On quantum cluster algebras of finite type

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Abstract We extend the definition of a quantum analogue of the Caldero-Chapoton map defined [17]. When Q is a quiver of finite type, we prove that the algebra $\mathscr{A}\mathscr{H}_{|k|}(Q)$ generated by all cluster characters (see Definition 1) is exactly the quantum cluster algebra $\mathscr{E}\mathscr{H}_{|k|}(Q)$.

Keywords cluster variable, quantum cluster algebra
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1 Introduction

Quantum cluster algebras were introduced by A. Berenstein and A. Zelevinsky [3] to study the canonical basis. When $q = 1$, the quantum cluster algebras are exactly the corresponding cluster algebras which were introduced and studied by S. Fomin and A. Zelevinsky in a series of papers [9][10][1]. A quantum analogue of the Caldero-Chapoton formula [4] was defined by D. Rupel [17] and the author conjectured that cluster variables could be expressed using this formula and proved it for the cluster variables in finite types as well as in almost acyclic clusters. Later this conjecture was confirmed for acyclic equally valued quivers in [16]. Quantum cluster algebra structures have been studied in a few cases, see for example [13][17][15][6][16][7].

The cluster category was introduced for its combinatorial similarities with cluster algebras. In contrast to the case of cluster algebras, for any objects M, N in the cluster category associated to a quantum cluster algebra, it does not generally hold that $X_N X_M = |k|^{\pm\frac{1}{2}n_{N\oplus M}} X_{N\oplus M}$ for any $n_{N\oplus M} \in \mathbb{Z}$. Thus the natural problem is to ask if $X_{N\oplus M}$ is in the corresponding quantum cluster algebra. Hence it becomes interesting to study the relation between the algebra generated by all cluster characters (see Definition 1) and the corresponding quantum cluster algebra. In the case of cluster algebras, these are equal for finite and affine types [5][8]. In [11][12], C. Geiss, B. Leclerc and J. Schröer have proved that a large class of cluster algebras always contain cluster characters of all objects in the cluster categories. The aim of this article is to prove that for any quiver Q of finite type, the algebra $\mathscr{A}\mathscr{H}_{|k|}(Q)$ generated by all cluster characters is still the quantum cluster algebra $\mathscr{E}\mathscr{H}_{|k|}(Q)$.

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2 Preliminaries and statement of the main result

2.1 Definition of quantum cluster algebras Let L be a lattice of rank m and $\Lambda : L \times L \to \mathbb{Z}$ a skew-symmetric bilinear form. Note that Λ can be identified with an $m \times m$ skew-symmetric matrix which still denoted by Λ if there is no confusion. Set a formal variable q and the ring of integer Laurent polynomials $\mathbb{Z}[q^{\pm 1/2}]$. Define the *based quantum torus* associated to the pair (L, Λ) to be the $\mathbb{Z}[q^{\pm 1/2}]$-algebra $\mathscr{T}$ with a distinguished $\mathbb{Z}[q^{\pm 1/2}]$-basis $\{X^e : e \in L\}$ and the multiplication

$$X^e X^f = q^{\Lambda(e,f)/2} X^{e+f}.$$

It is known that $\mathscr{T}$ is contained in its skew-field of fractions $\mathscr{F}$. A *toric frame* in $\mathscr{F}$ is a map $M : \mathbb{Z}^m \to \mathscr{F} \setminus \{0\}$ given by

$$M(\mathbf{c}) = \varphi(X^{\eta(\mathbf{c})})$$

where φ is an automorphism of $\mathscr{F}$ and $\eta : \mathbb{Z}^m \to L$ is an isomorphism of lattices. By the definition, the elements $M(\mathbf{c})$ form a $\mathbb{Z}[q^{\pm 1/2}]$-basis of the based quantum torus $\mathscr{T}_M := \varphi(\mathscr{T})$ and satisfy the following relations:

$$M(\mathbf{c})M(\mathbf{d}) = q^{\Lambda_M(\mathbf{c},\mathbf{d})/2} M(\mathbf{c}+\mathbf{d}),\ M(\mathbf{c})M(\mathbf{d}) = q^{\Lambda_M(\mathbf{c},\mathbf{d})} M(\mathbf{d})M(\mathbf{c}),$$

