulus

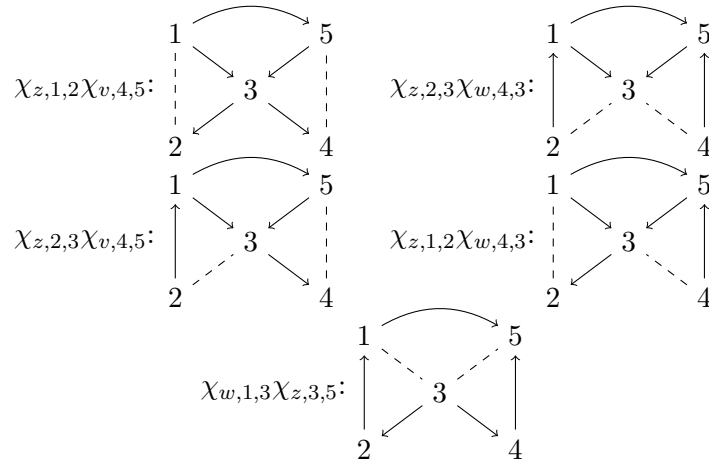


FIGURE 4.5. The surface algebras of (S, M, T) with AG-invariant given by $\{(2, 2), (3, 3)\}$

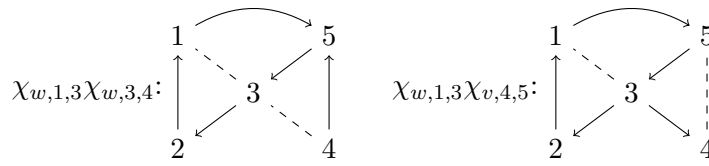


FIGURE 4.6. The surface algebras of (S, M, T) with AG-invariant given by $\{(1, 0), (4, 5)\}$

Note that from Corollary 4.10, these are also the derived equivalence classes for the surface algebras of type (S, M, T) coming from admissible cuts.

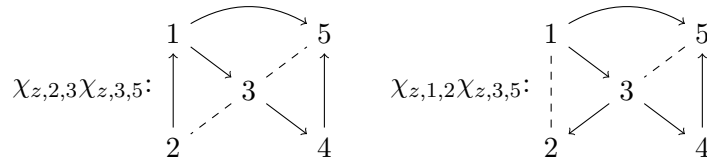


FIGURE 4.7. The surface algebras with of (S, M, T) AG-invariant given by $\{(3, 4), (2, 1)\}$

5. A GEOMETRIC DESCRIPTION OF THE MODULE CATEGORIES OF SURFACE ALGEBRAS

In this section, we study the module categories of surface algebras. Let (S, M, T) be a triangulated surface without punctures and let $(S, M^\dagger, T^\dagger)$ be a cut of (S, M, T) with corresponding surface algebra B_{T^\dagger} . We will associate a category \mathcal{E}^\dagger to $(S, M^\dagger, T^\dagger)$ whose objects are given in terms of curves on the surface (S, M^\dagger) , and we will show that there is a functor from \mathcal{E}^\dagger to $\text{mod } B_{T^\dagger}$ is faithful and dense in the category of string modules over B_{T^\dagger} . Consequently, the string modules over the surface algebra B_{T^\dagger} have a geometric description in terms of certain curves on (S, M^\dagger) .

In the special case where S is a disc, or more generally if B_{T^\dagger} is representation finite, we even have that \mathcal{E}^\dagger and B_{T^\dagger} are equivalent categories. We will use this fact in Section 6 to give a geometric description of tilted algebras of type \mathbb{A} and their module categories.

5.1. Permissible arcs.

Definition 5.1.

- A curve γ in $(S, M^\dagger, T^\dagger)$ is said to *consecutively cross* $\tau_1, \tau_2 \in T^\dagger$ if γ crosses τ_1 and τ_2 in the points p_1 and p_2 , and the segment of γ between the points p_1 and p_2 does not cross any other arc in T^\dagger .
- A generalized arc or closed curve in (S, M^\dagger) is called *permissible* if it is not consecutively crossing two non-adjacent sides of a quasi-triangle in T^\dagger .
- Two permissible generalized arcs γ, γ' in (S, M^\dagger) are called *equivalent* if there is a side δ of a quasi-triangle Δ^\dagger in T^\dagger such that
 - γ is homotopic to the concatenation of γ' and δ ,
 - both γ and γ' start at an endpoint of δ and their first crossing with T^\dagger is with the side of Δ^\dagger that is opposite to δ .

Examples of a non-permissible arc and of equivalent arcs are given in Figure 5.1.

