

Sum rules and three point functions

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ABSTRACT: Sum rules constraining the R-current spectral densities are derived holographically for the case of D3-branes, M2-branes and M5-branes all at finite chemical potentials. In each of the cases the sum rule relates a certain integral of the spectral density over the frequency to terms which depend both on long distance physics, hydrodynamics and short distance physics of the theory. The terms which depend on the short distance physics result from the presence of certain chiral primaries in the OPE of two R-currents which are turned on at finite chemical potential. Since these sum rules contain information of the OPE they provide an alternate method to obtain the structure constants of the two R-currents and the chiral primary. As a consistency check we show that the 3 point function derived from the sum rule precisely matches with that obtained using Witten diagrams.

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1. Introduction

Sum rules constraining spectral densities in any quantum field theory provide useful information of the theory. For example the shear and bulk sum rules in QCD is a consequence of the hydrodynamic behaviour at large distances and asymptotic freedom at short distances [1, 2, 3, 4, 5]. Similarly the Ferrell-Glover sum rule in BCS theory plays an important role in determining the skin depth of superconductors [6, 7].

Recently there has been interest in modeling systems that are studied in condensed matter physics as well as QCD using the gauge gravity duality. Clearly sum rules derived using the AdS/CFT framework for these systems can provide important constraints on these models. On a more conceptual level since sum rules are a consequence of unitarity and causality of the field theory, they will be useful to study

how these properties are encoded in the bulk. The study of sum rules in AdS/CFT was initiated in [3] in which the shear spectral sum rule was derived and verified numerically for the $\mathcal{N} = 4$ theory at strong coupling. The shear and bulk sum rules in non-conformal theories dual to the Chambilin-Real backgrounds were obtained in [8, 9]. The R-charge sum rule for $\mathcal{N} = 4$ Yang-Mills was derived holographically in [10].

Using the gauge/gravity framework, the derivation of the sum rules for the strongly coupled dual can be cast into a problem of establishing the analytic behaviour of solutions of the differential equations which determine the relevant retarded Green's function [11]. By studying properties of these differential equations, a proof for various sum rules including the shear sum rule for $\mathcal{N} = 4$ Yang-Mills was provided in [11]. In [12], this proof for the shear sum rules was developed and extended to systems with chemical potential. It was noticed there that the shear sum rules are modified due to the short distance properties of the theory. It was shown that appropriate scalar operators in the stress tensor operator product expansion (OPE) gain expectation values in the presence of the chemical potential. This results in the modified shear sum rule for the case of D3, M2 and M5-branes when compared to the situation in the absence of chemical potentials.

In the present paper we derive the R-current spectral sum rules at finite chemical potential for the dual of the D3, M2 and M5-brane theory. The differential equations which determine the R-current correlator are a set of coupled equations unlike the situation for the shear correlator which was determined by a single equation. We show that the R-charge sum rules contain both a term determined by hydrodynamics at long distance: the conductivity, as well as a term whose origin is from short distance physics: the OPE.. We then examine the term due to the presence of scalars in the OPE and obtain the structure constants of two R-currents and the scalar. We show that the structure constants evaluated from the sum rule agrees precisely with that evaluated using Witten diagrams.

We now briefly describe the structure of the R-current spectral sum rules derived in this paper. The formulae are of the form

$$\int_{-\infty}^{\infty} \frac{d\omega}{\pi\omega} (\rho_T^i(\omega) - \rho_{T=0}^i(\omega)) = \lim_{\omega \rightarrow 0} \omega \text{Im} \sigma^i(\omega) - C_{ii}^{\hat{k}} \langle \mathcal{O}^{\hat{k}} \rangle_T, \quad (1.1)$$

where $\rho^i(\omega) = \text{Im} G^i$ and G^i is the retarded Green's function of the x component of the R-current J_x^i . σ^i refers to the corresponding conductivity. $C_{ii}^{\hat{k}}$ is the structure constant of two R-currents and the scalar $\mathcal{O}^{\hat{k}}$. The index \hat{k} is summed over all the operators of appropriate dimensions in the theory. The expectation value of this scalar is taken in the thermal state at temperature T . Notice that the first term in the R-current sum rule is determined by hydrodynamics, while the second term is determined by the short distance physics of the theory. For the case of D3-branes the operators $\mathcal{O}^{\hat{k}}$ are chiral primaries of dimension 2. While for the case of M2,

M5-branes they are chiral primaries of dimension 1 and 4 respectively. It is clear from (1.1) that once the one knows the expectation values of these primaries, the structure constants can be extracted. In each of the situations we verify that the structure constants obtained from the sum rule agrees with that evaluated from Witten diagrams.

The organization of the paper is as follows: In the next section we first recall the general analytical properties of the retarded Green's function required to derive sum rules for the corresponding spectral densities. We then go over the details of the derivation of the R-current spectral sum rules for the dual of $\mathcal{N} = 4$ Yang-Mills at finite chemical potential. In section 3, we examine the origin of the high frequency contribution to the sum rules. We show that these terms occur due to presence of chiral primary operators of dimension 2 in the OPE of two R-currents. From this we derive the structure constants of two R-currents with these chiral primaries. As a consistency check, these constants are then compared to that obtained from Witten diagrams and shown to agree. In section 4, the analysis is repeated for the case of M2 and M5-branes. The discussion in this section is brief and only the results are highlighted. Section 6 contains our conclusions. Appendix A discussed the evaluation of the 3 point and 2 point functions using Witten diagrams which are necessary to compare with that obtained using the sum-rules. Appendix B provides the necessary details of the charged M2 and M5-brane solution.

2. R-charge sum rules for the D3-brane

Before we begin the derivation of the spectral sum rules for the D3-branes we briefly state the analytic properties of the Green's function in the complex ω -plane which are required to derive sum rules. Consider a function $G(\omega)$ which satisfies the following two properties in the complex ω -plane.

1. $G(\omega)$ is holomorphic in the upper half ω -plane, including the real axis.
2. $\lim_{|\omega| \rightarrow \infty} G(\omega) = 0$ for $\text{Im } \omega > 0$.

We refer to these properties as 'property 1' and 'property 2' respectively. Then one can show using Cauchy's theorem that

$$G(0) = \lim_{\epsilon \rightarrow 0^+} \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{\rho(\omega)}{\omega - i\epsilon}, \quad (2.1)$$

where $\rho(\omega) = \text{Im } G(\omega)$. The details of this analysis can be found in [12]. In this section we examine the retarded Green's function corresponding to the R-current correlators for $\mathcal{N} = 4$ Yang-Mills at strong coupling and construct a regulated Green's

function which satisfies both the above properties and thus derive the spectral sum rule.

We first introduce the gravity dual of $\mathcal{N} = 4$ super Yang-Mills at finite temperature and finite chemical potential. We examine the situation in which the chemical potentials corresponding to the three Cartans of the $SO(6)$ R-symmetry is turned on. The gravity dual of this system is given by the R-charged black hole of Behrndt, Cvetič and Sabra [13]. We study the retarded Green's function of the R-symmetry currents in this background. We will focus on the following diagonal correlator:

$$G^i(t, \vec{x}) = i\theta(t)\langle [J_x^i(t, \vec{x}), J_x^i(t, \vec{x})] \rangle, \quad (2.2)$$

where J_x^i is the x component of the i -th R-symmetry current and $i \in \{1, 2, 3\}$. This is done by first obtaining the coupled set of differential equations for the fluctuations of the gauge fields dual to the R-symmetry currents in the R-charged black hole. We then evaluate the retarded Green's function using the standard prescription developed in [14, 15]. Examining these set of differential equations we will show that the Green's function satisfies the two properties necessary for the derivation of the sum rule. We will show that the RHS of sum rule depends on two terms as given in (1.1). The first term depends on the zero frequency behaviour of the Green's function and can be understood in terms of the hydrodynamic properties of the field theory. The second term arises due to the high frequency behaviour of the Green's function. In the next section it will be shown that the high frequency terms in the sum rule result from scalars corresponding to chiral primary operators gaining expectation values at finite chemical potential.

2.1 Green's function from gravity

The R-charged D3-brane solution of [13] in 5 dimensions is given by

$$\begin{aligned} ds_5^2 &= -\mathcal{H}^{-2/3} \frac{(\pi T_0 L)^2}{u} f dt^2 + \mathcal{H}^{1/3} \frac{(\pi T_0 L)^2}{u} (dx^2 + dy^2 + dz^2) + \mathcal{H}^{1/3} \frac{L^2}{4fu^2} du^2, \\ f(u) &= \mathcal{H}(u) - u^2 \prod_{i=1}^3 (1 + k_i), \quad H_i = 1 + k_i u, \quad k_i \equiv \frac{q_i}{r_H^2}, \quad T_0 = \frac{r_+}{\pi L^2}, \\ u &= \frac{r_+^2}{r^2}, \quad \mathcal{H} = (1 + k_1 u)(1 + k_2 u)(1 + k_3 u). \end{aligned} \quad (2.3)$$

The solution also involves scalars and vectors which have the following background values.

$$X^i = \frac{\mathcal{H}^{1/3}}{H_i(u)}, \quad A_t^i = \frac{\tilde{k}_i u}{H_i(u)}, \quad \tilde{k}_i = \frac{\sqrt{q_i}}{L} \prod_{i=1}^3 (1 + k_i)^{1/2}. \quad (2.4)$$

This background solves the equation of motion of the action which given by

$$S = \frac{N^2}{8\pi^2 L^3} \int d^5x \sqrt{-g} \mathcal{L}, \quad (2.5)$$

$$\mathcal{L} = R + \frac{2}{L^2}V - \frac{1}{2}\tilde{G}_{ij}F_{\mu\nu}^i F^{\mu\nu j} - \tilde{G}_{ij}\partial_\mu X^i \partial^\mu X^j + \frac{1}{24\sqrt{-g}}\epsilon^{\mu\nu\rho\sigma\lambda}\mathcal{C}_{ijk}F_{\mu\nu}^i F_{\rho\sigma}^j A_\lambda^k, \quad (2.6)$$

where $F_{\mu\nu}^i, i = 1, 2, 3$ are the field-strengths for the three Abelian gauge fields corresponding to the R-currents of the boundary theory. The three scalar fields X^i 's are subject to the constraint $X^1 X^2 X^3 = 1$ and \mathcal{C}_{ijk} are coefficients totally symmetric in the R-symmetry indices. The metric on the scalar manifold is given by

$$\tilde{G}_{ij} = \frac{1}{2}\text{diag} \{(X^1)^{-2}, (X^2)^{-2}, (X^3)^{-2}\}. \quad (2.7)$$

The scalar potential is given by

$$\mathcal{V} = 2 \left(\frac{1}{X^1} + \frac{1}{X^2} + \frac{1}{X^3} \right). \quad (2.8)$$

The thermodynamic properties of this black hole have been obtained in [16]. Note that unlike in [16], we are working with units such that the gauge fields A^i are dimensionless. The Hawking temperature T_H , entropy density s , energy density ϵ , pressure P , charge densities ρ_i and the conjugate chemical potentials μ_i are given by

$$\begin{aligned} T_H &= \frac{2 + k_1 + k_2 + k_3 - k_1 k_2 k_3}{2\sqrt{(1+k_1)(1+k_2)(1+k_3)}} T_0, & s &= \frac{\pi^2 N^2 T_0^3}{2} \prod_{i=1}^3 (1+k_i)^{1/2}, \\ \epsilon &= \frac{3\pi^2 N^2 T_0^4}{8} \prod_{i=1}^3 (1+k_i), & P &= \frac{\pi^2 N^2 T_0^4}{8} \prod_{i=1}^3 (1+k_i), \\ \rho_i &= \frac{\pi N^2 T_0^3}{4L} \sqrt{k_i} \prod_{i=1}^3 (1+k_i), & \mu_i &= \frac{\pi T_0 L \sqrt{k_i}}{(1+k_i)} \prod_{i=1}^3 (1+k_i)^{1/2}. \end{aligned} \quad (2.9)$$

The thermodynamical stability condition of this black hole is given by

$$2 - k_1 - k_2 - k_3 + k_1 k_2 k_3 > 0. \quad (2.10)$$

It is also useful to write down the explicit equations of motion that follow from the action in (2.5). The Einstein equations are

$$R_{\mu\nu} - \left(F_{\mu\nu}^2 - \frac{1}{6}g_{\mu\nu}F^2 \right) - \partial_\mu X^i \partial_\nu X^j \tilde{G}_{ij} + \frac{4}{3L^2}g_{\mu\nu} \left(\frac{1}{X^1} + \frac{1}{X^2} + \frac{1}{X^3} \right) = 0. \quad (2.11)$$

with $F_{\mu\nu}^2 \equiv \tilde{G}_{ij}F_{\mu\rho}^i F_{\nu\lambda}^j g^{\rho\lambda}$. The equations of motion for the gauge fields are given by

$$\frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g} \tilde{G}_{ij} F^{j\mu\nu} \right) + 3\kappa \epsilon^{\nu\alpha\beta\gamma\lambda} F_{\alpha\beta}^j F_{\gamma\lambda}^k \mathcal{C}_{ijk} = 0, \quad (2.12)$$

where $\kappa = \frac{1}{48}$. Finally the equations of motion for the scalars are given by

$$\frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g} g^{\mu\nu} G_{ij} \partial_\nu X^j \right) - \frac{1}{4} \partial_i \tilde{G}_{jk} F_{\mu\nu}^j F^{k\mu\nu} + \frac{1}{L^2} \partial_i V(X) = 0, \quad (2.13)$$

Here ∂_i refers to derivative with respect to the scalar X^i .