$$M(\mathbf{0}) = 1,\ M(\mathbf{c})^{-1} = M(-\mathbf{c}),$$

where Λ_M is the skew-symmetric bilinear form on $\mathbb{Z}^m$ obtained from the lattice isomorphism η. Let Λ_M be the skew-symmetric $m \times m$ matrix defined by $\lambda_{ij} = \Lambda_M(e_i, e_j)$ where $\{e_1, \ldots, e_m\}$ is the standard basis of $\mathbb{Z}^m$. Given a toric frame M, let $X_i = M(e_i)$. Then we have

$$\mathscr{T}_M = \mathbb{Z}[q^{\pm 1/2}]\langle X_1^{\pm 1}, \ldots, X_m^{\pm 1} : X_i X_j = q^{\lambda_{ij}} X_j X_i \rangle.$$

An easy computation shows that:

$$M(\mathbf{c}) = q^{\frac{1}{2}\sum_{i<j} c_i c_j \lambda_{ji}} X_1^{c_1} X_2^{c_2} \cdots X_m^{c_m} =: X^{(\mathbf{c})} \quad (\mathbf{c} \in \mathbb{Z}^m).$$

Let Λ be an $m \times m$ skew-symmetric matrix and $\tilde{B}$ an $m \times n$ matrix with $n \leq m$. We call the pair $(\Lambda, \tilde{B})$ *compatible* if up to permuting rows and columns $\tilde{B}^T \Lambda = (D|0)$ with $D = diag(d_1, \cdots, d_n)$ where $d_i \in \mathbb{N}$ for $1 \leq i \leq n$. The pair $(M, \tilde{B})$ is called a *quantum seed* if the pair $(\Lambda_M, \tilde{B})$ is compatible. Define the $m \times m$ matrix $E = (e_{ij})$ as follows

$$e_{ij} = \begin{cases} \delta_{ij} & \text{if } j \neq k; \\ -1 & \text{if } i = j = k; \\ max(0, -b_{ik}) & \text{if } i \neq j = k. \end{cases}$$

For $n, k \in \mathbb{Z}$, $k \geq 0$, denote $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n - q^{-n})\cdots(q^{n-k+1} - q^{-n+k-1})}{(q^k - q^{-k})\cdots(q - q^{-1})}$. Let $k \in [1, n]$ where $[1, n] = \{1, \cdots, n\}$ and $\mathbf{c} = (c_1, \ldots, c_m) \in \mathbb{Z}^m$ with $c_k \geq 0$. Define the toric frame $M' : \mathbb{Z}^m \to \mathscr{F} \setminus \{0\}$ as follows

$$M'(\mathbf{c}) = \sum_{p=0}^{c_k} \begin{bmatrix} c_k \\ p \end{bmatrix}_{q^{d_k/2}} M(E\mathbf{c} + p\mathbf{b}^k), \quad M'(-\mathbf{c}) = M'(\mathbf{c})^{-1}. \tag{1}$$

where the vector $\mathbf{b}^k \in \mathbb{Z}^m$ is the kth column of $\tilde{B}$. Following [9], we say a real $m \times n$ matrix $\tilde{B}'$ is obtained from $\tilde{B}$ by matrix mutation in direction k if the entries of $\tilde{B}'$ are given by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k; \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{otherwise.} \end{cases}$$

Then the quantum seed $(M', \tilde{B}')$ is defined to be the mutation of $(M, \tilde{B})$ in direction k. Two quantum seeds are called mutation-equivalent if they can be obtained from each other by a sequence of mutations. Let $\mathscr{C} = \{M'(e_i) : i \in [1, n]\}$ where $(M', \tilde{B}')$ is mutation-equivalent to $(M, \tilde{B})$. The elements of $\mathscr{C}$ are called the *cluster variables*. Let $\mathbb{P} = \{M(e_i) : i \in [n+1, m]\}$ and the elements of $\mathbb{P}$ are called *coefficients*. Denote by $\mathbb{ZP}$ the ring of Laurent polynomials generated by $q^{\frac{1}{2}}, \mathbb{P}$ and their inverses. Then the *quantum cluster algebra* $\mathscr{A}_q(\Lambda_M, \tilde{B})$ is defined to be the $\mathbb{ZP}$-subalgebra of $\mathscr{F}$ generated by $\mathscr{C}$.

