# Derived equivalences in $n$-angulated categories

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**Abstract**

In this paper, we consider $n$-perforated Yoneda algebras for $n$-angulated categories, and show that, under some conditions, $n$-angles induce derived equivalences between the quotient algebras of $n$-perforated Yoneda algebras. This result generalizes some results of Hu, König and Xi. And it also establishes a connection between higher cluster theory and derived equivalences. Namely, in a cluster tilting subcategory of a triangulated category, an Auslander-Reiten $n$-angle implies a derived equivalence between two quotient algebras. This result can be compared with the fact that an Auslander-Reiten sequence suggests a derived equivalence between two algebras which was proved by Hu and Xi.

## 1 Introduction

Derived categories and derived equivalences occur widely in a number of mathematical fields. For example, algebraic geometry [3, 4, 28], differential equation [32, 21], the representation theory of algebras [7, 33]. In modern representation theory of finite groups, the famous Abelian defect conjecture of Broué is actually to predicate a derived equivalence between two block algebras. As is known, derived equivalences preserve many homological properties of algebras such as the number of simple modules, the finiteness of global dimension and finitistic dimension, the algebraic K-theory and Hochschild (co)homological groups (see [6, 10, 22, 30, 31, 29]). In this sense, derived equivalences provide us a bridge to compare properties of different algebras, and are helpful for us to understand some properties of algebras through the other ones. One of the fundamental problems on the study of derived equivalences of rings is

**How to construct derived equivalences between rings?**

Richard gave a theoretical solution to this problem which is well known as the Morita theorem for derived categories [30] (see also Keller [23]). The Richard's theorem for derived categories is that for two rings $A$ and $B$, the derived categories $D^b(A\text{-Mod})$ and $D^b(B\text{-Mod})$ are equivalent as triangulated categories if and only if there exists a special complex $T^\bullet$ in $D^b(A\text{-Mod})$, called " tilting complex", such that $B$ is the endomorphism ring of $T^\bullet$. However, it is difficult to construct all tilting complexes explicitly. And there are so many obstacles to determine the endomorphism ring of a complex. Consequently, it is necessary to give a systematic way to construct derived equivalences between rings.

In order to construct derived equivalences, one strategy is to develop a practical technique which can produce new derived equivalences from given ones. In [30, 31], Rickard used tensor product and trivial extension to produce derived equivalences. These results were generalized by Ladkani in the sense of triangular matrix ring arising from extension of tilting modules [25] and componentwise tensor products [26]. In [16], Hu and Xi presented a method to construct new derived equivalences between these $\Phi$-Auslander Yoneda algebras, or their quotient algebras, from given almost $\nu$-stable derived equivalences.

Another strategy is trying to construct derived equivalences from certain sequences. Recently, Hu and Xi introduced $\mathcal{D}$-split sequences and showed that each $\mathcal{D}$-split sequence gives rise to a derived equivalence via a tilting module [15]. Thus, every Auslander-Reiten sequence is a $\mathcal{D}$-split sequence

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and induces a derived equivalence via a BB-tilting module. This beautiful result presents a relation between Auslander-Reiten theory and derived equivalences. And later, Hu, König and Xi generalized the result in the context of triangulated categories, adding higher extensions and replacing the shift functor by any other auto-equivalence of triangulated categories [14]. Note that the derived equivalences are induced by tilting complexes of length 2. Meanwhile, Ladkani [27] and Dugas [9] discussed $\mathcal{D}$-split sequences in the version of mutations of algebras and algebraic triangulated categories, respectively.

In [11], Geiss, Keller and Oppermann introduced $n$-angulated categories which occur widely in cluster tilting theory and are closely related to algebraic geometry and string theory. A natural question is how to construct derived equivalences in $n$-angulated categories?

In this paper, we give an affirmative answer to this question. We construct derived equivalences in the context of $n$-angulated categories and generalize some results of Hu, König and Xi in [14]. By the result of Geiss, Keller, Oppermann [11], every $(n-2)$-cluster tilting subcategory which is closed under $\Sigma^{n-2}$ is an $n$-angulated category. Thus, we can construct derived equivalences which are induced by tilting complex of arbitrary length. This result generalizes the main result of Hu, König and Xi in [14]. At the same time, there is a high dimensional version of the fact that Auslander-Reiten sequences suggest a derived equivalence between two algebras which was proved in [15]. Namely, in some cluster tilting subcategory, any Auslander-Reiten $n$-angle implies a derived equivalence between two quotient algebras.

In order to describe the main result precisely, we fix some notations first. Let $R$ be a fixed commutative Artin ring, and let $k$ be a fixed field. Let $\mathcal{F}$ be an $n$-angulated $R$-category with suspension functor $\Sigma$, and let $X$ be an object in $\mathcal{F}$. Suppose that $\mathcal{F}$ has split idempotents. Let $\Phi$ be an admissible subset of $\mathbb{Z}$. Then we can define $n$-perforated Yoneda algebra $\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(X) := \oplus_{i\in\Phi}\mathrm{Hom}_{\mathcal{F}}(X, F^iX)$. Its multiplication is defined in a natural way. The left (right) $(\mathrm{add}(M), F, \Phi)$-approximation is extension of general approximation in the sense of Auslander and smalø, adding higher extension. For more details, we refer readers to section 2. The objects of $\mathscr{X}_{\mathcal{F}}^{\mathrm{F},\Phi}(M)$ and $\mathscr{Y}_{\mathcal{F}}^{\mathrm{F},\Phi}(\mathrm{M})$ satisfy some properties of orthogonal, i.e.,

$$\mathscr{X}_{\mathcal{F}}^{\mathrm{F},\Phi}(M) := \{X \in \mathcal{F} \mid \mathrm{Hom}_{\mathcal{F}}(X, F^iM) = 0 \text{ for all } i \in \Phi/\{0\}\}$$

$$\mathscr{Y}_{\mathcal{F}}^{\mathrm{F},\Phi}(\mathrm{M}) := \{\mathrm{Y} \in \mathcal{F} \mid \mathrm{Hom}_{\mathcal{F}}(\mathrm{M}, \mathrm{F}^{\mathrm{i}}\mathrm{Y}) = 0 \text{ for all } \mathrm{i} \in \Phi/\{0\}\}.$$

The sets $I$ and $J$ are ideals of $\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(X)$ and $\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(Y)$, respectively (see section 3 for details). The main result in this paper is the following:

**Theorem 1.1.** *Let $\Phi$ be an admissible subset of $\mathbb{Z}$, and let $\mathcal{F}$ be an $n$-angulated $R$-category with an auto-equivalence $F$. Suppose that $X \xrightarrow{\alpha_1} M_1 \xrightarrow{\alpha_2} M_2 \to \cdots \to M_{n-2} \xrightarrow{\alpha_{n-1}} Y \xrightarrow{\alpha_n} \Sigma X$ is an $n$-angle in $\mathcal{F}$ such that $\alpha_1 : X \to M_1$ is a left $(\mathrm{add}(M), F, \Phi)$-approximation of $X$ and $\alpha_{n-1} : M_{n-2} \to Y$ is a right $(\mathrm{add}(M), F, -\Phi)$-approximation of $Y$. If $X \in \mathscr{Y}_{\mathcal{F}}^{\mathrm{F},\Phi}(\mathrm{M})$ and $Y \in \mathscr{X}_{\mathcal{F}}^{\mathrm{F},\Phi}(M)$, then $\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(X \oplus M)/I$ and $\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(M \oplus Y)/J$ are derived equivalent.*

This theorem extends the main result of Hu, König and Xi in [11]. The following corollary establishes a connection between higher cluster theory and derived equivalences.

**Corollary 1.2.** *Let $\mathcal{T}$ be a Krull- Schmidt triangulated $k$-category with shift functor $\Sigma_3$, and let $\mathcal{S}$ be an $(n-2)$-cluster tilting subcategory of $\mathcal{T}$, which is closed under $\Sigma_3^{n-2}$. Suppose that*

$$X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} X_3 \to \cdots \to X_n$$

*is an Auslander-Reiten $n$-angle in $\mathcal{S}$ and $X_1, X_n \notin \oplus_{i=2}^{n-1} X_i$. Then the two rings $\mathrm{End}_{\mathcal{S}}(\oplus_{i=1}^{n-1} X_i)/I$ and $\mathrm{End}_{\mathcal{S}}(\oplus_{i=2}^{n} X_i)/J$ are derived equivalent, where $I, J$ are defined as in Theorem 1.1.*

This paper is organized as follows: In section 2, we make a preparation for the proof of the main result. We fix some notations and recall some basic definitions. In section 3, we give the proof of the main result and deduce some consequences of the main result. In section 4, we display an example to illustrate our main result.

## 2 Preliminaries

In this section, we will recall some basic definitions and facts which are needed in our proofs.

### 2.1 Notations and conventions

Throughout this paper, $R$ is a fixed commutative Artin ring with identity, and $k$ is a fix field.

Let $\mathcal{C}$ be an additive category. For an object $X$ in $\mathcal{C}$, we denote by $\mathrm{add}(X)$ the full subcategory of $\mathcal{C}$ consisting of all direct summands of finite direct sums of $X$. For two morphisms $f: X \to Y$ and $g: Y \to Z$ in $\mathcal{C}$, we write $fg$ for their composition which is a morphism from $X$ to $Z$. For two functors $F: \mathcal{C} \to \mathcal{D}$ and $G: \mathcal{D} \to \mathcal{E}$, we write $GF$ for the composition instead of $FG$.

Let $\mathcal{C}$ be an additive category with an endo-functor $F: \mathcal{C} \to \mathcal{C}$. Let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$, and let $\Phi$ be a non-empty subset of $\mathbb{N}$. If $F$ has an inverse, then $\Phi$ can be chosen to be a subset of $\mathbb{Z}$. Let $X$ be a object of $\mathcal{C}$. A morphism $f: X \to D$ in $\mathcal{C}$ is called a *left cohomological $\mathcal{D}$-approximation* of $X$ with respect to $(F,\Phi)$ (or left $(\mathcal{D},F,\Phi)$-approximation of $X$) if $D \in \mathcal{D}$, and for any morphism $g: X \to F^i(D')$ with $D' \in \mathcal{D}$ and $i \in \Phi$, there is a morphism $g': D \to F^i(D')$ such that $g = fg'$. Note that $F^0 = id_{\mathcal{C}}$. Dually, we have the notion of *right cohomological $\mathcal{D}$-approximation* of $X$ (or right $(\mathcal{D},F,\Phi)$-approximation of $X$) if for any $i \in \Phi$ and any morphism $g: F^iD' \to X$ with $D' \in \mathcal{D}$, there is a morphism $g': F^iD' \to D$ such that $g = g'f$ (see [14]). In particular, if $\Phi = \{0\}$, then left (resp., right)-$(\mathcal{D},F,\Phi)$-approximation of $X$ is left (resp., right) $\mathcal{D}$-approximation of $X$. The subcategory $\mathcal{D}$ is called *contravariantly finite* subcategory of $\mathcal{C}$ if any object $Y$ in $\mathcal{C}$ has a right $\mathcal{D}$-approximation. Dually, a covariantly finite subcategory of $\mathcal{C}$ is defined. The subcategory $\mathcal{D}$ is called *functorially finite* of $\mathcal{C}$ if $\mathcal{D}$ is contravariantly finite and covariantly finite in $\mathcal{C}$. We denote by $J_{\mathcal{C}}$ the Jacobson radical of $\mathcal{C}$. Let $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ be a morphism. We call $f$ a *sink map* of $Y$ if $f$ satisfies the following conditions: (1) if $g: X \to X$ satisfies $gf = f$, then $g$ is an automorphism. (2) $f \in J_{\mathcal{C}}$ and

$$\mathrm{Hom}_{\mathcal{C}}(-,X) \xrightarrow{\cdot f} J_{\mathcal{C}}(-,Y) \to 0$$

is exact as functors on $\mathcal{C}$. Dually, a source map is defined (see [20]).

