Spectral Properties of Descent Algebra Elements

Hery Randriamaro*
Fachbereich Mathematik und Informatik
Philipps-Universität Marburg
D-35032 Marburg
herand@mathematik.uni-marburg.de

May 3, 2014

Abstract

The descent algebra of finite Coxeter groups is studied by many famous mathematicians like Bergeron, Brown, Howlett, or Reutenauer. Blessenohl, Hohlweg, and Schocker, for example, proved a symmetry property of the descent algebra, when it is linked to the representation theory of its Coxeter group. The interest is particularly showed for the descent algebra of symmetric group. Thibon determined the eigenvalues and their multiplicities of the action on the group algebra of symmetric group of the descent algebra element, which is the sum over all permutations weighted by $q^{\mathtt{maj}}$. And even the author diagonalized the matrix of the action of the descent algebra element, which is the sum over all permutations weighted by the new introduced statistic $\mathtt{des}_X$. In this article, we give a more general result by determining the eigenvalues and their multiplicities of the action on the group algebra of finite Coxeter group of an element of its descent algebra.

1 Introduction

We keep the usual notations $\mathbb{K}$ for an algebraically closed field of characteristic 0, and (W, S) for a finite Coxeter system, that is to say, W is a finite group generated by the elements of S subject to the defining relations

$$(sr)^{m_{sr}} = e, \text{ for all } s, r \in S$$

where e is the neutral element of W, the m_{sr} are positive integers, and $m_{ss} = 1$ for all $s \in S$.

Let $J \subseteq S$. We naturally use the notations W_J for the parabolic subgroup of W generated by the elements of J, and c_J for a Coxeter element of W_J which is a product of the elements of J taken in some fixed order. We write $\overline{c_J}$ for the conjugacy class of c_J.

Let $J \subseteq S$. We write $\widetilde{J}$ for the set of subsets $K \subseteq S$ such that the parabolic subgroups W_J and W_K are conjugate. Let $\widetilde{J}_1, \dots, \widetilde{J}_p$ be pairwise different such that

$$\{\widetilde{J}_i\}_{i\in[p]} = \{\widetilde{J}\}_{J\subseteq S}.$$

Let $J, K \subseteq S$. We write ${}^JW^K$ for the distinguished cross section for the double coset space $W_J\backslash W/W_K$. If $J = \{\varnothing\}$ resp. $K = \{\varnothing\}$, we just write W^K resp. ${}^JW$.

*This research was supported by DAAD.

Let $J \subseteq S$. We write

$$x_J := \sum_{w \in W^J} w.$$

Let $\Xi_W := \{x_J \,|\, J \subseteq S\}$. We know that $\mathbb{K}[\Xi_W]$ is the descent algebra of W which is a subalgebra of the group algebra $\mathbb{K}[W]$ multiplicatively equipped with [18, Theorem 1]:

$$x_J x_K = \sum_{L \subseteq K} a_{JKL} x_L$$

for $J, K \subseteq S$, with

$$a_{JKL} := \big|\{x \in {}^J W^K \mid x^{-1} W_J \, x \cap W_K = W_L\}\big|.$$

Let $u = \sum_{w \in W} \lambda_w w \in \mathbb{K}[W]$. We write $R_W(u) = (\lambda_{ww'^{-1}})_{w,\, w' \in W}$ for the matrix of the regular representation of the left-multiplication action of u on $\mathbb{K}[W]$ relatively to the standard basis $\{w \,|\, w \in W\}$. The purpose of this article is to prove the following theorem:

Theorem 1.1. *Let $d = \sum_{J \subseteq S} \lambda_J x_J \in \mathbb{K}[\Xi_W]$. Then the spectrum of $R_W(d)$ is*

$$Sp\big(R_W(d)\big) = \Big\{\Delta_j = \sum_{i=1}^{p} a_{J_i J_j J_j}\big(\sum_{K_i \in \widetilde{J}_i} \lambda_{K_i}\big)\Big\}_{j \in [p]}$$

with corresponding multiplicities

$$\big\{m_{\Delta_j} = |\overline{c_{J_j}}|\big\}_{j \in [p]}.$$

Let $w \in W$. Recall that the set of left resp. right descent set of w is

$$\mathtt{DES_L}(w) := \{s \in S \,|\, \mathtt{l}(sw) < \mathtt{l}(w)\} \text{ resp. } \mathtt{DES_R}(w) := \{s \in S \,|\, \mathtt{l}(ws) < \mathtt{l}(w)\},$$

