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Additive Schwarz Methods for Integral Equations of the First Kind

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1.1 INTRODUCTION

This paper gives a survey on additive Schwarz methods applied to the h and p versions of the boundary element method. We consider weakly singular and hypersingular integral equations, namely the single layer potential operator and the normal derivative of the double layer potential operator. That means we have to consider weak formulations of the form

$$a(u,v) := \langle Au, v\rangle = \langle g, v\rangle \quad \text{for all } v \in \tilde{H}^{\alpha/2}(\Gamma), \tag{1.1}$$

where $A : \tilde{H}^{\alpha/2}(\Gamma) \to H^{-\alpha/2}(\Gamma)$ and $a(\cdot,\cdot)$ is symmetric and positive definite. Here, $\langle\cdot,\cdot\rangle$ denotes the $L^2(\Gamma)$ inner product. An approximation u_M of u is obtained by solving the Galerkin scheme: find $u_M \in S_M \subset \tilde{H}^{\alpha/2}(\Gamma)$ such that

$$\langle Au_M, v\rangle = \langle g, v\rangle \quad \text{for all } v \in S_M. \tag{1.2}$$

where S_M is a finite-dimensional subspace. The condition number of this linear system grows at least like $h^{-1}p^2$ if the h-p version is used for quasi-uniform meshes, and it grows exponentially for geometric meshes. Here h is the mesh size and p is the degree of the splines used in the Galerkin scheme. Therefore in order to use the conjugate gradient algorithm to solve efficiently the system we need a good preconditioner.

The hypersingular and weakly singular integral equations we consider are

$$Wu(x) := -\frac{1}{\pi} f.p. \int_\Gamma \frac{u(y)}{|x-y|^2}\, ds_y = g_1(x), \qquad x \in \Gamma, \tag{1.3}$$

and

$$Vu(x) := -\frac{1}{\pi}\int_\Gamma u(y)\log|x-y|\,ds_y = g_2(x), \qquad x\in\Gamma, \tag{1.4}$$

where f.p. denotes a finite part integral in the sense of Hadamard and $g_1 \in H^{-1/2}(\Gamma)$, $g_2 \in H^{1/2}(\Gamma)$. Note that the order α of the pseudodifferential operator W is 1 whereas V has order -1.

Different choices of the trial space S_M in (1.2) lead to different types of the Galerkin scheme, namely to the h, p, and h-p versions defined as follows. For simplicity we consider only $\Gamma = (-1,1)$ and a regular mesh of size h on Γ defined as

$$x_j = -1 + jh, \qquad h = \frac{2}{N}, \qquad j = 0,\dots,N. \tag{1.5}$$

In the h version we take on this mesh the space S^h of continuous piecewise-linear functions for the hypersingular operator W while piecewise-constant functions for the weakly singular operator V. For W in order that S^h is a subspace of $\tilde{H}^{1/2}(\Gamma)$ it is necessary that the functions vanish at the endpoints ± 1. For the h version we solve (1.2) in S^h and increase the accuracy by reducing h.

In the p version the space S^p on this mesh is defined as the space of functions on Γ whose restrictions to $\Gamma_j := (x_{j-1}, x_j)$, $j = 1,\dots,N$, are polynomials of degree at most p. For W it is required that these functions are continuous and vanish at the endpoints of Γ. For the p version we solve (1.2) by functions in S^p and increase the accuracy by increasing p.

The Galerkin discretization of boundary integral equations lead to very large systems of linear equations $Au = f$ with a full $M\times M$ matrix. The direct solution of these linear systems can be very expensive in terms of storage and computational work, i.e. a Gauss solver requires $\mathcal{O}(M^3)$ operations. So we need a good iterative method to approximate the Galerkin solution, keeping the error in the energy norm at the order of the Galerkin error. For a positive definite matrix A the conjugate gradient method needs $\mathcal{O}(M^{1/2}\log M)$ iterations and thus $\mathcal{O}(M^{5/2}\log M)$ operations to achieve this order of accuracy. A multigrid method for the hypersingular integral equation has been analyzed in [vPS90] and has been used in [vPS92] as preconditioner for the conjugate gradient method to reduce the number of operations to $\mathcal{O}(M^2)$.

Additive Schwarz methods were originally designed for finite element discretisations of differential equations (see e.g. [BCMP91, BLP94, BP93, BPX90, DW91, Lio88, Pav94, Wid89, Zha92]). Our analysis for the h version Galerkin boundary element method for the hypersingular integral equation extends the approach by Bramble et.al. [BLP94, BP93, BPX90] and Xu [Xu92] whose applications were on finite element methods. Since the boundary integral operators are non-local the finite element theory does not carry over directly to the boundary element method. The latter needs a new analysis which for the hypersingular integral equation is based on the fractional Sobolev space $H^{1/2}_\infty(\Gamma)$.

The applicability of Schwarz iterations with overlapping domains to first kind integral equations was recently investigated in [HS96] for the h version of the boundary element method. Multilevel methods for the h version were considered in [TS96] for the standard nodal basis, in [TSM] for the hierarchical basis and in [STZ] for a wavelet basis. The p version for weakly singular and hypersingular integral operators was analyzed in [ST95]. The results in [ST95, TS96, TSM, STZ] can be summarised as follows. For the h version the 2-level and multilevel additive Schwarz methods yield preconditioned systems which have bounded condition numbers when the standard nodal basis or a wavelet basis are used, whereas a hierarchical basis preconditioner gives logarithmically growing condition numbers. For the p version considered in [ST95] the condition number is proved to behave like $1 + \log^3 p$. In [HTS] we design a multilevel method for the p version, which is simply preconditioning by the inverse of the diagonal matrix and which results in a preconditioned system having condition numbers behaving like $p\,(1+\log^3 p)$. In [HTS] we also prove the efficiency of the 2-level and multilevel methods for the h-p version, with both quasi-uniform and geometric meshes. For the p-version we note that even though the condition number of the multilevel method increases faster than that of the 2-level method, which may lead to a bigger number of iterations to solve the linear system, the implementation of the multilevel method is recommended since for each iteration it is actually the diagonal preconditioner, and is therefore cheaper for each iteration.

The additive multilevel methods usually need slightly more iteration steps than the multigrid methods but the higher flexibility of these algorithms simplifies the use of parallel computing (the simple subspace corrections are not applied in a sequential order but in parallel). Another advantage is that it allows a simpler, more natural data structure, and is therefore much better for non-uniformly refined grids. Consequently it is highly efficient to combine additive multilevel methods with adaptive methods (cf. Section 1.5).

The paper is organized as follows. In Section 1.2 we report on the aditive Schwarz methods applied to the h and p-versions for positive definite symmetric problems resulting from the first kind integral equations (1.3) and (1.4). In Section 1.3 we consider indefinite problems resulting from integral equation formulations for the Helmholtz equation. In Section 1.4 we report on 3D problems, in Section 1.5 on adaptive methods and in Section 1.6 we present numerical experiments.

