PFAFFIAN GRAPHS, T-JOINS AND CROSSING NUMBERS

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Abstract. We prove a technical theorem about the numbers of crossings in T-joins in different drawings of a fixed graph. As a corollary we characterize Pfaffian graphs in terms of their drawings in the plane and give a new proof of a theorem of Kleitman on the parity of crossings in drawings of K_{2j+1} and $K_{2j+1,2k+1}$. This gives a new proof of the Hanani-Tutte theorem.

1. Introduction

All graphs considered in this paper are finite and have no loops or multiple edges. For a graph G we denote its vertex set by $V(G)$ and its edge set by $E(G)$. If u and v are vertices in a graph G, then uv denotes the edge joining u and v. A *perfect matching* is a set of edges in a graph that covers each vertex exactly once. For sets X and Y we denote their symmetric difference by $X \triangle Y$.

A pair (G, T) consisting of a graph G and a set $T \subseteq V(G)$ of even cardinality is called *a graft*. *A T-join* is a subset $J \subseteq E(G)$ such that every vertex $v \in V(G)$ is incident with an odd number of edges in J if and only if $v \in T$.

T-joins were first introduced in relation to the Chinese Postman problem, which can be reformulated as follows: find the minimum set of edges in a graph whose doubling results in an Eulerian graph. Note that such set of edges is a T-join, where T is the set of all vertices of odd degree. Perfect matching are other example of a T-join, where $T = V(G)$. Since their introduction T-joins have been extensively studied (see for example [15], sections 6.5 and 6.6 of [10], [3], section 2 of [2]).

By *a drawing* Γ of a graph G we mean an immersion of G in the plane such that edges are represented by homeomorphic images of $[0, 1]$, not containing vertices in their interiors. Edges are permitted to intersect, but there are only finitely many intersections and each intersection is a crossing. For edges e, f of a graph G drawn in the plane let $cr(e, f)$ denote the number of times the edges e and f cross. For a set $J \subseteq E(G)$ let $cr(J, \Gamma)$, or $cr(J)$ if the drawing is understood from context, denote $\sum cr(e, f)$, where the sum is taken over all unordered pairs of distinct edges $e, f \in J$.

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We say that an unordered pair $\{e, f\}$ of adjacent edges in G is *an angle*. We denote the set of all edges and angles in a graph G by Æ(G). If $J \subseteq E(G)$ we say that $e \in E(G)$ *lies* in J if $e \in J$, and we say that an angle $\{e, f\}$ *lies* in J if $e, f \in J$. For $J \subseteq E(G)$ and $S \subseteq$ Æ(G) we denote by $J \sqcap S$ the set of elements of S which lie in J.

The following theorem is the main result of this paper. While the theorem itself is rather technical, it has a number of interesting applications.

Theorem 1.1. *Let (G,T) be a graft and let Γ_1 and Γ_2 be two drawings of G in the plane. Then there exists $S = S(T, \Gamma_1, \Gamma_2) \subseteq$ Æ(G) such that for every T-join $J \subseteq E(G)$ the following identity holds modulo 2*

$$cr(J, \Gamma_1) = cr(J, \Gamma_2) + |J \sqcap S|. \tag{1}$$

We prove Theorem 1.1 in Section 2. In Section 3 Theorem 1.1 is used to characterize Pfaffian graphs in terms of their drawings in the plane. In Section 4 we apply Theorem 1.1 to give a new proof of a result of Kleitman on the parity of the number of crossings in a graph. A well-known theorem of Hanani and Tutte follows as a corollary.

2. Proof of the main theorem

Throughout this section all integer identities are modulo 2.

For any n and any two sequences $(a_1, a_2, .., a_n)$ and $(b_1, b_2, .., b_n)$ of pairwise distinct points in the plane, there clearly exists a homeomorphic transformation of the plane that takes a_i to b_i for all $1 \leq i \leq n$. Therefore without loss of generality we assume that the vertices of G are represented by the same points in the plane in both Γ_1 and Γ_2.

We say that the drawings Γ_1 and Γ_2 are *adjacent* if they differ only in the position of a single edge $e = u_1u_2$. We start by proving Theorem 1.1 for adjacent drawings.