To study the retarded Green's function for the currents J_x^i we need to obtain the equations of motion of the fluctuations for the dual gauge field A_x^i . Further more we are interested in the Green's function at zero momentum, therefore we can restrict our attention to fluctuations which have only time dependence. It can be shown that it is consistent to turn on the following fluctuations

$$\begin{aligned} A_x^i &= A_x^{i(0)} + a_x^i(u, t), & A_l^i &= A_l^{i(0)}, \\ g_{xt} &= g_{xt}^{(0)} + h_{xt}(u, t), & g_{lt} &= g_{lt}^{(0)}, \\ g_{lm} &= g_{lm}^{(0)}, & X^i &= X^{i(0)}. \end{aligned} \quad (2.14)$$

Here $l \in \{y, z, t, u\}$ and the superscript (0) refers to the background values given in (2.3) and (2.4). Note that u refers to the radial direction. The consistency of setting several of the fields to their background values in (2.14) can be shown by examining the equations of motion given in (2.11), (2.12) and (2.13), We now define

$$T(u)e^{i\omega t}, \quad a_x^i(u, t) = \phi^i(u)e^{i\omega t}. \quad (2.15)$$

Here we have used the time translational invariance of the problem to study fluctuations of a given frequency ω . Note that the same analysis can be carried out for any of the spatial directions. By the symmetry of the problem it is easy to see that the results will be independent of any of the direction. Then examining the xu component of the Einstein equation given in (2.11) to first order in the fluctuations we obtain the following constraint

$$T' + \sum_{i=1}^3 \left(\frac{u}{\mathcal{H}} (1 + k_i) \frac{m_i}{\pi T_0 L} \phi^i \right) = 0, \quad (2.16)$$

where the prime refers to derivative with respect to u . We now examine the x component of the equations of motion of the three gauge fields given in (2.12) to obtain

$$\mu_i (1 + k_i) T' + \left(\frac{f H_i^2}{\mathcal{H}} \frac{d\phi^i}{du} \right)' + \frac{\tilde{\omega}^2}{uf} H_i^2 \phi^i = 0, \quad (2.17)$$

where we have defined $\tilde{\omega} = \frac{\omega}{2\pi T_0}$. Note that in the equation in (2.17) there is no summation over the index i . Substituting the T' from (2.16) in the equation (2.17) we obtain the closed set of equations for the gauge field fluctuations. These are given by

$$\frac{d}{du} \left(\frac{f H_i^2}{\mathcal{H}} \frac{d\phi^i}{du} \right) - (1 + k_i) m_i \sum_{j=1}^3 \left(\frac{u}{\mathcal{H}} (1 + k_j) m_j \phi^j \right) + \frac{\tilde{\omega}^2}{uf} H_i^2 \phi^i = 0. \quad (2.18)$$

It is useful to rewrite the above equation in terms of the conventional radial coordinate r which is related to u by

$$u = \frac{r_+^2}{r^2}. \quad (2.19)$$

Then the equation (2.18) is given by

$$\phi^{i'''} + \left(\ln\left(\frac{FH_i^2}{\mathcal{H}}\right)' + \frac{1}{r} \right) \phi^{i''} + \frac{\omega^2 \mathcal{H}}{F^2} \phi^i - (1 + k_i) \frac{m_i}{H_i^2} \sum_{j=1}^3 \left(\frac{4r_+^6}{r^6 L^2 F} (1 + k_j) m_j \phi^j \right) = 0, \quad (2.20)$$

where the prime now denotes derivative with respect to r and

$$F = \frac{r^2}{L^2} f. \quad (2.21)$$

We will also need the fact that the equations of motion for the gauge field fluctuations given in (2.20) can be obtained from the following action

$$S_\phi = \int_{r_h}^{\infty} dr \frac{FrH_i^2}{\mathcal{H}} \left(\frac{d\phi^{i*}}{dr} \delta_{ij} \frac{d\phi^j}{dr} \right) + \phi^{*i} \left(M_{ij} - \frac{\omega^2 H_i^2 r}{F} \delta_{ij} \right) \phi^j, \quad (2.22)$$

where the matrix M_{ij} is given by

$$M_{ij} = \frac{4r_+^6}{L^2 r^5 \mathcal{H}} (1 + k_i) m_i (1 + k_j) m_j. \quad (2.23)$$

In (2.22) summation over the indices i, j is implied. It is easy to see that on variation of the action in (2.22) by ϕ^{i*} we obtain the equations given in (2.20).

Finally we will describe the behaviour of the solutions to the equations in (2.22) near the horizon and the boundary. Near the horizon r_+ the equations decouple and the fluctuations ϕ^i behave as

$$\phi^i(r) \sim (r - r_+)^{\pm i\alpha\omega}, \quad (2.24)$$

where α is given by

$$\alpha = \frac{1}{F_h} \sqrt{(1 + k_1)(1 + k_2)(1 + k_3)} \quad (2.25)$$

$$F = (r - r_+)F_h + \dots, \quad F_h = \frac{2r_+}{L^2} (2 + k_1 + k_2 + k_3 - k_1 k_2 k_3).$$

Near the boundary $r \rightarrow \infty$ the geometry reduces to that of AdS_4 . From the equation (2.20) it is easy the fluctuations decouple at $r \rightarrow \infty$. The behaviour of the two independent solutions of each of the fluctuations in the limit $r \rightarrow \infty$ is given by

$$\phi^i(r) \sim \frac{1}{irL^2\omega} I_1\left(\frac{iL^2\omega}{r}\right) \sim r^{-2}, \quad \phi^i(r) \sim \frac{1}{irL^2\omega} K_1\left(\frac{iL^2\omega}{r}\right) \sim \text{constant}. \quad (2.26)$$

Before we begin our analysis we will review the procedure to obtain the retarded Green's of the R-Currents given by

$$G^i(\omega, q) = i \int dt d^3x e^{i(\omega t - qx)} \theta(t) \langle [J_x^i(t, \vec{x}), J_x^i] \rangle, \quad (2.27)$$

where the expectation value is taken in the thermal state at finite chemical potential. We will restrict our attention to the Green's function with diagonal entries in the R-symmetry index. To obtain this Green's function from the gravity dual we will follow the procedure developed by [14, 15]. We will need to solve the coupled set of equations in (2.20) with ingoing boundary conditions at the horizon which is given by

$$\phi^i(r) \sim (r - r_+)^{-i\alpha\omega}, \quad (2.28)$$

With this ingoing boundary conditions at the horizon we obtain the solution at the boundary $r \rightarrow \infty$. Once this is done, the retarded Green's function in Fourier space is given by

$$\begin{aligned} G_T^i(\omega) &= \hat{G}^i(\omega, T) + G_{\text{counter}}(\omega, T), \\ \hat{G}^i(\omega, T) &= - \frac{N^2}{8\pi^2 L^3} \lim_{r \rightarrow \infty} \frac{r F \phi^{i'}}{L \phi^i} \Big|_{\phi_\infty^j = 0, j \neq i}, \end{aligned} \quad (2.29)$$

where we have dropped the momentum dependence since we are interested in $q = 0$. We have also explicitly denoted the temperature dependence by T in the Green's function. $G_{\text{counter}}(\omega, T)$ is the counter term needed to cancel the $\log(r)$ divergences. As it will be clear in our subsequent discussion we will not be requiring the explicit form of these terms. Thus the important properties of the Green's function is essentially contained in the function

$$g^i(\omega) = \lim_{r \rightarrow \infty} \frac{r F}{L \phi^i} \frac{d\phi^i}{dr} \Big|_{\phi_\infty^j = 0, j \neq i} \quad (2.30)$$

Note that since we are solving the coupled set of equations in (2.20) with the boundary conditions (2.28), there will be 3 independent constants. These are the values of ϕ^i at the boundary denoted by ϕ_∞^i . The i -th diagonal component of the correlator can be obtained by first evaluating the ratio given in (2.30) and then setting the boundary values of $\phi_\infty^j = 0$ for $j \neq i$. A simple way to understand this condition is that the boundary effective action from which the Green's function at frequency ω is evaluated will be a quadratic functional of ϕ_∞^i

$$S_{\text{Boundary}} = \sum_{i,j=1}^3 \phi_\infty^i G^{ij}(\omega) \phi_\infty^j. \quad (2.31)$$

Then to obtain the diagonal component G^{ii} we can examine the response in ϕ^i by setting all the other sources to $\phi^j, j \neq i$ to zero. Thus to study the behaviour of the retarded Green's function in the frequency plane, it is sufficient to study the function $g^i(\omega)$.

2.2 Green's function in the ω -plane

In this subsection we will discuss the analytic properties of the Greens function g^i in the complex ω plane. We will show that this function does not have any poles or branch cuts in the upper half ω plane. We then will obtain the asymptotic behaviour of this function in the $\omega \rightarrow i\infty$ limit. The analysis in this section is an extension of the one performed in [12] for shear correlator. For the shear correlator the behaviour of the Green's function is determined by a single differential equation. For the situation at hand we need to examine the properties of the coupled set of equations given in (2.20).

No poles for $\text{Im } \omega > 0$

It is known that poles or divergences in the retarded Green's function of an operator corresponds to the the quasi-normal modes of the differential equation satisfied by the corresponding dual field [15]. In fact by examining the definition of g^i given in (2.30) we see that poles will correspond to the situation when ϕ_∞^i also vanish. Thus poles in $g^i(\omega)$ corresponds to quasi-normal modes of the equation (2.20). Quasi-normal modes are solutions to the equation (2.20) with the following boundary conditions

$$\begin{aligned}\phi^i(r) &\sim (r - r_+)e^{-i\alpha\omega}, & r \rightarrow r_+, \\ \phi^i(r) &\sim r^{-2}, & r \rightarrow \infty.\end{aligned}\tag{2.32}$$

for all i . We will now show that such quasinormal modes do not exist with $\text{Im } \omega > 0$ by examining the equation (2.20) and the action (2.22). Note that all the coefficients in the equation (2.20) are real. Therefore if $\phi^i(r)$ is a quasinormal mode with frequency ω then $(\phi^i(r))^*$ is a quasinormal mode with frequency ω^* . This is easily demonstrated by taking the complex conjugate of the the set of equations in (2.20). Let us consider the equation $S_\phi - S_\phi = 0$. By substituting the solutions $\phi^i(r)$ and $(\phi^i(r))^*$ into this equation and integration by parts and then using the equations of motion given in (2.20). we obtain

$$0 = \sum_{i=1}^3 \left(Fr \frac{H_i^2}{\mathcal{H}} (\phi^{i*'} \phi^i - \phi^{i'} \phi^{i*})|_{r_h}^\infty + (\omega^{*2} - \omega^2) \int_{r_h}^\infty dr \frac{\mathcal{H}r}{F} \phi^i \phi^{i*} \right).\tag{2.33}$$

Here we have used the equations of motion of $(\phi^i)^*$ in the first S_ϕ and the equation of motion of ϕ^i in the second S_ϕ . We have also used the fact that the matrix M_{ij} in (2.23) is symmetric. From the conditions for the quasi-normal modes given in (2.32) and the fact that $F \sim (r - r_h)F_h$ as $r \rightarrow r_h$, we see that the first term in the above equation vanishes for $\text{Im}(\omega) > 0$. Since the second term is positive definite, we are led to the situation

$$\omega^2 = \omega^{*2}.\tag{2.34}$$

since the second term is positive definite. Thus we see that ω has to be either purely real or purely imaginary. This together with the condition $\text{Im}(\omega) > 0$ results in the fact that ω is restricted to lie on the upper imaginary axis.

We now assume that there exists a quasinormal mode on the upper imaginary axis with $\omega^2 < 0$. Then S_ϕ is given by

$$S_\phi = \int_{r_h}^{\infty} dr \frac{FrH_i^2}{\mathcal{H}} \left(\frac{d\phi^{i*}}{dr} \delta_{ij} \frac{d\phi^j}{dr} \right) + \phi^{i*} \left(M_{ij} + \frac{|\omega|^2 H_i^2 r}{F} \delta_{ij} \right) \phi^j. \quad (2.35)$$

The first and third term in the action are manifestly positive definite. The only term one needs to check is the following expression

$$I_\phi = \int_{r_h}^{\infty} dr \phi^{i*} M_{ij} \phi^j, \quad (2.36)$$

where the matrix is given in (2.23). Since the matrix M_{ij} is symmetric, to prove this term is positive definite, it is sufficient to show that the eigen values of this matrix is greater than equal to zero. From the form of the matrix M_{ij} it is easy to show that the eigen values are given by

$$\left(\frac{4r_+^6}{L^2 r^5 \mathcal{H}} \sum_{i=1}^3 (1 + k_i)^2 m_i^2, 0, 0 \right). \quad (2.37)$$

Thus the term in (2.36) is positive definite. Therefore the action S_ϕ for $\omega^2 < 0$ is positive. Now substituting the quasinormal mode with $\omega^2 < 0$ into S_a and integrating by parts we obtain

$$S_\phi = \sum_{i=1}^3 \frac{FrH_i^2}{\mathcal{H}} \phi^{i'} \phi^{i*} |_{r_h}^{\infty}. \quad (2.38)$$

From the boundary conditions of $\phi^i(r)$ for a quasinormal mode given in (2.32) and from the condition $\text{Im}(\omega) > 0$ we see that the boundary terms given above vanish. Thus we obtain $S_\phi = 0$, but since S_ϕ is positive definite we must have $\phi^i = 0$. Thus no quasinormal modes exist in the upper half plane which implies no poles or divergences exist in the Green's function in this domain.