Note that the arcs in T^\dagger are permissible by definition. Furthermore, note that the side δ in part (c) of the definition may be, but does not have to be, a boundary segment.

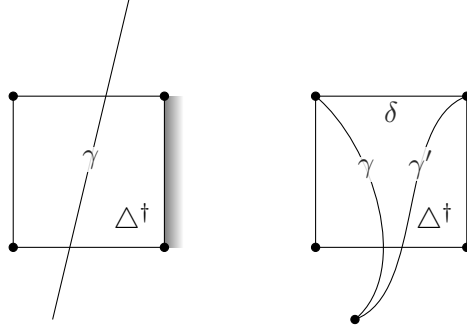


FIGURE 5.1. Example of a non-permissible arc γ on the left, and of two equivalent arcs γ, γ' on the right

5.2. Pivots.

Definition 5.2. Let γ be a permissible generalized arc, and let v and w be its endpoints. Let $f_v\gamma$ be the permissible generalized arc or boundary segment obtained from γ by fixing the endpoint v and moving the endpoint w as follows:

- (i) if w is a vertex of a quasi-triangle Δ^\dagger and w' is the counterclockwise neighbor vertex of w in Δ^\dagger such that γ is equivalent to an arc γ' with endpoints v, w' then move w to the counterclockwise neighbor of w' on the boundary of S . Thus $f_v\gamma$ is given by a curve homotopic to the concatenation of γ' and the boundary segment that connects w' to its counterclockwise neighbor.
- (ii) otherwise, move w to its counterclockwise neighbor on the boundary. Thus $f_v\gamma$ is given by a curve homotopic to the concatenation of γ and the boundary segment that connects w to its counterclockwise neighbor.

We call $f_v\gamma$ the *pivot of γ fixing v* .

Remark 5.3. In the special case of a trivial cut, that is $(S, M^\dagger, T^\dagger) = (S, M, T)$, the pivots are defined by condition (ii) only, since there are no quasi-triangles. In this case our pivots are the same as in [12], where it has been show that these pivots correspond to irreducible morphisms in $\text{mod } B_T$.

Definition 5.4. Let γ be a permissible generalized arc in $(S, M^\dagger, T^\dagger)$ and let v and w be its endpoints. Let $\tau^-\gamma$ be the permissible generalized arc given by $\tau^-\gamma = f_v f_w \gamma$. Dually, let $\tau^+\gamma$ be the permissible generalized arc in $(S, M^\dagger, T^\dagger)$ such that $\tau^-(\tau^+\gamma) = \gamma$.

Remark 5.5. Again, in the case where $(S, M^\dagger, T^\dagger) = (S, M, T)$, it has been show in [12] that τ^+ and τ^- correspond to the Auslander-Reiten translation and the inverse Auslander-Reiten translation in $\text{mod } B_T$, respectively.

5.3. The categories \mathcal{E}^\dagger and \mathcal{E} .

Definition 5.6. Let $\mathcal{E}^\dagger = \mathcal{E}(S, M^\dagger, T^\dagger)$ be the additive category whose indecomposable objects are the equivalence classes of permissible arcs in $(S, M^\dagger, T^\dagger)$ that are not in T^\dagger and whose morphisms between indecomposable objects are generated by the pivots $f_v \in \text{Hom}_{\mathcal{E}^\dagger}(\gamma, f_v\gamma)$ subject to the relations $f_v f_w = f_w f_v$. Here we use the convention that $f_v\gamma$ denotes the zero object in \mathcal{E}^\dagger whenever $f_v\gamma$ is a boundary segment in (S, M^\dagger) or an arc in T^\dagger .

Our first goal is to relate the category $\mathcal{E}^\dagger = \mathcal{E}(S, M^\dagger, T^\dagger)$ to the category $\mathcal{E} = \mathcal{E}(S, M, T)$ of the original triangulated surface, by constructing a functor $G: \mathcal{E}^\dagger \rightarrow \mathcal{E}$. Let γ be an indecomposable object in \mathcal{E}^\dagger ; thus γ is represented by a permissible generalized arc in (S, M^\dagger) . The generalized arc γ is determined by the sequence of its crossing points with T^\dagger together with the sequence of its segments between consecutive crossing points. Each of these segments lies entirely in a triangle or quasi-triangle of T^\dagger . Define $G(\gamma)$ to be the unique generalized arc in (S, M) determined by the same sequence of crossing points and segments, with the difference that segments of γ that were lying on a quasi-triangle become segments of $G(\gamma)$ which lie in the corresponding triangle.