$\mathtt{l}$ being the statistic which gives the length of the minimal expression of the elements of W in terms of elements of S.
Let $J \subseteq S$. We write D_J for the set of elements of W with right descent set J. We have the disjoint union

$$W^{S \setminus K} = \bigcup_{J \subseteq K} D_J.$$

Thus setting

$$y_J := \sum_{w \in D_J} w,$$

we get

$$x_{S \setminus K} = \sum_{J \subseteq K} y_J,$$

and hence by Möbius inversion formula

$$y_J = \sum_{K \subseteq J} (-1)^{|J \setminus K|} x_{S \setminus K}.$$

The descent algebra is a well studied object on the borderline of combinatorics and algebra ([1], [2], [4]). Especially the descent algebra relative to the Coxeter system of the symmetric group

$(A_n, S_{A_n} = \{t_1, \ldots, t_n\})$, where t_i is the transposition $(i\; i+1)$ ([7], [11]). Thibon determined the eigenvalues and their multiplicities of the action of the element [10, Theorem 56]

$$\sum_{J \subseteq S_{A_n}} q^{\mathtt{Maj}(J)}\, y_J \in \mathbb{R}(q)[\Xi_{A_n}],$$

where

$$\mathtt{Maj}(J) := \sum_{j \in \{i \in [n] \,|\, t_i \in J\}} j.$$

In [17], Schocker worked on the descent algebra of the symmetric group. Using hyperplanes arrangement and random walk properties ([5], [6]), Brown determined the condition of diagonalizability of the action of element of the descent algebra of finite Coxeter group, and gave a remarkable approach of the eigenvalues and corresponding multiplicities of its regular representation. In [12], the regular representation of the element

$$\sum_{J \subseteq S_{A_n}} \mathtt{Des}_X(J)\, y_J \in \mathbb{R}(X_1, \ldots, X_n)[\Xi_{A_n}],$$

where

$$\mathtt{Des}_X(J) := \sum_{j \in \{i \in [n] \,|\, t_i \in J\}} X_j,$$

was diagonalized

This paper is organized as follows. We begin with the calculation of the formula for the coefficients $a_{J_i J_j J_j}$. Then, we determine the eigenvalues and their corresponding multiplicities of the regular representation of an element of the descent algebra of a finite Coxeter group. In the appendix part, we treat the Coxeter group F_4 as complete example, and we give a counterexample of the formula in [2, Theorem 6.5].

2 Special Coefficients of the Descent Algebra

In this section, we determine a formula for the values of a_{JKK}. Through a slight modification, our formula corrects a mistake from the formula proposed in [2, Theorem 6.5].
Let $J \subseteq S$. We write N_J for the subgroup [8, Corollary 3]

$$N_J := \{w \in W \,|\, w^{-1} W_J w = W_J\} \cap W^J.$$

Let $w \in W$ and U, V be subgroups of W. We write

$${}^U w^V := \{x \in UwV \mid \mathtt{l}(x) \leq \mathtt{l}(uxv),\, \forall u \in U,\, \forall v \in V\}$$

for the set of minimal double coset representatives of w relative to U and V. If $U = \{e\}$ resp. $V = \{e\}$, we just write w^V resp. ${}^U w$. For the case of parabolic subgroups, we just write ${}^{W_J} w^{W_K} = {}^J w^K$ with $J, K \subseteq S$. This lemma can be read off from [18, 1.Introduction].

Lemma 2.1. *Let $w \in W$ and $J, K \subseteq S$. Then the set ${}^J w^K$ contains a unique element. In this case, we consider ${}^J w^K$ no more as a subset of W but as an element of W.*

Let $K \subseteq S$ and $K' \in \widetilde{K}$. We write

$$C_{K'K} := \{w \in W \mid w^{-1} W_{K'}\, w = W_K\}.$$

Lemma 2.2. *Let $c \in C_{K'K}$, and $c_{K'K} \in {}^{N_{K'}W_{K'}}c^{N_K W_K}$. Then $c_{K'K} \in C_{K'K}$.*

Proof. If $c = n'\,k'\,c_{K'K}\,n\,k$ with $n'k' \in N_{K'}W_{K'}$ and $nk \in N_K W_K$, then

$$\begin{aligned}
(n'\,k'\,c_{K'K}\,n\,k)^{-1}W_{K'}\,n'\,k'\,c_{K'K}\,n\,k &= W_K,\\
(c_{K'K}\,n\,k)^{-1}W_{K'}\,c_{K'K}\,n\,k &= W_K,\\
c_{K'K}^{-1}W_{K'}\,c_{K'K} &= n\,k\,W_K\,(n\,k)^{-1},\\
c_{K'K}^{-1}W_{K'}\,c_{K'K} &= W_K.
\end{aligned}$$

□

Let $E \subseteq W$ and U, V be subgroups of W. We write

$${}^U E^V := \bigcup_{w \in E} {}^U w^V.$$

For the case of parabolic subgroups, we just write ${}^{W_J}E^{W_K} = {}^J E^K$, with $J, K \subseteq S$.