Let e_1 and e_2 denote the images of e in Γ_1 and Γ_2 correspondingly. By changing these images within the regions of $\Gamma_1 \setminus e_1$ we can assume that e_1 and e_2 have finitely many intersections and each intersection is a crossing. Define $C = e_1 \cup e_2$. The closed curve C separates its complement into two sets P_1 and P_2 with the property that every simple curve with ends $a \in P_i$ and $b \in P_j$ crosses C an even number of times if and only if $i = j$.

For $x \in (V(G) \cup E(G)) \setminus \{e\}$ we will not distinguish between x and its representation in Γ_1 and Γ_2. Define F_i to be the set of all edges $f \in E(G) \setminus \{e\}$ such that f is adjacent to u_j for some $j \in \{1, 2\}$ and $f \cap U \subseteq P_i \cup \{u_j\}$ for every some neighborhood U of u_j in the plane. Define

$$S = \{\{e, f\} | f \in F_1\}$$

if $|T \cap P_1|$ is even, and

$$S = \{\{e, f\} | f \in F_1\} \cup \{e\}$$

if $|T \cap P_1|$ is odd.

If $e \notin J$ then $cr(J,\Gamma_1) = cr(J,\Gamma_2)$ and (1) trivially holds, so we assume $e \in J$. We have

$$cr(J,\Gamma_1) + cr(J,\Gamma_2) = 2 \sum_{\{f,g\} \subseteq J \setminus \{e\}} cr(f,g) + \sum_{f \in J \setminus \{e\}} (cr(f,e_1) + cr(f,e_2)) = \sum_{f \in J \setminus \{e\}} cr(f,C)$$

Therefore it suffices to prove that

$$|J \sqcap S| = \sum_{f \in J \setminus \{e\}} cr(f,C),$$

or equivalently that

$$|J \cap F_1| + |T \cap P_1| = \sum_{f \in J \setminus \{e\}} cr(f,C). \tag{2}$$

From the definition of T-join we can deduce that for any $X \subseteq V(G)$

$$|T \cap X| = |\{uv \in J | u \in X, v \notin X\}|.$$

In particular

$$|T \cap P_1| = |\{uv \in J | u \in P_1, v \notin P_1\}| = |\{uv \in J | u \in P_1, v \in P_2\}| + \\ + |\{uv \in J | u \in P_1, v \in \{u_1,u_2\}\}|. \tag{3}$$

Let $J_1 = \{uv \in J \cap F_2 | u \in P_1\}$ and $J_2 = \{uv \in J \cap F_1 | u \in P_2\}$. Note that

$$(J \cap F_1) \triangle \{uv \in J | u \in P_1, v \in \{u_1,u_2\}\} = J_1 \cup J_2,$$

and therefore

$$|J \cap F_1| + |\{uv \in J | u \in P_1, v \in \{u_1,u_2\}\}| = |J_1 \cup J_2|. \tag{4}$$

Let $J_3 = \{uv \in J | u \in P_1, v \in P_2\}$. The sets J_1, J_2 and J_3 are pairwise disjoint. From (3) and (4) we have

$$|J \cap F_1| + |T \cap P_1| = |J_1 \cup J_2 \cup J_3|. \tag{5}$$

But $J_1 \cup J_2 \cup J_3$ is exactly the set of those edges $f \in J \setminus \{e\}$ which cross C an odd number of times. Therefore (2) follows from (5) and the proof of Theorem 1.1 for adjacent drawings is complete.

For two arbitrary drawings Γ_1 and Γ_2 of G there always exist an integer n and a sequence of drawings $\Gamma_1 = \Gamma'_1, \Gamma'_2, \dots, \Gamma'_n = \Gamma_2$ of G such that Γ'_i is adjacent to Γ'_{i+1} for all $i \in \{1,2,\dots,n-1\}$. We have proved that there exist sets $S_i \subseteq Æ(G)$ for all $i \in \{1,2,\dots,n-1\}$ such that

$$cr(J,\Gamma'_i) = cr(J,\Gamma'_{i+1}) + |J \sqcap S_i| \tag{6}$$

for all T-joins J. Let $S = S_1 \triangle S_2 \triangle \dots \triangle S_{n-1}$. Summing up (6) over all $i \in \{1,2,\dots,n-1\}$ we get (1), thereby completing the proof of Theorem 1.1 for arbitrary drawings.

3. Pfaffian graphs

We say that a cycle C in a graph G is *central* if $G\setminus V(C)$ has a perfect matching. In a directed graph D we say that an even cycle C is *oddly oriented* if C has an odd number of edges oriented in the direction of each orientation of C. We say that a graph G is *Pfaffian* if there exists an orientation D of G such that every central cycle is oddly oriented in D, in which case we say that D is a *Pfaffian orientation* of G.