No poles for ω real and $\omega \neq 0$

To show that there are no poles for ω real and $\omega \neq 0$ we will first redefine the variables

$$\phi^i = \Phi_i(r) \varphi^i, \quad \Phi_i(r) = \sqrt{\frac{\mathcal{H}}{rFH_i^2}}. \quad (2.39)$$

Note that here there is no summation over the index i . Substituting this redefinition in the equation (2.20) we obtain the following equation

$$\varphi^{i'''} + \left(\frac{\Phi_i''}{\Phi_i} + \left(\ln \left(\frac{rFH_i^2}{\mathcal{H}} \right) \right)' \frac{\Phi_i'}{\Phi_i} \right) \varphi^{i'} + \omega^2 \frac{\mathcal{H}}{F^2} \varphi^i - \tilde{M}_{ij} \varphi^j = 0, \quad (2.40)$$

where

$$\tilde{M}_{ij} = \frac{4r_+^6}{r^6 L^2 F} (1 + k_i) \frac{m_i}{H_i} (1 + k_j) \frac{m_j}{H_j}. \quad (2.41)$$

If ω is real, then $(\varphi^i)^*$ is also another solution of the equations in (2.40). Now consider the quantity

$$W(r) = \sum_{i=1}^3 ((\varphi^{i'})^* - (\varphi^i)^{*\prime}) \varphi^i. \quad (2.42)$$

Using the equations in (2.40) and the fact that \tilde{M}_{ij} is symmetric and real it is easy to see that W satisfies the equation

$$W'(r) = 0, \quad W(r) = C, \quad (2.43)$$

where C is a constant. Now let us assume that a quasinormal mode with ω real and $\omega \neq 0$ exists. We can then determine the value of the constant by examining the boundary conditions for the quasinormal mode. In the limit $r \rightarrow r_+$ using (2.32) and the redefinition (2.39) we obtain

$$\varphi^i \rightarrow \varphi_{r_+}^i (r - r_+)^{-i\alpha\omega + \frac{1}{2}}, \quad r \rightarrow r_+, \quad (2.44)$$

where $\varphi_{r_+}^i$ are constants. Substituting this in the definition of W we get

$$W = -2i\omega\alpha \sum_{i=1}^3 |\varphi_{r_+}^i|^2. \quad (2.45)$$

Now let's examine W near the boundary. Again from the equations (2.32) and (2.39) we see that

$$\varphi^i \rightarrow r^{-1/2}, \quad r \rightarrow \infty. \quad (2.46)$$

Using this in the equation for W we see that $W \rightarrow \frac{1}{r^2}$ as $r \rightarrow \infty$. This contradicts our conclusion in (2.45). Thus the quasi-normal modes and hence poles of the Green's function do not exist for ω real and $\omega \neq 0$.

No poles for $\omega = 0$

For this analysis it is convenient to rewrite the equations in (2.20) as

$$\phi''^i + \left(\ln\left(\frac{fH_i^2}{H}\right)' + \frac{3}{r} \right) \phi^{i'} + \frac{L^4 \omega^2 \mathcal{H}}{r^4 f^2} \phi^i - \frac{4r_+^6}{r^8} \prod_{j=1}^3 (1 + k_j) \sqrt{k_i} \sum_l \left(\frac{\sqrt{k_l}}{H_i^2 f} \phi^l \right) = 0. \quad (2.47)$$

Here we have rewritten m_i and F in terms of their original definitions. For $\omega = 0$ the above equations can be exactly solved [17]. The solution which is finite at $r \rightarrow r_+$ is given by

$$\phi^i = \tau_{ij} \phi_\infty^j, \quad \tau_{ij} \equiv \delta_{ij} - \frac{1}{2H_i} \sqrt{k_i k_j} u \quad (2.48)$$

where ϕ_∞^i are the values of ϕ^i at the boundary. We will now determine the behaviour of the Greens's function in the limit $\omega \rightarrow 0$. Let the solution for the equation (2.20) for finite ω be

$$\phi^i(r) = \tau_{ij}\psi^j(r). \quad (2.49)$$

Substituting (2.49) in (2.20) we have

$$\tau_{ij}\psi''^j + 2\tau'_{ij}\psi'^j + \left(\ln \frac{fr^3H_i^2}{\mathcal{H}}\right)' \tau_{ij}\psi'^j + \frac{\omega^2\mathcal{H}L^4}{r^4f^2}\tau_{ij}\psi^j = 0. \quad (2.50)$$

From the definition in (2.49) and (2.48) we see that the boundary values ϕ_∞^i are related to the boundary values ψ_∞^i by

$$\psi_\infty^i = \phi_\infty^i. \quad (2.51)$$

Then from the equation (2.50) we see that in the $\omega \rightarrow 0$ limit, the solution is given by the constants

$$\psi^i(r)_{\omega \rightarrow 0} = \psi_\infty^i = \phi_\infty^i. \quad (2.52)$$

Since the ψ^i in the $\omega \rightarrow 0$ limit are constants, we see that the values of ϕ^i are the horizon in the $\omega \rightarrow 0$ limit are given by

$$\phi^i(r \rightarrow r_+)_{\omega \rightarrow 0} = \tau_{ij}(r_+)\psi_\infty^j = \tau_{ij}(r_+)\phi_\infty^j. \quad (2.53)$$

Now let us obtain the equation for the derivative

$$f_i = \tau_{ij}\psi^{j'}(r). \quad (2.54)$$

Using the differential equation (2.50) we see that f^i satisfy the equation

$$f'_i + \tau'_{ik}\tau_{kj}^{-1}f_j + \left(\ln \frac{fr^3H_i^2}{\mathcal{H}}\right)' f_i + \frac{\omega^2\mathcal{H}L^4}{r^4f^2}\tau_{ij}\psi^j = 0. \quad (2.55)$$

The equation (2.52) implies at in the $\omega \rightarrow 0$ limit $f^i \rightarrow 0$. We are interested in the behaviour of f^i to the linear order in ω . From (2.55) we see that to this order it is sufficient to solve the equation

$$f'_i + \tau'_{ik}\tau_{kj}^{-1}f_j + \left(\ln \frac{fr^3H_i^2}{\mathcal{H}}\right)' f_i = 0. \quad (2.56)$$

The initial conditions of this equation at the horizon can be obtained from the ingoing boundary conditions for ϕ^i which translates to ingoing boundary conditions for ψ^i . This is given by

$$f_i(r \rightarrow r_+) = (-i\omega\alpha)\tau_{ij}(r_+)(r - r_+)^{-i\alpha\omega-1}\psi^j(r_+). \quad (2.57)$$

Thus in the $\omega \rightarrow 0$ limit we see that the f^i has the following behaviour at the horizon

$$f_i(r \rightarrow r_+)_{\omega \rightarrow 0} = (-i\omega\alpha)\tau_{ij}(r_+)(r - r_+)^{-1}\phi^j(\infty), \quad (2.58)$$

where we have used the fact that in the $\omega \rightarrow 0$ limit the ψ^i 's are constants given by (2.52). We now need to solve (2.56) such that it obeys the initial conditions (2.58). This solution is given by

$$f_i = \frac{\mathcal{H}}{fr^3H_i^2}(\tau^T)^{-1}_{ij}f_j^{r_+}, \quad (2.59)$$

where

$$(\tau^T)^{-1}_{ij} = \delta_{ij} + \frac{\sqrt{k_i k_j} u}{2H_j(1 - \sum_{i=1}^3 \frac{k_i u}{2H_i})}, \quad (2.60)$$

and $f_j^{r_+}$ are determined by the initial conditions given in (2.58). This is given by

$$f_j^{r_+} = -\frac{i\omega r_+ L^2}{\sqrt{(1+k_1)(1+k_2)(1+k_3)}} \sum_{l=1}^3 \tau_{jl}^T(r_+)(1+k_l)^2 \tau_{lm}(r_+) \phi_\infty^m. \quad (2.61)$$

Now that we have the solution for f_j , we can obtain the leading behaviour of the Green's function in the $\omega \rightarrow 0$ limit.

$$\begin{aligned} g^i|_{\omega \rightarrow 0} &= \lim_{r \rightarrow \infty} \frac{rF}{L\phi^i} \frac{d\phi^i}{dr} \Big|_{\phi_\infty^j=0, j \neq i}, \quad (2.62) \\ &= \lim_{r \rightarrow \infty} \frac{r^3}{L^3} \tau'_{ii} + \frac{f_i^{r_+}}{L^3 \phi_i^\infty} \Big|_{\phi_\infty^j=0, j \neq i}, \\ &= \frac{k_i r_+^2}{L^3} - \frac{i\omega r_+}{4L\sqrt{(1+k_1)(1+k_2)(1+k_3)}} (4 + 4k_i + k_i \sum_{j=1}^3 (k_j)). \end{aligned}$$

In the second line we have used the definition in (2.49) and the solution for f_i given in (2.59). The same analysis can easily be extended to obtain the full retarded correlator in the $\omega \rightarrow 0$ limit. This results in

$$g^{ij}|_{\omega \rightarrow 0} = \frac{\sqrt{k_i k_j} r_+^2}{L^3} - \frac{i\omega r_+}{\sqrt{(1+k_1)(1+k_2)(1+k_3)}} \sum_{l=1}^3 \tau_{il}^T(r_+)(1+k_l)^2 \tau_{lj}. \quad (2.63)$$

From this it is easy to evaluate the real and imaginary part of the conductivity matrix at zero frequency. This is given by

$$\sigma_{\omega \rightarrow 0}^{ij} = -\frac{N^2}{8\pi^2 L^3} \lim_{\omega \rightarrow 0} \frac{1}{i\omega} g^{ij}(\omega), \quad (2.64)$$

where g^{ij} in the limit $\omega \rightarrow 0$ is given in (2.63). The real part of the conductivity matrix has been evaluated earlier by [17] using a slightly different approach. We have verified that the real part of the conductivity given in (2.64) agrees with the

expression given below equation (4.23) of [17]. account the fact that the gauge fields in this paper are dimensionless. Note that the imaginary part of the conductivity has a pole at $\omega \rightarrow 0$. The residue of the pole is the constant value of the Green's function in the $\omega \rightarrow 0$ limit. For future reference we will write this equation as

$$\lim_{\omega \rightarrow 0} \omega \text{Im} \sigma^i = \frac{1}{e^2} g^i |_{\omega \rightarrow 0} = \frac{1}{e^2} \frac{k_i r_+^2}{L^3}, \quad (2.65)$$

$$\frac{1}{e^2} = \frac{N^2}{8\pi^2 L^3}$$

Here we have just written the relation for the diagonal entries in the Green's function. Thus the above analysis which resulted in the equation (2.62) and (2.63) shows that the Greens' function g^i, g^{ij} has a well defined behaviour in the $\omega \rightarrow 0$ limit.

We can now show that the Green's function admits a systematic power series expansion around the origin. Let define

$$\tilde{g}_{ik}(r) = \frac{\tau_{ij} \psi^{j'}}{\omega \psi^k} = \frac{f_i}{\omega \psi^k}. \quad (2.66)$$

From our analysis of the solutions f_i and ϕ^i in the limit $\omega \rightarrow 0$ we see that \tilde{g}_{ik} has a well defined in the limit $\omega \rightarrow 0$. This is because f_i behaves linearly for small ω . Using (2.50) we see that \tilde{g}_{ik} satisfies the following equation

$$\tilde{g}'_{ik} + \tau_{ij} \tau_{jl}^{-1} \tilde{g}_{lk} + \left(\ln \frac{f H_i^2}{\mathcal{H}} \right)' \tilde{g}_{ik} + \frac{\omega \mathcal{H}}{3F^2} \tau_{ij} \hat{g}_{il} g_{lk} + \omega \tilde{g}_{ik} \tau_{kj}^{-1} \tilde{g}_{jk} = 0, \quad (2.67)$$

where

$$\hat{g}_{ki} = \frac{1}{\tilde{g}_{ik}}. \quad (2.68)$$

In (2.67) the indices i, k are free indices and are not summed over, all the other repeated indices are summed over. Note that the equation (2.67) is a non-linear equation for \tilde{g}_{ik} and it admits a power series expansion in ω . Since the solution is known in the $\omega \rightarrow 0$ limit, we can set up a power series expansion around this solution. From \tilde{g}_{ik} , it is easy to obtain an expansion of the Green's function as a power series in ω . The Green's function is given by

$$g^i = \lim_{r \rightarrow \infty} \frac{rF}{L \phi^i} \frac{d\phi^i}{dr} \Big|_{\phi_\infty^j=0, j \neq i}, \quad (2.69)$$

$$= \lim_{r \rightarrow \infty} \frac{rF}{L} \left(\tau'_{ii} + \omega \tilde{g}_{ii} |_{\phi_\infty^j=0, j \neq i} \right).$$

To conclude we have shown that the Green's function of interest has a well defined limit as $\omega \rightarrow 0$ and it admits a power series in ω at the origin in the complex ω plane.

Absence of branch cuts for $\text{Im } \omega \geq 0$

The argument for the absence of branch cuts is essentially same as the one developed in [12] for the shear tensor correlator. The important properties of the Green's function is determined by the function defined in (2.30). This is determined by solving the differential equation in (2.20) with the ingoing boundary condition at the horizon (2.28). Both the differential equation and the boundary conditions are smooth with respect to ω . Now the theory of ordinary differential equations ensures that a local Forbenius expansion of the solution is smooth with respect to the parameters of the differential equation provided the equation and the boundary conditions are smooth with respect to the parameters [18]. Thus ϕ^i and its radial derivative at the boundary must be smooth with respect to ω . Then from the definition of the Green's function (2.30), the only locations at which it or its n -th order derivative with respect to ω can be singular is when ϕ^i satisfies the quasinormal mode boundary conditions given in (2.32). This is because the denominator in (2.30) vanishes for the quasi-normal mode boundary conditions. But as we have already argued that these modes do not occur in the upper half plane. Thus we have the result that the Green's function is smooth with respect to ω in the upper ω -plane.