If $f_v: \gamma \rightarrow f_v\gamma$ is a morphism in \mathcal{E}^\dagger given by a pivot of type (ii) in Definition 5.2, then let $G(f_v): G(\gamma) \rightarrow G(f_v\gamma)$ be the pivot in \mathcal{E} fixing the marked point corresponding to v in M . If $f_v: \gamma \rightarrow f_v\gamma$ is given by a pivot of type (i) in Definition 5.2, then let $G(f_v): G(\gamma) \rightarrow G(f_v\gamma)$ be the minimal sequence of pivots in \mathcal{E} , each pivot fixing the marked point corresponding to v in M , which transforms $G(\gamma)$ into $G(f_v\gamma)$.

Proposition 5.7. *G is a full and faithful functor.*

Proof. The image of a composition of pivots under G is equal to the composition of the images of the pivots. Moreover, if v^\dagger, w^\dagger are marked points in M^\dagger and v, w are the corresponding points in M , then $G(f_{v^\dagger}) \circ G(f_{w^\dagger})(G(\gamma)) = G(f_{w^\dagger}) \circ G(f_{v^\dagger})(G(\gamma))$, which shows that G respects the relations on \mathcal{E}^\dagger .

To show that G is full, let γ, γ' be indecomposable objects in \mathcal{E}^\dagger and let $f \in \text{Hom}_{\mathcal{E}}(G(\gamma), G(\gamma'))$ be a nonzero morphism in \mathcal{E} . Then f is given by a sequence of pivots $f = f_{v_i} \circ f_{v_{i-1}} \circ \cdots \circ f_{v_1}$ with $v_1, \dots, v_i \in M$. Using the relations $f_v f_w = f_w f_v$ in \mathcal{E} , we may suppose without loss of generality that $v_1 = v_2 = \cdots = v_h \neq v_{h+1} = \cdots = v_{i-1} = v_i$, for some h with $1 \leq h \leq i$. If all intermediate generalized arcs $f_{v_j} \circ f_{v_{j-1}} \circ \cdots \circ f_{v_1}(G(\gamma))$ are in the image of G , then there is a corresponding sequence of pivots $f_{v_i^\dagger} \circ f_{v_{i-1}^\dagger} \circ \cdots \circ f_{v_1^\dagger}$ in \mathcal{E}^\dagger such that $G(f_{v_i^\dagger} \circ f_{v_{i-1}^\dagger} \circ \cdots \circ f_{v_1^\dagger}(\gamma)) = f_{v_i} \circ f_{v_{i-1}} \circ \cdots \circ f_{v_1}(G(\gamma))$. Otherwise, let $\alpha = f_{v_j} \circ f_{v_{j-1}} \circ \cdots \circ f_{v_1}(G(\gamma))$ be the first generalized arc in this sequence that does not lie in the image of G . Then α must cross an internal triangle Δ in T such that the corresponding (non-permissible) generalized arc α^\dagger in $(S, M^\dagger, T^\dagger)$ crosses the corresponding quasi-triangle in two opposite sides consecutively. Let $\delta = f_{v_{j-1}} \circ f_{v_{j-2}} \circ \cdots \circ f_{v_1}(G(\gamma))$ be the immediate predecessor of α in the sequence, and let $\beta = f_{v_k} \circ f_{v_{k-1}} \circ \cdots \circ f_{v_1}(G(\gamma))$ with

$k > j$ be the first arc after α in the sequence which is again in the image of G . We distinguish two cases.

- (1) Suppose that $v_j = v_{j+1} = \dots = v_k$. Thus β and α are both incident to v_j , and it follows that in \mathcal{E}^\dagger , we have $f_{v_j^\dagger} \delta^\dagger = \beta^\dagger$, where $\delta^\dagger = f_{v_{j-1}^\dagger} \circ f_{v_{j-2}^\dagger} \circ \dots \circ f_{v_1^\dagger}(\gamma)$ in \mathcal{E}^\dagger is the object corresponding to δ , and β^\dagger the object in \mathcal{E}^\dagger corresponding to β . Clearly,

$$G(f_{v_j^\dagger} \circ f_{v_{j-1}^\dagger} \circ \dots \circ f_{v_1^\dagger}(\gamma)) = G(\beta^\dagger) = \beta.$$

By induction on the number of generalized arcs in the sequence $f_{v_i} \circ f_{v_{i-1}} \circ \dots \circ f_{v_1}(G(\gamma))$ which are not in the image of G , it follows now that the morphism f is in the image of G .