Given an $R$-algebra $A$, we denote the opposite algebra of $A$ by $A^{op}$. By an $A$-module we mean a unitary left $A$-module; the category of all (resp., finitely generated) $A$-modules is denoted by $A$-Mod (resp., $A$-mod), the full subcategory of $A$-Mod consisting of all (resp., finitely generated) projective modules is denoted by $A$-Proj (resp., $A$-proj). Similarly, the full subcategory of $A$-Mod consisting of all (resp., finitely generated) injective $A$-modules is denoted by $A$-Inj (resp., $A$-inj). An algebra $A$ is called an Artin $R$-algebra if $A$ is finitely generated as an $R$-module. Let $A$ be an Artin $R$-algebra, we denote by $D$ the usual duality on $A$-mod. The functor $\nu_A := D\mathrm{Hom}_A(-,{}_AA): A\text{-proj} \to A\text{-inj}$ is Nakayama functor. We denote the syzygy functor by $\Omega$. Namely, for an $A$-module, we denote the first syzygy of $M$ by $\Omega_A(M)$. The stable category $A$-$\underline{\mathrm{mod}}$ is a quotient category of $A$-mod. The objects of $A$-$\underline{\mathrm{mod}}$ are the objects of $A$-mod. Let $X,Y$ be in $A$-mod. The homomorphism set $\underline{\mathrm{Hom}}(X,Y)$ is $\mathrm{Hom}(X,Y)$ modulo the submodule generated by homomorphism which can factorize through some projective $A$-module.

Let $A$ be an Artin algebra. A *complex* $X^\bullet = (X^i, d_X^i)$ of $A$-modules is a sequence of $A$-modules and $A$-module homomorphisms $d_X^i: X^i \to X^{i+1}$ such that $d_X^i d_X^{i+1} = 0$ for all $i \in \mathbb{Z}$. A *morphism* $f^\bullet: X^\bullet \to Y^\bullet$ between two complexes $X^\bullet$ and $Y^\bullet$ is a collection of homomorphisms $f^i: X^i \to Y^i$ of $A$-modules such that $f^i d_Y^i = d_X^i f^{i+1}$. The morphism $f^\bullet$ is said to be *null-homotopic* if there exists a homomorphism $h^i: X^i \to Y^{i-1}$ such that $f^i = d_X^i h^{i+1} + h^i d_Y^{i-1}$ for all $i \in \mathbb{Z}$. A complex $X^\bullet$ is called *bounded below* if $X^i = 0$ for all but finitely many $i < 0$, *bounded above* if $X^i = 0$ for all but finitely many $i > 0$, and *bounded* if $X^\bullet$ is bounded below and above. We denote by $C(A)$ (resp., $C(A\text{-Mod})$) the category of complexes of finitely generated (resp., all) $A$-modules. The homotopy category $K(A)$ is quotient category of $C(A)$ modulo the ideals generated by null-homotopic morphisms. We denote the derived category of $A$-mod by $D(A)$ which is the quotient category of $K(A)$ with respect to the subcategory of $K(A)$ consisting of all the acyclic complexes. The full subcategory of $K(A)$ and $D(A)$ consisting of bounded complexes over $A$-mod is denoted by $K^b(A)$ and $D^b(A)$, respectively. We denoted by $C^+(A)$ the category of complexes of bounded below, and by $K^+(A)$ the homotopy category of $C^+(A)$. The full subcategory of $D(A)$ consisting of bounded below complexes is denoted by $D^+(A)$.

Similarly, we have the category $C^-(A)$ of complexes bounded above, the homotopy category $K^-(A)$ of $C^-(A)$ and the derived category $D^-(A)$ of $C^-(A)$. If we focus on the category of left $A$-modules, then we have the homotopy category $K(A\text{-Mod})$ of $C(A\text{-Mod})$ and the derived category $D(A\text{-Mod})$ of $C(A\text{-Mod})$. Suppose that $X^\bullet = (X^i, d_X^i)$ and $Y^\bullet = (Y^i, d_Y^i)$ are two complexes. We define the *direct sum* of $X^\bullet$ and $Y^\bullet$ by the complex $Z^\bullet = (Z^i, d_Z^i)$ such that $Z^i = X^i \oplus Y^i$ and $d_Z^i = \begin{pmatrix} d_X^i & 0 \\ 0 & d_Y^i \end{pmatrix} : X^i \oplus Y^i \to X^{i+1} \oplus Y^{i+1}$. The complex $X^\bullet$ and the complex $Y^\bullet$ are called the *direct summands* of $Z^\bullet$.

The following result, due to Rickard (see [30, Theorem 6.4]), may be called the Morita theorem of derived categories.

**Lemma 2.1.** [30] *Let $\Lambda$ and $\Gamma$ be two rings. The following conditions are equivalent:*

(1) $K^-(\Lambda\text{-proj})$ *and* $K^-(\Gamma\text{-proj})$ *are equivalent as triangulated categories;*
(2) $D^b(\Lambda\text{-Mod})$ *and* $D^b(\Gamma\text{-Mod})$ *are equivalent as triangulated categories;*
(3) $K^b(\Lambda\text{-Proj})$ *and* $K^b(\Gamma\text{-Proj})$ *are equivalent as triangulated categories;*
(4) $K^b(\Lambda\text{-proj})$ *and* $K^b(\Gamma\text{-proj})$ *are equivalent as triangulated categories;*
(5) $\Gamma$ *is isomorphic to* $\mathrm{End}(T^\bullet)$, *where $T^\bullet$ is a complex in* $K^b(\Lambda\text{-proj})$ *satisfying:*
 (a) $T^\bullet$ *is self-orthogonal, that is,* $\mathrm{Hom}_{K^b(\Lambda\text{-proj})}(T^\bullet, T^\bullet[i]) = 0$ *for all* $i \neq 0$,
 (b) *add*$(T^\bullet)$ *generates* $K^b(\Lambda\text{-proj})$ *as a triangulated category.*

Two rings $\Lambda$ and $\Gamma$ are called *derived equivalent* if the above conditions (1)-(5) are satisfied. A complex $T^\bullet$ in $K^b(\Lambda\text{-proj})$ as above is called a *tilting complex* over $\Lambda$. It is also equivalent to say that the two rings $\Lambda$ and $\Gamma$ are derived equivalent if and only if there exists a complex $X^\bullet$ in $D(\Lambda\text{-Mod})$, isomorphic to a complex in $K^b(\Lambda\text{-proj})$ which satisfies [Lemma 2.1(5), (a) and (b)], such that the two rings $\Gamma$ and $\mathrm{End}_{D(\Lambda\text{-Mod})}(X^\bullet)$ are isomorphic. In particular, if the tilting complex $T^\bullet$ is isomorphic to a module $T$ in $D^b(\Lambda)$, then $T$ is called *tilting module*.

## 2.2 The $n$-angulated categories

In this part, we will recall the definition and some properties of $n$-angulated categories which are proposed by Geiss, Keller and Oppermann in [11]. For the convenience of the reader, we repeat the relevant material from [11].

Suppose that $\mathcal{F}$ is an additive category with an automorphism $\Sigma$, and $n$ $(\geq 3)$ is an integer. A sequence of objects and morphisms in $\mathcal{F}$ of the form

$$X_\bullet := X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\alpha_n} \Sigma X_1$$

is called an *$n$-$\Sigma$-sequence*. An $n$-$\Sigma$-sequence $X_\bullet$ is called *exact* if the following sequence of $\mathbb{Z}$-modules

$$\mathrm{Hom}_{\mathcal{F}}(Y, X_\bullet) : \cdots \to \mathrm{Hom}_{\mathcal{F}}(Y, X_1) \to \mathrm{Hom}_{\mathcal{F}}(Y, X_2) \to \cdots \to \mathrm{Hom}_{\mathcal{F}}(Y, X_n) \to \cdots$$

is exact for every object $Y \in \mathcal{F}$. The left rotation of $X_\bullet$ is the following $n$-$\Sigma$-sequence

$$X_\bullet[1] := (X_2 \xrightarrow{\alpha_2} X_3 \xrightarrow{\alpha_3} \cdots \xrightarrow{\alpha_n} \Sigma X_1 \xrightarrow{(-1)^n \Sigma\alpha_1} \Sigma X_2).$$

Similarly, the right rotation of $X_\bullet$ is the $n$-$\Sigma$-sequence

$$X_\bullet[-1] := (\Sigma^{-1} X_n \xrightarrow{(-1)^n \Sigma^{-1}\alpha_n} X_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{n-2}} X_n).$$

An $n$-$\Sigma$-sequence of the form $(TX)_\bullet := (X \xrightarrow{1_X} X \to 0 \to \cdots \to 0 \to \Sigma X)$ for $X \in \mathcal{F}$, or its rotation is called *trivial*. A *morphism* of two $n$-$\Sigma$-sequences is given by a sequence of morphisms $\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_n)$ in $\mathcal{F}$ such that the following diagram commutates:

$$\begin{tikzcd}
X_1 \arrow[r, "\alpha_1"] \arrow[d, "\varphi_1"'] & X_2 \arrow[r, "\alpha_2"] \arrow[d, "\varphi_2"'] & X_3 \arrow[r] \arrow[d, "\varphi_3"'] & \cdots \arrow[r] & X_n \arrow[r, "\alpha_n"] \arrow[d, "\varphi_n"'] & \Sigma X_1 \arrow[d, "\Sigma\varphi_1"] \\
Y_1 \arrow[r, "\beta_1"] & Y_2 \arrow[r, "\beta_2"] & Y_3 \arrow[r] & \cdots \arrow[r] & Y_n \arrow[r, "\beta_n"] & \Sigma Y_n.
\end{tikzcd}$$

The morphism $\varphi$ is called a *weak isomorphism* if $\varphi_i$ and $\varphi_{i+1}$ are isomorphisms, where $1 \leq i \leq n$, and $\varphi_{n+1}$ is denoted by $\Sigma\varphi_1$. Two $n$-$\Sigma$-sequences $X^1_\bullet$ and $X^n_\bullet$ are called *weakly isomorphic* if there is a chain of $n$-$\Sigma$-sequences

$$X^1_\bullet - X^2_\bullet - \cdots - X^{n-1}_\bullet - X^n_\bullet$$

satisfying that there is a weak isomorphism between $X^i_\bullet$ and $X^{i+1}_\bullet$ for $1 \leq i \leq n-1$.