Behaviour as $\omega \rightarrow \infty$

To obtain the behaviour of the Green's function at large ω we follow the procedure developed in [12]. We first rewrite the differential equation given in (2.20) by defining a dimensionless variables

$$z = \frac{r_+}{r}, \quad i\lambda = \frac{L^2}{r_+}\omega. \quad (2.70)$$

$$\phi^{i'''} + \left(\ln \frac{fH_i^2}{z\mathcal{H}} \right)' \phi^{i'} - \frac{\lambda^2 \mathcal{H}}{f^2} \phi^{i'} - (1 + k_i)m_i \sum_j \left(\frac{4z^4}{H_i^2 f^2} (1 + k_j)m_j \phi^j \right) = 0. \quad (2.71)$$

For convenience we have gone over to the Euclidean frequency labelled as λ . We are interested in obtaining the behaviour of the functions ϕ^i as $\lambda \rightarrow \infty$. For this purpose we rescale the co-ordinates as

$$y = \lambda z. \quad (2.72)$$

This leads to the following equations for the gauge field fluctuations as a $1/\lambda$ expansion.

$$\phi^{i'''} + \left(\frac{f'}{f} - \frac{1}{y} + \frac{2y}{\lambda^2} J_i \right) \phi^{i'} - \frac{2y^4}{H_i^2 \lambda^6 f} (1 + k_i)m_i \sum_j (1 + k_j)m_j \phi^j - \frac{H}{f^2} \phi^i = 0. \quad (2.73)$$

where

$$J_1 = \left(\frac{k_1}{H_1} - \frac{k_2}{H_2} - \frac{k_3}{H_3} \right), \quad J_2 = \left(\frac{k_2}{H_2} - \frac{k_3}{H_3} - \frac{k_1}{H_1} \right), \quad J_3 = \left(\frac{k_3}{H_3} - \frac{k_1}{H_1} - \frac{k_2}{H_2} \right), \quad (2.74)$$

and there is no sum over i in (2.73). In terms of the dimensionless variables the incoming boundary conditions (2.28) can be re-written as

$$\phi^i \sim \left(1 - \frac{y}{\lambda} \right)^{\frac{\lambda r + \alpha}{L^2}}, \quad y \rightarrow \lambda. \quad (2.75)$$

We can set up a Forbeinus expansion of the solutions to the set of equations in (2.73) obeying the boundary conditions (2.75). This expansion is valid in the domain $y \sim \lambda$. One can also set up an expansion for $y \rightarrow 0$ which can be organized as a systematic expansion in powers of $1/\lambda$. It is important to note when λ is strictly infinity the equations in (2.73) decouples and reduces to that of three vector fields in the background of pure AdS_5 . Thus the leading solutions to the equations in the large λ expansion will be identical to the zero temperature case. We will construct the 6 independent solutions of the coupled equations given in (2.73) as an expansion in $1/\lambda$ as follows. Let us define

$$a(y) = \frac{\phi^{1'}(y)}{\phi^1(y)}, \quad b(y) = \frac{\phi^{2'}(y)}{\phi^2(y)}, \quad c(y) = \frac{\phi^{3'}(y)}{\phi^3(y)}, \quad (2.76)$$

where the derivatives are with respect to y . We will now focus on obtaining a perturbative expansion for $a(y)$, a similar analysis can be performed for the quantities $b(y)$, $c(y)$. From (2.73) we can obtain the equation for $a(y)$ which can be given as

$$a' + (a)^2 + \left\{ \frac{f'}{f} - \frac{1}{y} + \frac{2y}{\lambda^2} \left(\frac{k_1}{H_1} - \frac{k_2}{H_2} - \frac{k_3}{H_3} \right) \right\} a - \frac{4y^4}{H_1^2 \lambda^6 f} (1 + k_1) m_1 \sum_j (1 + k_j) m_j \frac{\phi^j}{\phi^1} - \frac{H}{f^2} = 0. \quad (2.77)$$

From the equation (2.77) we see that the a admits an expansion of the form

$$a = a_0 + \frac{1}{\lambda^2} a_1 + \frac{1}{\lambda^4} a_2 + \mathcal{O}\left(\frac{1}{\lambda^6}\right). \quad (2.78)$$

By substituting this expansion for a into (2.77) and matching orders in $\frac{1}{\lambda^2}$ we can obtain equations which determine the functions a_i . The leading orders in the expansion are determined by the following equations

$$\begin{aligned} a'_0 + (a_0)^2 - \frac{1}{y} a_0 - 1 &= 0, \\ a'_1 + \left(2a_0 - \frac{1}{y} \right) a_1 + 4k_1 y a_0 + (k_1 + k_2 + k_3) y^2 &= 0, \\ a'_2 + \left(2a_0 - \frac{1}{y} \right) a_2 + (a_1)^2 + \left((1 + k_1)(1 + k_2)(1 + k_3) + k_1^2 \right) y^3 a_0 + 4k_1 y a_1 \\ - \left(2(1 + k_1)(1 + k_2)(1 + k_3) + k_1^2 + k_2^2 + k_3^2 \right) y^4 &= 0. \end{aligned} \quad (2.79)$$

The two independent solutions for the first equation in (2.79) are

$$a_0^{(1)} = -\frac{K_0(y)}{K_1(y)} = \frac{d}{dy} \ln(yK_1(y)), \quad a_0^{(2)} = \frac{I_0(y)}{I_1(y)} = \frac{d}{dy} \ln(yI_1(y)). \quad (2.80)$$

We now show how to obtain a systematic expansion around the first solution $a_0^{(1)}$. The first order correction is given by

$$a_1^{(1)} = -2k_1y - \left(\frac{k_1 + k_2 + k_3}{6} \right) y^3 \left(1 - \frac{K_2^2}{K_1^2} \right) + \frac{c_1}{yK_1^2}. \quad (2.81)$$

We set $c_1 = 0$, so as not to change the asymptotics of this solution at $y \rightarrow \infty$. Substituting this into the equation for $a_2^{(1)}$ it is easy to obtain the solution. However for us it is sufficient to obtain the behaviour near $y \rightarrow 0$. This is given by

$$a_2^{(1)}(y) = y^3 + O(y^5, y^5 \log y). \quad (2.82)$$

A similar expansion can be set up for the functions b and c defined in (2.76). Examining the equations for a_n it is easy to see that near the origin their behaviour is given by

$$a_n^{(1)}(y) \sim y^m, \quad m \geq 3, \quad \text{for } n \geq 2 \quad (2.83)$$

Note that the equation which determines a couples with the functions b and c at $O(\frac{1}{\lambda^6})$ and therefore the coupling can be treated perturbatively. Thus a systematic expansion for the first solution to a can be found. This is given by

$$\begin{aligned} \phi_{(1)}^1(y) &= \exp \left[\int_0^y dy \left(a_0 + \frac{a_1^{(1)}}{\lambda^2} + \frac{a_2^{(1)}}{\lambda^4} + \dots \right) \right], \\ &= yK_1(y) \left(1 + \frac{1}{\lambda^2} \int_0^y dy a_1^{(1)}(y) + \dots \right). \end{aligned} \quad (2.84)$$

Note that

$$\phi_{(1)}^1(y) \sim \text{constant}, \quad y \rightarrow 0. \quad (2.85)$$

From (2.81) we see that the $O(1/\lambda^2)$ term in (2.84) goes as y^2 near the origin. The higher order terms (2.82) are further suppressed as $y \rightarrow 0$. We now obtain the second solution starting with the zeroth order second solution given in (2.80). The first order correction about the second solution is given by

$$\begin{aligned} a_1^{(2)} &= -2k_1y + \left(\frac{k_1 + k_2 + k_3}{6} \right) y^3 \left(1 - \frac{I_2^2}{I_1^2} \right), \\ &= -2k_1y + \frac{k_1 + k_2 + k_3}{6} \left(y^3 - \frac{y^5}{16} \right) + O(y^7). \end{aligned} \quad (2.86)$$

Similarly it can be shown that $a_2^{(2)}(y)$ has the following behaviour

$$a_2^{(2)}(y) \sim y^3 + O(y^5). \quad (2.87)$$

Thus we can write the second solution as

$$\begin{aligned}\phi_{(2)}^1(y) &= \exp \left[\int_0^y dy \left(a_0^{(2)} + \frac{a_1^{(2)}}{\lambda^2} + \dots \right) \right], \\ &= yI_1(y) \left(1 + \frac{1}{\lambda^2} \int_0^y dy a_1^{(2)}(y) + \dots \right).\end{aligned}\tag{2.88}$$

Note that the leading behaviour of the second solution at the origin is given by

$$\phi_{(2)}^1(y) \sim y^2, \quad y \rightarrow 0.\tag{2.89}$$

In this way one can obtain the 6 independent solutions to the coupled set of equations given in (2.20). These solutions are all valid as expansions around the origin $y = 0$.

Thus the general solution for the ϕ^1 is the linear combination

$$\phi^1(y) = \alpha(\lambda)\phi_{(1)}^1(y) + \beta(\lambda)\phi_{(2)}^1(y).\tag{2.90}$$

Similar solutions can be written for ϕ^2, ϕ^3 . The coefficients $\alpha(\lambda), \beta(\lambda)$ are determined by extrapolating the Forbeinus expansions near the horizon $y \sim \lambda$ to $y \rightarrow 0$ and matching with the expansions in (2.90) extrapolated to $y \rightarrow \infty$. For our purpose we only need the large λ behaviour of these coefficients. We see that

$$\beta(\lambda) \rightarrow 0, \quad \alpha(\lambda) \rightarrow 1, \quad \lambda \rightarrow \infty.\tag{2.91}$$

The reason for this is that when λ is strictly ∞ , the equation (2.73) reduces to that of the zero temperature case as we have observed earlier. In this situation the solution which is finite as $y \rightarrow \infty$ is given by $yK_1(y)$ ¹. Thus we must have $\beta(\lambda) \rightarrow 0$ in the large λ limit.

We will now examine the implications of the properties of the solutions of the differential equation derived in (2.81), (2.82), (2.83) as well as (2.86), (2.87) and (2.89) on the retarded Green's function $g^1(\omega)$. In terms of the variable y the Green's function is given by

$$g^1(\lambda) = -\frac{r_+^2}{L^3} \lim_{y \rightarrow 0} \frac{\lambda^2}{y} \frac{1}{\alpha(\lambda)\phi_{(1)}^1(y) + \beta(\lambda)\phi_{(2)}^1(y)} \frac{d}{dy} (\alpha(\lambda)\phi_{(1)}^1(y) + \beta(\lambda)\phi_{(2)}^1(y)).\tag{2.92}$$

Taking the $y \rightarrow 0$ limit and using the behaviour of the functions near the origin we obtain

$$g^1(\lambda) = -\frac{r_+^2}{L^3} \left(\lim_{y \rightarrow 0} \frac{\lambda^2}{y} a_0^{(1)}(y) + \frac{2}{3}(-2k_1 + k_2 + k_3) + 2\frac{\beta(\lambda)}{\alpha(\lambda)} \right).\tag{2.93}$$

Note that the above expression is valid for all values of λ . Taking the $\lambda \rightarrow \infty$ limit we are left with

$$\lim_{\lambda \rightarrow \infty} g^1(\lambda) = -\frac{r_+^2}{L^3} \left(\lim_{y \rightarrow 0} \frac{\lambda^2}{y} a_0^{(1)}(y) + \frac{2}{3}(-2k_1 + k_2 + k_3) \right),\tag{2.94}$$

¹We have chosen the normalization of the solution $yK_1(y)$ to be 1 at the origin $y = 0$.

where we have used (2.91). The rest of the terms in g^1 is subleading in the $\lambda \rightarrow \infty$ limit. As we have mentioned earlier, the leading contribution in the λ expansion is identical to the zero temperature case. This has a $\log(r)$ divergence which has to be removed by the counter term. Since this divergence is independent of the temperature we have the relation

$$G_{\text{counter}}^1(\omega, T) = G_{\text{counter}}^1(\omega, 0). \quad (2.95)$$

There is also a constant term due to the first order correction $a_1^{(1)}$. Thus the Green's function does not satisfy 'property 2' and needs to be regulated.

To regulate the Green's function we consider

$$\delta G^1(\omega) = G^1(\omega, T) - G^1(\omega, 0) - \frac{1}{e^2} \frac{2r_+^2}{3L^3} (-2k_1 + k_2 + k_3). \quad (2.96)$$

where e^2 is defined in (2.65). Now using (2.94) as well as (2.95) we have

$$\delta G^1(\omega) \rightarrow 0, \quad \omega \rightarrow i\infty. \quad (2.97)$$

We have essentially subtracted the divergent and the constant term in $G_R(\omega)$ so that 'property 2' is true on $\delta G_R(\omega)$, which can then be used to obtain the sum rule.

2.3 The sum rule

From the above discussion we see that $\delta G_R(\omega)$ satisfies both 'property 1' and 'property 2'. Also since the subtracted constant in defining $\delta G^1(\omega)$ is real we have the relation

$$\text{Im} \delta G^1(\omega) = \rho^1(\omega)_T - \rho^1(\omega)_{T=0}, \quad (2.98)$$

where

$$\rho^1(\omega)_T = \text{Im} G^1(\omega, T), \quad \rho^1(\omega)_{T=0} = \text{Im} G^1(\omega, 0). \quad (2.99)$$

Thus using analyticity properties of $\delta G^1(\omega)$ we obtain the sum rule

$$\int_{-\infty}^{\infty} \frac{d\omega}{\pi\omega} (\rho^1(\omega) - \rho^1_{T=0}(\omega)) = \delta G^1(0), \quad (2.100)$$

where the RHS of the sum rule is given by

$$\delta G^1(0) = G^1(0, T) - \frac{1}{e^2} \frac{2r_+^2}{3L^3} (-2k_1 + k_2 + k_3). \quad (2.101)$$

The zero frequency Green's function can be related to the conductivity as given in equation (2.64). From the analysis of the zero frequency limit and the equation (2.65) we see that this is given by the residue of the pole of the imaginary part of the conductivity at zero frequency. Therefore we have

$$\begin{aligned} G_R^1(0, T) &= \lim_{\omega \rightarrow 0} \omega \text{Im} \sigma^1(\omega), \\ &= \frac{1}{e^2} \frac{r_+^2 k_1}{L^3}. \end{aligned} \quad (2.102)$$

Thus the sum rule can be written as

$$\int_{-\infty}^{\infty} \frac{d\omega}{\pi\omega} (\rho^1(\omega) - \rho_{T=0}^1(\omega)) = \lim_{\omega \rightarrow 0} \omega \text{Im} \sigma^1(\omega) - \frac{1}{e^2} \frac{2r_+^2}{3L^3} (-2k_1 + k_2 + k_3). \quad (2.103)$$

Similarly studying the Green's functions $G_R^2(\omega)$ and $G_R^3(\omega)$ we obtain the following sum rules for their spectral densities.

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{d\omega}{\pi\omega} (\rho^2(\omega) - \rho_{T=0}^2(\omega)) &= \lim_{\omega \rightarrow 0} \omega \text{Im} \sigma^2(\omega) - \frac{1}{e^2} \frac{2r_+^2}{3L^3} (k_1 - 2k_2 + k_3), \\ \int_{-\infty}^{\infty} \frac{d\omega}{\pi\omega} (\rho^3(\omega) - \rho_{T=0}^3(\omega)) &= \lim_{\omega \rightarrow 0} \omega \text{Im} \sigma^3(\omega) - \frac{1}{e^2} \frac{2r_+^2}{3L^3} (k_1 + k_2 - 2k_3). \end{aligned} \quad (2.104)$$

Note that at zero chemical potentials all terms in the RHS of the the sum rules in (2.103), (2.104) vanish. Thus they reduce to the ones derived in [10] and [11].