1.2 ADDITIVE SCHWARZ METHODS FOR POSITIVE DEFINITE PROBLEMS

Let

$$S_M = S_0 + S_1 + \cdots + S_N \tag{1.6}$$

denote a decomposition of S_M into subspaces S_j. Then the additive Schwarz method (ASM) consists in solving, by an iterative method, the equation

$$P u_M := (P_0 + P_1 + \cdots + P_N) u_M = g_M, \tag{1.7}$$

where the projections $P_j : S_M \to S_j$, $j = 0, \ldots, N$, are defined for any $v_M \in S_M$ by

$$a(P_j v_M, \phi_j) = a(v_M, \phi_j) \quad \text{for any } \phi_j \in S_j. \tag{1.8}$$

Note the right hand side of (1.7), $g_M = \sum_{j=0}^{N} g_{M,j}$, can be computed without knowing the solution u_M of (1.2) by

$$a(g_{M,j}, \phi_j) = \langle g, \phi_j \rangle \quad \text{for any } \phi_j \in S_j,\ j = 0, \ldots, N. \tag{1.9}$$

Note that equations (1.2) and (1.7) have the same solutions and an explicit form for P is not necessary in solving (1.7).

In fact, if we use the Richardson method to solve (1.7), then given an iterate u_M^n we compute u_M^{n+1} by

$$u_M^{n+1} = u_M^n - \tau(Pu_M^n - g_M),$$

where the residual $r^n := (Pu_M^n - g_M)$ is computed as $r^n = \sum_{j=0}^N r_j^n$ with $r_j^n := P_j u_M^n - g_{M,j}$ being solutions of

$$a(r_j^n, w_j) = a(u_M^n, w_j) - \langle g, w_j\rangle \quad \text{for any } w_j \in S_j. \tag{1.10}$$

Here τ is a damping parameter whose optimal value is given by

$$\tau_{opt} = \frac{2}{\lambda_{\max}(P) + \lambda_{\min}(P)}.$$

In the following we comment on the performance of the additive Schwarz operator as preconditioner for the CG or GMRES method, for the use of the additive Schwarz method as a linear solver, see [HS96].

Let A_M be the discrete form of the pseudodifferential operator A, i.e. $A_M : S_M \to S_M$ be defined for any $v_M \in S_M$ as

$$\langle A_M v_M, w\rangle = a(v_M, w) \quad \forall w \in S_M. \tag{1.11}$$

Then the additive Schwarz operator P can be written as $P = BA_M$, where the preconditioner B is given by

$$B = \sum_{j=0}^{M} A_j^{-1} Q_j.$$

Here Q_j is the L^2-projection from S_M onto S_j, and A_j is the restriction of A_M onto S_j, i.e.

$$\langle Q_j v_M, w_j\rangle = \langle v_M, w_j\rangle \quad \forall v_M \in S_M \text{ and } w_j \in S_j,$$

and

$$\langle A_j v_j, w_j\rangle = a(v_j, w_j) \quad \text{for any } v_j, w_j \in S_j.$$

This representation of $P = BA_M$ can be seen from the fact that $P_j = A_j^{-1} Q_j A_M$. In the implementation of the preconditioned conjugate gradient method, an explicit form for B is not important. Indeed, we only need to know the acting of B on any $v_M \in S_M$.

It is known that the rates of convergence of the above-mentioned methods to solve (1.7) depend on the condition number $\kappa(P)$ of P. Moreover, if there exists positive constants c_0 and c_1 such that

$$c_0 a(v_M, v_M) \le a(Pv_M, v_M) \le c_1 a(v_M, v_M) \quad \forall v_M \in S_M,$$

then $\kappa(P) \le c_1/c_0$.

The following lemma is standard in proving bounds for the maximum and minimum eigenvalues of the additive Schwarz operator P.

Lemma 1 *Let* $v_M = \sum_{j=0}^{N} v_{M,j}$, *where* $v_{M,j} \in S_j$, *be a representation of* $v_M \in S_M = S_0 + \cdots + S_N$.

(i) If a representation can be chosen such that, for some $C_1 > 0$,

$$\sum_{j=0}^{N} a(v_{M,j}, v_{M,j}) \leq C_1^{-1} a(v_M, v_M), \tag{1.12}$$

then $\lambda_{\min}(P) \geq C_1$.

(ii) If there exists $C_2 > 0$ *such that for any representation of* v_M

$$a(v_M, v_M) \leq C_2 \sum_{j=0}^{N} a(v_{M,j}, v_{M,j}) \tag{1.13}$$

then $\lambda_{\max}(P) \leq C_2$.

1.2.1 HYPERSINGULAR INTEGRAL EQUATIONS.

In this subsection we consider different additive Schwarz methods for the hypersingular integral operator. We recall the results for the 2-level and multilevel methods for the h-version from [TS96], for the p version from [ST95] and for the h-p version from [HTS]. Since the additive Schwarz method is generally defined by (1.6)–(1.9) it suffices to give for each version the decomposition of the ansatz space corresponding to (1.6).

h version

We decompose S_h, the space of continuous piecewise-linear functions, as

$$S_h = S_H + S_{h,1} + \cdots + S_{h,N_h-1}, \tag{1.14}$$

where S_H is the space of continuous piecewise-linear functions on the coarse mesh with size $H = 2h$ and $S_{h,j} = \text{span}\,\{\phi_{h,j}\}$. Here $\phi_{h,j}$ is the hat function which takes the value 1 at the mesh point x_j and 0 at the other mesh points. In order to show that the additive Schwarz operator for (1.14) has bounded condition number one proceeds as follows (see [TS96]).

In view of Lemma 1 one verifies (1.12) in order to get a good bound for $\lambda_{\min}(P)$. It is necessary to define an appropriate decomposition for any $v_h \in S_h$. This is done in [TS96] via a partition of unity consisting of suitable piecewise-linear functions $\{\Theta_j\}_{j=1,\ldots,N_h-1}$ with supp $\Theta_j = \bar{\Gamma}_j' := [x_{j-1}, x_j]$ and $\left|\frac{d\Theta_j}{dx}\right| \leq \frac{c}{h}$ with a constant $c > 0$. Then

$$v_h = v_H + v_{h,1} + \cdots + v_{h,N_h-1} \tag{1.15}$$

where $v_H := \tilde{P}_{S_H} v_h$ and $v_{h,j} := \Pi_h(\Theta_j w_h)$ with $w_h := v_h - w_H$. Here $\tilde{P}_{S_H}$ is the Galerkin projection from $H_\infty^{1/2}(\Gamma)$ onto S_H, and Π_h is the interpolation operator from $C(\Gamma)$ onto S_h. Then

$$\|v_{h,j}\|^2_{H_\infty^{1/2}(\Gamma_j')} \leq C\,\|\Theta_j w_h\|^2_{H_\infty^{1/2}(\Gamma_j')} \leq C\,\left(h^{-1}\|w_h\|^2_{L^2(\Gamma_j')} + h\|w_h\|^2_{H^1(\Gamma_j')}\right). \tag{1.16}$$