Lemma 2.3. *Let $K \subseteq S$, $K' \in \widetilde{K}$, and $c_{K'K} \in {}^{N_{K'}W_{K'}}C_{K'K}{}^{N_K W_K}$. Then*

$${}^{K'}(c_{K'K}N_K)^K = c_{K'K}N_K \text{ and } {}^{K'}(N_{K'}\,c_{K'K})^K = N_{K'}\,c_{K'K}.$$

Proof. It is clear that $(c_{K'K}N_K)^K = c_{K'K}N_K$. Effectively, since $c_{K'K}$ is a left coset representative of the subgroup $N_K W_K$, and the elements of N_K are left coset representatives of W_K, then the elements of $c_{K'K}N_K$ are left coset representatives of W_K.
We just then have to prove that ${}^{K'}(c_{K'K}N_K) = c_{K'K}N_K$. Let $c_{K'K}\,n \in c_{K'K}N_K$ and let us suppose that $c_{K'K}\,n = k'b$, where $k' \in W_{K'}$ and $b \in {}^{K'}W$. Then $\mathtt{l}(c_{K'K}\,n) = \mathtt{l}(k'b) = \mathtt{l}(k') + \mathtt{l}(b)$ i.e.

$$\mathtt{l}(c_{K'K}\,n) \geq \mathtt{l}(b).$$

On the other hand, we have $(k')^{-1}c_{K'K}\,n = c_{K'K}\,k_1 n = c_{K'K}\,n k_2 = b$ with $k_1, k_2 \in W_K$. Then $\mathtt{l}(c_{K'K}\,nk_2) = \mathtt{l}(c_{K'K}\,n) + \mathtt{l}(k_2) = \mathtt{l}(b)$, i.e.

$$\mathtt{l}(c_{K'K}\,n) \leq \mathtt{l}(b).$$

The only possibility is then $k_2 = k_1 = k' = e$, so we get the result.
The proof for ${}^{K'}(N_{K'}\,c_{K'K})^K = N_{K'}\,c_{K'K}$ is analogous. □

Lemma 2.4. *Let $K \subseteq S$, $K' \in \widetilde{K}$, and $c_{K'K} \in {}^{N_{K'}W_{K'}}C_{K'K}{}^{N_K W_K}$. Then*

$$\{w \in {}^{K'}W^K \mid w^{-1}W_{K'}\,w = W_K\} = c_{K'K}N_K = N_{K'}\,c_{K'K}.$$

Proof. It is clear that $\{w \in {}^K W^K \mid w^{-1}W_K\,w = W_K\} = N_K$ for all $K \subseteq S$.

- The map $\phi : \{w \in {}^{K'}W^{K'} \mid w^{-1}W_{K'}\,w = W_{K'}\} \to \{w \in {}^{K'}W^K \mid w^{-1}W_{K'}\,w = W_K\}$

  $$n \mapsto n\,c_{K'K}$$

  is clearly injective.

- The map $\phi' : \{w \in {}^{K'}W^K \mid w^{-1}W_{K'}\, w = W_K\} \to \{w \in {}^{K'}W^{K'} \mid w^{-1}W_{K'}\, w = W_{K'}\}$,

$$x \mapsto {}^{K'}(c_{K'K}\, x^{-1})^{K'}$$

is injective. Effectively, $\phi'(x) = \phi'(y)$ means $c_{K'K}\, x^{-1} = u_1 c_{K'K}\, y^{-1} u_2$ with $u_1, u_2 \in W_{K'}$. Then $c_{K'K}\, x^{-1} = c_{K'K} v y^{-1} u_2$ and $x^{-1} = v y^{-1} u_2$ with $v \in W_K$. The only possibility is $v = u_2 = e$.

Then we deduce that ϕ is bijective and $\{w \in {}^{K'}W^K \mid w^{-1}W_{K'}\, w = W_K\} = N_{K'}\, c_{K'K}$.
The proof is analogous for $\{w \in {}^{K'}W^K \mid w^{-1}W_{K'}\, w = W_K\} = c_{K'K} N_K$. □

Let $J, K \subseteq S$, and, for all $K' \in \widetilde{K}$, let us fixe an element $c_{K'K} \in {}^{N_{K'}W_{K'}}C_{K'K}{}^{N_K W_K}$. We write

$$H_{JK} := \{K' \in \widetilde{H} \mid c_{K'K} = {}^J c_{K'K}\}.$$

We can now give the formula to determine a_{JKK}.