**Definition 2.2.** ([11]) *A collection $\pentagon$ of $n$-$\Sigma$-sequences is called a (*pre-*) $n$-angulation of $(\mathcal{F}, \Sigma)$ and its elements* n-angles *if $\pentagon$ fulfills the following conditions:*

1. *(a) $\pentagon$ is closed under direct sums and under taking summands.*
   *(b) For all $X \in \mathcal{F}$, the trivial $n$-$\Sigma$-sequence $(TX)_\bullet$ belongs to $\pentagon$.*
   *(c) For each morphism $\alpha_1 : X_1 \to X_2$ is $\mathcal{F}$, there exists an $n$-angle starting with $\alpha_1$.*
2. *An $n$-$\Sigma$-sequence $X_\bullet$ belongs to $\pentagon$ if and only if $X_\bullet[1]$ belongs to $\pentagon$.*
3. *Each commutative diagram*

$$\begin{tikzcd}
X_1 \arrow[r, "\alpha_1"] \arrow[d, "\varphi_1"'] & X_2 \arrow[r, "\alpha_2"] \arrow[d, "\varphi_2"'] & X_3 \arrow[r] & \cdots \arrow[r] & X_n \arrow[r, "\alpha_n"] & \Sigma X_1 \arrow[d, "\Sigma\varphi_1"] \\
Y_1 \arrow[r, "\beta_1"] & Y_2 \arrow[r, "\beta_2"] & Y_3 \arrow[r] & \cdots \arrow[r] & Y_n \arrow[r, "\beta_n"] & \Sigma Y_n
\end{tikzcd}$$

*with rows in $\pentagon$ can be completed to a morphism of $n$-$\Sigma$-sequences.*

*Moreover, if $\pentagon$ fulfills the following condition, it is called an $n$-angulation of $(\mathcal{F}, \Sigma)$:*
*4. In the situation of 3 the morphisms $\varphi_3, \varphi_4, \cdots, \varphi_n$ can be chosen such that the cone $C(\varphi_\bullet)$:*

$$X_2 \oplus Y_1 \xrightarrow{\begin{pmatrix} -\alpha_2 & 0 \\ \varphi_2 & \beta_1 \end{pmatrix}} X_3 \oplus Y_2 \xrightarrow{\begin{pmatrix} -\alpha_3 & 0 \\ \varphi_3 & \beta_2 \end{pmatrix}} \cdots \xrightarrow{\begin{pmatrix} -\alpha_n & 0 \\ \varphi_n & \beta_{n-1} \end{pmatrix}} \Sigma X_1 \oplus Y_n \xrightarrow{\begin{pmatrix} -\Sigma\alpha_1 & 0 \\ \Sigma\varphi_1 & \beta_n \end{pmatrix}} \Sigma X_2 \oplus \Sigma Y_1$$

*belongs to $\pentagon$.*

**Definition 2.3.** *Suppose that $(\mathcal{F}, \Sigma, \pentagon)$ and $(\mathcal{F}', \Sigma', \pentagon')$ are two $n$-angulated categories. An additive functor $F : \mathcal{F} \to \mathcal{F}'$ is called* $n$-angle functor *if $F(\pentagon) = \pentagon'$, i.e., there exists an invertible natural transformation $\xi : F\Sigma \to \Sigma' F$ such that $(FX_1, FX_2, \cdots, FX_n, F\alpha_1, F\alpha_2, \cdots, F\alpha_n\xi_{X_1})$ is in $\pentagon'$ for $(X_1, X_2, \cdots, X_n, \alpha_1, \alpha_2, \cdots, \alpha_n)$ in $\pentagon$. Moreover, if $F$ is an equivalence of categories, then $F$ is called* $n$-angle equivalence.

**Remark.** If $n = 3$, then $F$ is well-known as triangle functor.

In [11], Geiss, Keller and Oppermann show how to construct $n$-angulated categories inside triangulated categories.

**Example 2.1.** [11] *Let $\mathcal{T}$ be a triangulated category with an $(n-2)$-cluster tilting subcatgory $\mathcal{F}$, which is closed under $\Sigma_3^{n-2}$, where $\Sigma_3$ denotes the suspension in $\mathcal{T}$. Then $(\mathcal{F}, \Sigma_3^{n-2}, \pentagon)$ is an $n$-angulated category, where $\pentagon$ is the class of all sequences*

$$X_1 \overset{\alpha_1}{\to} X_2 \overset{\alpha_2}{\to} \cdots \overset{\alpha_{n-1}}{\to} X_n \overset{\alpha_n}{\to} \Sigma_3^{n-2} X_1$$

*such that there exists a diagram*

$$\begin{tikzcd}
& X_2 \arrow[rr, "\alpha_2"] \arrow[dr] & & X_3 \arrow[rr] \arrow[dr] & & X_4 \arrow[r] \arrow[dr] & \cdots & \cdots \arrow[r] & X_{n-1} \arrow[dr, "\alpha_{n-1}"] & \\
X_1 \arrow[ur, "\alpha_1"] & & X_{2.5} \arrow[ll, dashed] \arrow[ur] & & X_{3.5} \arrow[ll, dashed] \arrow[ur] & & \cdots \arrow[ll, dashed] & X_{n-1.5} \arrow[l, dashed] \arrow[ur] & & X_n \arrow[ll, dashed]
\end{tikzcd}$$

*with $X_i \in \mathcal{T}$ for $i \notin \mathbb{Z}$, such that all oriented triangles are triangles in $\mathcal{T}$, all non-oriented triangles commute, and $\alpha_n$ is the composition along the lower edge of the diagram.*

In order to prove the main result, we should prove the following lemma.

**Lemma 2.4.** *Let $(\mathcal{F},\Sigma,\pentagon)$ be a pre-n-angulated category.*

*For $2 \leq i < n$. Each commutative diagram*

$$\begin{tikzcd}
X_1 \arrow[r, "\alpha_1"] \arrow[d, "\varphi_1"'] & X_2 \arrow[r, "\alpha_2"] \arrow[d, "\varphi_2"'] & \cdots \arrow[r] & X_i \arrow[r] \arrow[d, "\varphi_i"'] & X_{i+1} \arrow[r] \arrow[d, dashed, "\varphi_{i+1}"'] & \cdots \arrow[r] & X_n \arrow[r, "\alpha_n"] \arrow[d, dashed, "\varphi_n"'] & \Sigma X_1 \arrow[d, "\Sigma\varphi_1"'] \\
Y_1 \arrow[r, "\beta_1"] & Y_2 \arrow[r, "\beta_2"] & \cdots \arrow[r] & Y_i \arrow[r] & Y_{i+1} \arrow[r] & \cdots \arrow[r] & Y_n \arrow[r, "\beta_n"] & \Sigma Y_n
\end{tikzcd}$$

*with rows in $\pentagon$ can be completed to a morphism of n-$\Sigma$-sequences.*

**Proof.** The proof is similar with [11, Lemma 2.3].□

Suppose that $\mathcal{F}$ has split idempotents. If we denote this lemma by $(3')$, then we can modify the definition of pre-$n$-angulated category. That is, a collection $\pentagon$ of $n$-$\Sigma$-sequences is called a pre-$n$-angulation of $(\mathcal{F},\Sigma)$ if $\pentagon$ satisfies the following conditions: $(1(a)-1(c)),2,3'$. It is easy to prove that the two cases of definition are equivalent. However, the change is vital for the proof of the main result.

## 2.3 Admissible subsets and $n$-perforated Yoneda algebras

In this part, we will introduce a new class of algebras which are called $n$-perforated Yoneda algebras.

Let $\mathbb{N} = \{0,1,2,\cdots\}$ be the set of natural numbers, and let $\mathbb{Z}$ be the set of all integers. For a natural number $n$ or infinity, let $\mathbb{N}_n := \{i \in \mathbb{N} \mid 0 \leq i < n+1\}$.

Recall from [16] that a subset $\Phi$ of $\mathbb{Z}$ containing 0 is called an *admissible subset* of $\mathbb{Z}$ if the following condition is satisfied:

*If $i,j$ and $k$ are in $\Phi$ such that $i+j+k \in \Phi$, then $i+j \in \Phi$ if and only if $j+k \in \Phi$.*

Any subset $\{0,i,j\}$ of $\mathbb{N}$ is an admissible subset of $\mathbb{Z}$. Moreover, for any subset $\Phi$ of $\mathbb{N}$ containing zero and for any positive integer $m \geq 3$, the set $\{x^m \mid x \in \Phi\}$ is admissible in $\mathbb{Z}$. The intersection of a family of admissible subsets of $\mathbb{N}$ is admissible (for more examples, see [16]). Nevertheless, not every subset of $\mathbb{N}$ containing zero is admissible. Note that $\Phi^2$ is not necessary admissible in $\mathbb{N}$ even if $\Phi$ is an admissible subset of $\mathbb{N}$. For instance, $\{0,1,2,4\}$ is not admissible. In fact, this is the 'smallest' non-admissible subset of $\mathbb{N}$. For more details, we refer reader to [16].

Admissible sets were used to define the $\Phi$-Auslander Yoneda algebras in [16] and the perforated Yoneda algebra in [14], if we restrict to the case of an object in a triangulated category. However, in this paper, we will restrict to the case of objects in an $n$-angulated category.

The following is the most natural generalization of perforated Yoneda algebra, proposed by Hu, König and Xi in [14], for $n$-angulated categories.

Let $\Phi$ be an admissible subset of $\mathbb{Z}$, and let $\mathcal{F}$ be an $n$-angulated $R$-category with suspension functor $\Sigma$. Suppose that $F$ is an $n$-angle functor from $\mathcal{F}$ to $\mathcal{F}$. Note that $F^i = 0$ for $i < 0$ if the quasi-inverse of $F$ does not exist. Consider the $(\Phi,F)$-orbit category $\mathcal{F}^{F,\Phi}$, the extension of orbit category, whose object are the objects of $\mathcal{F}$. Suppose that $X$ and $Y$ are two objects in $\mathcal{F}^{F,\Phi}$, the homomorphism set in $\mathcal{F}^{F,\Phi}$ is defined as follows:

$$\mathrm{Hom}_{\mathcal{F}^{F,\Phi}}(X,Y) := \bigoplus_{i\in\Phi} \mathrm{Hom}_{\mathcal{F}}(X,F^iY) \in R\text{-Mod}$$

and the composition is defined in an obvious way. Since $\Phi$ is admissible, the $(\Phi,F)$-orbit category $\mathcal{F}^{F,\Phi}$ is an additive $R$-category. Let $X,Y$ be objects in $\mathcal{F}^{F,\Phi}$. Thus, $\mathrm{Hom}_{\mathcal{F}^{F,\Phi}}(X,X)$ is an $R$-algebra. It is called the *n-perforated Yoneda algebra* of $X$ with respect to $F$, and denoted by $\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(X)$. $\mathrm{Hom}_{\mathcal{F}^{F,\Phi}}(X,Y)$ is a $\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(X)$-$\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(Y)$-bimodule. For convenience, we denote $\mathrm{Hom}_{\mathcal{F}^{F,\Phi}}(X,Y)$ by $\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(X,Y)$.

The following lemma, which was essentially taken form [16, Lemma 3.5], [14, Lemma 2.2], describes the basic properties of the algebra $\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(X)$ where $X$ is an object in the $n$-angulated $R$-category $\mathcal{F}$, which can also be verified directly.

**Lemma 2.5.** *Let $\mathcal{F}$ be an n-angle R-category with an n-angle endo-functor $F$, and let $U$ be an object in $\mathcal{F}$. Suppose that $U_1, U_2, U_3$ are in* $\mathrm{add}(U)$*, and that $\Phi$ is an admissible subset of $\mathbb{Z}$. Then*

(1) *There is a natural isomorphism*

$$\mu : \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(U_1,U_2) \to \mathrm{Hom}_{\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(U)}(\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(U,U_1), \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(U,U_2)),$$

*which sends $x \in \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(U_1,U_2)$ to the morphism $a \mapsto ax$ for $a \in \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(U,U_1)$. Moreover, if $x \in \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(U_1,U_2)$ and $y \in \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(U_2,U_3)$, then $\mu(xy) = \mu(x)\mu(y)$.*

(2) *The functor $\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(U,-) : add(U) \to \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(U)$-proj is faithful.*

(3) *If $\mathrm{Hom}_{\mathcal{F}}(U_1, F^i U_2) = 0$ for all $i \in \Phi \setminus \{0\}$, then the functor $\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(U,-)$ induces an isomorphism of R-modules:*

$$\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(U,-) : \mathrm{Hom}_{\mathcal{F}}(U_1,U_2) \to \mathrm{Hom}_{\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(U)}(\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(U,U_1), \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(U,U_2)).$$

# 3 Proof of the main result

In this section, we will construct derived equivalences from an $n$-angle. Firstly, we will prove Theorem 1.1. Secondly, we will derive some consequences form the main result.