2.2 The quantum Caldero-Chapoton map and main result Let k be a finite field with cardinality $|k| = q$ and $m \geq n$ be two positive integers and $\widetilde{Q}$ an acyclic valued quiver with vertex set $\{1, \ldots, m\}$. Denote the subset $\{n+1, \ldots, m\}$ by C. The full subquiver Q on the vertices $1, \ldots, n$ is called the *principal part* of $\widetilde{Q}$. For $1 \leq i \leq m$, let S_i be the ith simple module for $k\widetilde{Q}$.

Let $\widetilde{B}$ be the $m \times n$ matrix associated to the quiver $\widetilde{Q}$ whose entry in position (i, j) given by

$$b_{ij} = |\{\text{arrows}\, i \longrightarrow j\}| - |\{\text{arrows}\, j \longrightarrow i\}|$$

for $1 \leq i \leq m$, $1 \leq j \leq n$. Denote by $\widetilde{I}$ the left $m \times n$ submatrix of the identity matrix of size $m \times m$. Assume that there exists some antisymmetric $m \times m$ integer matrix Λ such that

$$\Lambda(-\widetilde{B}) = \begin{bmatrix} I_n \\ 0 \end{bmatrix}, \tag{2}$$

where I_n is the identity matrix of size $n \times n$. Let $\widetilde{R} = \widetilde{R}_{\widetilde{Q}}$ be the $m \times n$ matrix with its entry in position (i, j) given by

$$\widetilde{r}_{ij} := \dim_k \mathrm{Ext}^1_{k\widetilde{Q}}(S_j, S_i) = |\{\text{arrows}\, j \longrightarrow i\}|.$$

for $1 \leq i \leq m$, $1 \leq j \leq n$. Set $\widetilde{R}^{tr} = \widetilde{R}_{\widetilde{Q}^{op}}$. Denote the principal $n \times n$ submatrices of $\widetilde{B}$ and $\widetilde{R}$ by B and R respectively. Note that $\widetilde{B} = \widetilde{R}^{tr} - \widetilde{R}$ and $B = R^{tr} - R$.

Let $\mathscr{C}_{\widetilde{Q}}$ be the cluster category of $k\widetilde{Q}$, i.e., the orbit category of the derived category $\mathscr{D}^b(\widetilde{Q})$ under the action of the functor $F = \tau \circ [-1]$ (see [2]). Let I_i be the indecomposable injective $k\widetilde{Q}$ module for $1 \leq i \leq m$. Then the indecomposable $k\widetilde{Q}$-modules and $I_i[-1]$ for $1 \leq i \leq m$ exhaust all indecomposable objects of the cluster category $\mathscr{C}_{\widetilde{Q}}$. Each object M in $\mathscr{C}_{\widetilde{Q}}$ can be uniquely decomposed as

$$M = M_0 \oplus I_M[-1]$$

where M_0 is a module and I_M is an injective module.

The Euler form on $k\widetilde{Q}$-modules M and N is given by

$$\langle M, N\rangle = \dim_k \mathrm{Hom}(M, N) - \dim_k \mathrm{Ext}^1(M, N).$$

Note that the Euler form only depends on the dimension vectors of M and N.

The quantum Caldero-Chapoton map of an acyclic quiver $\widetilde{Q}$ has been defined in [17] and [16]. In [17], the author defined the quantum Caldero-Chapoton map for $k\widetilde{Q}$-modules while in [16] for coefficient-free rigid object in $\mathscr{C}_{\widetilde{Q}}$. For our purpose, we need to extend these definitions to the following map

$$X_? : \mathrm{obj}\mathscr{C}_{\widetilde{Q}} \longrightarrow \mathscr{T}$$

defined by the following rules:
(1) If M is a kQ-module, then

$$X_M = \sum_{\underline{e}} |\mathrm{Gr}_{\underline{e}} M| q^{-\frac{1}{2}\langle \underline{e}, \underline{m} - \underline{e} - \underline{i}\rangle} X^{-\widetilde{B}\underline{e} - (\widetilde{I} - \widetilde{R}^{tr})\underline{m}};$$