Theorem 2.5. *Let $J, K \subseteq S$. We have*

$$a_{JKK} = \sum_{K' \in H_{JK} \cap 2^J} \frac{|N_K|}{|W_J \cap N_{K'}|}.$$

Proof. Recall that

$$a_{JKK} := \left|\{w \in {}^J W^K \mid W_J \cap wW_K w^{-1} = wW_K w^{-1}\}\right|,$$

with $wW_K w^{-1} = W_{K'}$ and $K' \in \widetilde{K} \cap 2^J$.
We have $w \in c_{K'K} N_K = N_{K'}\, c_{K'K}$. But we must also have $w = {}^J w$. That means:

- On the one hand, we must have $c_{K'K} = {}^J c_{K'K}$. Otherwise ${}^J(c_{K'K} N_K) \cap C_{K'K} = \varnothing$.
- On the other hand, if $c_{K'K} = {}^J c_{K'K}$, then $w \in {}^J(N_{K'}\, c_{K'K}) = ({}^J N_{K'})\, c_{K'K}$.

Since

$$\left|{}^J N_{K'}\right| = \frac{|N_{K'}|}{|W_J \cap N_{K'}|},$$

it follows that

$$\begin{aligned} a_{JKK} &= \sum_{K' \in H_{JK} \cap 2^J} \frac{|N_{K'}|}{|W_J \cap N_{K'}|} \\ &= \sum_{K' \in H_{JK} \cap 2^J} \frac{|N_K|}{|W_J \cap N_{K'}|}. \end{aligned}$$

since W_K and $W_{K'}$ are conjugate. □

Let $J' \in \widetilde{J}$ and $K' \in \widetilde{K}$. From Theorem 2.5, we deduce that

$$a_{JKK} = a_{J'K'K'}.$$

This result can also be found in [2, Theorem 6.2].

3 Eigenvalues and Multiplicities

We are now able to determine the eigenvalues and their corresponding multiplicities.
Let $d = \sum_{J\subseteq S} \lambda_J x_J \in \mathbb{K}[\Xi_W]$. We write $v_{\Xi_W}(d)$ for the column vector of d relative to the basis Ξ_W, and $M_{\Xi_W}(d)$ for the matrix of the left-multiplication action of d on $\mathbb{K}[\Xi_W]$ relative the basis Ξ_W i.e.

$$v_{\Xi_W}(d) = (\lambda_K)_{K\subseteq S} \ \text{ and } \ M_{\Xi_W}(d) = \big(\sum_{J\subseteq S} \lambda_J a_{JKL}\big)_{K,L\subseteq S}.$$

Let $n \geq 2$ and $(d_i)_{i\in[n]} \in \mathbb{K}[\Xi_W]^n$. We have

$$v_{\Xi_W}\big(\overrightarrow{\prod_{i\in[n]}} d_i\big) = \big(\overrightarrow{\prod_{i\in[n-1]}} M_{\Xi_W}(d_i)\big) \cdot v_{\Xi_W}(d_n).$$

Recall that the noncommutative multiplication is defined in the following way:

$$\overrightarrow{\prod_{i\in[n]}} \lambda_i = \lambda_1\lambda_2\ldots\lambda_n.$$

We write $M_{\Xi_W}(d)_{|\bullet,K}$ for the column of $M_{\Xi_W}(d)$ corresponding to the basis vector x_K, i.e.

$$M_{\Xi_W}(d)_{|\bullet,K} = \big(\sum_{J\subseteq S} \lambda_J a_{JKL}\big)_{L\subseteq S} = v_{\Xi_W}(d\cdot x_K).$$

Lemma 3.1. *Let $d \in \mathbb{K}[\Xi_W]$. Then $R_W(d)$ and $M_{\Xi_W}(d)$ have the same spectrum.*

Proof. It is clear that $Sp\big(M_{\Xi_W}(d)\big) \subseteq Sp\big(R_W(d)\big)$.
We write $0_{2^{|S|}}$ for the matrix with entry 0 of $\mathbb{K}^{2^{|S|}\times 2^{|S|}}$, and $I_{2^{|S|}}$ for the identity matrix of $\mathbb{K}^{2^{|S|}\times 2^{|S|}}$. Let $\sum_{i=0}^{2^{|S|}} \mu_i t^i$ be the characteristic polynomial of $M_{\Xi_W}(d)$ in the variable t with $(\mu_i)_{i\in\{0\}\cup[2^{|S|}]} \in \mathbb{K}^{2^{|S|}+1}$. We have $\sum_{i=0}^{2^{|S|}} \mu_i M_{\Xi_W}(d)^i = 0_{2^{|S|}}$, especially

$$\mu_0 I_{2^{|S|}|\bullet,S} + \sum_{i=1}^{2^{|S|}} \mu_i M_{\Xi_W}^{i-1}(d)\cdot M_{\Xi_W}(d)_{|\bullet,S} = 0_{2^{|S|}|\bullet,S}.$$