Let $\mathcal{F}$ be an $n$-angulated category with suspension functor $\Sigma$, and let $\pentagon$ be an $n$-angulation of $(\mathcal{F},\Sigma)$. Suppose that $\mathcal{F}$ has split idempotent and the functor $F : \mathcal{F} \to \mathcal{F}$ is an $n$-angle functor. Since $F$ is an $n$-angulated category, there is a natural isomorphism $\delta : F\Sigma \to \Sigma F$ associated with $F$. We denote the isomorphism $F^i(\Sigma^j X) \to \Sigma^j(F^i X)$ by $\delta(F,i,X,j)$. Note that there is an inclusion $\iota : \mathrm{Hom}_{\mathcal{F}}(X,Y) \to \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(X,Y)$. Given a morphism $f \in \mathrm{Hom}_{\mathcal{F}}(X,Y)$, $\iota(f)$ is an element of $\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(X,Y)$ concentrated in degree 0. For convenience, we denote $\iota(f)$ by $\underline{f}$.

Set

$$X \xrightarrow{\alpha_1} M_1 \xrightarrow{\alpha_2} M_2 \to \cdots \to M_{n-2} \xrightarrow{\alpha_{n-1}} Y \xrightarrow{\alpha_n} \Sigma X$$

be an $n$-angle in $\pentagon$.

For simplicity, we denote $\oplus_{i=1}^{n-2} M_i$ by $M$ and write $V, W$ instead of $X \oplus M, M \oplus Y$, respectively. Thus, we can get $M_i \in \mathrm{add}(M)$ for $i = 1, 2, \cdots, n-2$.

Since the direct sum of two $n$-angles is still an $n$-angle, there are two $n$-angles

$$X \xrightarrow{\alpha_1} M_1 \xrightarrow{\alpha_2} M_2 \to \cdots \to M_{n-3} \xrightarrow{\overline{\alpha_{n-2}}} M_{n-2} \oplus M \xrightarrow{\overline{\alpha_{n-1}}} W \xrightarrow{\overline{\alpha_n}} \Sigma X$$

$$\Sigma^{-1} Y \xrightarrow{(-1)^n \Sigma^{-1} \widetilde{\alpha_n}} V \xrightarrow{\widetilde{\alpha_1}} M_1 \oplus M \xrightarrow{\widetilde{\alpha_2}} \cdots \xrightarrow{\alpha_{n-2}} M_{n-2} \xrightarrow{\alpha_{n-1}} Y$$

We define

$$\overline{\alpha_{n-2}} := (\alpha_{n-2}, 0) : M_{n-3} \to M_{n-2} \oplus M \qquad \overline{\alpha_{n-1}} := \begin{pmatrix} 0 & \alpha_{n-1} \\ 1 & 0 \end{pmatrix} : M_{n-2} \oplus M \to M \oplus Y$$

$$\overline{\alpha_n} := \begin{pmatrix} 0 \\ \alpha_n \end{pmatrix} : M \oplus Y \to \Sigma X \qquad \widetilde{\alpha_1} := \begin{pmatrix} \alpha_1 & 0 \\ 0 & 1 \end{pmatrix} : X \oplus M \to M_1 \oplus M$$

$$\widetilde{\alpha_2} := \begin{pmatrix} \alpha_2 \\ 0 \end{pmatrix} : M_1 \oplus M \to M_2 \qquad \widetilde{\alpha_n} := \begin{pmatrix} \alpha_n & 0 \end{pmatrix} : Y \to \Sigma V$$

For a subset $\Phi$ of $\mathbb{Z}$, we define $-\Phi := \{-x \mid x \in \Phi\}$ and

$$\mathscr{X}_{\mathcal{F}}^{\mathrm{F},\Phi}(M) := \{X \in \mathcal{F} \mid \mathrm{Hom}_{\mathcal{F}}(X, F^i M) = 0 \text{ for all } i \in \Phi \setminus \{0\}\},$$

$$\mathscr{Y}_{\mathcal{F}}^{\mathrm{F},\Phi}(\mathrm{M}) := \{\mathrm{Y} \in \mathcal{F} \mid \mathrm{Hom}_{\mathcal{F}}(\mathrm{M}, \mathrm{F^i Y}) = 0 \text{ for all i} \in \Phi \setminus \{0\}\}.$$

$$I := \{x = (x_i) \in \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(X \oplus M) \mid \quad x_i = 0 \text{ for } 0 \neq i \in \Phi, \; x_0 \text{ factorizes through add(M) and } \Sigma^{-1}\widetilde{\alpha_n}\},$$

$$J := \{y = (y_i) \in \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(M \oplus Y) \mid \quad y_i = 0 \text{ for } 0 \neq i \in \Phi, \quad y_0 \text{ factorizes through add(M) and } \overline{\alpha_n}\}.$$

In order to prove Theorem 1.1, we prove the following lemmas.

**Lemma 3.1.** *The sets $I$ and $J$ are ideals of $\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V)$ and $\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(W)$, respectively.*

**Proof.** It is easily seen that the set $I$ is closed under addition. By the definition of $I$, we can write $x_0 = uv$ for $u : V \to M'$ and $v : M' \to V$, where $M'$ is an object in add($M$), and $x_0 = s(\Sigma^{-1}\widetilde{\alpha_n})$ for a morphism $s : V \to \Sigma^{-1}Y$. Suppose $x = (x_i)_{i\in\Phi} \in I, y = (y_i)_{i\in\Phi} \in \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V)$. In order to prove that the set $I$ is an ideal of $\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V)$, it suffices to prove that $xy = (x_0y_i)_{i\in\Phi} \in I, yx = (y_iF^i(x_0))_{i\in\Phi} \in I$.

It is clear that $x_0y_0$ factorizes through $\Sigma^{-1}\widetilde{\alpha_n}$ and some object in add($M$). Set $0 \neq i \in \Phi$. Note that $\widetilde{\alpha_1} : V \to M_1 \oplus M$ is a left (add($M$), F, $\Phi$)-approximation of $V$. Thus, for given $y_i : V \to F^iV$, there is a morphism $z_i : M_1 \oplus M \to F^i(M_1 \oplus M)$ such that $\widetilde{\alpha_1}z_i = y_iF^i(\widetilde{\alpha_1})$. Since $F$ is an $n$-angle functor, there is a commutative diagram between two $n$-angles.

$$\begin{tikzcd}
\Sigma^{-1}Y \arrow[r, "(-1)^n\Sigma^{-1}\widetilde{\alpha_n}"] \arrow[d] & V \arrow[r, "\widetilde{\alpha_1}"] \arrow[d, "y_i"] & M_1 \oplus M \arrow[r, "\widetilde{\alpha_2}"] \arrow[d, "z_i"] & M_2 \arrow[r] \arrow[d] & \cdots \arrow[r, "\alpha_{n-2}"] \arrow[d] & M_{n-2} \arrow[r, "\alpha_{n-1}"] \arrow[d] & Y \arrow[d] \\
\Sigma^{-1}F^iY \arrow[r] & F^iV \arrow[r, "F^i\widetilde{\alpha_1}"] & F^i(M_1 \oplus M) \arrow[r] & F^iM_2 \arrow[r] & \cdots \arrow[r] & F^iM_{n-2} \arrow[r, "F^i\alpha_{n-2}"] & F^iY
\end{tikzcd}$$

Let $p_X$ and $p_M$ be the projections of $V$ onto $X$ and $M$, respectively. Since $\widetilde{\alpha_1} : V \to M_1 \oplus M$ is a left $(\mathrm{add}(M), F, \Phi)$-approximation of $V$, $y_iF^ip_M$ factorizes through $\widetilde{\alpha_1}$. So there is a morphism $s_i : M_1 \oplus M \to F^iM$ such that $y_iF^ip_M = \widetilde{\alpha_1}s_i$. Hence $x_0y_iF^ip_M = s(\Sigma^{-1}\widetilde{\alpha_n})\widetilde{\alpha_1}s_i = 0$. By assumption $X \in \mathscr{Y}_{\mathcal{F}}^{\mathrm{F},\Phi}(\mathrm{M})$, we have $\mathrm{Hom}_{\mathcal{F}}(M, F^iX) = 0$. Then the composition $vy_iF^ip_X : M' \xrightarrow{v} V \xrightarrow{y_i} F^iV \xrightarrow{F^ip_X} F^iX$ belongs to $\mathrm{Hom}_{\mathcal{F}}(M', F^iX) = 0$, thus $x_0y_iF^ip_X = uvy_iF^ip_X = 0$. Altogether, we have shown that $x_0y_i = 0$ for $0 \neq i \in \Phi$. Hence $xy \in I$, and $I$ is a right ideal in $\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V)$.

Obviously, $y_0x_0$ factorizes through an object in $\mathrm{add}(M)$ and through $\Sigma^{-1}\widetilde{\alpha_n}$. Set $0 \neq i \in \Phi$. Note that $\widetilde{\alpha_1} : V \to M_1 \oplus M$ is a left $(\mathrm{add}(M), F, \Phi)$-approximation of $V$. Thus there is a morphism $h_i : M_1 \oplus M \to F^iM'$ such that $y_iF^iu = \widetilde{\alpha_1}h_i$. By assumption, we have $\mathrm{Hom}_{\mathcal{F}}(M, F^iX) = 0$ for $0 \neq i \in \Phi$. This implies that $h_iF^ivF^ip_X = 0$, and therefore $y_iF^ix_0F^ip_X = \widetilde{\alpha_1}h_iF^ivF^ip_X = 0$. Since $(\Sigma^{-1}\widetilde{\alpha_n})p_M = 0$, we have shown that $y_iF^ix_0F^ip_M = y_iF^isF^i(\Sigma^{-1}\widetilde{\alpha_n})F^ip_M = y_iF^isF^i(\Sigma^{-1}\widetilde{\alpha_n}p_M) = 0$. Thus, $y_iF^ix_0 = 0$ for $0 \neq i \in \Phi$. Hence $yx \in I$, and $I$ is a left ideal in $\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V)$. Thus $I$ is an ideal in $\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V)$.

In the same manner we can seen that $J$ is an ideal in $\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(W)$. $\square$

The following lemma is essentially taken from [14]. The proof remains valid for the present situation.

**Lemma 3.2.** *Then notations are the same as above. Then*

(1) $I \cdot \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V, M) = 0$.

(2) $I \cdot \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V, X) = \{(x_i)_{i\in\Phi} \in \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V, X) \mid x_i = 0$ *for* $0 \neq i \in \Phi$, $x_0$ *factorizes through* $\mathrm{add}(M)$ *and* $\Sigma^{-1}\alpha_n\}$

(3) *For* $x = (x_i)_{i\in\Phi} \in \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V', X)$ *with* $V' \in \mathrm{add}(V)$, *we have* $\mathrm{Im}(\mu(x)) \subseteq I \cdot \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V, X)$ *if and only if* $x_i = 0$ *for all* $0 \neq i \in \Phi$ *and* $x_0$ *factorizes through* $\mathrm{add}(M)$ *and* $\Sigma^{-1}\alpha_n$.