- (2) Suppose that $v_j \neq v_k$. Then the triangle Δ separates the two arcs $G(\gamma)$ and β , which implies that none of the arcs in T can cross both $G(\gamma)$ and β . Therefore $\text{Hom}_{\mathcal{E}}(G(\gamma), \beta) = 0$, hence $f = 0$.

This shows that G is full.

To prove that G is faithful, it suffices to show that G is faithful on pivots $\gamma \mapsto f_v \gamma$. For pivots of type (ii) in Definition 5.2, this is clear, so let $\gamma \mapsto f_v \gamma$ be a pivot of type (i). Recall that such a pivot occurs if γ is a permissible generalized arc in $(S, M^\dagger, T^\dagger)$ with endpoint $v, w \in M^\dagger$, where w is a vertex of a quasi-triangle Δ^\dagger in T^\dagger such that γ is equivalent to a permissible generalized arc γ' with endpoints v, w' , where w' is the counterclockwise neighbor of w in Δ^\dagger . Since γ and γ' are equivalent, they define the same object of \mathcal{E}^\dagger , so the morphism $\gamma \mapsto f_v \gamma$ can be represented by the pivot $\gamma' \mapsto f_v \gamma'$ which is of type (ii). This shows that the image of this morphism under G is non-zero, hence G is faithful. \blacksquare

5.4. Geometric description of module categories. It has been shown in [12] that there exists a faithful functor $F: \mathcal{E} \rightarrow \text{mod } B_T$ which sends pivots to irreducible morphisms and τ^- to the inverse Auslander-Reiten translation in $\text{mod } B_T$. This functor induces a dense and faithful as functor from \mathcal{E} to the category of B_T -string modules.

The composition $F \circ G$ is a faithful functor from \mathcal{E}^\dagger to $\text{mod } B_{T^\dagger}$. We will now define another functor $\text{res}: \text{mod } B_T \rightarrow \text{mod } B_{T^\dagger}$ and get a functor $H = \text{res} \circ F \circ G: \mathcal{E}^\dagger \rightarrow \text{mod } B_{T^\dagger}$ completing the following commutative diagram.

$$\begin{array}{ccc} \mathcal{E}^\dagger & \xrightarrow{G} & \mathcal{E} \\ \downarrow H & & \downarrow F \\ \text{mod } B_{T^\dagger} & \xleftarrow{\text{res}} & \text{mod } B_T \end{array}$$

Note that the inclusion map $i: B_{T^\dagger} \rightarrow B_T$ is an algebra homomorphism which sends $1_{B_{T^\dagger}}$ to 1_{B_T} . Thus B_{T^\dagger} is a subalgebra of B_T , and we can define res to be the functor given by the restriction of scalars.

On the other hand, B_{T^\dagger} is the quotient of B_T by the two-sided ideal generated by the arrows in the cut. Denoting the projection homomorphism by $\pi: B_T \rightarrow B_{T^\dagger}$, we see that $\pi \circ i$ is the identity morphism on B_{T^\dagger} . The extension of scalars functor $\iota = - \otimes_{B_{T^\dagger}} B_T: \text{mod } B_{T^\dagger} \rightarrow \text{mod } B_T$ is sending B_{T^\dagger} -modules M to B_T -modules $\iota(M) = M$ with the arrows in the cut acting trivially; and ι is the identity on morphisms. We have $res \circ \iota = 1_{B_{T^\dagger}}$.

Theorem 5.8. *Let $H = res \circ F \circ G: \mathcal{E}^\dagger \rightarrow \text{mod } B_{T^\dagger}$.*

- (a) *The functor H is faithful and induces a dense, faithful functor from \mathcal{E}^\dagger to the category of string modules over B_{T^\dagger} .*
- (b) *If $f_v: \gamma \rightarrow f_v\gamma$ is a pivot in \mathcal{E}^\dagger then $H(f_v): H(\gamma) \rightarrow H(f_v\gamma)$ is an irreducible morphism in $\text{mod } B_{T^\dagger}$.*
- (c) *The inverse Auslander-Reiten translate of $H(\gamma)$ is $H(\tau^-\gamma)$, and the Auslander-Reiten translate of $H(\gamma)$ is $H(\tau^+\gamma)$.*
- (d) *If $\gamma \in T^\dagger$ then $H(\tau^-\gamma)$ is an indecomposable projective B_{T^\dagger} -module and $H(\tau^+\gamma)$ is an indecomposable injective B_{T^\dagger} -module.*
- (e) *If the surface S is a disc, then H is an equivalence of categories.*
- (f) *If the algebra B_{T^\dagger} is of finite representation type, then H is an equivalence of categories.*

Proof. Part (a). The restriction of scalars functor is a faithful functor. The fact that F is faithful has been shown in [12], and G is faithful by Proposition 5.7. Thus H is faithful.