3. Structure constants from sum rules

From the derivation of the sum rule we see that the RHS of the sum rule in (2.103) and (2.104) consists of two terms. One term is the zero frequency dependence of the Green's function, this is determined by hydrodynamics. The zero frequency contribution is due to the residue of the pole in the imaginary part of the conductivity. The second term is from the finite term in the high frequency behaviour of the Green's function. As emphasized in [12] the finite terms that arise in the sum rule can be understood by examining the operator product expansion of the currents involved in the corresponding Green's function. In this section we will examine the OPE of the R-currents. We will show that the finite term in the sum rule which arises from the high frequency behaviour of the Green's function is due to the presence of chiral primary operators of dimension 2. These operators gain expectation values in the presence of chemical potential. From this analysis we obtain the structure constants involving two R-currents and the chiral primary. We then obtain these structure constants from Witten diagrams using the methods developed in [19] and show that they agree with that obtained from the sum rule.

3.1 OPE and high frequency behaviour

Here the OPE of interest is that of the R-currents which have conformal dimension 3 in four dimensions. The leading terms in this OPE are given by

$$J_\mu^i(x) J_\nu^j(0) \sim \frac{\mathcal{C} \delta_{ij} I_{\mu\nu}(x)}{x^6} + \mathcal{A}_{\mu\nu} C_{ij}^{\hat{k}} \mathcal{O}_{\hat{k}}(0) + \mathcal{B}_{\mu\nu;k}^{ij;\rho} J_\rho^k(0) + \dots, \quad (3.1)$$

where μ, ν and η are the space time indices and i, j and k are the R-symmetry indices. The hatted indices \hat{k} run over the scalar operators in the theory. The tensor $I_{\mu\nu}(x)$

is given by

$$I_{\mu\nu}(x) = \delta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}. \quad (3.2)$$

Note that for the purpose of analyzing the OPE we work in Euclidean signature. The OPE given in (3.1) can contain higher tensor operators for example, the stress tensor. But as we will see, these operators have higher dimensions and will not be relevant for our analysis. The tensor structure $\mathcal{A}_{\mu\nu}$ can be obtained by examining the three point function $\langle J_\mu^i J_\nu^j \mathcal{O}_{\hat{k}} \rangle$ and $C_{ij}^{\hat{k}}$ are the structure constants.

Before we determine the tensor structure $\mathcal{A}_{\mu\nu}$, it is easy to see that the $\mathcal{A}_{\mu\nu}$ must scale as $x^{\Delta-6}$ where Δ is the conformal dimension of the scalar operator $\mathcal{O}_{\hat{k}}$. Let us now take the Fourier transform on both sides of the OPE in (3.1) with spatial momentum $q = 0$. Then by a simple scaling analysis as done in [12], it is easy to see that the only operator which can contribute to the constant term as the frequency ω in the Fourier transform is scaled to infinity is $\Delta = 2$. For $\Delta \neq 2$, the term scales as $\omega^{2-\Delta}$. Thus these terms diverge for $\Delta < 2$ and are subleading when $\Delta > 2$. Note that the leading term in (3.1) is proportional to the central charge \mathcal{C} scales like x^{-6} . This diverges as ω^2 on taking the Fourier transform. It is this divergence which needs to be subtracted to regulate the Green's function.

Let us now examine the term which is proportional to the R-current in the RHS of the OPE. The tensor \mathcal{B} contains various space-time structures which can be found in [20] from the three point function of $\langle J_\mu^i J_\nu^j J_\rho^k \rangle$. From conformal invariance we see that \mathcal{B} scales as x^{-3} . Using the same reasoning we see that the term proportional to the current on the RHS of the OPE is subleading as $\omega \rightarrow \infty$. Similar reasoning allows us to conclude that tensor operators of higher dimension will not contribute in the large frequency limit.

We are interested in the expectation value of the Green's function in the thermal state and at finite chemical potential. Thus taking expectation values on both sides of the OPE in (3.1), we conclude that the term which is finite as $\omega \rightarrow \infty$ arises from the presence of expectation values of operators of dimension 2 in the OPE. From the structure of the OPE in (3.1), we see that given the finite terms in the large ω expansion we can extract the structure constant $C_{ij}^{\hat{k}}$ if one has the knowledge of the expectation value of the scalar $\mathcal{O}_{\hat{k}}$. Our goal is to use the sum rules derived in (2.103) and (2.104) and extract out the structure constants $C_{ij}^{\hat{k}}$. For this purpose we first will evaluate the expectation values of the operators of dimension 2 which are turned on in the theory at finite chemical potential. We will then compare these structure constants evaluated using the conventional approach of [19] and show that they indeed do agree.

To begin with, let us be more precise about the structure of the tensor $\mathcal{A}_{\mu\nu}$. Consider the three point function of two currents with a scalar operator of dimension

2. Conformal invariance constrains this to the form given by [21]

$$\langle J_\mu^i(x) J_\nu^j(y) \mathcal{O}_{\hat{k}}(z) \rangle = C_{ijk} \frac{4I_{\mu\nu}(x-y) - 2I_{\mu\rho}(x-z)I_{\rho\nu}(z-y)}{(x-y)^4(x-z)^4}. \quad (3.3)$$

The tensor $I_{\mu\nu}$ is defined in (3.2). The leading term in the short distance expansion when $x \rightarrow y$ of this three point function is given by

$$\begin{aligned} \langle J_\mu^i(x) J_\nu^j(y) \mathcal{O}_{\hat{k}}(z) \rangle &= C_{ijk} \frac{4I_{\mu\nu}(x-y) - 2I_{\mu\rho}(x-z)I_{\rho\nu}(z-y)}{(x-y)^4(x-z)^4}, \\ &= C_{ijk} \frac{\hat{\mathcal{A}}_{\mu\nu}(s)}{(x-z)^4} + \dots, \end{aligned} \quad (3.4)$$

where $s^\mu = (x-y)^\mu$ and

$$\begin{aligned} \hat{\mathcal{A}}_{\mu\nu}(s) &= -\frac{8}{s^4} \left(\frac{s_\mu s_\nu}{s^2} - \frac{1}{4} \delta_{\mu\nu} \right), \\ &= -\partial_\mu \partial_\nu \frac{1}{s^2}. \end{aligned} \quad (3.5)$$

Note that since $\hat{\mathcal{A}}_{\mu\nu}(s) = O(1/s^4)$ it is not well defined as a distribution in 4 dimensions. Thus we need to regularize this by introducing a delta function following the methods of [20, 22, 23, 24]. Therefore we consider

$$\mathcal{A}_{\mu\nu}(s) = - \left(\partial_\mu \partial_\nu \frac{1}{s^2} + 4\pi^2 K \delta_{\mu\nu} \delta^4(s) \right). \quad (3.6)$$

This expression is identical to $\hat{\mathcal{A}}_{\mu\nu}$ for $s \neq 0$. We now determine the constant K using Ward identities. As a consequence of invariance under the $U(1)^3$ R-symmetry we obtain the Ward identity [20]²

$$\partial^\mu \langle J_\mu^i \rangle = 0 \quad (3.7)$$

Here A_μ^i are the sources corresponding to the currents J_μ^i . Differentiating this identity with respect to $A_\nu^j(y)$ and the source corresponding to the operator $\mathcal{O}_{\hat{k}}(z)$ we obtain the Ward identity

$$\partial^\mu \langle J_\mu^i(x) J_\nu^j(y) \mathcal{O}_{\hat{k}}(z) \rangle = 0. \quad (3.8)$$

Note that the operator $\mathcal{O}_{\hat{k}}$ is uncharged under the R-symmetry. Thus we must have

$$\partial^\mu \mathcal{A}_{\mu\nu}(s) = -(K-1)4\pi^2 \partial_\nu \delta^4(s) = 0. \quad (3.9)$$

This implies $K = 1$ and the regularized expression for $\hat{\mathcal{A}}_{\mu\nu}(s)$ is given by

$$\mathcal{A}_{\mu\nu}(s) = - \left(\partial_\mu \partial_\nu \frac{1}{s^2} + 4\pi^2 \delta_{\mu\nu} \delta^4(s) \right). \quad (3.10)$$

²Here we ignore the $U(1)^3$ anomaly since all our computations will be at zero momentum.

Thus contribution of the scalar operator of dimension 2 in that the OPE of the R-currents is given by

$$J_\mu^i(x)J_\nu^j(y) \sim C_{ij}^{\hat{k}}\mathcal{A}_{\mu\nu}(s)\mathcal{O}_{\hat{k}}(x) + \dots \quad (3.11)$$

Note that now \hat{k} runs over all operators of dimension 2 in the theory. Inserting the operator $\mathcal{O}_{\hat{l}}(z)$ on both sides of the above equation and comparing it with the three point function in (3.4), we obtain

$$C_{ij\hat{l}} = C_{ij}^{\hat{k}}g_{\hat{k}\hat{l}}, \quad (3.12)$$

where the constants $g_{\hat{k}\hat{l}}$ are defined by the two point function

$$\langle\mathcal{O}_{\hat{k}}(x)\mathcal{O}_{\hat{l}}(z)\rangle = \frac{g_{\hat{k}\hat{l}}}{(x-z)^4}. \quad (3.13)$$

Let us now examine the contribution of all the operators of dimension 2 that appear in the OPE to the finite terms in the $\omega \rightarrow \infty$ of the Euclidean Green's function. For this we evaluate the following

$$\begin{aligned} \lim_{\omega \rightarrow \infty} \delta G_E^i(\omega) &= \int d^4x e^{-i\omega t} (\langle J_x^i(x)J_x^i(0)\rangle_{T \neq 0} - \langle J_x^i(x)J_x^i(0)\rangle_{T=0}), \quad (3.14) \\ &= C_{ii}^{\hat{k}}\langle\mathcal{O}_{\hat{k}}(0)\rangle_{T \neq 0} \int d^3x e^{-i\omega t} \mathcal{A}_{xx}(x), \\ &= -4\pi^2 C_{ii}^{\hat{k}}\langle\mathcal{O}_{\hat{k}}(0)\rangle_T. \end{aligned}$$

To obtain the second line we have substituted the OPE derived in (3.11) and the form for \mathcal{A} given in (3.10). We have also used the fact that expectation value of the identity operator is normalized to 1 in both at zero and at finite temperature, this leads to the cancellation of the ω^2 divergence from the term proportional to the central charge \mathcal{C} in the OPE. Note that the equation in (3.14) is a relation for the Euclidean propagator. The Euclidean propagator can be analytically continued to the retarded Minkowski propagator in the whole of the upper half ω plane [25] using the relation

$$G(i\omega, q) = -G_E(\omega, q) \quad (3.15)$$

This leads to the following behaviour for the retarded two point function of interest in the limit $\omega \rightarrow i\infty$

$$\lim_{\omega \rightarrow i\infty} G_T^i(\omega) - G_{T=0}^i(\omega) = 4\pi^2 C_{ii}^{\hat{k}}\langle\mathcal{O}_{\hat{k}}(0)\rangle_T. \quad (3.16)$$

It is important to point out the analysis in this subsection has been performed entirely based on the conformal invariance of the dual theory. It does not rely on holography.

3.2 Reading the structure constants from the sum rule

From the analysis of the behaviour of the OPE we have concluded that the finite terms in the sum rule from $\omega \rightarrow i\infty$ limit arises because of certain operators of dimension 2 in the OPE. From the equation in (3.16), we see that once we know the finite term in the Greens' function in the $\omega \rightarrow i\infty$, and the expectation values of the operators of dimension 2 in the thermal state, we can read out the structure constants involved. We will now perform this analysis from the sum rules in (2.103) and (2.104) derived holographically.

The operators of dimension 2 which are relevant for our analysis are dual to the 2 independent scalars present in the background given in (2.4). We parametrize X^i 's in terms of these independent scalars by

$$\begin{aligned} X^1 &= \exp\left(-\frac{1}{\sqrt{6}}\vartheta_1 - \frac{1}{\sqrt{2}}\vartheta_2\right), & X^2 &= \exp\left(-\frac{1}{\sqrt{6}}\vartheta_1 + \frac{1}{\sqrt{2}}\vartheta_2\right), \\ X^3 &= \exp\left(\frac{2}{\sqrt{6}}\vartheta_1\right). \end{aligned} \quad (3.17)$$

Substituting these field redefinitions in the action (2.5) and expanding the scalar potential

$$V = \frac{4}{L^2} \sum_{i=1}^3 \frac{1}{X_i}, \quad (3.18)$$

to quadratic order in ϑ_i it can be shown that the masses of both is given by

$$m^2 L^2 = -4. \quad (3.19)$$

Using the mass-dimension relation for scalars

$$\Delta(\Delta - 4) = m^2 L^2, \quad (3.20)$$

we see that the fields ϑ_i corresponds to operators of dimension 2 in the field theory and saturate the Breitenlohner-Freedman bound. From the $\mathcal{N} = 4$ field theory point of view these operators correspond to two linear combinations of the following chiral primaries

$$\text{Tr}(X\bar{X}), \quad \text{Tr}(Y\bar{Y}), \quad \text{Tr}(Z\bar{Z}), \quad (3.21)$$

where X, Y, Z are 3 complex fields constructed out of the 6 scalars in $\mathcal{N} = 4$ Yang-Mills. These are the three scalars which are un-charged under the Cartans of $SO(6)$. Note that the combination $\text{Tr}(X\bar{X}) + \text{Tr}(Y\bar{Y}) + \text{Tr}(Z\bar{Z})$ is the Konishi scalar and therefore not a chiral primary and is to be excluded. We will denote the two chiral primaries dual to the field ϑ_1, ϑ_2 as \mathcal{O}_1 and \mathcal{O}_2 respectively.

it is easy to read out the expectation values of the operators \mathcal{O}_1 and \mathcal{O}_2 at finite chemical potential from the background given in (2.3) and (2.4) following the

procedure detailed in [26]. See [27, 28] for earlier work. Details of this has been repeated in [12]. Here we write down the results for the expectation values.

