

THE LINEAR ORDERING POLYTOPE VIA REPRESENTATIONS

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ABSTRACT. Let P_n denote the n -th linear ordering polytope. We define projections from P_n to the n -th permutahedron and to the $(n-1)$ -st linear ordering polytope. Both projections are equivariant with respect to the natural S_n -action and they project to orthogonal subspaces. In particular the second projection defines an S_n -action in P_{n-1} .

1. INTRODUCTION

Let S_n denote the symmetric group. For a permutation $\pi \in S_n$ and $1 \leq i < j \leq n$ we set

$$k_{ij}(\pi) := \begin{cases} 1 & \text{if } \pi(i) > \pi(j) \\ 0 & \text{if } \pi(i) < \pi(j) \end{cases}$$

The n -th *Linear Ordering Polytope* P_n is defined as the convex hull of the $n!$ vectors $p_\pi := (k_{ij}(\pi))_{1 \leq i < j \leq n} \in \mathbb{R}^{\binom{n}{2}}$. In this we follow the definition given in [4]. The linear ordering polytope is an important and well-studied object in combinatorial optimization, see for example Chapter 6 of [6] and the references therein. For its study we will also consider the n -th *permutahedron*; this is the polytope with $n!$ vertices $(\pi(i))_{1 \leq i \leq n} \in \mathbb{R}^n$ for $\pi \in S_n$. Both polytopes carry a natural S_n -action and a \mathbb{Z}_2 -action, which we will explain in Section 3.

This is the main result of this note:

Theorem 1. *There is a scalar product on $\mathbb{R}^{\binom{n}{2}}$ such that there is an orthogonal decomposition $\mathbb{R}^{\binom{n}{2}} = V_1 \perp V_2$ into subspaces of dimensions $n-1$ and $\binom{n-1}{2}$, such that:*

- (1) *The orthogonal projection of P_n onto V_1 is the n -th permutahedron.*
- (2) *The orthogonal projection onto V_2 is P_{n-1} .*
- (3) *Both projections are equivariant with respect to the $\mathbb{Z}_2 \times S_n$ -action on P_n . This gives rise to a $\mathbb{Z}_2 \times S_n$ -action on P_{n-1} .*

If we apply the theorem repeatedly, we get

Corollary 1. *There is a scalar product on $\mathbb{R}^{\binom{n}{2}}$ such that there is an orthogonal decomposition $\mathbb{R}^{\binom{n}{2}} = V_n \perp V_{n-1} \perp \dots \perp V_2$ satisfying*

- (1) $\dim V_i = i - 1$.
- (2) *The orthogonal projection of P_n onto V_i is the i -th permutahedron.*

The pure existence of the vector space decomposition $V_1 \perp V_2$ in Theorem 1 was known before. First, it seems to have been known to experts in combinatorial optimization. There, $\mathbb{R}^{\binom{n}{2}}$ is considered as the space of functions on the edges of

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a complete directed graph. Then V_1 is the subspace of potentials (all functions f such that $f(i, j) = g(i) - g(j)$, $1 \leq i < j \leq n$ for some function g defined on the nodes). Moreover V_2 is the space of circulations (all functions f such that $\sum_i f(i, j) = \sum_k f(j, k)$ for all j). We thank S. Fiorini for bringing this fact to our attention.

Second, these spaces arise in the context of the analysis of rank data, see [5]. There the $\mathbb{R}^{\binom{n}{2}}$ is considered as the *space of pairs* and V_1 is the *space of means*. They correspond to the *Babington Smith Model* resp. the *Bradley/TerryMallows model*. In Section 7.4.3 in [5], there are also a basis of V_1 and an generating set of V_2 given. The later contains the basis we give in Section 3.

Third, the decomposition arises from representation theoretic considerations in [7]. In that paper it is also shown to be unique.

We also note that the existence of the $\mathbb{Z}_2 \times S_n$ -action on P_{n-1} is known from [2], where this action is constructed from the $\mathbb{Z}_2 \times S_{n-1}$ -action and a certain class of automorphisms borrowed from [1].

The projection $P_n \rightarrow P_{n-1}$ maps n vertices to one. In [2, Lemma 2] the *trivial* and *3-cycle* facets are shown to be the only facets having the maximal number of $n!/2$ vertices. Thus the preimages of these facets under the projection are also facets of the same type. Since facets inequalities for P_n are a major research topic, it might be interesting to consider the other known families of facets (see for example [6]) from this point of view.

The construction of the linear ordering polytope can be generalized to arbitrary finite Coxeter groups. See [3] for Type B and [7, Section 3.5] for the general case. There is still a decomposition into two invariant subspaces as in Theorem 1. The space V_1 generalizes as expected: The action of the group is the geometric representation, so it is still irreducible and the polytope can be described as the dual of the Coxeter complex. We know less about the other space V_2 : We do not know if the representation is still irreducible and we also do not know the polytope. One could expect the polytope to be the polytope associated to the Coxeter group where one generator from one end of the Dynkin diagram was removed. This would generalize the $(n-1)$ st linear ordering polytope encountered in Theorem 1. But this fails for dimensional reasons. The representation of the group on this space is isomorphic to the one called ψ_c in [8, Theorem 37]. In the prove of Theorem 36 in that article, a generating system for the Types A , D and E is given, but no basis seems to be known.