(2) If M is a kQ-module and I is an injective $k\widetilde{Q}$-module, then

$$X_{M \oplus I[-1]} = \sum_{\underline{e}} |\mathrm{Gr}_{\underline{e}} M| q^{-\frac{1}{2}\langle \underline{e}, \underline{m} - \underline{e} - \underline{i}\rangle} X^{-\widetilde{B}\underline{e} - (\widetilde{I} - \widetilde{R}^{tr})\underline{m} + \underline{\dim} socI},$$

where $\underline{\dim} I = \underline{i}, \underline{\dim} M = \underline{m}$ and $\mathrm{Gr}_{\underline{e}} M$ denotes the set of all submodules V of M with $\underline{\dim} V = \underline{e}$. We note that

$$X_{P[1]} = X_{\tau P} = X^{\underline{\dim} P/radP} = X^{\underline{\dim}\mathrm{soc} I} = X_{I[-1]} = X_{\tau^{-1} I}.$$

for any projective $k\widetilde{Q}$-module P and injective $k\widetilde{Q}$-module I with $\mathrm{soc} I = P/\mathrm{rad} P$. In the following, we denote by the corresponding underlined lower case letter $\underline{x}$ the dimension vector of a kQ-module X and view $\underline{x}$ as a column vector in $\mathbb{Z}^n$.

Definition 1. X_L *is called* the corresponding cluster character*, if L is a kQ-module or $L = M \oplus I[-1] \in \mathscr{C}_{\widetilde{Q}}$ satisfying that M is a kQ-module and I is an injective $k\widetilde{Q}$-module.*

For a quiver Q, denote by $\mathscr{AH}_{|k|}(Q)$ the $\mathbb{ZP}$-subalgebra of $\mathscr{F}$ generated by all the cluster characters and by $\mathscr{EH}_{|k|}(Q)$ the corresponding quantum cluster algebra, i.e, the $\mathbb{ZP}$-subalgebra of $\mathscr{F}$ generated by all the cluster variables. Note that here we are working over a finite field, the definition of quantum cluster algebra in section 2.1 remains valid (see [16]). The main result of this article is the following theorem:

Theorem 1. *For any quiver Q of finite type, we have* $\mathscr{EH}_{|k|}(Q) = \mathscr{AH}_{|k|}(Q)$.

We conjecture that Theorem 1 holds for any quiver of affine type.

Conjecture 1. *For any quiver Q of affine type, we have* $\mathscr{EH}_{|k|}(Q) = \mathscr{AH}_{|k|}(Q)$.

3 Proof of the main theorem

In this section, we fix a quiver Q of finite type with n vertices. Firstly, we recall some notations. For any $k\widetilde{Q}-$modules M, N and E, denote by ε_{MN}^{E} the cardinality of the set $\mathrm{Ext}^1_{k\widetilde{Q}}(M,N)_E$ which is the subset of $\mathrm{Ext}^1_{k\widetilde{Q}}(M,N)$ consisting of those equivalence classes of short exact sequences with middle term isomorphic to E ([14, Section 4]). Let F_{AB}^{M} be the number of submodules U of M such that U is isomorphic to B and M/U is isomorphic to A. Then by definition, we have

$$|\mathrm{Gr}_{\underline{e}}(M)| = \sum_{A,B;\underline{\dim}B=\underline{e}} F_{AB}^{M}.$$

Denote by $[M,N]^1 = \dim_k \mathrm{Ext}^1_{k\widetilde{Q}}(M,N)$ and $[M,N] = \dim_k \mathrm{Hom}_{k\widetilde{Q}}(M,N)$. The following Theorem 2 proved in [7] and Proposition 1 give the explicit relations between $X_N X_M$ and $X_{N\oplus M}$.

Theorem 2. *[7] Let M and N be kQ-modules. Then*

$$q^{[M,N]^1} X_M X_N = q^{\frac{1}{2}\Lambda((\widetilde{I}-\widetilde{R}^{tr})\underline{m},(\widetilde{I}-\widetilde{R}^{tr})\underline{n})} \sum_{E} \varepsilon_{MN}^{E} X_E.$$

Let M be any $kQ-$module and I any injective $k\widetilde{Q}-$module. Define

$$\mathrm{Hom}_{k\widetilde{Q}}(M,I)_{BI'} := \{f : M \longrightarrow I | kerf \cong B, cokerf \cong I'\}.$$

Note that I' is an injective $k\widetilde{Q}-$module. The following result, together with Theorem 2, is essential for us to prove Theorem 1.