$$\begin{aligned}\langle \mathcal{O}_1 \rangle &= \frac{N^2 r_+^2}{8\pi^2 L^4} \frac{2}{\sqrt{6}} (k_1 + k_2 - 2k_3), \\ \langle \mathcal{O}_2 \rangle &= \frac{N^2 r_+^2}{8\pi^2 L^4} \frac{2}{\sqrt{2}} (k_1 - k_2).\end{aligned}\tag{3.22}$$

Note that these expectation values have mass dimension 2 since the operators $\mathcal{O}_1, \mathcal{O}_2$ have mass dimension 2 and they are constant in space time. We have normalized the operators following [29], that is by dividing scalar operators by $\Delta - d/2$ where Δ is the conformal dimension of the operator and d is the space time dimension³. This explains the additional factor of 2 in (3.22) when compared to equation (4.70) of [12]. We can now compare the large frequency terms in the sum rule (2.103), (2.104) and the expectation values of the scalars in (3.22) to rewrite the RHS of the sum rules as

$$\begin{aligned}\delta G^1(0) &= \lim_{\omega \rightarrow 0} \omega \text{Im} \sigma^1(\omega) + \frac{1}{\sqrt{6}L^2} \langle \mathcal{O}_1 \rangle + \frac{1}{\sqrt{2}L^2} \langle \mathcal{O}_2 \rangle, \\ \delta G^2(0) &= \lim_{\omega \rightarrow 0} \omega \text{Im} \sigma^2(\omega) + \frac{1}{\sqrt{6}L^2} \langle \mathcal{O}_1 \rangle - \frac{1}{\sqrt{2}L^2} \langle \mathcal{O}_2 \rangle, \\ \delta G^3(0) &= \lim_{\omega \rightarrow 0} \omega \text{Im} \sigma^3(\omega) - \frac{2}{\sqrt{6}L^2} \langle \mathcal{O}_1 \rangle.\end{aligned}\tag{3.23}$$

where we have used the expression for e^2 given in (2.65). Comparing the contribution from the higher frequency part in (3.23) and (3.16) we find the following values for the structure constants⁴

$$\begin{aligned}C_{11}^{\hat{1}} &= -\frac{1}{(2\pi)^2 L^2 \sqrt{6}}, & C_{11}^{\hat{2}} &= -\frac{1}{(2\pi)^2 L^2 \sqrt{2}}, \\ C_{22}^{\hat{1}} &= -\frac{1}{(2\pi)^2 L^2 \sqrt{6}}, & C_{22}^{\hat{2}} &= \frac{1}{(2\pi)^2 L^2 \sqrt{2}}, \\ C_{33}^{\hat{1}} &= \frac{2}{(2\pi)^2 L^2 \sqrt{6}}, & C_{33}^{\hat{2}} &= 0.\end{aligned}\tag{3.24}$$

Note that here the first two indices in the subscripts of C are the R-symmetry indices, while the third hatted index labels the operators of dimension 2.

3.3 Structure constants from Witten diagrams

In this section we use the the standard AdS/CFT prescription put forward in [30] to evaluate the the three point function $\langle J_\mu^i J_\nu^j \mathcal{O}_{\hat{k}} \rangle$ This will enable to us to obtain

³See discussion below equation (2.21) of [29].

⁴The finite terms in the Green's function at the high frequency is negative of that in the sum rule (3.23) since this term is subtracted in the renormalized Green's function.

the structure constants C_{ijk} . We can then compare them to that obtained from the sum rule listed in (3.24). To obtain the structure constants using the conventional AdS/CFT prescription we follow the methods developed in [23]. From the action given in (2.5) we see that the three point function of two R-currents with the operators of dimension 2 can be evaluated by examining the cubic interaction of the scalars with the gauge fields. This interaction occurs in the gauge kinetic term. Rewriting the scalars X^i in terms of the two independent scalars ϑ_1, ϑ_2 using the equations in (3.17) we obtain the following interaction

$$\frac{1}{4e^2} \vec{a}^i \cdot \int d^5x \sqrt{g} g^{\mu\rho} g^{\nu\sigma} \vec{\vartheta} F_{\mu\nu}^i F_{\nu\rho}^i, \quad (3.25)$$

where we have organized the two scalars ϑ_1, ϑ_2 into a two dimensional vector and the two dimensional vectors \vec{a}^i with $i = 1, 2, 3$ are given by

$$\vec{a}^1 = 2 \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}} \right), \quad \vec{a}^2 = 2 \left(\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{2}} \right), \quad \vec{a}^3 = 2 \left(-\frac{2}{\sqrt{6}}, 0 \right). \quad (3.26)$$

In (3.25) summation over i is implied. The three point function of two currents with an operator of dimension Δ in a d dimensional conformal field theory is obtained holographically using the methods of [23] in appendix A. Applying the equation (A.7) to the cubic interaction given in (3.25) together with $d = 4$ and $\Delta = 2$ we obtain the following values for the structure constants.

$$C_{i\hat{j}} = -a_j^i \mathcal{K}_4(2), \quad (3.27)$$

where \mathcal{K} is defined in (A.7) and the subscript with the hatted index refers to the component of the vector \vec{a}^i defined in (3.26). The structure constants evaluated from the sum rules given in (3.24) have one one raised index which are related to the ones in (3.27) by contraction with the constants that appear in the two point function of the scalars as given in (3.12). Given the action for the scalars in (2.5), the two point function of the corresponding with dimension Δ have been evaluated holographically in (A.10). This is given by

$$g_{i\hat{j}} = \delta_{i\hat{j}} \mathcal{G}_4(2). \quad (3.28)$$

The Kronecker delta results from the fact that the scalars $\vec{\vartheta}$ have a canonically normalized Kinetic term. Therefore using (3.12) together with (3.27) and (3.28) we obtain

$$C_{i\hat{j}}^{\hat{j}} = -a_j^i \frac{\mathcal{K}_4(2)}{\mathcal{G}_4(2)}. \quad (3.29)$$

Substituting the values of the normalizations and the components of the vector \vec{a}^i we obtain

$$C_{11}^{\hat{1}} = -\frac{1}{(2\pi)^2 L^2 \sqrt{6}}, \quad C_{11}^{\hat{2}} = -\frac{1}{(2\pi)^2 L^2 \sqrt{2}},$$

$$\begin{aligned}
C_{22}^{\hat{1}} &= -\frac{1}{(2\pi)^2 L^2 \sqrt{6}}, & C_{22}^{\hat{2}} &= \frac{1}{(2\pi)^2 L^2 \sqrt{2}}, \\
C_{33}^{\hat{1}} &= \frac{2}{(2\pi)^2 L^2 \sqrt{6}}, & C_{33}^{\hat{2}} &= 0.
\end{aligned}
\tag{3.30}$$

Comparing (3.24) and (3.30) we see that the structure constants derived from the sum rule precisely coincides with that evaluated using the Witten diagrams.

4. R-charge sum rules for M2 and M5-branes

In this section we will holographically derive the R-charge sum rules for the case of M2-branes and M5-branes. We will not go through all the details as was done for the D3-brane, but the same analysis can be repeated for these situations. We will only highlight the important points in the derivation of the sum rules so that the results can be presented. For the case of the M2 and the M5-branes there are finite terms in the sum rules which arises due to the high frequency behaviour of the relevant Green's function. We show that these finite terms are due to expectation values of operators of dimension 1 and dimension 4 respectively. We then compute the structure constants of the R-currents with these operators form the finite terms in the sum rule and show that they indeed agree with that evaluated from the corresponding Witten diagrams.

4.1 M2-branes

The theory on the M2-brane has an $SO(8)$ R-symmetry. Thus there are at the most 4 chemical potentials corresponding to the Cartans of $SO(8)$ which can be turned on. The gravity dual of this solution is given in (B.1) and (B.2). As for the case of D3-brane, the object of interest is the R-current retarded Green's function. To obtain this Green's function holographically we need to study the fluctuations of the 4 gauge fields dual to the R-currents corresponding to the Cartan of $SO(8)$. These equations are given by

$$\phi^{i'''} + \left(\frac{f'}{f} + \frac{2H'_i}{H_i} - \frac{\mathcal{H}'}{\mathcal{H}} \right) \phi^{i''} + \frac{L^4 \omega^2}{r_+^2 f} \mathcal{H} \phi^{i'} - (1 + k_i) \mu_i \sum_{j=1}^4 \frac{L^2}{r_+^2} \frac{u^2}{H_i^2 f} \mu_j (1 + k_j) \phi^j = 0.
\tag{4.1}$$

Here the prime is the derivative with respect to

$$u = \frac{r_+}{r},
\tag{4.2}$$

the indices i now run from $1, \dots, 4$. The functions $f, H_i, \mu_i, \mathcal{H}$ are defined in (B.1). $\phi^i(u)$ is the fluctuation of the gauge field in the x direction for the Fourier mode with frequency ω . We have set the spatial momentum of the fluctuations to zero.

To obtain the equations for the fluctuations we have followed the same procedure as that in the case of the D3-brane. One can consistently turn on fluctuations of the gauge field in the x direction and the metric component in the xt direction. Then we obtain a constraint equation if one examines the fluctuation in the xu component of the Einstein equation. This can be used to eliminate the fluctuation in the metric and obtain the set of equations given in (4.1).

The important term in the sum rule for our analysis is the finite term in the $\omega \rightarrow i\infty$ limit. This can be obtained by the same analysis as that developed for the D3-brane. We first rescale the coordinates as

$$y = \lambda u, \quad i\lambda = \frac{L^2}{r_+} \omega. \quad (4.3)$$

Then the set of coupled equations in (4.1) becomes

$$\phi^{i'''} + \left(\frac{f'}{f} + \frac{2H'_i}{H_i} - \frac{\mathcal{H}'}{\mathcal{H}} \right) \phi^{i''} - \frac{\mathcal{H}}{f^2} \phi^{i'} - (1 + k_i) \mu_i \sum_{j=1}^4 \frac{L^2}{r_+^2} \frac{y^2}{\lambda^4 H_i^2 f} \mu_j (1 + k_j) \phi^j = 0. \quad (4.4)$$

where, now the derivatives are with respect to the variable y . Note that the equations decouple in the strict $\lambda \rightarrow \infty$ limit and they reduce to that of gauge fields in pure AdS_4 . Thus the leading term in the large λ limit is again identical to the zero temperature situation. The leading independent solutions are

$$\phi^i \sim e^{\pm y} \quad (4.5)$$

Demanding that the solution is well behaved at the origin of AdS picks out e^{-y} as the zero temperature solution. To obtain the asymptotic behaviour of the Green's function we define the variables

$$a(y) = \frac{\phi^{1'}(y)}{\phi^1(y)}, \quad b(y) = \frac{\phi^{2'}(y)}{\phi^2(y)}, \quad c(y) = \frac{\phi^{3'}(y)}{\phi^3(y)}, \quad d(y) = \frac{\phi^{4'}(y)}{\phi^4(y)}. \quad (4.6)$$

We can then set up a perturbative expansion of each of the above functions as series in $1/\lambda$. We will discuss the leading equations for the function $a(y)$. We first expand $a(y)$ as a series in $1/\lambda$ as

$$a(y) = a_0 + \frac{a_1}{\lambda} + \frac{a_2}{\lambda^2} + \dots \quad (4.7)$$

Substituting this expansion for a in the non-linear equations determined from (4.4) and matching powers of $1/\lambda$ we obtain the the following equations for the leading coefficients.

$$\begin{aligned} a'_0 + a_0^2 - 1 &= 0, \\ a'_1 + 2a_0 a_1 + 2k_1 a_0 + (k_1 + k_2 + k_3 + k_4) y &= 0 \end{aligned} \quad (4.8)$$

The two independent solutions for the first equation in (4.8) are

$$a_0^{(1)} = -1, \quad a_0^{(2)} = 1. \quad (4.9)$$

The second solution corresponds to the growing solution $\phi^i \sim e^y$. But as we have mentioned above, to have a well behaved solution at the origin in the zero temperature limit we must chose the first solution $a_0^{(1)} = -1$. Evaluating the leading correction to $a_0^{(1)}$ we obtain

$$a_1^{(1)} = \frac{1}{4}(-3k_1 + k_2 + k_3 + k_4) + \frac{y}{2}(k_1 + k_2 + k_3 + k_4). \quad (4.10)$$

The retarded Green's function is given by

$$G^i(\omega, T) = -\frac{1}{e^2} \lim_{r \rightarrow \infty} \frac{r^2 \phi^{i'}}{L^2 \phi^i} \Big|_{\phi_\infty^j=0, j \neq i} + G_{\text{counter}}(\omega, T), \quad (4.11)$$

with

$$\frac{1}{e^2} = \frac{N^{\frac{3}{2}} \sqrt{2}}{24\pi L^2}. \quad (4.12)$$

Substituting the expressions for fluctuations ϕ^1 in the $\omega \rightarrow i\infty$ limit we obtain

$$G^1(\omega, T)|_{\omega \rightarrow i\infty} = \frac{1}{e^2} \lim_{y \rightarrow 0} \frac{r_+}{L^2} \left(\lambda a_0^{(1)}(y) + a_1^{(1)}(y) \right) + G_{\text{counter}}(\omega, T). \quad (4.13)$$

From our discussion we see that the divergent term in the frequency is identical to the zero temperature limit of the the Green's function. This motivates us to define the regularized Green's function as

$$\delta G^1(\omega) = G^1(\omega, T) - G^1(\omega, 0) - \frac{r_+}{4e^2 L^2} (-3k_1 + k_2 + k_3 + k_4). \quad (4.14)$$

With this regularization, the Green's function satisfies the properties necessary to obtain the sum rule. Thus the sum rule for this Green's function is given by

$$\int_{-\infty}^{\infty} \frac{d\omega}{\pi\omega} (\rho^1(\omega) - \rho_{T=0}^1(\omega)) = \lim_{\omega \rightarrow 0} \omega \text{Im} \sigma^1(\omega) - \frac{r_+}{4e^2 L^2} (-3k_1 + k_2 + k_3 + k_4). \quad (4.15)$$

where $\rho^i = \text{Im} G^i$ and σ^i is the corresponding conductivity. Similarly, the sum rules for the other diagonal components of the Green's function are given by

