On periodic p-harmonic functions on Cayley tree.

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Abstract: We show that any periodic with respect to normal subgroups (of the group representation of the Cayley tree) of finite index p-harmonic function is a constant. For some normal subgroups of infinite index we describe a class of (non-constant) periodic p-harmonic functions. If $p \neq 2$, the p-harmonicity is non-linear, i.e., the linear combination of p-harmonic functions need not be p-harmonic. In spite of this, we show that linear combinations of the p-harmonic functions described for normal subgroups of infinite index are also p-harmonic.

1 Introduction

A tree is connected acyclic graph. One special case of this is a Cayley tree, i.e., an infinite tree in which each vertex has exactly $k+1$ incident edges. The Cayley tree can be represent as the group G_k which is the free product of $k+1$ second order cyclic groups [1],[2],[6]. The group representation of the Cayley tree was used in [1],[2],[5]-[8] to study models of statistical mechanics and describe the sets of periodic Gibbs measures, also to study random walk trajectories in a random medium on the Cayley tree. These problems are related to description of harmonic functions on trees. In [5] a natural generalization of the notion of harmonic functions on a Cayley tree is introduced. Using some properties of G_k a set of harmonic functions described. Harmonic functions play an important role in probability theory, dynamical systems, statistical mechanics, and theory of electrical circuits. Note that p-harmonic functions are zeros of the p-Laplacian defined by

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right), \quad 1 < p < \infty.$$

If $p \neq 2$, then p-harmonicity is non-linear. We consider the equation $\Delta_p u = 0$ on (discrete) Cayley tree. Since there is no theory of functional equations with the unknown functions defined on a tree, the problem of finding all solution of $\Delta_p u = 0$ on tree is by no means easy. On the other hand, the theory of functional equations is not very developed even for case in which the unknown function is defined on R, only the solutions of functional equations of some special forms are known (e.g. see [3], [4]). Thus, it is natural to find periodic (simple) solutions of $\Delta_p u = 0$ first.

The main goal of this paper is to describe the set of p-harmonic functions which are periodic with respect to subgroups of the group G_k.

In section 2 we give some preliminary definitions and results which needed when describing p-harmonic (periodic) functions. Section 3 devoted to description of periodic p-harmonic functions (with respect to normal subgroups of finite index). We prove that only constant functions can be such periodic. Section 4 devoted to the problem in the case of infinite index. We find (explicitly) some p-harmonic functions which are periodic with respect to a subgroup of infinite index. In the last section we show that linear combinations of these functions are also p-harmonic.

2 Definitions and statement of the problem

2.1 Cayley tree.

A Cayley tree is an infinite tree in which each vertex has exactly $k+1$ incident edges (the Cayley tree of order $k \geq 1$). Let $\Gamma^k = (V, L)$ be the Cayley tree of order $k \geq 1$, where V and L are the vertex set and the edge set, respectively, of Γ^k.

If $x, y \in V$ are the endpoints of an edge $l \in L$, then x and y are said to be adjacent; in this case, we write $l =< x, y >$. The Cayley tree is equipped with a distance $d(x, y), x, y \in V$, given by the formula

$$d(x,y) = \min\{d : \ \exists x = x_0, x_1, ..., x_{d-1}, x_d = y \in V \text{ where } \\ < x_0, x_1 >, ..., < x_{d-1}, x_d > \text{ are adjacent}\}.$$

The sequence $\pi = \{x = x_0, x_1, ..., x_{d-1}, x_d = y \in V\}$ at which the minimum is attained is called the path form x to y. Let G_k be the group representation of the Cayley tree; i.e., G_k is the free product of $k+1$ second - order cyclic groups with generators $a_1, ..., a_{k+1}$ such that $a_i^2 = e, i = 1, ..., k+1$,where $e \in G_k$ is the identity element. Let $x \in G^k$. We denote by

$$S(x) = \{y \in G_k :< x, y >\}$$

the set of vertices adjacent to x.