On the other hand (see e.g. [SW90])

$$\|w_h\|_{L^2(\Gamma)} \le c\, H^{1/2} \|v_h\|_{H^{1/2}_\infty(\Gamma)}, \qquad \|w_h\|_{H^{1/2}_\infty(\Gamma)} \le c\,\|v_h\|_{H^{1/2}_\infty(\Gamma)}. \tag{1.17}$$

Hence summing over j yields with the inverse property

$$\sum_{j=1}^{N_h-1} \|v_{h,j}\|_{H^{1/2}_\infty(\Gamma'_j)} \le c\left(h^{-1}\|w_h\|^2_{L^2(\Gamma)} + h\|w_h\|^2_{\overset{\circ}{H}{}^1(\Gamma)}\right) \le c\|v_h\|^2_{H^{1/2}_\infty(\Gamma)} \tag{1.18}$$

implying (1.12) since $\|v_H\|_{H^{1/2}_\infty(\Gamma)} \le c\|v_h\|_{H^{1/2}_\infty(\Gamma)}$.

In order to show the boundedness of $\lambda_{\max}(P)$ one checks on (1.13). In order to do so one splits $P = P_0 + T$ with $T = \sum_{j=1}^{N_h-1} P_j$ and since P_0 is bounded in $H^{1/2}_\infty(\Gamma)$ one has left to consider

$$a(Tv_h, v_h) = \sum_{i=1}^{N_h-1} \frac{a(v_h, \phi_{hi})^2}{a(\phi_{hi}, \phi_{hi})} \le c\, a(v_h, v_h). \tag{1.19}$$

Let W_h be the discrete form of the hypersingular operator W, i.e. $W_h : S_h \to S_h$ defined $\forall v_h \in S_h$ as

$$\langle W_h v_h, w_h\rangle = a(v_h, w_h) \quad \forall w_h \in S_h. \tag{1.20}$$

Then with supp $\phi_{h,i} = \bar{\Gamma}'_i = [x_{i-1}, x_{i+1}]$

$$a(v_h, \phi_{hi}) \le \|W_h v_h\|_{L^2(\Gamma'_i)}\|\phi_{hi}\|_{L^2(\Gamma'_i)} \le h^{1/2}\|W_h v_h\|_{L^2(\Gamma'_i)}\|\phi_{hi}\|_{H^{1/2}_\infty(\Gamma'_i)} \tag{1.21}$$

Hence

$$a(Tv_h, v_h) \le c\, h\|W_h v_h\|^2_{L^2(\Gamma)}. \tag{1.22}$$

But $\|W_h v_h\|^2_{L^2(\Gamma)} \le c\, h^{-1}\|v_h\|^2_{H^{1/2}_\infty(\Gamma)}$ yielding (1.19).

If one continues the 2-level method for the global problem on S_H

$$S_h = S_{h_1} + \sum_{l=2}^{L}\left(S_{h_l,1} + \cdots + S_{h_l,N_{h_l}-1}\right) \tag{1.23}$$

where $h_{l-1} = 2h_l$ then one has the multilevel method.

For brevity we only comment on the boundedness of $\lambda_{\max}(P)$, for the lower bound for $\lambda_{\min}(P)$ see [TS96]. The multilevel additive Schwarz operator is now defined as

$$P = \sum_{l=1}^{L}\sum_{i=1}^{N_{h_l}-1} P_{S_{h_l,i}} \tag{1.24}$$

where $P_{S_{h_l,i}} : S_h \to S_{h_l,i}$ is defined for any $v \in S_h$ by

$$a(P_{S_{h_l,i}} v, w) = a(v, w) \quad \forall w \in S_{h_l,i}. \tag{1.25}$$

For $T_l := \sum_{i=1}^{N_{h_l}-1} P_{S_{h_l,i}}$ the strengthend Cauchy-Schwarz inequality holds (see [TS96]): There exists a constant $c > 0$ and $\gamma \in (0,1)$ such that for any $v \in S^k = \sum_{i=1}^{N_{h_k}-1} S_{h_k,i}$ where $1 \le k \le l \le L$ there holds

$$a(T_l v, v) \le c\,\gamma^{2(l-k)} a(v, v). \tag{1.26}$$

Then application of a general argument by Bramble and Pasciak [BP93, Theorem 3.1] yields for $P = \sum_{l=1}^{L} T_l$ (cf. Lemma 2.8 in [TS96]) with a constant c_1

$$a(Pv, v) \le c_1 a(v, v) \text{ for any } v \in S_h. \tag{1.27}$$

Altogether we have the boundedness of the condition number of the 2-level additive and the multilevel additive Schwarz operators.

Theorem 1 [TS96, STZ] *The additive Schwarz operator corresponding to* (1.14) *or* (1.23) *has bounded condition number independent of* h*, the mesh size of the finest level, and the number of the levels* L*.*

p version

In the 2 level method the ansatz space S^p is decomposed as follows. We denote by S^1 the space of continuous piecewise-linear functions which vanish at the endpoints ± 1. This space serves the same purpose as the coarse grid space in the h version. To each subinterval $\Gamma_j = (x_{j-1}, x_j)$, $j = 1, \dots, N_0$, we associate the space S_j^p which is the affine image on Γ_j of the space span $\{\mathcal{L}^2, \dots, \mathcal{L}^p\}$, where $\mathcal{L}^k(x) = \int_{-1}^{x} L^{k-1}(y)\, dy$ with the Legendre polynomial L^{k-1} of degree $k-1$. Note that $\mathcal{L}^k$ vanishes at ± 1 and therefore functions in S_j^p vanish at the endpoints of Γ_j. We can decompose S^p as a direct sum

$$S^p = S^1 \oplus S_1^p \oplus \dots \oplus S_{N_0}^p. \tag{1.28}$$

Theorem 2 *[ST95] For the additive Schwarz operator* P *according to* (1.28) *there holds*

$$\lambda_{\min}(P) \ge C_1 \,(1 + \log^3 p)^{-1} \quad \textit{and} \quad \lambda_{\max}(P) \le C_2,$$

with positive constants C_1 *and* C_2 *independent of* p *and* N_0 *and hence* $\kappa(P) \le C_2 C_1^{-1}(1 + \log^3 p)$.