(4) *Let* $f : M' \to X$ *with* $M' \in \mathrm{add}(M)$. *Then* $\mathrm{Im}(\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V, f)) \subseteq I \cdot \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V, X)$ *if and only if* $f$ *factorizes through* $\Sigma^{-1}\alpha_n$.

Now, we turn to prove Theorem 1.1.

**Proof of Theorem 1.1.** In order to prove Theorem 1.1, Our strategy is trying to find out a tilting complex over $\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V)/I$ and compute its endomorphism ring. For convenience, we define

$$\Lambda := \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V), \quad \Gamma := \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(W), \quad \overline{\Lambda} := \Lambda/I, \quad \overline{\Gamma} := \Gamma/J.$$

Set

$$\widetilde{T^{\bullet}}: 0 \to \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,X) \overset{(V,\alpha_1)}{\to} \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,M_1) \overset{(V,\alpha_2)}{\to} \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,M_2) \overset{(V,\alpha_3)}{\to} \cdots$$
$$\overset{(V,\overline{\alpha_{n-2}})}{\to} \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,M_{n-2}\oplus M) \to 0.$$

Note that $\widetilde{T^{\bullet}}$ is a complex in $K^b(\Lambda\text{-proj})$. However, by easy computation, $\widetilde{T^{\bullet}}$ is not a tilting complex over $\Lambda$.

Pick $x = (x_i)_{i\in\Phi} \in I \cdot \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,X)$. By the definition, $\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_1)(x) = (x_i F^i\alpha_1)_{i\in\Phi}$. Note that $x_i = 0$ for $0 \neq i \in \Phi$ and $x_0$ factorizes through $\Sigma^{-1}\alpha_n$. So $\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_1)(x) = 0$. Hence the morphism $\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_1) : \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,X) \to \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,M_1)$ induces a morphism

$$q : \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,X)/I\cdot \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,X) \to \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,M_1).$$

Let $P = \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,X)/I\cdot \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,X)$, and $p : \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,X) \to \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,X)/I\cdot \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,X)$ be the canonical surjective map. Then we can write $\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha) = pq$.

Thus, we have a complex

$$T^{\bullet} : 0 \to P \overset{q}{\to} \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,M_1) \overset{(V,\alpha_2)}{\to} \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,M_2) \overset{(V,\alpha_3)}{\to} \cdots \overset{(V,\overline{\alpha_{n-2}})}{\to} \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,M_{n-2}\oplus M) \to 0.$$

in $D^b(\overline{\Lambda})$. We will prove that $T^{\bullet}$ is a tilting complex over $\overline{\Lambda}$.

Note that $\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,X)$ is a finitely generated projective left $\Lambda$-module and $I\cdot \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,M) = 0$. Then $P$ and $\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,M)$ are finitely generated projective left $\overline{\Lambda}$-modules. Hence $T^{\bullet}$ is a complex in $K^b(\overline{\Lambda}\text{-proj})$. Clearly, $\mathrm{add}(T^{\bullet})$ generates $K^b(\Lambda\text{-proj})$ as a triangulated category. So it suffices to prove that $\mathrm{Hom}_{K^b(\overline{\Lambda})}(T^{\bullet}, T^{\bullet}[i]) = 0$ for $i \neq 0$.

(1) $\mathrm{Hom}_{K^b(\overline{\Lambda})}(T^{\bullet}, T^{\bullet}[i]) = 0$ for $i = 1,2,\cdots,n-2$.

The first case: $i = 1,2,\cdots,n-3$.

Let $f^{\bullet}$ be a morphism in $\mathrm{Hom}_{K^b(\overline{\Lambda})}(T^{\bullet}, T^{\bullet}[i])$. For simplicity, throughout the proof, we denote $\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(X,Y)$ by $(X,Y)$ in commutative diagrams.

$$\begin{tikzcd}
0 \arrow[r] & P \arrow[r, "q"] \arrow[d, "f^0"'] & (V,M_1) \arrow[r, "{(V,\alpha_2)}"] \arrow[d, "f^1"'] \arrow[dl, dashed, "\mu(s^1)"'] & (V,M_2) \arrow[r, "{(V,\alpha_3)}"] \arrow[d, "f^2"'] \arrow[dl, dashed, "\mu(s^2)"'] & \cdots \arrow[r] & (V,M_{n-2-i}) \arrow[r] \arrow[d, "f^{n-2-i}"] \arrow[dl, dashed, "\mu(s^{n-2-i})"'] & \cdots \\
\cdots \arrow[r] & (V,M_i) \arrow[r, "{(V,\alpha_{i+1})}"] & (V,M_{i+1}) \arrow[r] & (V,M_{i+2}) \arrow[r] & \cdots \arrow[r] & (V,M_{n-2}\oplus M) \arrow[r] & 0
\end{tikzcd}$$

By Lemma 2.5(1), we can assume that

$$\mu(x^0) = pf^0,\ f^{n-2-i} = \mu(x^{n-2-i}),\ f^j = \mu(x^j)$$

with

$$x^0 = (x_k^0)_{k\in\Phi} \in \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(X,M_i),$$
$$x^{n-2-i} = (x_k^{n-2-i})_{k\in\Phi} \in \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(M_{n-2-i},M_{n-2}\oplus M),$$
$$x^j = (x_k^j)_{k\in\Phi} \in \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(M_j,M_{i+j})$$

for $j = 1,2,\cdots,n-3-i$.

Note that $\alpha_1 : X \to M_1$ is a left $(\mathrm{add}(M),F,\Phi)$-approximation of $X$. Then there are morphisms $y_j^0 : M_1 \to F^j M_i$ such that $x_j^0 = \alpha_1 y_j^0$ for $j \in \Phi$. We denote $(y_j^0)_{j\in\Phi}$ by $y^0$.

Since

$$\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_1)\mu(y^0) = \mu(\underline{\alpha_1})\mu(y^0) = \mu(\underline{\alpha_1}y^0) = \mu((\alpha_1 y_j^0)_{j\in\Phi}) = \mu((x_j^0)_{j\in\Phi}) = \mu(x^0),$$

we can get $pq\mu(y^0) = \mu(x^0) = pf^0$. This implies that $q\mu(y^0) = f^0$ since $p$ is surjective. We denote $f^1 - \mu(y^0)\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_{i+1})$ by $s^1$.

Thus,

$$s^1 = f^1 - \mu(y^0)\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_{i+1}) = \mu(x^1) - \mu(y^0\underline{\alpha_{i+1}}) = \mu(x^1 - y^0\underline{\alpha_{i+1}}).$$

We denote $x_j^1 - y_j^0 F^j\alpha_{i+1}$ by $s_j^1$. Note that

$$\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_1)f^1 = pqf^1 = pf^0\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_{i+1}) = \mu(x^0)\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_{i+1}),$$

i.e., $\mu(\underline{\alpha_1}x^1) = \mu(x^0\underline{\alpha_{i+1}})$. This implies that $\alpha_1 x_j^1 = x_j^0 F^j\alpha_{i+1}$ for $j \in \Phi$.

It follows that

$$\alpha_1 s_j^1 = \alpha_1(x_j^1 - y_j^0F^j\alpha_{i+1}) = \alpha_1 x_j^1 - \alpha_1 y_j^0 F^j\alpha_{i+1} = \alpha_1 x_j^1 - x_j^0F^j\alpha_{i+1} = 0$$

for $j \in \Phi$. Then there exists $y_j^1 : M_2 \to M_{i+1}$ such that $s_j^1 = \alpha_2 y_j^1$ for $j \in \Phi$. For convenience, we denote $(y_j^1)_{j\in\Phi}$ by $y^1$. Now, we check that $\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_2)\mu(y^1) + \mu(y^0)\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_{i+1}) = f^1$.

$$\begin{array}{rcl}
\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_2)\mu(y^1) + \mu(y^0)\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_{i+1}) & = & \mu(\underline{\alpha_2})\mu(y^1) + \mu(y^0)\mu(\underline{\alpha_{i+1}}) \\
& = & \mu(\underline{\alpha_2}y^1 + y^0\underline{\alpha_{i+1}}) \\
& = & \mu((\alpha_2 y_j^1 + y_j^0\overline{F^j}\alpha_{i+1})_{j\in\Phi}) \\
& = & \mu((x_j^1)_{j\in\Phi}) \\
& = & f^1.
\end{array}$$

We denote $f^2 - \mu(y^1)\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_{i+2})$ by $s^2$. Thus,

$$s^2 = f^2 - \mu(y^1)\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_{i+2}) = \mu(x^2) - \mu(y^1)\mu(\underline{\alpha_{i+2}}) = \mu(x^2 - y^1\underline{\alpha_{i+2}}).$$

We denote $x_j^2 - y_j^1F^j\alpha_{i+2}$ by $s_j^2$ for $j \in \Phi$. Note that $\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_2)\mu(x^2) = \mu(x^1)\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_{i+2})$, we can get $\alpha_2 x_j^2 = x_j^1 F^j(\alpha_{i+2})$ for $j \in \Phi$.

It follows that

$$\begin{array}{rcl}
\alpha_2 s_j^2 & = & \alpha_2(x_j^2 - y_j^1F^j\alpha_{i+2}) \\
& = & \alpha_2 x_j^2 - s_j^1F^j\alpha_{i+2} \\
& = & \alpha_2 x_j^2 - (x_j^1 - y_j^0F^j\alpha_{i+1})F^j\alpha_{i+2} \\
& = & \alpha_2 x_j^2 - x_j^1F^j\alpha_{i+2} \\
& = & 0.
\end{array}$$

Hence there are $y_j^2 : M_3 \to F^jM_{i+2}$ such that $\alpha_3 y_j^2 = s_j^2$ for $j \in \Phi$. Similarly, we can check that $f^2 = \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_3)\mu(y^2) + \mu(y^1)\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_2)$. By induction, we can prove that $f^\bullet$ is null-homotopic. Hence $\mathrm{Hom}_{K^b(\overline{\Lambda})}(T^\bullet, T^\bullet[i]) = 0$ for $i = 1,2,\cdots,n-3$. The second case: $i = n-2$. It is easy to check $\mathrm{Hom}_{K^b(\overline{\Lambda})}(T^\bullet, T^\bullet[n-2]) = 0$.

(2) $\mathrm{Hom}_{K^b(\overline{\Lambda})}(T^\bullet, T^\bullet[-i]) = 0$ for $i = 1,\cdots,n-2$.