2.2 Normal subgrups of G_k.

Let $\omega_x(a_i)$ be the number of occurrences of the letter $a_i, i = 1, ..., k+1$, in the redused word $x \in G_k$. Let $A \subseteq N_k = \{1, ..., k+1\}$. The set $H_A = \{x \in G_k : \sum_{i \in A} \omega_x(a_i) \text{ is even}\}$ is a normal subgroup of index 2 of G_k. Some set of normal subgroups with index 2^m can be obtained by intersection $H_{A_1} \bigcap H_{A_2} \bigcap ... \bigcap H_{A_m}$ for suitable $A_1, ..., A_m \subseteq N_k$ (see. [2]). The following theorem gives possible values of the (finite) index of normal subgroups.

THEOREM 1. [5] *The group G_k does not have normal subgroups of odd index ($\neq 1$). It has a normal subgroup of arbitrary even index.*

Also there are normal subgroups of infinite index. Some of them can be described as following [7]. Fix $M \subseteq N_k$ such that $|M| > 1$. $|\bullet|$ is the cordinality of $\bullet$. Let the mapping $\pi_M : \{a_1, ..., a_{k+1}\} \longrightarrow \{a_i,\ i \in M\} \cup \{e\}$ be defined by

$$\pi_M(a_i) = \begin{cases} a_i, & \text{if } \ i \in M \\ e, & \text{if } \ i \notin M. \end{cases}$$

Denote by G_M the free product of cyclic groups $\{e, a_i\},\ i \in M$. Consider

$$f_M(x) = f_M(a_{i_1}a_{i_2}...a_{i_m}) = \pi_M(a_{i_1})\pi_M(a_{i_2})...\pi_M(a_{i_m}).$$

Then it is easy to see that f_M is a homomorphism and hence $H_M = \{x \in G_k :\ f_M(x) = e\}$ is a normal subgroup of infinite index.

2.3 p-Harmonic functions.

The resistances r is a positive function on L such that $r(y, x) = r(x, y)$. Denote $\mathcal{R} = \{r(x, y) : (x, y) \in L\}$. We define the discrete ∇u and the discrete p-Laplacian $\Delta_p u$ for a function u on $V\ (= G_k)$ by

$$\nabla u(x, y) = r(x, y)^{-1} \left(u(y) - u(x)\right),$$

$$\Delta_p u(x) = \sum_{y \in S(x)} |\nabla u(x, y)|^{p-2} \nabla u(x, y),$$

where $1 < p < \infty$. Let $D \subset V$. If $\Delta_p u = 0$ in D, then we say that u is p-harmonic in D [9]-[11]. Let $\{u_1, ..., u_n\}$ be an m-tuple of p-harmonic functions in D. If $p \neq 2$, then the linear combination of p-harmonic functions $u_1, ..., u_m$ need not be p-harmonic. The m-tuple of p-harmonic functions in D has a linear relation if every linear combination $\sum_{j=1}^{m} t_j u_j$ is p-harmonic in D. Also we say that $\{u_1, ..., u_m\}$ has a partial linear relation in D if $\sum_{j=1}^{m} t_j u_j$ p-harmonic in D for some $t_1, ..., t_m \in R \setminus \{0\}$.

DEFINITION 1. Let $H \subset G_k$ be a subgroup. A function $\{u(x), x \in G_k\}$ is said to be H-periodic if $u(x) = u(yx)$ for any $x \in G_k$ and $y \in H$.

If H is a normal subgroup of finite index, then the description of H-periodic solutions of $\Delta_p u$ can be reduced to solving a system of equations with finitely many unknowns.

The main goal of this paper is to describe H-periodic p-harmonic functions for any normal subgroup H and $\forall p \in (1, \infty)$.

3 The case of finite index.

Let G_k^* be a normal subgroup of (finite) index $m \geq 1$ for G_k. Denote by $\mathcal{H}_{p,k,m}(G_k^*, \mathcal{R})$ the set of all G_k^*-periodic, p-harmonic functions for given $p \in (1, \infty)$, $\mathcal{R}$, $m \geq 1$ and $k \geq 1$.

The following theorem completely describes the set of all periodic p-harmonic functions for finite $m \geq 1$.