Since $I_{2^{|S|}|\bullet,S} = v_{\Xi_W}(e)$, and $M_{\Xi_W}(d)_{|\bullet,S} = v_{\Xi_W}(d)$, then

$$\mu_0 v_{\Xi_W}(e) + \sum_{i=1}^{2^{|S|}} \mu_i\, M_{\Xi_W}^{i-1}(d)\cdot v_{\Xi_W}(d) = v_{\Xi_W}(0).$$

This means $\mu_0 e + \sum_{i=1}^{2^{|S|}} \mu_i d^i = 0$ and $Sp\big(R_W(d)\big) \subseteq Sp\big(M_{\Xi_W}(d)\big)$. □

For the rest of the section, we need a total order $\succ$ on the subsets of $S = \{s_i\}_{i\in\big[|S|\big]}$ which was introduced by F. and N. Bergeron [1]: We define $\min J := \min\{i \in \big[|S|\big] \mid s_i \in J\}$, and assume that $\min\varnothing = |S|+1$. Let $J, K \subseteq S$ such that $J \neq K$.

- If $\min J > \min K$ then $J \succ K$.

- Otherwise $J \succ K$ if and only if $J \setminus \{s_{\min J}\} \succ K \setminus \{s_{\min K}\}$.

We have already seen the definition of the set $\{\widetilde{J}_i\}_{i\in[p]}$ in the introduction.
Let K_i be the element of $\widetilde{J}_i$ such that $L_i \succ K_i$ for all $L_i \in \widetilde{J}_i \setminus \{K_i\}$. We order the sets $\{\widetilde{J}_i\}_{i\in[p]}$ such that $K_i \succ K_j$ if $i < j$.

Proposition 3.2. *Let $d = \sum_{J\subseteq S} \lambda_J x_J \in \mathbb{K}[\Xi_W]$. Then the spectrum of the matrix $R_W(d)$ is*

$$Sp\big(R_W(d)\big) = \Big\{ \sum_{i=1}^{p} a_{K_iK_jK_j} \big(\sum_{L_i\in\widetilde{J}_i} \lambda_{L_i} \big) \Big\}_{j\in[p]}.$$

Proof. We order the basis $(x_J)_{J\subseteq S}$ of $\mathbb{K}[\Xi_W]$ according to the total order $\succ$ of F. and N. Bergeron that means we get a new ordered basis $(x_{L_i})_{i\in[2^{|S|}]}$ such that $L_i \succ L_j$ if $i < j$. We assume that the rows and columns of the matrix $M_{\Xi_W}(d)$ are ordered in increasing order by the order $\succ$. We get $d \cdot x_{L_j} \in \big\langle \{x_{L_i}\}_{i\in[j]} \big\rangle$. Then the matrix of d on the basis $(x_{L_i})_{i\in[2^{|S|}]}$ is an upper triangular matrix. The characteristic polynomial of this matrix in the variable t is

$$\prod_{K\subseteq S} \big((\sum_{J\subseteq S} \lambda_J a_{JKK}) - t \big),$$

and

$$Sp(M_{\Xi_W}(d)) = \big\{ \sum_{J\subseteq S} \lambda_J a_{JKK} \big\}_{K\subseteq S}.$$

Since $a_{JKK} = a_{J'K'K'}$ for all $J, J' \in \widetilde{J}_i$, and all $K, K' \in \widetilde{J}_j$, we get the result. ◻

Proposition 3.3. *Let $d = \sum_{J\subseteq S} \lambda_J x_J \in \mathbb{K}[\Xi_W]$, and $\Delta_j = \sum_{i=1}^{p} a_{K_iK_jK_j} \big(\sum_{L_i\in\widetilde{J}_i} \lambda_{L_i} \big)$. Then the multiplicity of the eigenvalue Δ_j of $R_W(d)$ is*

$$m_{\Delta_j} = |\overline{c_{J_j}}|.$$

Proof. Let $A = (a_{K_iK_jK_j})_{i,j\in[p]}$, $m = (m_{\Delta_j})_{j\in[p]}$, $u = (|W|)_{j\in[p]}$, and $c = (|\overline{c_{J_j}}|)_{j\in[p]}$. At the end of the sixth section of [2], it is proved that $A^{-1}u = c$.
Let tr be the trace map of square matrix. We have $tr\big(R_W(d)\big) = |W| \sum_{J\subseteq S} \lambda_J$. Then, $\sum_{i=1}^{j} a_{K_jK_iK_i} m_{\Delta_i} \lambda_{L_j} = |W|\lambda_{L_j}$ for $L_j \in \widetilde{J}_j$ i.e.

$$\sum_{i=1}^{j} a_{K_jK_iK_i} m_{\Delta_i} = |W|.$$

In matrix form, we get $Am = u$. Thus $A^{-1}u = m$. ◻

We note that with the matrix relation $Am = u$, we can also get the cardinalities of the conjugacy classes of W.