Proof. In order to prove the estimate for $\lambda_{\min}(P)$ it suffices in view of Lemma 1 to prove that for any $u_p = u_1 + \sum_{j=1}^{N_0} u_{p,j}$ where $u_1 \in S^1$, and $u_{p,j} \in S_j^p$ there holds

$$\|u_1\|^2_{\tilde{H}^{1/2}_\infty(\Gamma)} + \sum_{j=1}^{N_0} \|u_{p,j}\|^2_{\tilde{H}^{1/2}_\infty(\Gamma_j)} \le C_1^{-1} \,(1 + \log^3 p)\, \|u_p\|^2_{\tilde{H}^{1/2}_\infty(\Gamma)}. \tag{1.29}$$

First we note that since $u_{p,j}$ vanishes at the mesh points, its linear interpolant is identically zero, and therefore u_1 is the linear interpolant of u_p. Hence there exists a positive constant c independent of p and N_0 such that

$$\|u_1\|^2_{\tilde{H}^{1/2}_\infty(\Gamma)} \le c\,(1 + \log p) \|u_p\|^2_{\tilde{H}^{1/2}_\infty(\Gamma)}. \tag{1.30}$$

Next let $w_p := u_p - u_1 = \sum_{j=1}^{N_0} u_{p,j}$. Then for any $j = 1, \dots, N_0$ since $w_p|_{\Gamma_j} = u_{p,j}$ we have

$$\sum_{j=1}^{N_0} \|u_{p,j}\|^2_{H^{1/2}(\Gamma_j)} = \sum_{j=1}^{N_0} \|w_p\|^2_{H^{1/2}(\Gamma_j)} \le \|w_p\|^2_{H^{1/2}(\Gamma)} \le \|w_p\|^2_{\tilde{H}^{1/2}_\infty(\Gamma)}, \tag{1.31}$$

where the first inequality characterizing the property of the $H^{1/2}$-norm was proved in [vP89]. On the other hand, [BCMP91, Theorem 6.5] gives

$$\|\psi\|_{\tilde{H}^{1/2}_\infty(\Gamma_j)} \leq c\,(1+\log p)\,\|\psi\|_{H^{1/2}_\infty(\Gamma_j)} \quad \text{for any } \psi \in S^p_j, j=1,\ldots,N_0, \tag{1.32}$$

where c is a constant independent of p and ψ. Hence (1.30)–(1.32) yield (1.29). □

We get a multilevel method when we further decompose S^p_j by $S^p_j = \oplus^p_{k=2}\tilde{S}^k_j$, where $\tilde{S}^k_j = \text{span}\,\{\mathcal{L}^k_j\}$ and $\mathcal{L}^k_j$ is the affine image of $\mathcal{L}^k$ onto Γ_j. Hence

$$S^p = S^1 \oplus \left(\oplus^p_{k=2} \oplus^{N_0}_{j=1} \tilde{S}^k_j\right). \tag{1.33}$$

Therefore a function u_p in S^p can be represented as

$$u_p = u_1 + \sum_{k=2}^{p}\sum_{j=1}^{N_0} u_{k,j}, \tag{1.34}$$

where $u_1 \in S^1$ and $u_{k,j} \in \tilde{S}^k_j$ for $k=2,\ldots,p$, $j=1,\ldots,N_0$.

With the following result one obtains a bound for the additive Schwarz operator of the multilevel p-version

Lemma 2 *[HTS] Let $u_k = c_k\mathcal{L}^*_k$ where $\mathcal{L}^*_k = \mathcal{L}_k/\|\mathcal{L}_k\|_{L^2(I)}$ with $I=(-1,1)$, and $u = \sum_{k=2}^p u_k$. Then there exist positive constants C_1 and C_2 independent of the polynomial degrees such that*

$$C_1\|u\|^2_{H^{1/2}_\infty(I)} \leq \sum_{k=2}^{p}\|u_k\|^2_{H^{1/2}_\infty(I)} \leq C_2 p\,\|u\|^2_{H^{1/2}_\infty(I)}.$$

In view of Lemma 1 one shows

$$\|u_1\|^2_{H^{1/2}_\infty(\Gamma)} + \sum_{j=1}^{N_0}\sum_{k=2}^{p}\|u_{k,j}\|^2_{H^{1/2}_\infty(\Gamma_j)} \leq C_1^{-1}p(1+\log^3 p)\|u_p\|^2_{H^{1/2}_\infty(\Gamma)} \tag{1.35}$$

which yields a lower bound for $\lambda_{\min}(P)$. Due to Lemma 2.4 in [ST95] we have

$$\|u_1\|^2_{H^{1/2}_\infty(\Gamma)} \leq C(1+\log p)\|u_p\|^2_{H^{1/2}_\infty(\Gamma)} \tag{1.36}$$

and with Lemma 2 we obtain

$$\sum_{k=2}^{p}\|u_{k,j}\|^2_{H^{1/2}_\infty(\Gamma_j)} \leq c\,p\|\sum_{k=2}^{p}u_{k,j}\|^2_{H^{1/2}_\infty(\Gamma_j)}. \tag{1.37}$$

Summing over j and using (1.36) one obtains (1.35). Since (1.33) is a direct sum decomposition the representation (1.34) is unique and application of Lemma 1 yields the boundedness of $\lambda_{\max}(P)$.

Theorem 3 *[HTS] There exist positive constants C_1, C_2 independent of p and N_0 such that $\lambda_{\min}(P) \geq C_1p^{-1}(1+\log^3 p)^{-1}$ and $\lambda_{\max}(P) \leq C_2$ for the multilevel additive Schwarz operator defined via (1.33). Hence $\kappa(P) \leq \frac{C_2}{C_1}p(1+\log^3 p)$.*

h-p version

In [TS] we consider a preconditioner based on a two-level mesh for the h-p version of the Galerkin boundary element method for the hypersingular equation. We show that the condition number of the additive Schwarz operator is of order $\max_i \left(1+\log \frac{H_i p_i}{h_i}\right)(1+\log p_i)$ where p_i is the maximal polynomial degree used in the i-th subdomain Γ_i. Here the possibly irregular coarse mesh is obtained by dividing Γ into disjoint subdomains Γ_i of length H_i whereas the quasi-uniform fine mesh is obtained by further dividing Γ_i into subintervalls Γ_{ij}, the maximal length of the subintervals Γ_{ij} is denoted by h_i.

For preconditioning on a geometric mesh we take for simplicity $\Gamma = (0,1)$ and assume that the solution of (1.3) has a singularity only at the origin and belongs to the countably normed space $B^2_\beta(\Gamma)$, $0 < \beta < 1$ (see [Heu92]). The geometric mesh Γ^n_σ contains layers Γ_i, $1 \le i \le n$, according to the distance to the origin, and $H_i = h_i = \operatorname{diam}(\Gamma_i) \sim \sigma^{n-i+1}$, $1 \le i \le n$. In this case the 2-level h-p version becomes just the 2-level p-version on the geometric mesh and the corresponding additive Schwarz operator has condition number of order $(1+\log n)^2$ with $p_i \sim i$ [HTS].