The first case: $i = 1,\cdots,n-3$. Let $f^\bullet$ be a morphism in $\mathrm{Hom}_{K^b(\overline{\Lambda})}(T^\bullet, T^\bullet[i])$. We have the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & (V,M_i) \arrow[r, "{(V,\alpha_{i+1})}"] \arrow[d, "f^0"] \arrow[ddl, dashed, "\mu(g)"'] & (V,M_{i+1}) \arrow[r, "{(V,\alpha_{i+2})}"] \arrow[d, "f^1"] \arrow[dl, dashed, "\mu(s^1)"'] & (V,M_{i+2}) \arrow[r] \arrow[d, "f^2"] \arrow[dl, dashed, "\mu(s^2)"'] & \cdots \arrow[r] & (V,M_{n-2}\oplus M) \arrow[r] \arrow[d, "f^{n-i-2}"] \arrow[dl, dashed, "\mu(s^{n-i-2})"'] & 0 \\
0 \arrow[r] & P \arrow[r, "q"] \arrow[dl, dashed, "p"'] & (V,M_1) \arrow[r, "{(V,\alpha_2)}"] & (V,M_2) \arrow[r, "{(V,\alpha_3)}"] & \cdots \arrow[r] & (V,M_{n-i-2}) \arrow[r, "{(V,\alpha_{n-i-1})}"] & \cdots \\
(V,X) \arrow[ur] \arrow[urr, "{(V,\alpha_1)}"'] & & & & & &
\end{tikzcd}$$

By Lemma 2.5(1), we assume $f^j = \mu(x^j), f^{n-i-2} = \mu(x^{n-i-2})$ with $x^j = (x_k^j)_{k\in\Phi} \in \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(M_{i+j}, M_j), x^{n-i-2} = (x_k^{n-i-2})_{k\in\Phi} \in \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(M_{n-2}\oplus M, M_{n-i-2})$ for $j = 1,2,\cdots,n-i-3$. From the above commutative diagram, we can get $f^{n-i-2}\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_{n-i-1})=0$. This implies $x_j^{n-i-2}F^j\alpha_{n-i-1}=0$ for $j\in\Phi$. So there are morphisms

$$s_j^{n-i-2}:M_{n-2}\oplus M\to F^j(M_{n-i-3})$$

such that

$$x_j^{n-i-2}=s_j^{n-i-2}F^j\alpha_{n-i-2}$$

for $j\in\Phi$. We denote $(s_j^{n-i-2})_{j\in\Phi}$ by $s^{n-i-2}$. So

$$\begin{array}{rcl}\mu(s^{n-i-2})\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_{n-i-2}) & = & \mu(s^{n-i-2}\underline{\alpha_{n-i-2}}) \\ & = & \mu((s_j^{n-i-2}F^j\alpha_{n-i-2})_{j\in\Phi}) \\ & = & \mu((x_j^{n-i-2})_{j\in\Phi}) \\ & = & f^{n-i-2}.\end{array}$$

We denote $f^{n-i-3}-\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\overline{\alpha_{n-2}})\mu(s^{n-i-2})$ by $t^{n-i-3}$, and we write $t_j^{n-i-3}$ instead of $x_j^{n-i-3}-\overline{\alpha_{n-2}}s_j^{n-i-2}$ for $j\in\Phi$. Note that

$$\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\overline{\alpha_{n-2}})f^{n-i-2}=f^{n-i-3}\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_{n-i-2}).$$

Then

$$(\overline{\alpha_{n-2}}x_j^{n-i-2})_{j\in\Phi}=(x_j^{n-i-3}F^j\alpha_{n-i-2})_{j\in\Phi}.$$

We can deduce

$$\begin{array}{rcl}t_j^{n-i-3}F^j\alpha_{n-i-2} & = & (x_j^{n-i-3}-\overline{\alpha_{n-2}}s_j^{n-i-2})F^j\alpha_{n-i-2} \\ & = & x_j^{n-i-3}F^j\alpha_{n-i-2}-\overline{\alpha_{n-2}}x_j^{n-i-2} \\ & = & 0.\end{array}$$

So there exist morphisms $s_j^{n-i-3}:M_{n-3}\to F^jM_{n-i-4}$ such that

$$s_j^{n-i-3}F^j\alpha_{n-i-3}=t_j^{n-i-3}$$

for $j\in\Phi$.

We denote $(s_j^{n-i-3})_{j\in\Phi}$ by $s^{n-i-3}$. We can deduce

$$\begin{array}{rl} & \mu(s^{n-i-3})\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_{n-i-3})+\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\overline{\alpha_{n-2}})\mu(s^{n-i-2}) \\ = & \mu(s^{n-i-3})\mu(\underline{\alpha_{n-i-3}})+\mu(\overline{\alpha_{n-2}})\mu(s^{n-i-2}) \\ = & \mu((s_j^{n-i-3}F^j\alpha_{n-i-3}+\overline{\alpha_{n-2}}s_j^{n-i-2})_{j\in\Phi}) \\ = & \mu((x_j^{n-i-3})_{j\in\Phi}) \\ = & f^{n-i-3}.\end{array}$$

By induction, there are morphisms

$$\mu(s^1):\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,M_{i+1})\to\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,X),\ \mu(s^k):\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,M_{i+k})\to\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,M_k-1)$$

such that

$$f^1=\mu(s^1)\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_1)+\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_{i+2})\mu(s^2),\ f^k=\mu(s^k)\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_k)+\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_{i+k+1})\mu(s^{k+1})$$

for $k=2,\cdots,n-i-2$. Here we define $\mu(s^{n-i-1})=0$.

Hence it suffices to prove that $f^0=\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_{i+1})\mu(s^1)p$. Note that $p:\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,X)\to P$ is surjective and $\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,M_i)$ is a projective $\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V)$-module. Then there exists a morphism $\mu(g):\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,M_i)\to\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,X)$ such that $f^0=\mu(g)p$. Since $X\in\mathscr{Y}_{\mathcal{F}}^{\mathrm{F},\Phi}(\mathrm{M})$, we can get $g_j=0,s_j^1=0$ for $0\neq j\in\Phi$. Note that $f^0q=\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_{i+1})f^1$, this implies $g_0\alpha_1=\alpha_{i+1}x_0^1$. So

$$(\alpha_{i+1}s_0^1-g_0)\alpha_1=\alpha_{i+1}(x_0^1-\alpha_{i+2}s_0^2)-g_0\alpha_1=\alpha_{i+1}x_0^1-g_0\alpha_1=0.$$

This implies that $\alpha_{i+1}s_0^1 - g_0$ can factorizes through $\Sigma^{-1}\alpha_n$. By Lemma 3.2(4), we can get $\mathrm{Im}\mu(\underline{\alpha_{i+1}s_0^1 - g_0}) \subseteq I \cdot \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,X)$. So $(\mu(g) - \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_{i+1})\mu(s^1))p = 0$. This implies $f^0 = \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_{i+1})\mu(s^1)p$. Hence $f^\bullet$ is null-homotopic. The second case: $i = n-2$. We can verify similarly.

Hence $T^\bullet$ is a tilting complex over $\overline{\Lambda}$.

Clearly, the homotopy category $K^b(\overline{\Lambda})$ can be viewed as a full subcategory of $K^b(\Lambda)$. Thus, we have a ring isomorphism $\mathrm{End}_{K^b(\overline{\Lambda}\text{-Proj})}(T^\bullet) \simeq \mathrm{End}_{K^b(\Lambda)}(T^\bullet)$. Now, we will determine the endomorphism ring $\mathrm{End}_{K^b(\Lambda)}(T^\bullet)$.

Let $f^\bullet \in \mathrm{End}_{K^b(\Lambda)}(T^\bullet)$. There is an $\Lambda$-homomorphism $u^0 : \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,X) \to \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,X)$ such that $u^0 p = p f^0$, because $p : \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,X) \to P$ is an epimorphism and $\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,X)$ is a projective $\Lambda$-module. By Lemma 2.5(1), we can assume

$$u^0 = \mu(x^0),\, f^{n-2} = \mu(x^{n-2}),\, f^i = \mu(x^i)$$

with

$$x^0 = (x_i^0)_{i\in\Phi} \in \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(X), x^i = (x_j^i)_{j\in\Phi} \in \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(M_i,M_i),$$

$$x^{n-2} = (x_j^{n-2})_{j\in\Phi} \in \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(M_{n-2}\oplus M)$$

for $i = 1,\cdots,n-3$.

$$\begin{tikzcd}
(V,X) \arrow[dd, dashed, "u^0=\mu(x^0)"'] \arrow[dr, "p"] \arrow[drr, "{(V,\alpha_1)}"] & & & & & & \\
{} \arrow[r, "p"] & P \arrow[r, "q"'] \arrow[dd, "f^0"'] & (V,M_1) \arrow[r, "{(V,\alpha_2)}"] \arrow[dd, "f^2=\mu(x^1)"] & (V,M_2) \arrow[r] \arrow[dd, "f^2=\mu(x^2)"] & \cdots \arrow[r, "{(V,\overline{\alpha_{n-2}})}"] & (V,M_{n-2}\oplus M) \arrow[r] \arrow[dd, "f^{n-2}=\mu(x^{n-2})"] & 0 \\
(V,X) \arrow[dr, "p"] \arrow[drr, "{(V,\alpha_1)}"] & & & & & & \\
{} \arrow[r, "p"] & P \arrow[r, "q"'] & (V,M_1) \arrow[r, "{(V,\alpha_2)}"] & (V,M_2) \arrow[r] & \cdots \arrow[r, "{(V,\overline{\alpha_{n-2}})}"] & (V,M_{n-2}\oplus M) \arrow[r] & 0
\end{tikzcd}$$

By the commutativity of the above diagram, we have

$$\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_1) f^1 = \mu(x^0)\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_1),$$

$$\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_i) f^i = f^{i-1}\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_i) \text{ for } i = 2,\cdots,n-3,$$

$$\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\overline{\alpha_{n-2}}) f^{n-2} = f^{n-3}\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\overline{\alpha_{n-2}}).$$

It follows that

$$\alpha_1 x_j^1 = x_j^0 F^j \alpha_1,$$

$$\alpha_i x_j^i = x_j^{i-1} F^j \alpha_i \text{ for } i = 2,\cdots,n-3,$$

$$\overline{\alpha_{n-2}} x_j^{n-2} = x_j^{n-3} F^j \overline{\alpha_{n-2}}$$

for $j \in \Phi$ from Lemma 2.5(1).

By Lemma 2.4, we can form the following commutative diagram in $\mathcal{F}$:

$$\begin{tikzcd}
X \arrow[r, "\alpha_1"] \arrow[d, "x_i^0"'] & M_1 \arrow[r, "\alpha_2"] \arrow[d, "x_i^1"'] & M_2 \arrow[r, "\alpha_3"] \arrow[d, "x_i^2"'] & \cdots \arrow[r, "\overline{\alpha_{n-2}}"] & M_{n-2}\oplus M \arrow[r, "\overline{\alpha_{n-1}}"] \arrow[d, "x_i^{n-2}"'] & W \arrow[r, "\overline{\alpha_n}"] \arrow[d, dashed, "h_i"'] & \Sigma X \arrow[d, "\Sigma x_i^0"] \\
F^iX \arrow[r, "F^i\alpha_1"] & F^iM_1 \arrow[r, "F^i\alpha_2"] & F^iM_2 \arrow[r, "F^i\alpha_3"] & \cdots \arrow[r, "F^i(\overline{\alpha_{n-2}})"] & F^i(M_{n-2}\oplus M) \arrow[r, "F^i\overline{\alpha_{n-1}}"] & F^iW \arrow[r] & \Sigma(F^iX)
\end{tikzcd} \qquad (\star)$$

where $h_i \in \mathrm{Hom}_{\mathcal{F}}(W,F^iW)$. Thus, for each $f^\bullet \in \mathrm{End}_{K^b(\Lambda)}(T^\bullet)$, we can get an element $h := (h_i)_{i\in\Phi} \in \Gamma$. Define the following correspondence:

$$\Theta : \mathrm{End}_{K^b(\Lambda)}(T^\bullet) \to \overline{\Gamma} = \Gamma/J,$$

$$f^\bullet \mapsto h + J.$$

Now, we will prove that the correspondence $\Theta$ is a ring homomorphism. The proof is divided into four steps.