Pfaffian orientations have been introduced by Kasteleyn [4, 5, 6]. He demonstrated that one can enumerate perfect matchings in a Pfaffian graph in polynomial time.

Pfaffian bipartite graphs were characterized in terms of forbidden subgraphs by Little [9]. A structural characterization of Pfaffian bipartite graphs was given by Robertson, Seymour and Thomas [14] and independently by McCuaig [11]. They also provided a polynomial time algorithm for recognition of Pfaffian bipartite graphs. The problem of recognition of Pfaffian bipartite graphs is equivalent to many interesting problems, e.g. the Pólya permanent problem [13], the even cycle problem for directed graphs [18] and the problem of determining which real square matrices are sign non-singular [7].

No satisfactory characterization is known for general Pfaffian graphs. While attempting to find such a characterization I was able to obtain the following result. A self-contained proof of it will also appear in [12].

Theorem 3.1. *A graph G is Pfaffian if and only if there exists a drawing of G in the plane such that $cr(M)$ is even for every perfect matching M of G.*

The "if" part of this theorem was known to Kasteleyn [6] and was proved by Tesler [16]; however our proof of this part is different. We derive Theorem 3.1 from a more general result. To state it we need a definition.

Let Γ be a drawing of a graph G in the plane. We say that $S \subseteq E(G)$ is a *marking* of Γ if $cr(M)$ and $|M \cap S|$ have the same parity for every perfect matching M of G.

Theorem 3.2. *For a graph G the following are equivalent:*
(a) G is Pfaffian;
(b) some drawing of G in the plane has a marking;
(c) every drawing of G in the plane has a marking;
(d) there exists a drawing of G in the plane such that $cr(M)$ is even for every perfect matching M of G.

We say that Γ is *a standard drawing* of a labeled graph G if the vertices of Γ are arranged on a circle in order and every edge of Γ is drawn as a straight line.

The equivalence of conditions (a), (b) and (c) of Theorem 3.2 immediately follows from the next two lemmas.

Lemma 3.3. *Let Γ be a standard drawing of a labeled graph G. Then G is Pfaffian if and only if Γ has a marking.*

Proof. Let D be an orientation of G. Let $M = \{u_1v_1, u_2v_2, \ldots, u_kv_k\}$ be a perfect matching of D. The sign of M is the sign of the permutation

$$P = \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & 2k-1 & 2k \\ u_1 & v_1 & u_2 & v_2 & \ldots & u_k & v_k \end{pmatrix}.$$

Let $i(P)$ denote the number of inversions in P. We have

$$sgn(M) = sgn(P) = (-1)^{i(P)} = \prod_{1\le i<j\le 2k} sgn(P(j) - P(i)) =$$

$$= \prod_{1\le i<j\le k} sgn((u_j - u_i)(v_j - u_i)(u_j - v_i)(v_j - v_i))\times$$

$$\times \prod_{1\le i\le k} sgn(v_i - u_i). \tag{7}$$

In Γ edges u_iv_i and u_jv_j cross if and only if, in the circle containing the vertices of Γ, each of the two arcs with ends u_i and v_i contains one of the vertices u_j and v_j, in other words if and only if

$$sgn((u_j - u_i)(v_j - u_i)(u_j - v_i)(v_j - v_i)) = -1.$$

Define $S_D = \{uv \in E(D) | u > v\}$. From (7) we deduce that

$$sgn(M) = (-1)^{cr(M)} \times (-1)^{|M\cap S_D|}.$$

Therefore M has a positive sign if and only if $cr(M)$ and $|M \cap S_D|$ have the same parity. It follows that D is a Pfaffian orientation of G if and only if S_D is a marking of the standard drawing of G. $\square$

Notice that we have in fact shown that there exists a one-to-one correspondence between Pfaffian orientations of a labeled graph and markings of its standard drawing.