It also follows from [12] that B_T -modules in the image of F are string modules and that F is a dense and faithful functor from \mathcal{E} to the category of string B_T -modules. Consequently, the B_{T^\dagger} -modules in the image of H are string modules. Now let M be any indecomposable string module over B_{T^\dagger} , then $\iota(M)$ is a string module in $\text{mod } B_T$, and hence there exists an indecomposable $\gamma \in \mathcal{E}$ such that $F(\gamma) \cong \iota(M)$. Since the arrows in the cut act trivially on $\iota(M)$, the generalized arc γ in (S, M, T) lifts to a permissible generalized arc γ^\dagger in $(S, M^\dagger, T^\dagger)$. Moreover $H(\gamma^\dagger) = res \circ F(\gamma) \cong M$. Thus H is dense to the category of string B_{T^\dagger} -modules.

Part (b). It has been shown in [15] that the irreducible morphisms starting at an indecomposable string module $M(w)$ can be described by adding hooks or deleting cohooks at the endpoints of the string w . If w is a string such that there exists an arrow a such that aw is a string, then let v_a be the maximal non-zero path starting at $s(a)$ whose initial arrow is different from a . Then there is an irreducible morphism $M(w) \rightarrow M(w_h)$, where $w_h = v_a^{-1}aw$ is obtained from w by adding the ‘‘hook’’ $v_a^{-1}a$ at the starting point of w . On the other hand, if w is a string such that there is no arrow a such that aw is a string, then there are two possibilities:

- (1) either w contains no inverse arrow, in which case there is no irreducible morphism corresponding to the starting point of w ,
- (2) or w is of the form $w = u_a a^{-1} w_c$, where a is an arrow and u_a is maximal nonzero path ending at $t(a)$. In this case, there is an irreducible morphism $M(w) \rightarrow M(w_c)$ and w_c is said to be obtained by deleting the ‘‘cohook’’ $u_a a^{-1}$ at the starting point of the string w .

In a similar way, there are irreducible morphisms associated to adding a hook or deleting a cohook at the terminal point of the string w .

The inverse Auslander-Reiten translation of the string module $M(w)$ corresponds to the string obtained from w by performing the two operations of adding a hook, respectively deleting a cohook, at both endpoints of the string w .

Brüstle and Zhang have shown that for the Jacobian algebra B_T of a triangulated unpunctured surface (S, M, T) , the irreducible morphisms between indecomposable string modules are precisely given by the pivots of the generalized arcs. Adapting their construction to cuts, we show now that the same is true for surface algebras.

Let γ^\dagger be a generalized permissible arc in $(S, M^\dagger, T^\dagger)$ which is not in T^\dagger , let v^\dagger and w^\dagger be the endpoints of γ^\dagger and consider the pivot $f_{v^\dagger} \gamma^\dagger$. Applying the functor H , we obtain a homomorphism between B_{T^\dagger} -string modules

$$H(f_{v^\dagger}): H(\gamma^\dagger) \longrightarrow H(f_{v^\dagger} \gamma^\dagger),$$

and we must show that this morphism is irreducible. Clearly, if the image under the extension of scalars functor $\iota H(f_{v^\dagger})$ is an irreducible morphism in $\text{mod } B_T$, then $H(f_{v^\dagger})$ is an irreducible morphism in $\text{mod } B_{T^\dagger}$. By [12], this is precisely the case when $\iota H(f_{v^\dagger} \gamma^\dagger) = F(f_v \gamma)$, where f_v is the pivot in (S, M, T) at the vertex v corresponding to the vertex v^\dagger , and $\gamma = G(\gamma^\dagger)$.