Proposition 1. *With the above notations, we have*

$$q^{[M,I]} X_M X_{I[-1]} = q^{\frac{1}{2}\Lambda((\widetilde{I}-\widetilde{R}^{tr})\underline{m},-\underline{\dim}socI)} \sum_{B,I'} |\mathrm{Hom}_{k\widetilde{Q}}(M,I)_{BI'}| X_{B\oplus I'[-1]}.$$

Proof. We calculate

$$\begin{aligned}
& X_M X_{I[-1]} \\
= & \sum_{G,H} q^{-\frac{1}{2}\langle H,G\rangle} F^M_{GH} X^{-\widetilde{B}\underline{h}-(\widetilde{I}-\widetilde{R}^{tr})\underline{m}} X^{\underline{\dim} socI} \\
= & \sum_{G,H} q^{-\frac{1}{2}\langle H,G\rangle} F^M_{GH} q^{\frac{1}{2}\Lambda(-\widetilde{B}\underline{h}-(\widetilde{I}-\widetilde{R}^{tr})\underline{m},\underline{\dim} socI)} X^{-\widetilde{B}\underline{h}-(\widetilde{I}-\widetilde{R}^{tr})\underline{m}+\underline{\dim} socI} \\
= & q^{\frac{1}{2}\Lambda(-(\widetilde{I}-\widetilde{R}^{tr})\underline{m},\underline{\dim} socI)} \sum_{G,H} q^{-\frac{1}{2}\langle H,G\rangle} q^{\frac{1}{2}\Lambda(-\widetilde{B}\underline{h},\underline{\dim} socI)} F^M_{GH} X^{-\widetilde{B}\underline{h}-(\widetilde{I}-\widetilde{R}^{tr})\underline{m}+\underline{\dim} socI} \\
= & q^{\frac{1}{2}\Lambda((\widetilde{I}-\widetilde{R}^{tr})\underline{m},-\underline{\dim} socI)} \sum_{G,H} q^{-\frac{1}{2}\langle H,G\rangle} q^{-\frac{1}{2}[H,I]} F^M_{GH} X^{-\widetilde{B}\underline{h}-(\widetilde{I}-\widetilde{R}^{tr})\underline{m}+\underline{\dim} socI}.
\end{aligned}$$

Here we use the fact that

$$\Lambda(-\widetilde{B}\underline{h}, \underline{\dim} socI) = -\underline{h}^{tr}\widetilde{B}^{tr}\Lambda(\underline{\dim} socI) = -\underline{h}^{tr}(\underline{\dim} socI) = -[H,I].$$

Note that if we have the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & & \\
 & Y \arrow[r, equal] \arrow[d] & Y \arrow[d] & & & \\
0 \arrow[r] & B \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & I \arrow[r] & I' \arrow[r] & 0 \\
0 \arrow[r] & X \arrow[r] \arrow[d] & G \arrow[d] & & & \\
 & 0 & 0 & & &
\end{tikzcd}$$

and short exact sequences

$$0 \longrightarrow B \longrightarrow M \longrightarrow A \longrightarrow 0$$

$$0 \longrightarrow A \longrightarrow I \longrightarrow I' \longrightarrow 0,$$

then by [14] it follows that

$$\sum_B F^B_{XY} F^M_{AB} = \sum_G F^G_{AX} F^M_{GY}, \ |\mathrm{Hom}_{k\widetilde{Q}}(M,I)_{BI'}| = \sum_A |\mathrm{Aut}(A)| F^M_{AB} F^I_{I'A}$$

and

$$\sum_{A,I',X} |\mathrm{Aut}(A)| F^I_{I'A} F^G_{AX} = \sum_{I',X} |\mathrm{Hom}_{k\widetilde{Q}}(G,I)_{XI'}| = q^{[G,I]} = q^{\langle G,I\rangle}.$$