2. GENERAL CONSIDERATIONS

Consider the vector space \mathbb{R}^N with the usual scalar product $\langle \cdot, \cdot \rangle$ and the standard orthonormal basis e_1, \dots, e_N . Let $\Delta = \{ \sum_i \lambda_i e_i \mid 0 \leq \lambda_i \leq 1, 1 \leq i \leq N \}$ denote the standard simplex.

To every subspace $U \subset \mathbb{R}^N$ we can associate a polytope $P(U)$ as follows: Let $P(U)$ be the image of Δ under the orthogonal projection $\mathbb{R}^N \rightarrow U$. If $V \subset U$ is another subspace, the projection $\mathbb{R}^N \rightarrow V$ factors through U , giving a linear projection $P(U) \rightarrow P(V)$.

There is a second way to associate a polytope to a subspace $U \subset \mathbb{R}^N$: Choose a generating system b_1, \dots, b_l of U and write the vectors as rows into a matrix \mathcal{B} . Then we define $\mathfrak{P}(b_1, \dots, b_l)$ to be the convex hull of the column vectors of \mathcal{B} in the \mathbb{R}^l .

Lemma 1. *The polytopes $P(U)$ and $\mathfrak{P}(b_1, \dots, b_l)$ are affinely isomorphic.*

Proof. The map $\mathbb{R}^N \rightarrow \mathbb{R}^l$ defined by the matrix \mathcal{B} maps $P(U)$ bijectively onto $\mathfrak{P}(b_1, \dots, b_l)$. This can be checked by a short calculation. \square

Now we specialize to the situation where a finite group G acts on the set of basis vectors e_1, \dots, e_N of \mathbb{R}^N . We extend this action to a permutation representation of G . Note that the scalar product and the simplex Δ are invariant under this action. We further assume that $U \subset \mathbb{R}^N$ is an invariant subspace. Then $P(U)$ is invariant as well and the orthogonal projection is an equivariant mapping. Therefore, the action of G induces automorphisms of $P(U)$. If $V \subset U$ are both invariant subspaces, then the projection $P(U) \rightarrow P(V)$ is equivariant.

3. THE LINEAR ORDERING POLYTOPE

Consider the space of functions $\mathbb{R}^{S_n} = \{f : S_n \rightarrow \mathbb{R}\}$. There is a natural left action of the S_n on this space by $(\pi f)(\tau) := f(\tau\pi)$. We choose the canonical basis $e_\pi, \pi \in S_n$ defined by $e_\pi(\tau) = \delta_{\pi, \tau}$. This basis is permuted by the action: $\tau e_\pi = e_{\pi\tau^{-1}}$, hence we are in the situation considered at the end of Section 2. The vector space \mathbb{R}^{S_n} also carries a natural invariant scalar product $\langle f, g \rangle = \sum_{\pi \in S_n} f(\pi)g(\pi)$. Let $\mathbb{1}$ denote the constant function with value 1 on S_n .

Regard the k_{ij} defined in the introduction as functions in \mathbb{R}^{S_n} . Then the linear ordering polytope is $\mathfrak{P}(k_{ij}, 1 \leq i < j \leq n)$. We compute the action of the S_n on the k_{ij} :

$$\pi k_{ij} = \begin{cases} k_{\pi(i)\pi(j)} & \text{if } \pi(i) > \pi(j) \\ \mathbb{1} - k_{\pi(j)\pi(i)} & \text{if } \pi(i) < \pi(j) \end{cases}$$

Therefore, the vector space spanned by the k_{ij} is not invariant, but the space U_{Inv} generated by the k_{ij} together with $\mathbb{1}$ is. The polytope $\mathfrak{P}(\mathbb{1}, k_{ij}, 1 \leq i < j \leq n)$ is clearly affinely isomorphic to P_n because the $\mathbb{1}$ only adds a new coordinate with constant value 1 to the polytope, thus moving it into an affine hyperplane. Hence we can regard the linear ordering polytope as $P(U_{Inv})$. The left action of S_n is what is called *relabeling* action in [2]. There is an additional symmetry on this space: Let $\omega_0 \in S_n$ denote the reversal permutation defined by $\omega_0(1) = n, \omega_0(2) = n-1, \dots, \omega_0(n) = 1$. It defines a \mathbb{Z}_2 -action on \mathbb{R}^{S_n} via $(\omega_0 f)(\tau) := f(\omega_0\tau)$ which commutes with the S_n -action mentioned before. We compute $\omega_0 k_{ij} = \mathbb{1} - k_{ij}$, hence the space U_{Inv} is also invariant under ω_0 . This defines the *duality* automorphism of [2].