Thus we must show the result in the case where $\iota H(f_{v^\dagger} \gamma^\dagger) \neq F(f_v \gamma)$. In this case, we have that $f_v \gamma$ consecutively crosses two sides α, β of a triangle Δ in T , which corresponds to a quasi-triangle Δ^\dagger in T^\dagger in which the sides α, β give rise to two opposite sides $\alpha^\dagger, \beta^\dagger$, and w is a common endpoint of α and β , which gives rise to two points w' and w'' in Δ^\dagger , see Figure 5.2. Now in this situation, $f_{v^\dagger} \gamma^\dagger$ is obtained from γ by moving the endpoint w' along the boundary segment of the quasi-triangle Δ^\dagger to the point w'' . The generalized arc $f_{v^\dagger} \gamma^\dagger$ crossed every arc in T^\dagger that is crossed by γ^\dagger and, in addition, $f_{v^\dagger} \gamma^\dagger$ also crosses every arc in T^\dagger that is in the complete fan $\beta_1, \beta_2, \dots, \beta_j = \beta^\dagger$ at w^\dagger . In particular, its last crossing is with the arc β^\dagger . On the level of the string modules, this corresponds to obtaining the string modules $H(f_{v^\dagger} \gamma^\dagger)$ by adding the hook $\bullet \leftarrow \beta_1 \rightarrow \beta_2 \rightarrow \dots \rightarrow \beta^\dagger$ to the string module $H(\gamma^\dagger)$. Thus the morphism $H(f_{v^\dagger})$ is irreducible.

Part (c). This follows directly from (b) and from the description of the Auslander-Reiten translation for string modules in [15].

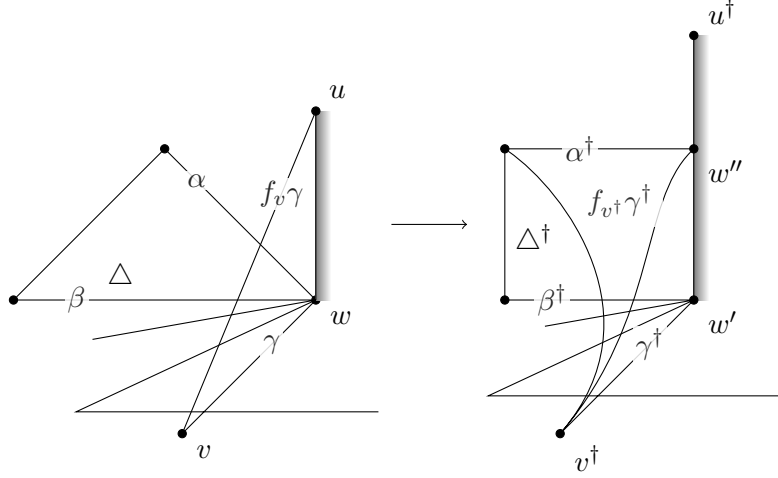


FIGURE 5.2. Proof that irreducible morphisms are given by pivots

Part (d). The statement about the projective modules follows from (c) and the fact that the indecomposable projective modules are the indecomposable string modules $M(w)$ whose Auslander-Reiten translate is zero. Similarly, the statement about the injective modules follows from (c) and the fact that the indecomposable injective modules are the indecomposable string modules $M(w)$ whose inverse Auslander-Reiten translate is zero.

Part (e). If S is a disc, then it has been shown in [16] that the functor F is an equivalence of categories. In particular, all B_T -modules, and therefore all B_{T^\dagger} -modules, are string modules, which shows that $H: \mathcal{E}^\dagger \rightarrow \text{mod } B_T$ is dense.

It remains to show that H is full. Let γ, γ' be indecomposable objects in \mathcal{E}^\dagger and let $f \in \text{Hom}_{B_{T^\dagger}}(H(\gamma), H(\gamma'))$. Applying the functor ι yields $\iota(f) \in \text{Hom}_{B_T}(\iota \circ H(\gamma), \iota \circ H(\gamma'))$. Since γ and γ' are objects in \mathcal{E}^\dagger , their images under $F \circ G$ are B_T -modules on which the arrows in the cut act trivially. Thus, applying $\iota \circ \text{res}$ to $F \circ G(\gamma)$ and $F \circ G(\gamma')$ gives back $F \circ G(\gamma)$ and $F \circ G(\gamma')$, hence $\iota \circ H(\gamma) = F \circ G(\gamma)$ and $\iota \circ H(\gamma') = F \circ G(\gamma')$. Since the composition $F \circ G$ is full, it follows that there exists a morphism $f^\dagger \in \text{Hom}_{\mathcal{E}^\dagger}(\gamma, \gamma')$ such that $F \circ G(f^\dagger) = \iota(f)$. Applying res now yields $H(f^\dagger) = f$. Thus H is full.

Part (f). If B_{T^\dagger} is of finite representation type, then all B_{T^\dagger} -modules are string modules. Indeed, a band module would give rise to infinitely many isoclasses of indecomposable band modules. Thus H is dense by part (a). Since B_{T^\dagger} is of finite representation type it follows that the Auslander-Reiten quiver of B_{T^\dagger} has only one connected component and that each morphism between indecomposable B_{T^\dagger} -modules is given as a composition of finitely many irreducible morphisms. It follows from part (b) that H is full. \blacksquare

Remark 5.9. Since B_{T^\dagger} is a string algebra, it has two types of indecomposable modules: string modules and band modules, see [15]. The theorem shows that the string modules are given by the equivalence classes of permissible generalized arcs.