Lemma 3.4. *Let Γ_1 and Γ_2 be two drawings of a labeled graph G in the plane. Then Γ_1 has a marking if and only if Γ_2 has one.*

Proof. The integer identities throughout the proof are modulo 2. Perfect matchings are T-joins in the graft $(G, V(G))$. Therefore by Theorem 1.1 there exists $S \subseteq Æ(G)$ such that for every perfect matching M of G we have

$$cr(M, \Gamma_1) = cr(M, \Gamma_2) + |M \sqcap S|.$$

Let $S' = S \cap E(G)$. As no perfect matching contains an angle we have

$$cr(M, \Gamma_1) = cr(M, \Gamma_2) + |M \cap S'|$$

for every perfect matching M of G. Let S_1 be a marking of Γ_1. Then

$$cr(M, \Gamma_2) = cr(M, \Gamma_1) - |M\cap S'| = |M\cap S_1| - |M\cap S'| = |M\cap(S' \triangle S_1)|$$

for every perfect matching M of G. Therefore $S' \triangle S_1$ is a marking of Γ_2. $\square$

Since clearly (d) implies (b), to finish the proof of Theorem 3.2 it remains to show that (b) implies (d). Suppose G satisfies (b) and consider a drawing of G in the plane with a marking S. Suppose there exists $e \in S$. We change the way e is drawn, so that the closed curve C which is composed from the old and the new drawing of e separates one vertex of G from the rest. From the proof of Theorem 1.1 it follows that $S \backslash \{e\}$ is a marking in the new drawing. By repeating the procedure we produce a drawing of G such that the empty set is a marking, therefore demonstrating that G satisfies condition (d) of Theorem 3.2.

4. Parity of the number of crossings

We say that a set $\mathcal{J}$ of T-joins in a graft (G, T) is *nice* if every $x \in Æ(G)$ lies in an even number of elements of $\mathcal{J}$.

Lemma 4.1. *Let $\mathcal{J}$ be a nice set of T-joins in a graft (G, T). Then the parity of*

$$\sum_{J \in \mathcal{J}} cr(J, \Gamma) \tag{8}$$

is independent of the choice of a drawing Γ of G in the plane.

Proof. By Theorem 1.1 it suffices to prove that

$$\sum_{J \in \mathcal{J}} |J \sqcap S|$$

is even for any $S \subseteq Æ(G)$. This is true by the definition of a nice set of T-joins. □

We derive the next theorem from Lemma 4.1.

Theorem 4.2. *(Kleitman [8]) Let $G = K_{2j+1}$ or $G = K_{2j+1,2k+1}$ for some positive integers j and k. Then the parity of the total number of crossings of non-adjacent edges is independent of the choice of a drawing of G in the plane.*

Proof. By Lemma 4.1 it suffices to find $T \subseteq V(G)$ and a nice set $\mathcal{J}$ of T-joins such that

$$|\{J \in \mathcal{J} | \{e, f\} \subseteq J\}|$$

is odd for every two non-adjacent edges e, f of G. (By the definition of a nice set, $|\{J \in \mathcal{J} | \{e, f\} \subseteq J\}|$ is even for every angle $\{e, f\}$.)

For $G = K_{2j+1,2k+1}$ we choose $T = \emptyset$ and we choose $\mathcal{J}$ to be the set of all cycles of length 4 in G.

For $G = K_{2j+1}$ the construction is slightly more complicated. Choose $v \in V(G)$ and let $T = V(G) \setminus \{v\}$. Let $\mathcal{J}_1$ be the set of all perfect matchings of $G \setminus \{v\}$. For distinct vertices $u_1, u_2 \in T$ let

$$J_{u_1 u_2} = \{vw | w \in T \setminus \{u_1, u_2\}\} \cup \{u_1 u_2\}$$

and let $\mathcal{J}_2 = \{J_{u_1u_2} | \{u_1, u_2\} \subseteq T, u_1 \neq u_2\}$. Let $J_3 = \{vw | w \in T\}$. Finally, if j is odd let $\mathcal{J} = \mathcal{J}_1 \cup \mathcal{J}_2$ and if j is even let $\mathcal{J} = \mathcal{J}_1 \cup \mathcal{J}_2 \cup \{J_3\}$.

In both cases by straightforward counting we can check that $\mathcal{J}$ is as required. □

Kuratowki's theorem states that every non-planar graph has a subgraph isomorphic to a subdivision of K_5 or $K_{3,3}$. One can therefore easily deduce the following well-known theorem from Theorem 4.2 and Kuratowski's theorem.

Theorem 4.3. *(Hanani* [1]*, Tutte* [17]*) Let* Γ *be a drawing of a non-planar graph* G *in the plane. Then there exist distinct non-adjacent edges* $e, f \in E(G)$ *such that* $cr(e, f)$ *is odd.*

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