THEOREM 2. *For* $\forall p \in (1, \infty)$, $\forall m \geq 1$, $\forall k \geq 1$, $\forall G_k^* \subset G_k$ *with index* m *and any fixed* $\mathcal{R} = \{r(x,y) > 0 : (x,y) \in L\}$ *the set* $\mathcal{H}_{p,k,m}(G_k^*, \mathcal{R})$ *contains only constant functions.*

Proof. Let $G_k|G_k^* = \{G_k^1, ..., G_k^m\}$ be the factor group. Any G_k^*-periodic p-harmonic function $u(x)$ has the following form

$$u(x) = u_i \text{ if } x \in G_k^i, i = 1, ..., m \tag{1}$$

where u_i, $i = 1, ..., m$ are a solution to the following system

$$\sum_{j=1}^{m} \sum_{y \in S_j(x)} \frac{|u_j - u_i|^{p-2}}{r^{p-1}(x,y)} (u_j - u_i) \mathbf{1}(S_j(x) \neq \emptyset) = 0, \quad i = 1, ..., m \tag{2}$$

where $x \in G_k^i$, $S_j(x) = S(x) \cap G_k^j$, $j = 1, ..., m$. Note that some $S_j(x)$ can be empty set, so

$$\mathbf{1}(S_j(x) \neq \emptyset) = \begin{cases} 1, & \text{if } S_j(x) \neq \emptyset \\ 0, & \text{if } S_j(x) = \emptyset \ . \end{cases}$$

We shall prove that the system (2) has only solutions $(u_1^*, ..., u_m^*)$ with $u_1^* = ... = u_m^*$. Assume that there is a solution $u^* = (u_1^*, ..., u_m^*)$ with

$$u_i^* \neq u_j^* \text{ for some } i, j \in \{1, ..., m\} \tag{3}$$

Denote $u_{i_0}^* = \max\{u_i^*, \quad i = 1, ..., m\}$.

From i_0th equation of (2) we have

$$\sum_{j=1}^{m} \sum_{y \in S_j(x)} \frac{|u_j^* - u_{i_0}^*|^{p-2}}{r^{p-1}(x,y)} \mathbf{1}(S_j(x) \neq \emptyset)(u_j^* - u_{i_0}^*) =$$

$$\sum_{\substack{j=1: \\ S_j(x) \neq \emptyset}}^{m} \sum_{y \in S_j(x)} \frac{|u_j^* - u_{i_0}^*|^{p-2}(u_j^* - u_{i_0}^*)}{r^{p-1}(x,y)} < 0, \tag{4}$$

since $u_j^* - u_{i_0}^* \leq 0$ and by (3) $u_j^* - u_{i_0}^* < 0$ for at least one $j \in \{1, ..., m\} \setminus \{i_0\}$. The inequality (4) contradicts to the assumption that u^* is a solution of (2). Theorem is proved.

4 The Case of infinite index.

In this section we show that for some subgroups of infinite index there are non - constant p-harmonic functions. Consider $M = \{i, j\}$, $i \neq j \in N_k$ and $H_M = H_{ij} = \{x \in G_k :$

$f_M(x) = e\}$ (see section 2.2). It is known that H_{ij} is a normal subgroup of infinite index and the corresponding factor group can be written as (see [7]):

$$G_k|H_{ij} = \{..., H_{-2}, H_{-1}, H_0, H_1, H_2, ...\}.$$

LEMMA 1. *If* $x \in H_n$ *then* $S(x) \subset H_{n-1} \cup H_n \cup H_{n+1}$. *Moreover* $|S(x) \cap H_{n-1}| = 1,\ |S(x) \cap H_n| = k-1,\ |S(x) \cap H_{n+1}| = 1$.