The band modules can be parameterized by triples (γ, ℓ, ϕ) where γ is a permissible closed curve corresponding to a generator of the fundamental group of S , ℓ is a positive integer, and ϕ is an automorphism of k^ℓ . The Auslander-Reiten structure for band modules is very simple: the irreducible morphisms are of the form $M(\gamma, \ell, \phi) \rightarrow M(\gamma, \ell + 1, \phi)$ and, if $\ell > 1$, $M(\gamma, \ell, \phi) \rightarrow M(\gamma, \ell - 1, \phi)$; whereas the Auslander-Reiten translation is the identity $\tau M(\gamma, \ell, \phi) = M(\gamma, \ell, \phi)$.

5.5. Finite representation type.

Corollary 5.10. *The surface algebra B_{T^\dagger} is of finite representation type if and only if no simple closed non-contractible curve in $(S, M^\dagger, T^\dagger)$ is permissible.*

Proof. Every permissible non-contractible closed curve defines infinitely many indecomposable band modules. Conversely, if no simple non-contractible closed curve is permissible, then B_{T^\dagger} has no band modules. It follows that B_{T^\dagger} is of finite representation type. ■

Corollary 5.11.

- (a) *If S is a disc, then B_{T^\dagger} is of finite representation type.*
- (b) *If S is an annulus, then B_{T^\dagger} is of finite representation type if and only if there is a quasi-triangle in T^\dagger with two vertices on one boundary component and two vertices on the other boundary component.*

Proof. Part (a). If S is a disc, then there are no simple closed non-contractible curves in S , and the result follows from Corollary 5.10.

Part (b). The fundamental group of S is generated by a closed curve γ that goes around the inner boundary component exactly once. This curve γ is permissible precisely when it does not cross two opposite sides of a quasi-triangle in T^\dagger , and this is precisely the case if there is no quasi-triangle with two vertices on the interior boundary component and two vertices on the exterior boundary component. ■

Example 5.12. Let (S, M, T) and $(S, M^\dagger, T^\dagger)$ be as in Figure 5.3. By Corollary 5.11, B_{T^\dagger} is of finite representation type. The Auslander-Reiten quiver is given in Figure 5.4, where modules are given by their Loewy series in the upper picture and by the corresponding generalized permissible arc on the surface in the lower picture.

6. A GEOMETRIC DESCRIPTION OF ITERATED TILTED ALGEBRAS OF TYPE \mathbb{A}_n OF GLOBAL DIMENSION AT MOST 2

In this section, we apply the results in section 3 to obtain a description of the admissible cut surface algebras of type (S, M, T) where S is a disc.

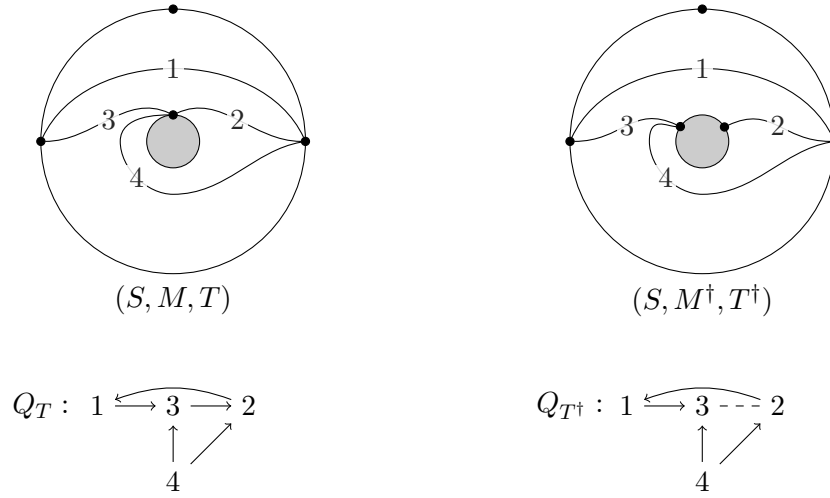


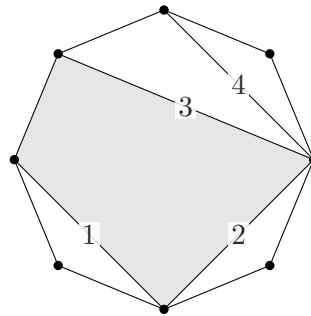
FIGURE 5.3. An admissible cut of the annulus whose corresponding surface algebra is of finite representation type

In this case, the Jacobian algebra B_T is a cluster-tilted algebra of type \mathbb{A} and B_{T^\dagger} is a quotient of B_T by an admissible cut. It has been show in [10] that the algebras B_{T^\dagger} obtained in this way are precisely the iterated tilted algebras of type \mathbb{A} of global dimension at most two.