1.2.2 WEAKLY SINGULAR INTEGRAL EQUATIONS

In this subsection we consider different additive Schwarz methods for the weakly singular integral operator recalling results from [ST95, TS96, HTS].

h-version

In the 2-level method the ansatz space S_M in (1.2) is now the space $\bar{S}_h$ of piecewise-constant functions on Γ discretised by a mesh of width h. We decompose

$$\bar{S}_h = \bar{S}_H + \bar{S}_{h,1} + \cdots + \bar{S}_{h,N_h-1}, \tag{1.38}$$

where $H = 2h$ and $S_{h,j} = \operatorname{span}\{\chi_{h,j}\}$. Here the Haar function $\chi_{h,j}$ is the derivative of the hat function $\phi_{h,j}$ and has therefore support on $\bar{\Gamma}_j \cup \bar{\Gamma}_{j+1}$.

If we continue the 2-level method for the global problem on $\bar{S}_H$, we then have the multilevel method which was analysed in [TS96]. The decomposition of the ansatz space can then be described by

$$\bar{S}_h = \bar{S}_{h_1} + \sum_{l=2}^{L} \left(\bar{S}_{h_l,1} + \cdots + \bar{S}_{h_l,N_{h_l}-1}\right) \tag{1.39}$$

where $\bar{S}_{h_1}$ is the space with the coarsest mesh including the functions being constant on Γ and $h_{l-1} = 2h_l$.

Theorem 4 [TS96, STZ] *There exist constants C_1 and C_2 independent of h (and the number of levels L) such that $\lambda_{\min}(P) \ge C_1$ and $\lambda_{\max}(P) \le C_2$ and $\kappa(P) \le (C_2/C_1)$ with P defined via* (1.38) *or* (1.39).

The proof of Theorem 4 is similar to the proof of Theorem 1. It uses i) that the antiderivative operator $J_{1/2}$ defined on $\tilde{H}_0^{-1/2}(I) = \{v \in \tilde{H}^{-1/2}(I) \,:\, \int_I v(x)\,dx = 0\}$

satisfies (cf. [HS96]) for $u \in \tilde{H}_0^{-1/2}(\Gamma)$

$$\|u\|_{\tilde{H}^{-1/2}(I)} \simeq \|J_{1/2}u\|_{\tilde{H}^{1/2}(I)} \tag{1.40}$$

and ii) that the Haar function $\chi_{h,j}$ has zero integral mean.

p-version

We define on the mesh (1.5) the space $\bar{S}^0$ of piecewise-constant functions on Γ. To each subinterval Γ_j, we associate the space $\bar{S}_j^p$ of polynomials of degree $p > 0$ with support in $\overline{\Gamma}_j$. The space $\bar{S}_j^p$ is the affine image onto Γ_j of the space spanned by the Legendre polynomials of degree p. We have

$$\bar{S}^p = \bar{S}^0 \oplus \bar{S}_1^p \oplus \cdots \oplus \bar{S}_{N_0}^p. \tag{1.41}$$

Then Theorem 2 holds with P defined via (1.41) instead of (1.28) (see [ST95]).

For a multilevel method we introduce on each subinterval Γ_j the space $\bar{S}_j^k = \operatorname{span}\{L_j^k\}$ where L_j^k is the affine image onto Γ_j of the Legendre polynomial L^k of degree k. Then

$$\bar{S}^p = \bar{S}^0 \oplus \left(\oplus_{k=1}^p \oplus_{j=1}^{N_0} \bar{S}_j^k \right). \tag{1.42}$$

and Theorem 3 holds with P defined via (1.42) instead of (1.33). Note that the Legendre polynomials L^k have zero integral mean and therefore the norm equivalence (1.40) holds for $\bar{S}_j^k$.

Hierarchical bases

Alternatively to the subspace decomposition (23) we consider now the following hierarchical basis decomposition of S_h:

$$S_h \;=\; S_{h_1} \oplus \bigoplus_{l=2}^{L} \bigoplus_{j=1}^{N_l-1}{}' S_{h_l,j}$$

where $'$ denotes the sum over odd values of j only. The corresponding multilevel additive Schwarz operator is defined as

$$P_{HB} \;=\; \sum_{l=1}^{L} \sum_{j=1}^{N_l-1}{}' P_{S_{h_l},j} \,.$$

In [TSM] it could be shown that the largest eigenvalue of P_{HB} is bounded independently of L and the smallest eigenvalue behaves like L^{-3}. This correponds in some way to the hierarchical basis result in [Yse88] for the 2D-FEM. By using a different basis (wavelets) it could be shown in [STZ] that the corresponding multilevel additive Schwarz operator has bounded condition number (independently of L). Corresponding results could also be obtained for the weakly singular case [TSM, STZ]. It can be shown by a simple example that the hierarchical basis preconditioner fails for the three-dimensional hypersingular case. However, the situation for the weakly singular case in 3D is currently under research and the numerical results indicate that in this case a hierarchical basis preconditioner reduces the condition number of the linear systems considerably.

1.3 ADDITIVE SCHWARZ METHODS FOR INDEFINITE PROBLEMS

Now let $B = A + K$ be a pseudodifferential operator of order $\alpha = 1$ or $\alpha = -1$ where A is positive definite and K is a compact operator from $\tilde{H}^{\alpha/2}(\Gamma)$ into $H^{-\alpha/2}(\Gamma)$. Then B satisfies a Gårding inequality, i.e., there exist $\gamma > 0$ and $\eta > 0$ such that the real part of $\langle Bu, u\rangle$ satisfies $\forall u \in \tilde{H}^{\alpha/2}(\Gamma)$,

$$\Re(B(u,u)) \geq \gamma \|u\|^2_{\tilde{H}^{\alpha/2}(\Gamma)} - \eta \|u\|^2_{\tilde{H}^{\alpha/2-1/2}(\Gamma)}. \tag{1.43}$$

Again with $f \in H^{-\alpha/2}(\Gamma)$ we solve $Bu = f$ by Galerkins method, i.e. find $u \in S \subset \tilde{H}^{\alpha/2}(\Gamma)$ such that

$$\langle Bu, v\rangle = \langle f, v\rangle \quad \forall v \in S. \tag{1.44}$$

The following operators map the boundary element space S onto the subspaces S_i, $i = 0, \ldots, N$, and are defined in terms of the bilinear forms $b(u,v) := \langle Bu, v\rangle$ and $a(u,v) := \langle Au, v\rangle$.

Definition 1.3.1 *For any $w \in V$, $Q_0 w \in S_0$ is the solution of $b(Q_0 w, v_0) = b(w, v_0) \quad \forall v_0 \in S_0$. For $i = 1, \ldots, N$ and for any $w \in V$, $P_i w \in S_i$ is the solution of $a(P_i w, v_i) = b(w, v_i) \quad \forall v_i \in S_i$.*

The additive Schwarz operator is now defined as $Q = Q_0 + P_1 + \cdots + P_N$ and the additive Schwarz method consists in solving

$$Qu_M = b_M \tag{1.45}$$

with RHS $b_M = \sum_{j=0}^N b_j$ where

$$b(b_0, v_0) = \langle f, v_0\rangle \quad \forall v_0 \in S_0, \quad a(b_j, v_j) = \langle f, v_j\rangle \quad \forall v_j \in S_j, j = 1, \ldots, N. \tag{1.46}$$

It is shown in [STa, STb] that this additive Schwarz algorithm when used with the GMRES method gives an efficient solver for the Galerkin scheme (1.44). The rate of convergence is bounded for the h-version and logarithmically growing in p for the p-version, if the mesh size of the coarse space is sufficiently small.