Step 1. we will prove that $\Theta$ is well-defined. Suppose that $f^\bullet \in \mathrm{End}_{K^b(\Lambda)}(T^\bullet)$ is null-homotopic, that is, there are

$$r_1 : \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,M_1) \to P,\ r_i : \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,M_i) \to \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,M_{i-1}), i = 2,\cdots,n-3,$$

$$r_{n-2} : \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,M_{n-2}\oplus M) \to \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,M_{n-3}),$$

such that

$$f^0 = qr_1,\ f^1 = r_1 q + \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_2) r_2,$$

$$f^i = r_i \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_i) + \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_{i+1} r_{i+1}) r_{i+1} \text{ for } i = 2,\cdots,n-3,$$

$$f^{n-3} = r_{n-3}\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_{n-3}) + \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\overline{\alpha_{n-2}}) r_{n-2},\ f^{n-2} = r_{n-2}\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\overline{\alpha_{n-2}}).$$

Since $p$ is surjective and $\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,M_1)$ is projective, there is a morphism $s : \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,M_1) \to \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,X)$ such that $r_1 = sp$. By Lemma 2.5(1), we can assume

$$s = \mu(t), r_{n-2} = \mu(l)$$

with

$$t = (t_i)_{i\in\Phi} \in \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(M_1,X), l = (l_i)_{i\in\Phi} \in \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(M_{n-2}\oplus M, M_{n-3}).$$

$$\begin{tikzcd}
(V,X) \arrow[dd, "u^0=\mu(x^0)"'] \arrow[dr, "p"] & & & & & & \\
& P \arrow[r, "q"] \arrow[dd, "f^0"'] & (V,M_1) \arrow[r, "{(V,\alpha_2)}"] \arrow[dll, dashed, "s=\mu(t)"'] \arrow[ddl, "r_1"] \arrow[dd, "f^2=\mu(x^1)"] & (V,M_2) \arrow[r] \arrow[ddl, "r_2"] \arrow[dd, "f^2=\mu(x^2)"] & \cdots \arrow[r, "{(V,\overline{\alpha_{n-2}})}"] & (V,M_{n-2}\oplus M) \arrow[r] \arrow[ddl, "r_{n-2}=\mu(l)"'] \arrow[dd, "f^{n-2}=\mu(x^{n-2})"] & 0 \\
(V,X) \arrow[dr, "p"] & & & & & & \\
& P \arrow[r, "q"'] & (V,M_1) \arrow[r, "{(V,\alpha_2)}"'] & (V,M_2) \arrow[r] & \cdots \arrow[r, "{(V,\overline{\alpha_{n-2}})}"'] & (V,M_{n-2}\oplus M) \arrow[r] & 0
\end{tikzcd}$$

By the definition of $\mathscr{Y}_{\mathcal{F}}^{\mathrm{F},\Phi}(\mathrm{M})$, we have $t_i = 0$ for $0 \neq i \in \Phi$. It follows that

$$\mu(x^0 - \underline{\alpha_1 t_0})p = (u^0 - pqs)p = 0,\ \mu(x^{n-2}) = \mu(l)\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\overline{\alpha_{n-2}}).$$

It follows immediately that

$$\mathrm{Im}\mu(x^0 - \underline{\alpha_1 t_0}) \subseteq I \cdot \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,X),\ (x_i^{n-2})_{i\in\Phi} = (l_i F^i \overline{\alpha_{n-2}})_{i\in\Phi}.$$

By Lemma 3.2(2), we can get that $x_i^0 = 0$ for $0 \neq i \in \Phi$ and $x_0^0 - \alpha_1 t_0$ factorizes through $\mathrm{add}(M)$ and $\Sigma^{-1}\alpha_n$. So $x_0^0 - \alpha_1 t_0 = ab$ for some morphisms $a : X \to M'$ and $b : M' \to X$ with $M' \in \mathrm{add}(M)$. Since $\alpha_1 : X \to M_1$ is a left $(\mathrm{add}(M), F, \Phi)$-approximation of $X$, there is a morphism $c : M_1 \to M'$ such that $a = \alpha_1 c$. It follows that

$$x_0^0 = ab + \alpha_1 t_0 = \alpha_1 cb + \alpha_1 t_0 = \alpha_1(cb + t_0).$$

Since $\overline{\alpha_{n-1}} h_i = x_i^{n-2} F^i \overline{\alpha_{n-1}} = l_i F^i \overline{\alpha_{n-2}} F^i \overline{\alpha_{n-1}} = 0$, $h_i$ factorizes through $\overline{\alpha_n}$. So $h_i|_M = 0$ since $\overline{\alpha_n}|_M = 0$. Since $x_i^0 = 0$ for $0 \neq i \in \Phi$ and $Y \in \mathscr{X}_{\mathcal{F}}^{\mathrm{F},\Phi}(M)$, we deduce $h_i|_Y = 0$. It follows that $h_i = 0$ for $0 \neq i \in \Phi$.

We have $\overline{\alpha_{n-1}} h_0 = x_0^{n-2}\overline{\alpha_{n-1}} = l_0 \overline{\alpha_{n-2}\alpha_{n-1}} = 0$ which implies that $h_0$ factorizes through $\overline{\alpha_n}$. Since $h_0 \overline{\alpha_n} = \overline{\alpha_n}\Sigma x_0^0 = \overline{\alpha_n}(\Sigma\alpha_1)\Sigma(cb + t) = 0$, the morphism $h_0$ factorizes through $M_{n-2}\oplus M$ which is in $\mathrm{add}(M)$. Thus, $h$ is an element in $J$. So $\Theta$ is well-defined.

Step 2. we will prove that the map $\Theta$ is injective. Suppose that $\Theta(f^\bullet) = h + J = J$. It suffices to prove that $f^\bullet$ is null-homotopic. By the definition of $J$, we have that $h_i = 0$ for $0 \neq i \in \Phi$, and $h_0$ factorizes through $\mathrm{add}(M)$ and $\overline{\alpha_n}$. Since $h_i = 0$ for $0 \neq i \in \Phi$ and $h_0$ factorizes through $\overline{\alpha_n}$, we have $x_i^{n-2} F^i \overline{\alpha_{n-1}} = 0$, by the commutativity of $(\star)$. Thus, there is a morphism

$r_i^{n-2}: M_{n-2}\oplus M\to F^iM_{n-3}$ such that $x_i^{n-2}=r_i^{n-2}F^i\overline{\alpha_{n-2}}$ for $i\in\Phi$. Let us denote $r^{n-2}$ the morphism $(r_i^{n-2})_{i\in\Phi}$. Then $\mu(r^{n-2})\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\overline{\alpha_{n-2}})=\mu(x^{n-2})$. And we will denote $s^{n-3}$ the morphism $f^{n-3}-\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\overline{\alpha_{n-2}})\mu(r^{n-2})$. Thus $s_i^{n-3}=x_i^{n-3}-\overline{\alpha_{n-2}}r_i^{n-2}$ for $i\in\Phi$. Since $f^{n-3}\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\overline{\alpha_{n-2}})=\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\overline{\alpha_{n-2}})f^{n-2}$, that is, $(x_i^{n-3}F^i\overline{\alpha_{n-2}})_{i\in\Phi}=(\overline{\alpha_{n-2}}x_i^{n-2})_{i\in\Phi}$, we can deduce

$$\begin{aligned}(x_i^{n-3}-\overline{\alpha_{n-2}}r_i^{n-2})F^i(\overline{\alpha_{n-2}}) &= x_i^{n-3}F^i\overline{\alpha_{n-2}}-\overline{\alpha_{n-2}}r_i^{n-2}F^i\overline{\alpha_{n-2}}\\ &= x_i^{n-3}F^i\overline{\alpha_{n-2}}-\overline{\alpha_{n-2}}x_i^{n-2}\\ &= 0.\end{aligned}$$

Thus, there are morphisms $r_i^{n-3}: M_{n-3}\to F^iM_{n-4}$ such that $x_i^{n-3}-\overline{\alpha_{n-2}}r_i^{n-2}=r_i^{n-3}F^i\alpha_{n-3}$ for $i\in\Phi$. We denote $(r_i^{n-3})_{i\in\Phi}$ by $r^{n-3}$.

Note that $x_i^{n-3}-\overline{\alpha_{n-2}}r_i^{n-2}=r_i^{n-3}F^i\alpha_{n-3}$ for $i\in\Phi$. Then

$$\begin{aligned}\mu(r^{n-3})\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_{n-3})+\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\overline{\alpha_{n-2}})\mu(r^{n-2}) &= \mu((r_i^{n-3}F^i\alpha_{n-3}+\overline{\alpha_{n-2}}r_i^{n-2})_{i\in\Phi})\\ &= \mu((x_i^{n-3})_{i\in\Phi})\\ &= \mu(x^{n-3})\\ &= f^{n-3}.\end{aligned}$$

By induction, we can construct

$$r^i:=(r_j^i)_{j\in\Phi}\in\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(M_i,M_{i-1})$$

and

$$s^i:=f^i-\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_{i+1})\mu(s^{i+1})=(f_j^i-\alpha_{i+1}s_j^{i+1})_{j\in\Phi}\in\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(M_i,M_i)$$

satisfying that

$$f^i=\mu(s^i)\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_i)+\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_{i+1})\mu(s^{i+1})$$

for $i=2,\cdots,n-4$. Let us denote $s^1$ the morphism

$$f^1-\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_2)\mu(r^2)=(f_i^1-\alpha_2r_i^2)_{i\in\Phi}\in\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(M_1,M_1).$$

Note that $\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_2)f^2=f^1\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_2)$, that is $(\alpha_2x_i^2)_{i\in\Phi}=(x_i^1F^i\alpha_2)_{i\in\Phi}$. Then

$$\begin{aligned}s_i^1F^i\alpha_2 &= (x_i^1-\alpha_2r_i^2)F^i\alpha_2\\ &= x_i^1F^i\alpha_2-\alpha_2r_i^2F^i\alpha_2\\ &= x_i^1F^i\alpha_2-\alpha_2(x_i^2-\alpha_3r_i^3)\\ &= x_i^1F^i\alpha_2-\alpha_2x_i^2\\ &= 0.\end{aligned}$$

Thus, there are morphisms $r_i^1: M_1\to F^iX$ such that $r_i^1F^i\alpha_1=s_i^1=x_i^1-\alpha_2r_i^2$ for $i\in\Phi$. We define $r^1:=(r_i^1)_{i\in\Phi}$. Since $X\in\mathscr{Y}_{\mathcal{F}}^{\mathrm{F},\Phi}(\mathrm{M})$, we have $r_i^1=0$ for $0\neq i\in\Phi$. Consequently,

$$f^1=\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_2)\mu(r^2)+\mu(r^1)\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_1).$$

We can get $\overline{\alpha_n}\Sigma x_i^0=0$ by the assumption that $h_i=0$ for $0\neq i\in\Phi$. Thus, $x_i^0$ factorizes through $\alpha_1$. Since $X\in\mathscr{Y}_{\mathcal{F}}^{\mathrm{F},\Phi}(\mathrm{M})$, we can obtain $x_i^0=0$ for $0\neq i\in\Phi$.

Note that $u^0\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_1)=\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_1)f^1$. Then

$$\begin{aligned}(x_0^0-\alpha_1r_0^1)\alpha_1 &= x_0^0\alpha_1-\alpha_1r_0^1\alpha_1\\ &= x_0^0\alpha_1-\alpha_1(x_0^1-\alpha_2r_0^2)\\ &= x_0^0\alpha_1-\alpha_1x_0^1\\ &= 0.\end{aligned}$$

This implies that $x_0^0-\alpha_1r_0^1$ factorizes through $\Sigma^{-1}\overline{\alpha_n}$.