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{d\omega}{\pi\omega} (\rho^2(\omega) - \rho_{T=0}^2(\omega)) &= \lim_{\omega \rightarrow 0} \omega \text{Im} \sigma^2(\omega) - \frac{r_+}{4e^2 L^2} (k_1 - 3k_2 + k_3 + k_4), \\ \int_{-\infty}^{\infty} \frac{d\omega}{\pi\omega} (\rho^3(\omega) - \rho_{T=0}^3(\omega)) &= \lim_{\omega \rightarrow 0} \omega \text{Im} \sigma^3(\omega) - \frac{r_+}{4e^2 L^2} (k_1 + k_2 - 3k_3 + k_4), \\ \int_{-\infty}^{\infty} \frac{d\omega}{\pi\omega} (\rho^4(\omega) - \rho_{T=0}^4(\omega)) &= \lim_{\omega \rightarrow 0} \omega \text{Im} \sigma^4(\omega) - \frac{r_+}{4e^2 L^2} (k_1 + k_2 + k_3 - 3k_4). \end{aligned} \quad (4.16)$$

As discussed in the case of the D3-brane let us now examine the terms in the OPE of the R-currents to understand the high frequency terms in the RHS of the sum rule. The OPE of two R-currents in a 3 dimensional CFT is given by

$$J_\mu^i(x)J_\nu^j(0) \sim \frac{\mathcal{C}\delta_{ij}I_{\mu\nu}(x)}{x^4} + \mathcal{A}_{\mu\nu}C_{ij}^{\hat{k}}\mathcal{O}_{\hat{k}}(0) + \mathcal{B}_{\mu\nu;k}^{ij;\rho}J_\rho^k(0) + \dots, \quad (4.17)$$

The R-currents have dimension 2 and therefore from a similar scaling analysis discussed for the case of the D3-branes we see that the finite terms in the high frequency limit arise from scalar operators of dimension 1 in the OPE. For operators of dimension 1, the regularized tensor \mathcal{A} is given by

$$\mathcal{A}_{\mu\nu}(s) = -2 \left(\partial_\mu \partial_\nu \frac{1}{s} + 4\pi \delta_{\mu\nu} \delta^3(s) \right). \quad (4.18)$$

Taking the Fourier transform of the OPE and going over to Minkowski space we obtain the following relation for the high frequency behaviour of the retarded Green's function

$$\lim_{\omega \rightarrow \infty} (G^i(\omega, T) - G^i(\omega, 0)) = 8\pi C_{ii}^{\hat{k}} \langle \mathcal{O}_{\hat{k}}(0) \rangle_T, \quad (4.19)$$

where $\mathcal{O}_{\hat{k}}$ are the operators of dimension 1 in the theory.

For the M2-brane theory, there are 3 operators of dimension 1 corresponding to the scalars $\vartheta^{\hat{i}}$ given in (B.4). Note that in this case the scalars obey alternate quantization. The expectation values of these operators in the charged M2-brane background is given by

$$\begin{aligned} \langle \mathcal{O}_1 \rangle &= \left(\frac{N^{\frac{3}{2}}\sqrt{2}}{24\pi} \right) \frac{r_+}{L^2} (k_1 + k_2 - k_3 - k_4), \\ \langle \mathcal{O}_2 \rangle &= \left(\frac{N^{\frac{3}{2}}\sqrt{2}}{24\pi} \right) \frac{r_+}{L^2} (k_1 - k_2 + k_3 - k_4), \\ \langle \mathcal{O}_3 \rangle &= \left(\frac{N^{\frac{3}{2}}\sqrt{2}}{24\pi} \right) \frac{r_+}{L^2} (k_1 - k_2 - k_3 + k_4). \end{aligned} \quad (4.20)$$

We can now rewrite the high frequency terms in the RHS of the sum rules given in (4.15) and (4.15) as

$$\begin{aligned} \delta G^1(0, T) &= \lim_{\omega \rightarrow 0} \omega \text{Im} \sigma^1(\omega) + \frac{1}{4L^2} (\langle \mathcal{O}_1 \rangle + \langle \mathcal{O}_2 \rangle + \langle \mathcal{O}_3 \rangle), \\ \delta G^2(0, T) &= \lim_{\omega \rightarrow 0} \omega \text{Im} \sigma^2(\omega) + \frac{1}{4L^2} (\langle \mathcal{O}_1 \rangle - \langle \mathcal{O}_2 \rangle - \langle \mathcal{O}_3 \rangle), \\ \delta G^3(0, T) &= \lim_{\omega \rightarrow 0} \omega \text{Im} \sigma^3(\omega) + \frac{1}{4L^2} (-\langle \mathcal{O}_1 \rangle + \langle \mathcal{O}_2 \rangle - \langle \mathcal{O}_3 \rangle), \\ \delta G^4(0, T) &= \lim_{\omega \rightarrow 0} \omega \text{Im} \sigma^4(\omega) + \frac{1}{4L^2} (-\langle \mathcal{O}_1 \rangle - \langle \mathcal{O}_2 \rangle + \langle \mathcal{O}_3 \rangle). \end{aligned} \quad (4.21)$$

Comparing the equation (4.19) with the high energy contribution in the sum rules in (4.21) we read out the following values for the structure constants

$$\begin{aligned}
C_{11}^{\hat{1}} &= \frac{-1}{32\pi L^2}, & C_{11}^{\hat{2}} &= \frac{-1}{32\pi L^2}, & C_{11}^{\hat{3}} &= \frac{-1}{32\pi L^2}, \\
C_{22}^{\hat{1}} &= \frac{-1}{32\pi L^2}, & C_{22}^{\hat{2}} &= \frac{1}{32\pi L^2}, & C_{22}^{\hat{3}} &= \frac{1}{32\pi L^2}, \\
C_{33}^{\hat{1}} &= \frac{1}{32\pi L^2}, & C_{33}^{\hat{2}} &= \frac{-1}{32\pi L^2}, & C_{33}^{\hat{3}} &= \frac{1}{32\pi L^2}, \\
C_{44}^{\hat{1}} &= \frac{1}{32\pi L^2}, & C_{44}^{\hat{2}} &= \frac{1}{32\pi L^2}, & C_{44}^{\hat{3}} &= \frac{-1}{32\pi L^2},
\end{aligned} \tag{4.22}$$

We will now compare these values for the structure constants to that evaluated using the Witten's prescription. Following the same procedure as in the case of the D3-brane we obtain

$$C_{ii}^{\hat{j}} = -a_j^i \frac{\mathcal{K}_3(1)}{\mathcal{G}_3(1)} \tag{4.23}$$

where the a_j^i refer to the \hat{j} th component of the vector \vec{a}^i given in (B.4). The normalization constants in (4.23) are read out from (A.7) and (A.10). Substituting these constants, we indeed obtain the structure constants given in (4.22).

4.2 M5-brane

The theory on the M7-brane has a $SO(5)$ R-symmetry. Therefore there are at most 2 chemical potentials corresponding to the 2 Cartans that can be turned on. The gravity background dual to this system is given in (B.6) and (B.7). To obtain the R-current retarded correlators we study the fluctuations of the two gauge fields in this background. They obey the following equations

$$\phi^{i'''} + \left(\log \frac{H_i^2 f}{z^3 \mathcal{H}} \right)' \phi^{i''} + \frac{L^4 \omega^2}{r_+^2} \mathcal{H} f^2 \phi^i - \frac{16 \mu_i z^8 (1 + k_i)}{H_i^2 f} \sum_{j=1}^2 \mu_j (1 + k_j) \phi^j = 0, \tag{4.24}$$

where the prime is with respect to z defined as

$$z = \frac{r_+}{r}. \tag{4.25}$$

The functions $f, \mathcal{H}, H_i, \mu_i$ are defined in (B.6) and (B.8). Here just as in the case of the D3-brane and the M2-brane, ϕ^i are the fluctuations of the gauge field in the x direction for the Fourier mode with frequency ω . We have set all fluctuations except that of the metric in the xt direction to zero. This metric fluctuation is eliminated using the constraints to obtain the equations in (4.24).

To study the $\omega \rightarrow i\infty$ limit we rescale the coordinates as

$$y = \lambda z, \quad i\lambda = \frac{L^2}{r_+} \omega. \tag{4.26}$$

Then the equations in (4.24) become

$$\phi^{i'''} + \left(\log \frac{H_i^2 f}{y^3 \mathcal{H}} \right)' \phi^{i'} - \lambda^2 \mathcal{H} f^2 \phi^i - \frac{16 \mu_i y^8 (1 + k_i)}{\lambda^{10} H_i^2 f} \sum_{j=1}^2 \mu_j (1 + k_j) \phi^j = 0, \quad (4.27)$$

where now the derivatives are with respect to y . Here again the equations decouple in the limit $\lambda \rightarrow \infty$ and reduce to the zero temperature pure AdS_7 situation. To obtain the large frequency behaviour of the Green's function we define

$$a = \frac{\phi^{1'}(y)}{\phi^1(y)}, \quad b = \frac{\phi^{2'}(y)}{\phi^2(y)}. \quad (4.28)$$

One can then set up a perturbative expansion for each of these functions in powers of $1/\lambda$. We will briefly discuss the leading equations for the function $a(y)$. We expand this function as

$$a(y) = a_0 + \frac{1}{\lambda^4} a_1 + \frac{1}{\lambda^6} a_2 + \dots \quad (4.29)$$

Substituting this expansion in the non-linear equation for a determined from (4.27), we obtain the following equations for the leading orders

$$\begin{aligned} a_0' + a_0^2 - \frac{3}{y} a_0 - 1 &= 0, \\ a_1' + 2a_0 a_1 - \frac{3}{y} a_1 + 8y^3 k_1 a_0 + y^4 (k_1 + k_2) &= 0. \end{aligned} \quad (4.30)$$

The two independent solutions for the first equation in 4.30 are

$$a_0^{(1)} = -\frac{K_1(y)}{K_2(y)}, \quad a_0^{(2)} = \frac{I_1(y)}{I_2(y)}. \quad (4.31)$$

Demanding that the solution is well behaved at the origin, $y \rightarrow \infty$, at the leading order in λ picks out the first solution $a_0^{(1)}$. The leading correction to $a_0^{(1)}$ is given by

$$a_1^{(1)} = -4k_1 y^3 - (k_1 + k_2) \frac{y^5}{10} \left(1 - \frac{K_3^2(y)}{K_2^2(y)} \right). \quad (4.32)$$

The retarded Green's function is given by

$$G^i(\omega, T) = -\frac{1}{e^2} \lim_{r \rightarrow \infty} \frac{r^5 \phi^{i'}}{L^5 \phi^i} \Big|_{\phi_\infty^j=0, j \neq i} + G_{\text{counter}}(\omega, T), \quad (4.33)$$

with

$$\frac{1}{e^2} = \frac{N^3}{6\pi^3 L^5}. \quad (4.34)$$

Substituting the expressions for fluctuations ϕ^1 in the $\omega \rightarrow i\infty$ limit we obtain

$$G^1(\omega, T)|_{\omega \rightarrow i\infty} = \frac{1}{e^2} \lim_{y \rightarrow 0} \frac{r_+^5}{L^5 y^3} \left(\lambda^4 a_0^{(1)}(y) + a_1^{(1)}(y) \right) + G_{\text{counter}}^1(\omega, T). \quad (4.35)$$

As discussed for the D3-brane and the M2-brane we define the regularized Green's function as

$$\delta G^i(\omega, T) = G^1(\omega, T) - G^1(\omega, 0) - \frac{r_+^4}{e^2 L^5} \frac{4}{5} (-3k_1 + 2k_2). \quad (4.36)$$

The constant term which is subtracted is obtained by examining the limit of the first order correction $a_1^{(1)}$ given in (4.32). Following the same arguments developed for the D3-brane case, we can apply Cauchy's theorem for the regulated Green's function to derive the following sum rule

$$\int_{-\infty}^{\infty} \frac{d\omega}{\pi\omega} (\rho^1(\omega) - \rho_{T=0}^1(\omega)) = \lim_{\omega \rightarrow 0} \omega \text{Im} \sigma^1(\omega) - \frac{r_+^4}{e^2 L^5} \frac{4}{5} (-3k_1 + 2k_2), \quad (4.37)$$

where again the spectral density $\rho^i = \text{Im} G^i$ and σ^i is the corresponding conductivity. Carrying out the same procedure for the second component of the gauge field we obtain the sum rule

$$\int_{-\infty}^{\infty} \frac{d\omega}{\pi\omega} (\rho^2(\omega) - \rho_{T=0}^2(\omega)) = \lim_{\omega \rightarrow 0} \omega \text{Im} \sigma^2(\omega) - \frac{r_+^4}{e^2 L^5} \frac{4}{5} (2k_1 - 3k_2). \quad (4.38)$$

To understand the terms due to the high frequency limit of the Green's function we examine the OPE of 2 R-currents in a 6 dimensional CFT. This is given by

$$J_\mu^i(x) J_\nu^j(0) \sim \frac{\mathcal{C} \delta_{ij} I_{\mu\nu}(x)}{x^{10}} + \mathcal{A}_{\mu\nu} C_{ij}^{\hat{k}} \mathcal{O}_{\hat{k}}(0) + \mathcal{B}_{\mu\nu; k}^{ij; \rho} J_\rho^k(0) + \dots \quad (4.39)$$

In this case, the R-currents have dimension 5, and by a simple scaling analysis it is easy to see that the finite terms at high frequency arise from dimension 4 operators in the OPE. For operators of dimension 4 the regularized tensor \mathcal{A} is given by

$$\mathcal{A}_{\mu\nu}(s) = -2 \left(\frac{1}{4} \partial_\mu \partial_\nu \frac{1}{s^4} + \pi^3 \delta_{\mu\nu}(s) \right). \quad (4.40)$$

Taking the Fourier transform of the OPE and going over to Minkowski space we obtain the following relation for the high frequency behaviour of the retarded Green's function

$$\lim_{\omega \rightarrow \infty} (G^i(\omega, T) - G^i(\omega, 0)) = 2\pi^3 C_{ii}^{\hat{k}} \langle \mathcal{O}_{\hat{k}}(0) \rangle_T, \quad (4.41)$$

where $\mathcal{O}_{\hat{k}}$ are the operators of dimension 4 in the theory.