Proof. If $x \in H_n$ then $f_M(x) = \underbrace{a_i a_j \dots a_i}_{n}$ (or $a_i a_j ... a_j$ depending on n). $S(x) = \{xa_s,\ s = 1, ..., k+1\}$. We have

$$f_M(xa_s) = f_M(x) f_M(a_s) = a_i a_j \dots a_i f_M(a_s) = \begin{cases} \underbrace{a_i a_j \dots a_j}_{n-1} & \text{if } a_s = a_i \\ \underbrace{a_i a_j \dots}_{n} & \text{if } a_s \neq a_i, a_j \\ \underbrace{a_i a_j \dots a_i a_j}_{n+1} & \text{if } a_s = a_j \end{cases}$$

This relation complete the proof.

Note that any H_{ij}-periodic function $u(x)$ has the form

$$u(x) = u_n \text{ if } x \in H_n \tag{5}$$

Consider H_{ij}-periodic collection $\mathcal{R}$ of resistance functions $r(x,y)$ i.e. $r(x,y) = r_{nm}$ if $x \in H_n, y \in H_m$. By lemma 1 the equation $\Delta_p u = 0$ can be written as

$$\frac{|u_{n+1} - u_n|^{p-2}}{r_{n,n+1}^{p-1}}(u_{n+1} - u_n) + \frac{|u_{n-1} - u_n|^{p-2}}{r_{n-1,n}^{p-1}}(u_{n-1} - u_n) = 0 \tag{6}$$

for any $n \in Z$.

Denote $a_n = u_{n+1} - u_n$ then from (6) we see that a_n, a_{n-1} must have the same sign i.e., $a_n \cdot a_{n-1} > 0,\ n \in Z$.

Thus $u_n,\ n \in Z$ must be a monotone sequence.

Put

$$X_n = \frac{|a_n|^{p-2}}{r_{n,n+1}^{p-1}} a_n.$$

From (6) we get

$$X_n - X_{n-1} = 0,\ \forall n \in Z. \tag{7}$$

Hence $X_n = C = \text{Const}$ for any $n \in Z$.

Since u_n must be monotone, we have two possibility:

1) $u_{n+1} > u_n$ then $C > 0$. From $X_n = C$ we get

$$u_{n+1} = C^{\frac{1}{p-1}} \cdot r_{n,n+1} + u_n,\ n \in Z. \tag{8}$$

From (8) we obtain

$$u_n = C^{\frac{1}{p-1}} \sum_{s=-\infty}^{n-1} r_{s,s+1}, \ n \in Z. \tag{9}$$

2) $u_{n+1} < u_n$ then $-C > 0$. In this case we get

$$u_n = (-C)^{\frac{1}{p-1}} \sum_{s=n}^{+\infty} r_{s,s+1}, \ n \in Z. \tag{10}$$

Thus we have proved the following

THEOREM 3. *Let $\mathcal{R}$ be H_{ij}-periodic, and $\sum_{s=-\infty}^{\infty} r_{s,s+1} < +\infty$. Then there are two family U_1, U_2 of H_{ij}-periodic p-harmonic functions on the Cayley tree of $k \geq 1$ such that*

$$U_1 = \{u: \ u(x) = u_n = C^{\frac{1}{p-1}} \sum_{s=-\infty}^{n-1} r_{s,s+1} \text{ if } x \in H_n, \ n \in Z, \ C \geq 0\};$$

$$U_2 = \{u: \ u(x) = u_n = C^{\frac{1}{p-1}} \sum_{s=n}^{+\infty} r_{s,s+1} \text{ if } x \in H_n, \ n \in Z, \ C \geq 0\}.$$

Remarks.

1. Theorem 3 is also true for more general class of $\mathcal{R}$ than the class of H_{ij}-periodic resistance functions. For example, one can prove theorem 3 for resistance functions:

$$r(x,y) = \begin{cases} r_{nm}, & \text{if } \ x \in H_n, y \in H_m, \ n \neq m \\ r(x,y), & \text{if } \ x, y \in H_n \end{cases}$$

i.e. r takes arbitrary values on $(x,y) \in L$ such that $x, y \in H_n$.

2. The role of p in U_1 and U_2 is not important: one can denote $C^{\frac{1}{p-1}}$ by C.

3. Note that for $M \subset N_k$ the problem of describing of H_M-periodic p-harmonic functions on Cayley tree of order k is equivalent to describing of arbitrary (non - periodic) p-harmonic functions on a Cayley tree of order $|M| - 1$. Thus for $M \subset N_k$ with $|M| \geq 3$ this problem is difficult.