As proved in [EES83] the rate of convergence of the GMRES method when used to solve (1.45) is given as $1 - \frac{C_0^2}{C_1^2}$, where

$$C_0 := \inf_{v \in S} \frac{a(v, Qv)}{a(v,v)} \text{ and } C_1 := \sup_{v \in S} \frac{a(Qv, Qv)}{a(v,v)}. \tag{1.47}$$

With this result we show in [STa, STb] C_0 is independent of h in the h-version and behaves like $1 + \log^3 p$ in the p-version whereas C_1 is always constant.

Fitting in the above setting does the weakly singular integral equation

$$V_k v(x) := \frac{i}{2} \int_\Gamma H_0^1(k|x-y|)\, v(y) ds_y = f(x), \quad x \in \Gamma, \tag{1.48}$$

from time harmonic acoustic scattering on a screen Γ where H_0^1 is the Hankel function of the first kind and order 0 with $k \in \mathbb{C}$. While (1.48) corresponds to a Dirichlet problem in $\mathbb{R}^2 \backslash \bar{\Gamma}$ the corresponding Neumann problem leads to

$$W_k v(x) := \frac{-i}{2} \frac{\partial}{\partial n_x} \int_\Gamma \frac{\partial}{\partial n_y} \left[H_0^1(k|x-y|)\right] v(y)\, ds_y = f(x), \quad x \in \Gamma. \tag{1.49}$$

1.4 ADDITIVE SCHWARZ METHODS FOR POSITIVE DEFINITE INTEGRAL OPERATORS IN THREE DIMENSIONS

In this section we briefly summarize results on the preconditioning of the hypersingular and weakly singular integral operators in $\mathbb{R}^3$ which are defined by

$$Wu(x) := \frac{1}{4\pi}\frac{\partial}{\partial n_x}\int_\Gamma u(y)\frac{\partial}{\partial n_y}\frac{1}{|x-y|}\,dS_y, \quad x \in \Gamma$$

and

$$Vu(x) := \frac{1}{4\pi}\int_\Gamma \frac{u(y)}{|x-y|}\,ds_y, \quad x \in \Gamma,$$

respectively. Here, for ease of presentation, $\Gamma \subset \mathbb{R}^3$ is a plane open surface and all the theoretical results are also valid for polyhedrons with rectangular sides. Again, W and V are positive definite operators of orders 1 and -1, respectively. As in the two–dimensional case we solve first kind integral equations involving W and V by the Galerkin method. For the hypersingular operator we have to use boundary element spaces $S^1(\Gamma) \subset \tilde{H}^{1/2}(\Gamma)$ consisting of continuous functions and for the weakly singular operator it suffices to take boundary element spaces $S^0(\Gamma) \subset \tilde{H}^{-1/2}(\Gamma)$ of discontinuous functions.

1.4.1 H VERSION FOR THE HYPERSINGULAR OPERATOR

We use quasi–uniform rectangular meshes of size h on Γ and take piecewise bilinear functions corresponding to these meshes to construct $S^1(\Gamma) = S^1_h(\Gamma)$. Just as described in §1.2.1 we perform 2–level and multilevel additive Schwarz methods in an analogous way, i.e. we decompose

$$S^1_h = S^1_H + S^1_{h,1} + \cdots + S^1_{h,N_h-1} \tag{1.50}$$

in the 2–level case and

$$S^1_h = S^1_{h_1} + \sum_{l=2}^{L}\left(S^1_{h_l,1} + \cdots + S^1_{h_l,N_{h_l}-1}\right) \tag{1.51}$$

in the multilevel case. We obtain a result similar to the two–dimensional case.

Theorem 5 [Heua] *The additive Schwarz operator corresponding to* (1.50) *or* (1.51) *has a condition number which is bounded by*

$$\kappa(P) \le Ch^{-\epsilon}$$

with arbitrary $\epsilon > 0$. The constant C is independent of h, the mesh size of the finest level, and of the number of the levels L.

To get rid of the constraint $H = 2h$ one can also consider general 2–level methods where one has a coarse mesh Γ_H which is almost independent of the fine mesh Γ_h, the only restriction being the compatibility. The used decomposition is given by

$$S^1_h(\Gamma) = S^1_H(\Gamma) \cup S^1_{h,H}(\Gamma) \cup \cup_{j=1}^J S^1_h(\Gamma_j). \tag{1.52}$$

The space $S^1_H(\Gamma)$ consists of the usual continuous piecewise bilinear functions on the mesh Γ_H of size H. $S^1_{h,H}(\Gamma)$ is the so–called wire basket space which is spanned by the piecewise bilinear hat functions of $S^1_h(\Gamma)$ which are concentrated at the nodes lying on the element boundaries of the mesh of size H. The spaces $S^1_h(\Gamma_j)$ are spanned by the piecewise bilinear hat functions concentrated at the nodes interior to the restricted meshes $\Gamma_h|_{\Gamma_j} = \Gamma_{j,h}$, $j = 1, \ldots, J$. The result is the following.

Theorem 6 [HS] *The condition number of the additive Schwarz operator P which is defined by the decomposition* (1.52) *is bounded by*

$$\kappa(P) \le C(1 + \log \frac{H}{h})$$

where the constant $C > 0$ is independent of the coarse and fine mesh sizes H and h.

Thus, for this preconditioner, we have bounded condition numbers if the ratio H/h is fixed.

1.4.2 P VERSION FOR THE HYPERSINGULAR OPERATOR

We consider a fixed rectangular mesh Γ_h and take tensor products of piecewise linear functions and antiderivatives of Legendre polynomials as basis functions. For the following overlapping decomposition of the boundary element space $S^1(\Gamma) = S^1_p(\Gamma)$ we obtain bounded condition numbers of the corresponding additive Schwarz operator:

$$S^1_p = S^1_1 \cup S^1_{p,1} \cup \cdots \cup S^1_{p,N_h}. \tag{1.53}$$

The so–called coarse grid space is that of the h version, $S^1_1 = S^1_h(\Gamma)$, and the remaing spaces are subspaces localized at the neighborhood of each interior node. More precisely $S^1_{p,j}$ is the space of piecewise polynomials of degree p which are globally continuous and which have support contained in the elements adjacent to the node with number j. Therefore, subspaces for adjacent nodes have common functions.