A Example of the Symmetry Group of 24-cell

Recall that the Coxeter system of the symmetry group of 24-call with cardinality 1152 is $(F_4, S_{F_4} = \{s_i\}_{i\in[4]})$, and its Coxeter graph is

$$s_1 \longleftrightarrow s_2 \overset{4}{\longleftrightarrow} s_3 \longleftrightarrow s_4$$

Using Theorem 2.5 and the values of $|N_K|$ in [8, page 74], we get the following values of a_{JKK} for the case of F_4:

	$\varnothing$	$\{s_1\}$	$\{s_4\}$	$\{s_1,s_2\}$	$\{s_2,s_3\}$	$\{s_3,s_4\}$	$\{s_1,s_4\}$	$\{s_1,s_2,s_3\}$	$\{s_2,s_3,s_4\}$	$\{s_1,s_3,s_4\}$	$\{s_1,s_2,s_4\}$	S_{F_4}
$\varnothing$	1152	0	0	0	0	0	0	0	0	0	0	0
$\{s_1\}$	576	48	0	0	0	0	0	0	0	0	0	0
$\{s_4\}$	576	0	48	0	0	0	0	0	0	0	0	0
$\{s_1,s_2\}$	192	48	0	12	0	0	0	0	0	0	0	0
$\{s_2,s_3\}$	144	24	24	0	8	0	0	0	0	0	0	0
$\{s_3,s_4\}$	192	0	48	0	0	12	0	0	0	0	0	0
$\{s_1,s_4\}$	288	24	24	0	0	0	4	0	0	0	0	0
$\{s_1,s_2,s_3\}$	24	24	6	12	4	0	0	2	0	0	0	0
$\{s_2,s_3,s_4\}$	24	6	24	0	4	12	0	0	2	0	0	0
$\{s_1,s_3,s_4\}$	96	8	24	0	0	6	4	0	0	2	0	0
$\{s_1,s_2,s_4\}$	96	24	8	6	0	0	4	0	0	0	2	0
S_{F_4}	1	1	1	1	1	1	1	1	1	1	1	1

We consider the element

$$d = \sum_{J\subseteq S_{F_4}} \lambda_J x_J \in \mathbb{K}[\Xi_{F_4}].$$

The eigenvalues of $R_{F_4}(u)$ are

$$\begin{aligned}
\varDelta_1 &= 1152\lambda_\varnothing + 576(\lambda_{\{s_1\}} + \lambda_{\{s_2\}} + \lambda_{\{s_3\}} + \lambda_{\{s_4\}}) + 192(\lambda_{\{s_1,s_2\}} + \lambda_{\{s_3,s_4\}}) + 144\lambda_{\{s_2,s_3\}} \\
&\quad +288(\lambda_{\{s_1,s_3\}} + \lambda_{\{s_1,s_4\}} + \lambda_{\{s_2,s_4\}}) + 24(\lambda_{\{s_1,s_2,s_3\}} + \lambda_{\{s_2,s_3,s_4\}}) + 96(\lambda_{\{s_1,s_3,s_4\}} + \lambda_{\{s_1,s_2,s_4\}}) \\
&\quad +\lambda_{S_{F_4}} \\
\varDelta_2 &= 48(\lambda_{\{s_1\}} + \lambda_{\{s_2\}} + \lambda_{\{s_1,s_2\}}) + 24(\lambda_{\{s_2,s_3\}} + \lambda_{\{s_1,s_3\}} + \lambda_{\{s_1,s_4\}} + \lambda_{\{s_2,s_4\}} + \lambda_{\{s_1,s_2,s_3\}} + \lambda_{\{s_1,s_2,s_4\}}) \\
&\quad +6\lambda_{\{s_2,s_3,s_4\}} + 8\lambda_{\{s_1,s_3,s_4\}} + \lambda_{S_{F_4}} \\
\varDelta_3 &= 48(\lambda_{\{s_3\}} + \lambda_{\{s_4\}} + \lambda_{\{s_3,s_4\}}) + 24(\lambda_{\{s_2,s_3\}} + \lambda_{\{s_1,s_3\}} + \lambda_{\{s_1,s_4\}} + \lambda_{\{s_2,s_4\}} + \lambda_{\{s_2,s_3,s_4\}} + \lambda_{\{s_1,s_3,s_4\}}) \\
&\quad +6\lambda_{\{s_1,s_2,s_3\}} + 8\lambda_{\{s_1,s_2,s_4\}} + \lambda_{S_{F_4}} \\
\varDelta_4 &= 12\lambda_{\{s_1,s_2\}} + 12\lambda_{\{s_1,s_2,s_3\}} + 6\lambda_{\{s_1,s_2,s_4\}} + \lambda_{S_{F_4}} \\
\varDelta_5 &= 8\lambda_{\{s_2,s_3\}} + 4(\lambda_{\{s_1,s_2,s_3\}} + \lambda_{\{s_2,s_3,s_4\}}) + \lambda_{S_{F_4}} \\
\varDelta_6 &= 12(\lambda_{\{s_3,s_4\}} + \lambda_{\{s_2,s_3,s_4\}}) + 6\lambda_{\{s_1,s_3,s_4\}} + \lambda_{S_{F_4}} \\
\varDelta_7 &= 4(\lambda_{\{s_1,s_3\}} + \lambda_{\{s_1,s_4\}} + \lambda_{\{s_2,s_4\}}) + \lambda_{\{s_1,s_3,s_4\}} + \lambda_{\{s_1,s_2,s_4\}}) + \lambda_{S_{F_4}} \\
\varDelta_8 &= 2\lambda_{\{s_1,s_2,s_3\}} + \lambda_{S_{F_4}} \\
\varDelta_9 &= 2\lambda_{\{s_2,s_3,s_4\}} + \lambda_{S_{F_4}} \\
\varDelta_{10} &= 2\lambda_{\{s_1,s_3,s_4\}} + \lambda_{S_{F_4}} \\
\varDelta_{11} &= 2\lambda_{\{s_1,s_2,s_4\}} + \lambda_{S_{F_4}} \\
\varDelta_{12} &= \lambda_{S_{F_4}}
\end{aligned}$$