By [14, Lemma 1], we have $(\widetilde{I}-\widetilde{R}^{tr})\underline{i}=\underline{\dim}socI$. Now we can calculate the term

$$\begin{aligned}
&\sum_{B,I'}|\mathrm{Hom}_{k\widetilde{Q}}(M,I)_{BI'}|X_{B\oplus I'[-1]}\\
=&\sum_{A,B,I',X,Y}|\mathrm{Aut}(A)|F^{M}_{AB}F^{I}_{I'A}q^{-\frac{1}{2}\langle Y,X-I'\rangle}F^{B}_{XY}X^{-\widetilde{B}\underline{y}-(\widetilde{I}-\widetilde{R}^{tr})\underline{b}+\underline{\dim}socI'}\\
=&\sum_{A,G,I',X,Y}q^{-\frac{1}{2}\langle Y,X-I'\rangle}|\mathrm{Aut}(A)|F^{I}_{I'A}F^{G}_{AX}F^{M}_{GY}X^{-\widetilde{B}\underline{y}-(\widetilde{I}-\widetilde{R}^{tr})\underline{b}+\underline{\dim}socI'}.
\end{aligned}$$

Note that we have the following facts

$$\underline{i}'+\underline{a}=\underline{i},\ \underline{x}+\underline{a}=\underline{g}\Longrightarrow\underline{x}-\underline{i}'=\underline{g}-\underline{i},$$

and

$$\begin{aligned}
&-\widetilde{B}\underline{y}-(\widetilde{I}-\widetilde{R}^{tr})\underline{b}+\underline{\dim}socI'\\
=&-\widetilde{B}\underline{h}-(\widetilde{I}-\widetilde{R}^{tr})(\underline{m}-\underline{i}-\underline{i}')+\underline{\dim}socI'\\
=&-\widetilde{B}\underline{h}-(\widetilde{I}-\widetilde{R}^{tr})\underline{m}+(\widetilde{I}-\widetilde{R}^{tr})(\underline{i}-\underline{i}')+\underline{\dim}socI'\\
=&-\widetilde{B}\underline{h}-(\widetilde{I}-\widetilde{R}^{tr})\underline{m}+(\widetilde{I}-\widetilde{R}^{tr})\underline{i}\\
=&-\widetilde{B}\underline{h}-(\widetilde{I}-\widetilde{R}^{tr})\underline{m}+\underline{\dim}socI.
\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{B,I'}|\mathrm{Hom}_{k\widetilde{Q}}(M,I)_{BI'}|X_{B\oplus I'[-1]}\\
=&\sum_{G,H}q^{\langle G,I\rangle}q^{-\frac{1}{2}\langle H,G-I\rangle}F^{M}_{GH}X^{-\widetilde{B}\underline{h}-(\widetilde{I}-\widetilde{R}^{tr})\underline{m}+\underline{\dim}socI}\\
=&\sum_{G,H}q^{\langle M,I\rangle}q^{-\frac{1}{2}\langle H,I\rangle}q^{-\frac{1}{2}\langle H,G\rangle}F^{M}_{GH}X^{-\widetilde{B}\underline{h}-(\widetilde{I}-\widetilde{R}^{tr})\underline{m}+\underline{\dim}socI}\\
=&\,q^{[M,I]}\sum_{G,H}q^{-\frac{1}{2}[H,I]}q^{-\frac{1}{2}\langle H,G\rangle}F^{M}_{GH}X^{-\widetilde{B}\underline{h}-(\widetilde{I}-\widetilde{R}^{tr})\underline{m}+\underline{\dim}socI}.
\end{aligned}$$

This finishes the proof. □

Remark 1. *Proposition 1 holds for any acyclic quiver.*

The following lemma is well-known. Here we give a sketch of the proof following [5, Lemma 8(b)].

Lemma 1. *Let*

$$M\longrightarrow E\longrightarrow N\xrightarrow{\epsilon}M[1]$$

be a non-split triangle in $\mathscr{C}_{\widetilde{Q}}$. *Then*

$$\dim_k\mathrm{Ext}^1_{\mathscr{C}_{\widetilde{Q}}}(E,E)<\dim_k\mathrm{Ext}^1_{\mathscr{C}_{\widetilde{Q}}}(M\oplus N,M\oplus N).$$

Proof. For any object $L \in \mathscr{C}_{\widetilde{Q}}$, applying the functor $\mathrm{Ext}^1_{\mathscr{C}_{\widetilde{Q}}}(-, L)$ to the above non-split triangle gives rise to the exact sequence

$$0 \longrightarrow kerf_L \longrightarrow \mathrm{Ext}^1_{\mathscr{C}_{\widetilde{Q}}}(N, L) \xrightarrow{f_L} \mathrm{Ext}^1_{\mathscr{C}_{\widetilde{Q}}}(E, L) \xrightarrow{g_L} \mathrm{Ext}^1_{\mathscr{C}_{\widetilde{Q}}}(M, L) \longrightarrow cokerg_L \longrightarrow 0$$