Theorem 7 [Heua] *The condition number of the additive Schwarz operator P which is defined by the decomposition* (1.53) *has bounded condition number.*

Since the subspaces $S^1_{p,j}$ are rather large for large polynomial degree p one is interested in further splitting the decomposition. Due to the tensor product structure of the basis functions one has a natural decomposition into subspaces which consists of functions concentrated at the nodes, edges, and interior to the elements, seperately. However, it is well known that one cannot take the usual nodal hat functions seperately for such a splitting in higher dimensions. This would result in large condition numbers. Therefore, in order to use a non–overlapping decomposition, one has to consider well behaved basis functions, i.e. functions with small energy. In [Heuc] the minimizing polynomial of degree p defined by

$$||\varphi_0||_{L^2(-1,1)} = \min_{\varphi} ||\varphi||_{L^2(-1,1)}, \quad \varphi_0(1) = 1, \ \varphi_0(-1) = 0.$$

was taken to construct the nodal basis functions and as the component of the edge functions perpendicular to the edges. Further, basis functions for the edges and for the functions interior to the elements are defined as discrete tensor product solutions in the weak sense of the Laplacian. The decomposition is as follows.

$$S^1_p(\Gamma_h) = X_0 \oplus X_1 \oplus \cdots \oplus X_{J_h}. \tag{1.54}$$

Here $X_j = S^1_p(\Gamma_h) \cap \tilde{H}^{1/2}(\Gamma_j)$, $j = 1, \ldots, J_h$, where Γ_j is an element of the mesh. X_0 is the global space of the remaining functions which are concentrated at the nodes and the edges of the mesh. This space is called the wire basket space.

Theorem 8 [Heuc] *The condition number of the additive Schwarz operator P defined by the decomposition* (1.54) *is bounded by*

$$\kappa(P) \le C(1 + \log p)^3.$$

The constant C is independent of the mesh size h and the polynomial degree p.

As shown in [Heuc] a similar result holds even for the diagonal preconditioner if one takes the special discretely harmonic basis functions.

1.4.3 P VERSION FOR THE WEAKLY SINGULAR OPERATOR

We use quasi–uniform rectangular meshes of size h on Γ and take discontinuous piecewise polynomials of degree p for the boundary element space $S^0(\Gamma) = S^0_p(\Gamma)$. We decompose

$$S^0_p(\Gamma) = S^0_p(\Gamma_1) \cup \cdots \cup S^0_p(\Gamma_J). \tag{1.55}$$

The spaces $S^0_p(\Gamma_j)$ are the restriction of $S^0_p(\Gamma)$ onto a subdomain Γ_j where $\bar{\Gamma} = \cup_{j=1}^J \bar{\Gamma}_j$ is a, possibly oberlapping, decomposition of Γ. We obtain

Theorem 9 [Heub] *For any $\epsilon > 0$ there exists a constant $C > 0$ such that the condition number of the additive Schwarz operator defined by the decomposition* (1.55) *is bounded with arbitrary $\epsilon > 0$ by*

$$\kappa(P) \le C p^\epsilon.$$

1.5 ADAPTIVE TWO–LEVEL BOUNDARY ELEMENT METHODS

In this section we show how to derive *a posteriori* error estimates from the properties of two-level additive Schwarz operators. If the corresponding subspace decomposition is chosen carefully we may obtain local error indicators which are easily computable and which can be used for adaptive refinement strategies. We derive the main result for general two-level decompositions and for brevity apply the theory to the hypersingular integral operator with a subspace decomposition which was analyzed in [Cao95] and [TSM] in the hierarchical basis.

Let S_H be a finite dimensional space and let $u_H \in S_H$ be the Galerkin solution, i.e.

$$a(u_H, v) \;=\; \langle g, v \rangle \qquad \forall\, v \in S_H\,.$$

Let S_h be a refinement of S_H with discretization parameter $h = h(H)$ and let $u_h \in S_h$ be the Galerkin solution in S_h. We assume the following saturation condition:

There exists a constant $0 < \gamma < 1$ independent of H such that

$$||u - u_h|| \ \leq \ \gamma ||u - u_H|| \tag{1.56}$$

where u is the exact solution of the problem under consideration and $||\cdot||^2 = a(\cdot,\cdot)$ is the energy norm.

By using the triangle inequality it can be shown from (1.56) that

$$(1-\gamma)\,||u - u_H|| \ \leq \ ||u_h - u_H|| \ \leq \ (1+\gamma)\,||u - u_H||\,. \tag{1.57}$$

Let

$$S_h \ = \ S_H \oplus D_h\,, \qquad D_h \ = \ S_h^1 + S_h^2 + \ldots + S_h^N \tag{1.58}$$

be a subspace decomposition of the space S_h. If it can be shown that the correponding two-level additive Schwarz operator $P = P_{S_H} + \sum_{j=1}^N P_{S_h^j}$ has bounded condition number, i.e. if there are constants $\lambda_i > 0$ independent of H such that

$$\lambda_1 a(v,v) \ \leq \ a(Pv,v) \ \leq \ \lambda_2 a(v,v) \qquad \forall\, v \in S_h \tag{1.59}$$

then, by using

$$a(Pv,v) \ = \ a(P_{S_H}v,v) + \sum_{j=1}^N a(P_{S_h^j}v,v) \ = \ ||P_H v||^2 + \sum_{j=1}^N ||P_{S_h^j}v||^2$$

and the Galerkin property $||P_{S_H}(u_h - u_H)||^2 \ = \ 0$ we obtain the following result:

Theorem 10 *If (1.56) and (1.59) hold then*

$$\frac{1}{\lambda_2(1+\gamma)} \sum_{j=1}^N ||P_{S_h^j}(u_h - u_H)||^2 \ \leq \ ||u - u_H||^2 \ \leq \ \frac{1}{\lambda_1(1-\gamma)} \sum_{j=1}^N ||P_{S_h^j}(u_h - u_H)||^2$$

In [TSM], [Mun96] the h-version Galerkin method for the hypersingular integral operator W was analyzed. The spaces S_h and S_H were chosen as in Section 1.2.1 and

$$D_h \ = \ S_{h,1} \oplus S_{h,3} \oplus S_{h,5} \oplus \ldots \oplus S_{h,N_h-1}$$

with the same notation as in (1.14). It could be shown in [Mun96] that (1.59) holds. Hence, from the above theorem, we obtain local error indicators

$$\eta_j \ = \ ||P_{S_{h,j}}(u_h - u_H)|| \ = \ \frac{|a(u_h - u_H, \phi_{h,j})|}{a(\phi_{h,j},\phi_{h,j})^{1/2}} \ = \ \frac{|\langle Wu_h - g, \phi_{h,j}\rangle|}{\langle W\phi_{h,j},\phi_{h,j}\rangle^{1/2}}$$

with $j \in \{1,3,\ldots,N_h-1\}$ which can be used in an adaptive algorithm [Cao95]. Corresponding results for the weakly singular operator were obtained in [Fun96]. For an extension to surfaces in $\mathbb{R}^3$ see [MS96].

1.6 IMPLEMENTATION AND NUMERICAL RESULTS.

For the implementation of the additive Schwarz method we have to distinguish two basically different cases, namely when the subspaces form a direct sum or when they are nested.