For the M5-branes, the operators of dimension 4 correspond to the two scalars $\vartheta^{\hat{i}}$ in the theory. The expectation values of these scalars in the background (B.7) are given by

$$\begin{aligned} \langle \mathcal{O}_1 \rangle &= \frac{N^3 r_+^4}{6\pi^2 L^8} \frac{2}{\sqrt{2}} (k_1 - k_2), \\ \langle \mathcal{O}_2 \rangle &= \frac{N^3 r_+^4}{6\pi^2 L^8} \frac{2}{\sqrt{10}} (k_1 + k_2). \end{aligned} \quad (4.42)$$

Rewriting the high frequency contribution in the RHS of the sum rules given in (4.37) and (4.37) we obtain

$$\begin{aligned}\delta G^1(0) &= \lim_{\omega \rightarrow 0} \omega \text{Im} \sigma^1(\omega) + \frac{\sqrt{2}}{L^2} \langle \mathcal{O}_1 \rangle + \frac{\sqrt{2}}{L^2 \sqrt{5}} \langle \mathcal{O}_2 \rangle, \\ \delta G_J^2(0, T) &= \lim_{\omega \rightarrow 0} \omega \text{Im} \sigma^2(\omega) - \frac{\sqrt{2}}{L^2} \langle \mathcal{O}_1 \rangle + \frac{\sqrt{2}}{L^2 \sqrt{5}} \langle \mathcal{O}_2 \rangle.\end{aligned}\tag{4.43}$$

We now extract the structure constants by comparing (4.41) with the high frequency behaviour of the Green's function given in (4.43). This results in the following values for the structure constants.

$$\begin{aligned}C_{11}^{\hat{1}} &= -\frac{\sqrt{2}}{2\pi^3 L^2}, & C_{11}^{\hat{2}} &= -\frac{1}{2\pi^3 L^2} \sqrt{\frac{2}{5}}, \\ C_{22}^{\hat{1}} &= \frac{\sqrt{2}}{2\pi^3 L^2}, & C_{22}^{\hat{2}} &= -\frac{1}{2\pi^3 L^2} \sqrt{\frac{2}{5}}.\end{aligned}\tag{4.44}$$

Let us now compare the structure constants obtained from the sum rule in (4.44) with that using the conventional AdS/CFT prescription. As discussed for the case of the D3-brane and the M2-brane these are given by

$$C_{ii}^{\hat{j}} = -a_j^i \frac{\mathcal{K}_6(4)}{\mathcal{G}_6(4)}\tag{4.45}$$

where the a_j^i refer to the \hat{j} th component of the vector \vec{a}^i given in (B.10). The normalization constants in (4.45) are read out from (A.7) and (A.10). Substituting these constants, we again obtain the structure constants which were obtained from the sum rule that are given in (4.44).

5. Conclusions

We have derived the R-current spectral sum rules holographically for the theories dual to that of the D3-brane, M2-brane and M5-brane at finite chemical potential. The sum rule as we have seen is a consequence of the analytic behaviour of the corresponding retarded Green's function. It contains information about both long distance hydrodynamical property as well as the short distance property of the theory. We examine the term which occur due to the short distance effects and obtained the relevant structure constants. As a consistency check, this was then compared to that obtained using the conventional Witten diagrams and shown to agree.

Thus we see sum rules provide crucial information of the theory and provide important constraints on the spectral densities. It will be useful to examine the putative holographic duals of QCD in the literature and obtain various sum rules. This may provide tight constraints on the validity of the holographic models when compared with expectations of real QCD sum rules.

Anomalies are other phenomena in any quantum field theory which are quite robust and present both at short distance and long distances in the theory. It will be interesting to determine spectral densities sensitive to the $U(1)^3$ anomaly present in $\mathcal{N} = 4$ Yang-Mills hydrodynamics [31] and obtain sum rules which capture this anomaly.

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A. 2pt and 3pt functions from Witten diagrams

The $\langle JJO \rangle$ correlator

In this section we evaluate the three point function of two currents with a scalar operator of dimension Δ for a conformal field theory in d dimensions using the AdS/CFT correspondence.

$$\frac{1}{4e^2} \int d^{d+1}w \sqrt{g} g^{\mu\rho} g^{\nu\sigma} \vartheta \partial_{[\mu} A_{\nu} \partial_{\rho]} A_{\sigma}. \quad (\text{A.1})$$

Here ϑ is the scalar dual to the operator of dimension Δ and the gauge field A_{μ} is dual to a current. The current in the dual theory has the dimension $d - 1$. The metric for AdS_{d+1} is given by

$$ds^2 = \frac{L^2}{z_0^2} (d\vec{x}^2 + dz_0^2). \quad (\text{A.2})$$

We have chosen the radius of AdS_{d+1} as L in agreement with the asymptotic form of the metrics given for the charged D3-brane, M2-brane and the M5-brane which are given in (2.3), (B.1) and (B.6). The bulk to boundary Green's function for the scalar is given by [19]

$$K_{\Delta}(z_0, \vec{z}, \vec{x}) = \frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}} \Gamma(\Delta - \frac{d}{2})} \left(\frac{z_0}{z_0^2 + (\vec{z} - \vec{x})^2} \right)^{\Delta}. \quad (\text{A.3})$$

Note that though here we are following the normalization in which conflicts with a Ward identity [19] we rescale the final results by using the normalization given in [29]. The bulk to boundary Green's function for the vectors is given by [19]

$$G_{\mu i}(z, \vec{x}) = C^d \left(\frac{z_0}{(z - \vec{x})^2} \right)^{d-2} \partial_{\mu} \left(\frac{(z - \vec{x})_i}{(z - \vec{x})^2} \right), \quad (\text{A.4})$$

where

$$C_d = \frac{\Gamma(d)}{2\pi^{\frac{d}{2}}\Gamma(\frac{d}{2})}. \quad (\text{A.5})$$

Now substituting these in the interaction given in (A.1) and evaluating the integral using the methods of [19] we obtain the following expression for the three point function

$$\langle J_\mu(x) J_\nu(y) \mathcal{O}(z) \rangle = -\mathcal{K}^d(\Delta) \left(\frac{-(\Delta - 2(d-1))I_{\mu\nu}(x-y) - \Delta I_{\mu\rho}(x-z)I_\rho(z-y)}{(x-y)^{2(d-1)-\Delta}(x-z)^\Delta(y-z)^\Delta} \right). \quad (\text{A.6})$$

The normalization constant \mathcal{K} in (A.6) is given by

$$\begin{aligned} \mathcal{K}_d(\Delta) &= \frac{L^{d-3}}{e^2} \frac{(d-2)^2}{16\pi^{\frac{d}{2}}} \left(\frac{\Gamma(\frac{\Delta+d}{2}-1)\Gamma(d-1-\frac{\Delta}{2})\Gamma(\frac{\Delta}{2}+1)\Gamma(\frac{\Delta}{2})}{\Gamma(\Delta)(\Gamma(\frac{d}{2}))^2} \right) \\ &\quad \times \left(\frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}}\Gamma(\Delta-\frac{d}{2}+1)} \right). \end{aligned} \quad (\text{A.7})$$

Note that we have finally divided by $\Delta - d/2$ following the normalizations of [29]. Note that the expression in (A.7) is valid all the way down to the unitary bound $d/2 - 1$ and thus includes scalars which saturate the Breitenlohner-Freedman bound.

The $\langle \mathcal{O}\mathcal{O} \rangle$ correlator

The two point function of an scalar operator \mathcal{O} of conformal dimension Δ in a d dimensional theory is obtained by considering the following Lagrangian in AdS_{d+1} .

$$S = \frac{1}{2e^2} \int d^{d+1}z \sqrt{g} (g^{\mu\nu} \partial_\mu \vartheta \partial_\nu \vartheta + m^2 \vartheta^2). \quad (\text{A.8})$$

Substituting the bulk to boundary Green's function for the scalar given in (A.3) and evaluating the two point function using the standard AdS/CFT rules and following [19] we obtain

$$\langle \mathcal{O}(x) \mathcal{O}(y) \rangle = \mathcal{G}_d(\Delta) \frac{1}{(x-y)^{2\Delta}}, \quad (\text{A.9})$$

where the normalization \mathcal{G} is given by

$$\mathcal{G}_d(\Delta) = \frac{L^{d-1}}{e^2} \frac{2\Gamma(\Delta)}{\pi^{\frac{d}{2}}\Gamma(\Delta-\frac{d}{2}+1)}. \quad (\text{A.10})$$

Note that here we have divided by $(\Delta - d/2)^2$ following the normalizations of [29]. The expression in (A.10) agrees with that given in equation (2.21) of [29]. Here again the expression in (A.10) is valid all the way down to the unitary bound.

B. The M2 and M5-brane solution

M2-brane solution

The metric and the gauge field for the R-charged M2-brane with all the four charged turned on is given by

$$ds_4^2 = \frac{16(\pi T_0 L)^2}{9u^2} \mathcal{H}^{1/2} \left(-\frac{f}{\mathcal{H}} dt^2 + dx^2 + dz^2 \right) + \frac{L^2}{f u^2} \mathcal{H}^{1/2} du^2, \quad (\text{B.1})$$

$$A_t^i = \frac{4}{3} \pi T_0 L \sqrt{k_i \prod_{i=1}^4 (1 + k_i)} \frac{u}{H_i}, \quad u = \frac{r_+}{r}, \quad H_i = 1 + k_i u,$$

$$\mathcal{H} = \prod_{i=1}^4 H_i, \quad f = \mathcal{H} - \prod_{i=1}^4 (1 + k_i) u^3, \quad T_0 = \frac{3r_+}{4\pi L^2}.$$

The scalars are given by

$$X^i = \frac{\mathcal{H}^{1/4}}{H_i(u)}. \quad (\text{B.2})$$

The four scalars are not independent and are constrained by $X_1 X_2 X_3 X_4 = 1$. The charged M2-brane is the solution of the equation of motion of the following action

$$S = \frac{N^{3/2} \sqrt{2}}{24\pi L^2} \int d^4 x \sqrt{-g} \mathcal{L}, \quad (\text{B.3})$$

$$\mathcal{L} = R - \frac{1}{2} (\partial \vec{\vartheta})^2 + V(\vartheta) - \frac{1}{4} \sum_{i=1}^4 e^{\vec{a}^i \cdot \vec{\vartheta}} (F^i)^2.$$

where the fields four fields X_i are related to the three independent fields $\vec{\phi}_i = (\theta_1, \theta_2, \theta_3)$ by

$$X_i = \exp\left(-\frac{1}{2} \vec{a}^i \cdot \vec{\vartheta}\right), \quad (\text{B.4})$$

$$\vec{a}^1 = (1, 1, 1), \quad \vec{a}^2 = (1, -1, -1), \quad \vec{a}^3 = (-1, 1, -1), \quad \vec{a}^4 = (-1, -1, 1).$$

The scalar potential is given by

$$V(\vartheta) = \frac{2}{L^2} (\cosh \vartheta_1 + \cosh \vartheta_2 + \cosh \vartheta_3). \quad (\text{B.5})$$

The chemical potentials are given by

$$\mu_i = \frac{4\pi T_0}{3} L \frac{1}{1 + k_i} \sqrt{k_i \prod_{i=1}^4 (1 + k_i)}.$$

M5-brane

The metric and the gauge field for the R-charged M5-brane with all the charges turned on is given by [32]

$$\begin{aligned}
ds_7^2 &= \frac{4(\pi T_0 L)^2}{9u} \mathcal{H}^{1/5} \left(-\frac{f}{\mathcal{H}} dt^2 + dx_1^2 + \cdots + dx_4^2 + dz^2 \right) + \frac{L^2}{4fu^2} \mathcal{H}^{1/5} du^2, \\
A_t &= \frac{2}{3} \pi T_0 \sqrt{2k_i \prod_{i=1}^2 (1+k_i)} \frac{u^2}{H_i}, \quad H_i = 1 + k_i u^2, \\
T_0 &= \frac{3r_+}{2\pi L^2}, \quad u = \frac{r_+^2}{r^2} \\
H_i &= 1 + k_i u^2, \quad \mathcal{H} = \prod_{i=1}^2 H_i, \quad f = \mathcal{H} - \prod_{i=1}^2 (1+k_i) u^3.
\end{aligned} \tag{B.6}$$

The background solution for the two scalars are given by

$$X^i = \frac{\mathcal{H}^{2/5}}{H_i(u)}. \tag{B.7}$$

The chemical potentials for the solution in (B.6) are given by

$$\mu_i = \frac{2}{3} \pi T_0 \sqrt{2k_i \prod_{i=1}^2 (1+k_i)} \frac{1}{1+k_i}. \tag{B.8}$$

The charged M5-brane background is a solution of the equation of motion of the following action

$$\begin{aligned}
S &= \frac{N^3}{6\pi^3 L^5} \int d^7x \sqrt{-g} \mathcal{L}, \\
\mathcal{L} &= R - \frac{1}{2} (\partial \vec{\vartheta})^2 + V(\phi) - \frac{1}{4} \sum_{i=1}^2 e^{\vec{a}^i \cdot \vec{\vartheta}} (F^i)^2.
\end{aligned} \tag{B.9}$$

Where the two scalar fields X_i 's are related to $\vec{\vartheta} = (\vartheta_1, \vartheta_2)$ by

$$\begin{aligned}
X^i &= e^{-\frac{1}{2} \vec{a}^i \cdot \vec{\vartheta}}, \\
\vec{a}^1 &= (\sqrt{2}, \sqrt{\frac{2}{5}}), \quad \vec{a}^2 = (-\sqrt{2}, \sqrt{\frac{2}{5}}).
\end{aligned} \tag{B.10}$$

Note that the two scalars X_i are independent here, unlike the situation in the case of the D3-branes and M2-branes. The scalar potential V is given by

$$V = \frac{4}{L^2} \left(-4X_1 X_2 - 2X_1^{-1} X_2^{-2} - 2X_1^{-2} X_2^{-1} + \frac{1}{2} (X_1 X_2)^{-4} \right). \tag{B.11}$$

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