Thus we have

$$\dim_k kerf_L + \dim_k \mathrm{Ext}^1_{\mathscr{C}_{\widetilde{Q}}}(E, L) + \dim_k cokerg_L = \dim_k \mathrm{Ext}^1_{\mathscr{C}_{\widetilde{Q}}}(N, L) + \dim_k \mathrm{Ext}^1_{\mathscr{C}_{\widetilde{Q}}}(M, L)$$

Hence

$$\dim_k \mathrm{Ext}^1_{\mathscr{C}_{\widetilde{Q}}}(E, N) \leq \dim_k \mathrm{Ext}^1_{\mathscr{C}_{\widetilde{Q}}}(N, N) + \dim_k \mathrm{Ext}^1_{\mathscr{C}_{\widetilde{Q}}}(M, N)$$

$$\dim_k \mathrm{Ext}^1_{\mathscr{C}_{\widetilde{Q}}}(E, E) \leq \dim_k \mathrm{Ext}^1_{\mathscr{C}_{\widetilde{Q}}}(N, E) + \dim_k \mathrm{Ext}^1_{\mathscr{C}_{\widetilde{Q}}}(M, E).$$

Note that $0 \neq \epsilon \in kerf_M$, so we have

$$\dim_k \mathrm{Ext}^1_{\mathscr{C}_{\widetilde{Q}}}(E, M) < \dim_k \mathrm{Ext}^1_{\mathscr{C}_{\widetilde{Q}}}(N, M) + \dim_k \mathrm{Ext}^1_{\mathscr{C}_{\widetilde{Q}}}(M, M).$$

Therefore

$$\begin{aligned} \dim_k \mathrm{Ext}^1_{\mathscr{C}_{\widetilde{Q}}}(M \oplus N, M \oplus N) &> \dim_k \mathrm{Ext}^1_{\mathscr{C}_{\widetilde{Q}}}(E, N) + \dim_k \mathrm{Ext}^1_{\mathscr{C}_{\widetilde{Q}}}(E, M) \\ &= \dim_k \mathrm{Ext}^1_{\mathscr{C}_{\widetilde{Q}}}(N, E) + \dim_k \mathrm{Ext}^1_{\mathscr{C}_{\widetilde{Q}}}(M, E) \\ &\geq \dim_k \mathrm{Ext}^1_{\mathscr{C}_{\widetilde{Q}}}(E, E). \end{aligned}$$

This proves our assertion. □

Proof of Theorem 1: We need to prove that for any cluster character $X_L \in \mathscr{A}\mathscr{H}_{|k|}(Q)$, then $X_L \in \mathscr{E}\mathscr{H}_{|k|}(Q)$.

Let $L \cong \bigoplus_{i=1}^{l} L_i^{\oplus n_i}, n_i \in \mathbb{N}$ where L_i $(1 \leq i \leq l)$ are indecomposable objects in $\mathscr{C}_{\widetilde{Q}}$. Thus X_{L_i} $(1 \leq i \leq l)$ are in $\mathscr{E}\mathscr{H}_{|k|}(Q)$. By Theorem 2, Proposition 1 and Lemma 1, we have that

$$X_{L_1}^{n_1} X_{L_2}^{n_2} \cdots X_{L_l}^{n_l} = q^{\frac{1}{2} n_L} X_L + \sum_{\dim_k \mathrm{Ext}^1_{\mathscr{C}_{\widetilde{Q}}}(E,E) < \dim_k \mathrm{Ext}^1_{\mathscr{C}_{\widetilde{Q}}}(L,L)} f_{n_E}(q^{\pm\frac{1}{2}}) X_E$$

where $n_L \in \mathbb{Z}$ and $f_{n_E}(q^{\pm\frac{1}{2}}) \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$. Thus by induction, we can prove that $X_L \in \mathscr{E}\mathscr{H}_{|k|}(Q)$ which implies $\mathscr{E}\mathscr{H}_{|k|}(Q) = \mathscr{A}\mathscr{H}_{|k|}(Q)$.

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