(a) When the subspace decomposition is given as a direct sum we proceed as follows. Let

$$S = \operatorname{span}\{\phi_j \ : j = 1,\ldots,\dim S\},$$

and

$$S_i = \operatorname{span}\{\phi_j \ : j = N_{i-1}+1,\ldots,N_i\}.$$

Let A_i denote the Galerkin matrix corresponding to S_i, $i = 0,\ldots,N$. Then we have $P = \sum_i P_i = \sum_i C_i^T A_i^{-1} C_i A_S$, where the projection matrices $C_i = (c_{l,k}^{(i)})$ are diagonal matrices with $c_{l,k}^{(i)} = 1$ when $l = k = N_{i-1}+1,\ldots,N_i$ and $c_{l,k}^{(i)} = 0$ otherwise. We also have $A_i = C_i^T A_S C_i$, where A_S is the Galerkin matrix corresponding to S. The simple form of the projection matrices C_i is due to the fact that the same basis functions are used in S and in the subspaces S_i. Note this is exactly the situation of the p-version. If the matrices A_i are simply the diagonal blocks of A_S, the preconditioner P can be written as

$$P = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_N \end{pmatrix}^{-1} A_S.$$

(b) When the subspaces form a nested sequence we proceed with the implementation as follows. Again let A_i denote the Galerkin matrix belonging to S_i, $i = 0,\ldots,N$. Note that now the subspaces S_i are spanned by different basis functions, $S_i = \operatorname{span}\{\phi_j^i \ : \ j = 1,\ldots,\dim S_i\}$. Here the projection matrices C_i are defined by $C_i = (c_{j,l}^i)$. Hence the projection to the lower level is given by

$$\phi_j^i = \sum_{l=1}^{\dim S_{i+1}} c_{j,l}^i \phi_l^{i+1},$$

P can be written in matrix form

$$\Big\{(\operatorname{diag} A_N)^{-1} + \ldots + C_2^T((\operatorname{diag} A_2)^{-1} + C_1^T((\operatorname{diag} A_1)^{-1} + C_0^T A_0^{-1} C_0)C_1)C_2 \cdots C_{N-1}\Big\} A_S$$

where $A_i = C_i A_{i+1} C_i^T$ and $i = 0,\ldots,N-1$ and $A_N := A_S$.

Note that C_i are sparse matrices and therefore the action of C_i and C_i^T can be implemented in a very efficient way; therefore their multiplication with a vector costs only $\mathcal{O}(N)$ operations in contrary to $\mathcal{O}(N^2)$ in case of the full matrix.

In the following we list tables for the additive Schwarz methods described above. The numerical experiments were performed by Dr. M. Maischak on a SUN-Sparcstation 4/470 at the Institute for Applied Mathematics at University of Hannover.

Note that in case of the hypersingular equation the multilevel additive Schwarz preconditioner differs only by a scaling factor from the BPX preconditioner (cf.(1.19)) whereas for the weakly singular operator these preconditioners behave completely differently, since in this case $a(\chi_{h_i}, \chi_{h_i}) := \langle V\chi_{h_i}, \chi_{h_i}\rangle$ depends on the step size h_i of the Haar basis function χ_{h_i} (cf. Table 1).

	Condition number				Number of iterations			
N	A_N	2-level	multilevel	BPX	CG	2-lev	m-lev	BPX
16	15.7545	2.1310	5.1005	4.0543	8	8	8	8
32	32.6847	2.6175	6.0171	4.8520	20	12	11	12
64	65.1292	2.9767	6.9188	8.8454	33	16	14	17
128	129.6566	3.1926	7.8182	10.4593	45	19	16	19
256	259.8891	3.3153	8.7199	12.1835	66	21	17	22
512	517.1460	3.3777	9.6253	14.0330	85	21	19	23
1024	1036.1733	3.4108	10.5344	16.0186	125	22	19	24
2048	2066.6248	3.4255	11.4468	18.1483	168	22	19	26
4096	4119.1740	3.4338	12.3619	20.4271	226	23	19	29
8192		3.4361	13.2791	22.8595		23	19	30
16384		3.4370	14.1980	25.4481		23	19	31
32768		3.4379	15.1182	28.1890		23	19	32
65536		3.4406	16.0395	31.1033		24	19	33

Table 1 Weakly singular integral equation (1.4) with $g_2(x) \equiv 1$: h-version, using CG and 2-level and multilevel additive Schwarz and BPX preconditioner

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	Condition number			Number of iterations		
p	A_N	2-level	multilevel	CG	2-level	multilevel
1	6.22	1.89	1.86	4	2	2
2	19.32	2.09	2.08	6	3	3
3	41.01	2.22	2.41	9	4	4
4	72.47	4.89	2.64	12	5	5
5	115.18	4.12	2.94	15	6	6
6	170.82	2.33	3.20	17	6	7
7	241.21	2.37	3.51	21	6	8
8	328.23	2.38	3.81	24	6	9
9	433.82	2.40	4.14	27	6	10
10	559.88	8.01	4.46	33	6	11

Table 2 Weakly singular integral equation (1.4) with $g_2(x) \equiv 1$: p-version with $N_0 = 2$, using CG and PCG with 2-level and multilevel additive Schwarz methods.

	Condition number			Number of iterations		
N	A_N	2-level	multilevel	CG	2-level	multilevel
15	7.7629	2.1475	3.0353	8	7	8
31	15.5445	2.2072	3.4613	11	11	11
63	31.1092	2.2162	3.7561	17	12	14
127	62.4163	2.2276	3.9714	26	13	16
255	125.0924	2.2299	4.1335	38	13	17
511	250.4733	2.2262	4.2578	55	12	17
1023	501.2394	2.2250	4.3545	78	12	17
2047	1002.7757	2.2236	4.4308	109	12	17
4095	2005.8634	2.2225	4.4917	154	12	18
8191		2.2160	4.5408		11	18
16383		2.2153	4.5808		11	18
32767		2.2150	4.6138		11	18
65535		2.2067	4.6413		10	18

Table 3 Hypersingular integral equation (1.3) with $g_1(x) \equiv 1$: h-version, using CG and PCG with 2-level and multilevel additive Schwarz preconditioners

	Condition number			Number of iterations		
p	A_N	2-level	multilevel	CG	2-level	multilevel
2	4.44	2.31	2.31	2	2	2
3	12.75	3.55	3.55	3	3	3
4	26.04	4.16	4.85	4	4	4
5	45.88	4.84	6.14	5	5	5
6	73.32	5.31	7.41	7	6	6
7	109.94	5.84	8.69	8	7	7
8	156.98	6.21	9.95	9	7	8
9	215.97	6.61	11.24	12	7	9
10	288.23	6.95	12.50	13	7	10

Table 4 Hypersingular integral equation (1.3) with $g_1(x) \equiv 1$: p-version with $N_0 = 2$, using CG and PCG with 2-level and multilevel additive Schwarz methods

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