Proposition 6.1. *Every iterated tilted algebra C of type \mathbb{A}_n of global dimension at most two is isomorphic to the endomorphism algebra of a partial cluster-tilting object in the cluster category of type $\mathbb{A}_{n+\ell}$, where ℓ is the number of relations in a minimal system of relations for C .*

Proof. Let $(S, M^\dagger, T^\dagger)$ be the admissible cut corresponding to C . Then $C \cong \text{End}_{\mathcal{C}(S, M^\dagger)}(T^\dagger)$ and (S, M^\dagger) is a disc with $n + \ell$ marked points. ■

Example 6.2. Let C be the tilted algebra given by the bound quiver $1 \overset{\dashrightarrow}{\leftarrow} 2 \overset{\dashrightarrow}{\rightarrow} 3 \rightarrow 4$. Then we have $n = 4$ and $\ell = 1$, so C is the endomorphism algebra of a partial cluster-tilting object in the cluster category of type \mathbb{A}_5 . By [16] this category can be seen as the category of diagonals in an octagon, and the partial cluster-tilting object corresponds to the following partial triangulation.



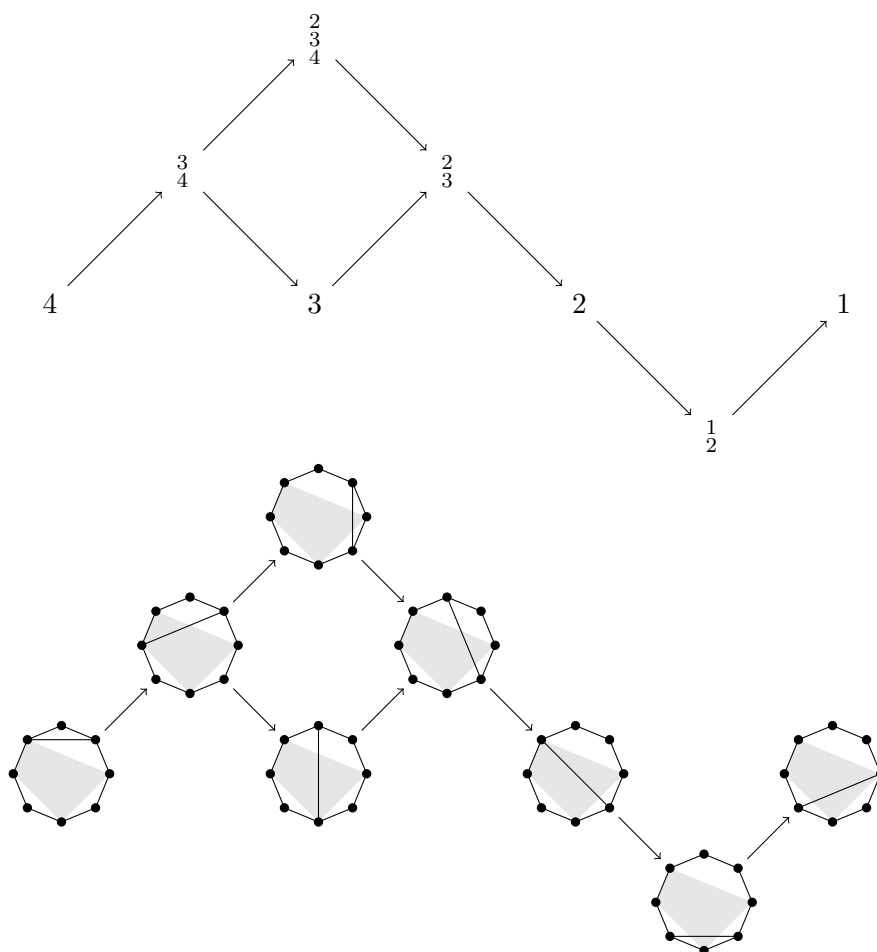


FIGURE 6.1. The Auslander-Reiten quiver of a tilted algebra of type A_4

quasi-triangle. Also note that each of the modules $\frac{3}{4}$, 3, 2 and 1, can be represented by two equivalent permissible arcs.

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