4. By definition a constant is a p-harmonic function and the linear combination of an arbitrary p-harmonic function and a constant is also p-harmonic. Thus the classes U_1 and U_2 generate two parametrized classes of p-harmonic functions: $U_1 + C_1$, $U_2 + C_2$ where $U_i + C_i$ means to each functions of U_i a constant C_i is added.

5 Linear relations.

In this section we describe p-harmonic linear combinations of p-harmonic functions from $U_1, \ U_2$.

THEOREM 4. *Let $\{v_1,...,v_{q_1},v_{q_1+1},...,v_q\}$ be a q-tuple of p-harmonic functions with $\{v_1,...,v_{q_1}\}\subset U_1$ and $\{v_{q_1+1},...,v_q\}\subset U_2$. Then $\{v_1,...,v_q\}$ has a linear relation on Cayley tree.*

Proof. Note that $v=\sum_{i=1}^{q} t_i v_j,\ t_i\in R\setminus\{0\}$ has the following form

$$v(x)=\sum_{i=1}^{q_1} t_i\left(C_i\sum_{s=-\infty}^{n-1} r_{s,s+1}\right)+\sum_{i=q_1+1}^{q} t_i\left(C_i\sum_{s=n}^{+\infty} r_{s,s+1}\right) \tag{11}$$

if $x\in H_n,\ n\in Z$. Hence v is a H_{ij}-periodic function. We shall prove that $\Delta_p v=0$ i.e. v is p-harmonic.

Denote $\varphi_p(t)=|t|^{p-2}t$. By definition and (11) we have

$$\Delta_p v=\sum_{y\in S(x)}\frac{|v(y)-v(x)|^{p-2}}{r^{p-1}(x,y)}(v(y)-v(x))=\sum_{y\in S(x)}\frac{\varphi_p(v(y)-v(x))}{r^{p-1}(x,y)}=$$

$$\frac{\varphi_p\left(\sum_{i=1}^{q_1} t_iC_i\left(\sum_{s=-\infty}^{n-2} r_{s,s+1}-\sum_{s=-\infty}^{n-1} r_{s,s+1}\right)+\sum_{i=q_1+1}^{q} t_iC_i\left(\sum_{s=n-1}^{\infty} r_{s,s+1}-\sum_{s=n}^{\infty} r_{s,s+1}\right)\right)}{r_{n-1,n}^{p-1}}+$$

$$\frac{\varphi_p\left(\sum_{i=1}^{q_1} t_iC_i\left(\sum_{s=-\infty}^{n} r_{s,s+1}-\sum_{s=-\infty}^{n-1} r_{s,s+1}\right)+\sum_{i=q_1+1}^{q} t_iC_i\left(\sum_{s=n+1}^{\infty} r_{s,s+1}-\sum_{s=n}^{\infty} r_{s,s+1}\right)\right)}{r_{n,n+1}^{p-1}}=$$

$$\frac{\varphi_p\left(r_{n-1,n}\left(-\sum_{i=1}^{q_1} t_iC_i+\sum_{i=q_1+1}^{q} t_iC_i\right)\right)}{r_{n-1,n}^{p-1}}+\frac{\varphi_p\left(r_{n+1,n}\left(\sum_{i=1}^{q_1} t_iC_i-\sum_{i=q_1+1}^{q} t_iC_i\right)\right)}{r_{n,n+1}^{p-1}} \tag{12}$$

It is easy to see that $\varphi_p(st)=\varphi_p(s)\varphi_p(t)$; $\varphi_p(t)=t^{p-1}$ if $t>0$ and $\varphi_p(-t)=-\varphi_p(t)$. Using these properties from (12) we get $\Delta_p v=0$. The theorem is proved.

Remark.

The H_{ij}-periodic functions described in Theorem 3 and 4 do not depend on $i,j\in N_k,\ i\neq j$.

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