Next we decompose U_{Inv} into invariant subspaces. First we remove the 1-dimensional subspace spanned by $\mathbb{1}$. Denote it by $V_0 := \mathbb{R}\mathbb{1}$. A new basis for the complement \tilde{U}_{Inv} is given by

$$\tilde{k}_{ij}(\pi) := \begin{cases} 1 & \text{if } \pi(i) > \pi(j) \\ -1 & \text{if } \pi(i) < \pi(j) \end{cases}$$

for $1 \leq i < j \leq n$. Thus, we have $\tilde{k}_{ij} = 2k_{ij} - \mathbb{1}$. It is convenient to define also $\tilde{k}_{ii} := 0$ and $\tilde{k}_{ji} := -\tilde{k}_{ij}$ for $i < j$. The action of $\mathbb{Z}_2 \times S_n$ is

$$\begin{aligned} \pi \tilde{k}_{ij} &= \tilde{k}_{\pi(i)\pi(j)} \\ \omega_0 \tilde{k}_{ij} &= -\tilde{k}_{ij}. \end{aligned}$$

Thus the subspace \tilde{U}_{Inv} spanned by the \tilde{k}_{ij} without $\mathbb{1}$ is indeed an invariant subspace. Using the same argument as above, one sees that $\mathfrak{P}(\tilde{k}_{ij}, 1 \leq i < j \leq n)$ is isomorphic to P_n . Since every subspace of \tilde{U}_{Inv} is invariant under the ω_0 -action, we only need consider the S_n -action.

To understand the structure of \tilde{U}_{Inv} we consider an additional space: Let F be an n -dimensional vector space with basis e_1, \dots, e_n and the standard S_n -representation $\pi e_i := e_{\pi(i)}$. Then the mapping $\psi : F \wedge F \rightarrow \tilde{U}_{Inv}$, $e_i \wedge e_j \mapsto \tilde{k}_{ij}$ is an equivariant isomorphism.

It is well-known that F decomposes into a direct sum of two invariant subspaces: F_0 with basis $\sum e_i$ and F_1 with basis $e_i - e_n, 1 \leq i \leq n-1$. A short computation yields the decomposition of $\tilde{U}_{Inv} \cong F \wedge F = F_1 \wedge F_0 \oplus F_1 \wedge F_1$. For both summands it is known that they are irreducible, so we found the decomposition of \tilde{U}_{Inv} . Since the two subspaces are irreducible and non-isomorphic as representations, they are orthogonal with respect to the invariant scalar product. Let V_1 and V_2 denote the spaces isomorphic to $F_0 \wedge F_1$ reps. $F_1 \wedge F_1$. We consider these spaces separately:

3.1. The space V_1 . We can get a basis for V_1 by applying the isomorphism ψ to any basis of F_1 . However, to see that $P(V_1)$ is a permutahedron, we use a particular generating system:

$$v_i := \sum_{j=1}^n \tilde{k}_{ij}$$

It is easy to see that $\tau v_i = v_{\tau(i)}$. This is the same S_n -representation as on V_1 . Since we already know that \tilde{U}_{Inv} has only one invariant subspace with this representation, we conclude that the subspace generated by $v_j, 1 \leq j \leq n$ is V_1 . A short calculation reveals that $v_i(\pi) = 2\pi(i) - (n+1)$. Therefore, $\mathfrak{P}(v_1, \dots, v_n)$ is affinely isomorphic to a permutahedron.

3.2. The space V_2 . Recall that $e_i - e_n, 1 \leq i \leq n-1$ is a basis for F_1 . Hence $(e_i - e_n) \wedge (e_j - e_n), 1 \leq i < j \leq n-1$ is a basis of $F_1 \wedge F_1$. We apply ψ and get the following basis of V_2 :

$$\tilde{w}_{ij} := \tilde{k}_{ij} - \tilde{k}_{in} + \tilde{k}_{jn}$$

for $1 \leq i < j \leq n-1$. Define \mathfrak{c} to be the cycle $(12\dots n) \in S_n$ and let C_n be the subgroup generated by \mathfrak{c} . Note that $\tilde{w}_{ij}(\mathfrak{c}\pi) = \tilde{w}_{ij}(\pi)$ for every $i < j$ and $\pi \in S_n$. This can be checked directly from the definition of the \tilde{w}_{ij} . Thus vertices of $\mathfrak{P}(\tilde{w}_{ij}, 1 \leq i < j \leq n)$ can be labelled with the right cosets of C_n . For each coset $C_n\pi$ there is a unique representative $\pi' \in S_n$ with $\pi(n) = n$ and $\tilde{w}_{ij}(\pi) = \tilde{w}_{ij}(\pi')$. Therefore, the polytope does not change if we restrict to the subspace of permutations fixing n . This can be naturally identified with $\mathbb{R}^{S_{n-1}}$. But for $\pi \in S_n$ with $\pi(n) = n$, we have $\tilde{w}_{ij}(\pi) = \tilde{k}_{ij}(\pi)$. Hence, the polytope is the $(n-1)$ -st linear ordering polytope.

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