with corresponding multiplicities

$$\begin{array}{rcccc} m_{\Delta_1} & = & 1 & = & |\overline{e}| \\ m_{\Delta_2} & = & 12 & = & |\overline{s_1}| \\ m_{\Delta_3} & = & 12 & = & |\overline{s_4}| \\ m_{\Delta_4} & = & 32 & = & |\overline{s_1 s_2}| \\ m_{\Delta_5} & = & 54 & = & |\overline{s_2 s_3}| \\ m_{\Delta_6} & = & 32 & = & |\overline{s_3 s_4}| \\ m_{\Delta_7} & = & 72 & = & |\overline{s_1 s_4}| \\ m_{\Delta_8} & = & 84 & = & |\overline{s_1 s_2 s_3}| \\ m_{\Delta_9} & = & 84 & = & |\overline{s_2 s_3 s_4}| \\ m_{\Delta_{10}} & = & 96 & = & |\overline{s_1 s_3 s_4}| \\ m_{\Delta_{11}} & = & 96 & = & |\overline{s_1 s_2 s_4}| \\ m_{\Delta_{12}} & = & 577 & = & |\overline{s_1 s_2 s_3 s_4}| \end{array}$$

B Counterexample on the Special Coefficients

The following formula is proposed in [2, Theorem 6.5]: For $J, K \subseteq S$,

$$a_{JKK} = \sum_{K' \in \widetilde{K} \cap 2^J} \frac{|N_K|}{|W_J \cap N_{K'}|}.$$

Recall that the Coxeter system of the symmetry group of the dodecahedron with cardinality 120 is $(H_3, S_{H_3} = \{s_i\}_{i \in [3]})$, and its Coxeter graph is

$$s_1 \overset{5}{\longleftrightarrow} s_2 \longleftrightarrow s_3$$

We have the values of $|N_K|$ for H_3 in [8, page 79]. If we use the formula in [2, Theorem 6.5], we get the following values of a_{JKK} for H_3:

	$\varnothing$	$\{s_1\}$	$\{s_1, s_2\}$	$\{s_2, s_3\}$	$\{s_1, s_3\}$	S_{H_3}
$\varnothing$	120	0	0	0	0	0
$\{s_1\}$	60	4	0	0	0	0
$\{s_1, s_2\}$	12	8	2	0	0	0
$\{s_2, s_3\}$	20	8	0	2	0	0
$\{s_1, s_3\}$	30	4	0	0	2	0
S_{H_3}	1	1	1	1	1	1

Let $\mathsf{A} = \begin{pmatrix} 120 & 0 & 0 & 0 & 0 & 0 \\ 60 & 4 & 0 & 0 & 0 & 0 \\ 12 & 8 & 2 & 0 & 0 & 0 \\ 20 & 8 & 0 & 2 & 0 & 0 \\ 30 & 4 & 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$. We know from Proposition 3.3 that $\mathsf{A}^{-1} \begin{pmatrix} 120 \\ 120 \\ 120 \\ 120 \\ 120 \\ 120 \end{pmatrix}$ gives the cardinalities of the conjugacy classes of H_3. However, we get

$$\mathsf{A}^{-1}\begin{pmatrix}120\\120\\120\\120\\120\\120\end{pmatrix}=\begin{pmatrix}1\\15\\-6\\-10\\15\\105\end{pmatrix}$$

which is absurd.
But if we use Theorem 2.5, the values of a_{JKK} calculated for H_3 are:

	$\varnothing$	$\{s_1\}$	$\{s_1, s_2\}$	$\{s_2, s_3\}$	$\{s_1, s_3\}$	S_{H_3}
$\varnothing$	120	0	0	0	0	0
$\{s_1\}$	60	4	0	0	0	0
$\{s_1, s_2\}$	12	4	2	0	0	0
$\{s_2, s_3\}$	20	4	0	2	0	0
$\{s_1, s_3\}$	30	4	0	0	2	0
S_{H_3}	1	1	1	1	1	1

Let $\mathsf{B}=\begin{pmatrix}120&0&0&0&0&0\\60&4&0&0&0&0\\12&4&2&0&0&0\\20&4&0&2&0&0\\30&4&0&0&2&0\\1&1&1&1&1&1\end{pmatrix}$. Then we get the following cardinalities of the conjugacy classes of H_3

$$\mathsf{B}^{-1}\begin{pmatrix}120\\120\\120\\120\\120\\120\end{pmatrix}=\begin{pmatrix}1\\15\\24\\20\\15\\45\end{pmatrix}$$

which are the correct values.

References

[1] F. Bergeron, and N. Bergeron, *Symbolic Manipulation for the Study of the Descent Algebra of Finite Coxeter Groups*, Journal of Symbolic Computation **14** (1992) 127-139

[2] F. Bergeron, N. Bergeron, R. B. Howlett, and D. E. Taylor, *A Decomposition of the Descent Algebra of a Finite Coxeter Group*, Journal of Algebraic Combinatorics **1** (1992) 23-44

[3] A. Björner, and F. Brenti, *Combinatorics of Coxeter Groups*, Graduate Texts in Mathematics (2004)

[4] D. Blessenohl, C. Hohlweg, and M. Schocker, *A Symmetry of the Descent Algebra of a Finite Coxeter Group*, Advances in Mathematics **193** (2005) 416-437

[5] K. Brown, *Semigroups, Rings, and Markov Chains*, Journal of Theoretical Probability, 13 **4** (2000) 871–938

[6] K. Brown, *Semigroup and Ring Theoretical Methods in Probability*, Fields Institute Communications, **00** (2003)

[7] I. M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V. S. Retakh, and J.-Y. Thibon, *Noncommutative Symmetric Functions*, Advances in Mathematics **112** (1995) 218-348

[8] R. Howlett, *Normalizers of Parabolic Subgroups of Reflection Groups*, Journal of the London Mathematical Society (2) **21** (1980) 62-80

[9] J. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge Studies in Advanced Mathematics (1990)

[10] C. Krattenthaler, *Advanced Determinant Calculus*, Séminaire Lotharingien de Combinatoire **42** (1999) Article B42q, 67 pp

[11] D. Krob, B. Leclerc, and J.-Y. Thibon, *Noncommutative Symmetric Functions II: Transformations of Alphabets*, International Journal of Algebra and Computation **7** (1997) 181-264

[12] H. Randriamaro, *Diagonalization of the Matrices of the Multinomial Descent and Multinomial Inversion Statistics on the Symmetric Group*, http://arxiv.org/abs/1104.4099 (2011)

[13] V. Reiner, F. Saliola, and V. Welker, *Spectra of Symmetrized Shuffling operators*, Memoirs of the American Mathematical Society to appear, http://arxiv.org/abs/1102.2460 (2011)

[14] P. Renteln, *The Distance Spectra of Cayley Graphs of Coxeter Groups*, Discrete Mathematics **311** (2011) 738-755

[15] C. Reutenauer, *Free Lie Algebras*, Oxford Science Publications (1993)

[16] B. Sagan, *The Symmetric Group Representations, Combinatorial Algorithms and Symmetric Functions*, Graduate Texts in Mathematics (2001)

[17] M. Schocker, *The Descent Algebra of the Symmetric Group*, Fields Institute Communications **00**, http://www.mathematik.uni-bielefeld.de/∼ringel/schocker-neu.ps

[18] L. Solomon, *A Mackey Formula in the Group Ring of a Coxeter Group*, Journal of Algebra **41** (1976) 255-268

[19] R. Stanley, *Enumerative Combinatorics*, Cambridge University Press (1997)

[20] A. Varchenko, *Bilinear Form of Real Configuration of Hyperplanes*, Advances in Mathematics **97** (1993) 110-144

[21] D. Zagier, *Realizability of a Model in Infinite Statistics*, Communications in Mathematical Physics **147** (1992) 199-210