Now, we prove that $x_0^0 - \alpha_1 r_0^1$ factorizes through $\mathrm{add}(M)$. Since $\alpha_1 r_0^1$ factorizes through $\mathrm{add}(M)$, it suffices to prove that $x_0^0$ can factorize through $\mathrm{add}(M)$. By assumption, $h_0$ can factorize through $\mathrm{add}(M)$. So there are morphisms $a: W \to M'$ and $b: M' \to W$ such that $h_0 = ab$ for $M' \in \mathrm{add}(M)$. Since $\overline{\alpha_{n-1}}$ is right $(\mathrm{add}(M), F, -\Phi)$-approximation of $W$, there is a morphism $c: M' \to M_{n-2} \oplus M$ such that $b = c\overline{\alpha_{n-1}}$. Consequently,

$$\overline{\alpha_n}\Sigma x_0^0 = h_0\overline{\alpha_n} = ac\overline{\alpha_{n-1}\alpha_n} = 0.$$

This implies that $x_0^0$ can factorize through $M_1$ which belongs to $\mathrm{add}(M)$. By Lemma 3.2(3), we deduce $\mathrm{Im}\mu(x^0 - \underline{\alpha_1 r_0^1}) \subseteq I \cdot \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,X)$. Thus,

$$p(f^0 - q\mu(\underline{r_0^1})p) = pf^0 - pq\mu(\underline{r_0^1})p = (u^0 - pq\mu(\underline{r_0^1}))p = 0.$$

Hence $f^0 = q\mu(\underline{r_0^1})p$. Altogether, we have proven that $f^\bullet$ is null-homotopic.

Step 3. we will prove that the map $\Theta$ is surjective. Let $h = (h_i)_{i\in\Phi}$ with $h_i : W \to F^iW$ for $i \in \Phi$. Since $\overline{\alpha_{n-1}}$ is a right $(\mathrm{add}(M), F, -\Phi)$-approximation of $W$, there is a commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "\alpha_1"] \arrow[d, "x_i^0"'] & M_1 \arrow[r, "\alpha_2"] \arrow[d, "x_i^1"'] & \cdots \arrow[r, "\alpha_{n-3}"] & M_{n-3} \arrow[r, "\overline{\alpha_{n-2}}"] \arrow[d, "x_i^{n-3}"'] & M_{n-2}\oplus M \arrow[r, "\overline{\alpha_{n-1}}"] \arrow[d, "x_i^{n-2}"'] & W \arrow[r, "\overline{\alpha_n}"] \arrow[d, "h_i"'] & \Sigma X \arrow[d, "\Sigma x_i^0"] \\
F^iX \arrow[r, "F^i\alpha_1"] & F^iM_1 \arrow[r, "F^i\alpha_2"] & \cdots \arrow[r] & F^iM_{n-3} \arrow[r, "F^i\overline{\alpha_{n-2}}"] & F^i(M_{n-2}\oplus M) \arrow[r, "F^i\overline{\alpha_{n-1}}"] & F^iW \arrow[r] & \Sigma F^iX
\end{tikzcd}$$

We denote $(x_i^j)_{i\in\Phi}$ by $x^j$ for $j = 0, 1, \cdots, n-2$. From the commutative diagram, we have $\overline{\alpha_{n-2}}x_i^{n-2} = x_i^{n-3}F^i\overline{\alpha_{n-2}}$ and $\alpha_j x_i^j = x_i^{j-1}F^i\alpha_j$ for $j = 1, 2, \cdots, n-2$. This implies

$$\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V, \overline{\alpha_{n-2}})\mu(x^{n-2}) = \mu(x^{n-3})\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V, \overline{\alpha_{n-2}}),$$

$$\mu(\underline{\alpha_j})\mu(x^j) = \mu(x^{j-1})\mu(\underline{\alpha_j}) \text{ for } j = 1, \cdots, n-2.$$

So we have the following commutative diagram

$$\begin{tikzcd}
(V,X) \arrow[dr, "p"] \arrow[drr, "{(V,\alpha_1)}"] \arrow[dd, "u^0=\mu(x^0)"'] & & & & & & \\
{} \arrow[r, "p"] & P \arrow[r, "q"'] \arrow[dd, dashed, "f^0"'] & (V,M_1) \arrow[r, "{(V,\alpha_2)}"] \arrow[dd, "f^2=\mu(x^1)"] & (V,M_2) \arrow[r] \arrow[dd, "f^2=\mu(x^2)"] & \cdots \arrow[r, "{(V,\overline{\alpha_{n-2}})}"] & (V,M_{n-2}\oplus M) \arrow[r] \arrow[dd, "f^{n-2}=\mu(x^{n-2})"] & 0 \\
(V,X) \arrow[dr, "p"] \arrow[drr, "{(V,\alpha_1)}"] & & & & & & \\
{} \arrow[r, "p"] & P \arrow[r, "q"'] & (V,M_1) \arrow[r, "{(V,\alpha_2)}"] & (V,M_2) \arrow[r] & \cdots \arrow[r, "{(V,\overline{\alpha_{n-2}})}"] & (V,M_{n-2}\oplus M) \arrow[r] & 0
\end{tikzcd}$$

We conclude from $\mu(x^0)(I \cdot \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,X)) \subseteq I \cdot \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,X)$, that $\mu(x^0)$ induces a morphism $f^0 : P \to P$ satisfying that $pf^0 = \mu(x^0)p$, and finally that

$$p(f^0q - q\mu(x^1)) = \mu(x^0)\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_1) - \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,\alpha_1)\mu(x^1) = 0.$$

Note that $p$ is surjective. Then $f^0q = q\mu(x^1)$. Define $f^i = \mu(x^i)$ for $i = 1, \cdots, n-2$. Hence $\Theta$ is surjective.

Step 4. we will prove that the map $\Theta$ is a ring homomorphism. Take $f^\bullet$ and $g^\bullet$ in $\mathrm{End}_{K^b(\Lambda)}(T^\bullet)$. Since $p$ is surjective and $\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,X)$ is projective as left $\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V)$-module, there is a map $\mu(x^0)$ : $\mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,X) \to \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,X)$ such that $\mu(x^0)p = pf^0$. Similarly, there is a map $\mu(y^0) : \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,X) \to \mathrm{E}_{\mathcal{F}}^{\mathrm{F},\Phi}(V,X)$ such that $\mu(y^0)p = pg^0$. Suppose that $f^i = \mu(x^i), g^i = \mu(y^i)$ for $i = 1, \cdots, n-2$. Define $h := (h_i)_{i\in\Phi}$ and $h' := (h'_i)_{i\in\Phi}$ be in $\Gamma$ such that

$$\overline{\alpha_{n-1}}h_i = x_i^{n-2}F^i\overline{\alpha_{n-1}}, \quad \overline{\alpha_n}\Sigma x_i^0 = h_i(F^i\overline{\alpha_n})\delta(F,i,X,1)$$
$$\overline{\alpha_{n-1}}h'_i = y_i^{n-2}F^i\overline{\alpha_{n-1}}, \quad \overline{\alpha_n}\Sigma y_i^0 = h'_i(F^i\overline{\alpha_n})\delta(F,i,X,1)$$

for $i \in \Phi$. By definition, we have $\Theta(f^\bullet) = h + J, \Theta(g^\bullet) = h' + J$ and

$$\Theta(f^\bullet)\Theta(g^\bullet) = (\sum_{\substack{i,j\in\Phi\\ i+j=k}} h_i F^i h'_j)_{k\in\Phi} + J.$$

Now, we calculate $\Theta(f^\bullet g^\bullet)$.

$$x^{n-2}y^{n-2} = (\sum_{\substack{i,j\in\Phi\\ i+j=k}} x_i^{n-2}F^i y_j^{n-2})_{k\in\Phi},\ x^0y^0 = (\sum_{\substack{i,j\in\Phi\\ i+j=k}} x_i^0 F^i y_j^0)_{k\in\Phi}.$$

For each $k \in \Phi$

$$\begin{aligned}
\overline{\alpha_{n-1}}(\textstyle\sum_{\substack{i,j\in\Phi\\ i+j=k}} h_iF^ih'_j) &= \textstyle\sum_{\substack{i,j\in\Phi\\ i+j=k}} \overline{\alpha_{n-1}}h_iF^ih'_j\\
&= \textstyle\sum_{\substack{i,j\in\Phi\\ i+j=k}} x_i^{n-2}F^i(y_j^{n-2}F^j\overline{\alpha_{n-1}})\\
&= \textstyle\sum_{\substack{i,j\in\Phi\\ i+j=k}} x_i^{n-2}F^iy_j^{n-2}F^k\overline{\alpha_{n-1}}.
\end{aligned}$$

$$\begin{aligned}
\overline{\alpha_n}\Sigma(x^0y^0)_k &= \overline{\alpha_n}\Sigma(\textstyle\sum_{\substack{i,j\in\Phi\\ i+j=k}} x_i^0F^iy_j^0)\\
&= \textstyle\sum_{\substack{i,j\in\Phi\\ i+j=k}} \overline{\alpha_n}(\Sigma x_i^0)(\Sigma F^iy_j^0)\\
&= \textstyle\sum_{\substack{i,j\in\Phi\\ i+j=k}} h_iF^i\overline{\alpha_n}\delta(F,i,X,1)(\Sigma F^iy_j^0)\\
&= \textstyle\sum_{\substack{i,j\in\Phi\\ i+j=k}} h_iF^jh'F^k\overline{\alpha_n}\delta(F,k,X,1).
\end{aligned}$$

So $\Theta(f^\bullet g^\bullet) = \Theta(f^\bullet)\Theta(g^\bullet)$. Thus, $\Theta$ is a ring homomorphism. $\square$

If $\mathcal{F}$ is a triangulated $R$-category, we can get the main result in [14]. Combined with [11, Theorem 3.1], we can get the following corollary.

**Corollary 3.3.** *Let $\Phi$ be an admissible subset of $\mathbb{N}$. Let $\mathcal{F}_3$ be a triangulated $k$-category with an $(n-2)$-cluster tilting subcategory $\mathcal{F}$, which is closed under $\Sigma_3^{n-2}$, where $\Sigma_3$ denotes the suspension functor in $\mathcal{F}_3$. Suppose that there exists a diagram*

$$\begin{tikzcd}[column sep=small]
& X_2 \arrow[rr, "\alpha_2"] \arrow[dr] & & X_3 \arrow[rr] \arrow[dr] & & X_4 \arrow[r] \arrow[dr] & \cdots & & \cdots \arrow[rr] & & X_{n-1} \arrow[dr, "\alpha_{n-1}"] & \\
X_1 \arrow[ur, "\alpha_1"] & & X_{2.5} \arrow[ll, dashed] \arrow[ur] & & X_{3.5} \arrow[ll, dashed] \arrow[ur] & & \cdots \arrow[ll, dashed] & & & X_{n-1.5} \arrow[lll, dashed] \arrow[ur] & & X_n \arrow[ll, dashed]
\end{tikzcd}$$

*in $\mathcal{F}_3$, satisfying that*

(1) $\alpha_1 : X_1 \to X_2$ *is a left* $(\mathrm{add}(X), F, \Phi)$*-approximation of* $X_1$

(2) $\alpha_{n-1} : X_{n-1} \to X_n$ *is a right* $(\mathrm{add}(X), F, -\Phi)$*-approximation of* $X_n$*,*

(3) $X_1 \in \mathscr{Y}_{\mathcal{F}}^{\Sigma_3^{n-2},\Phi}(X)$*,* $X_{n-1} \in \mathscr{X}_{\mathcal{F}}^{\Sigma_3^{n-2},\Phi}(X)$*,*

*where $X$ is the direct sum of $X_i$ for $i = 2,3,\cdots,n-1$.*

*Then we can get that the two algebras $\mathrm{E}_{\mathcal{F}}^{\Sigma_3^{n-2},\Phi}(X_1 \oplus X)/I$ and $\mathrm{E}_{\mathcal{F}}^{\Sigma_3^{n-2},\Phi}(X_{n-1} \oplus X)/J$ are derived equivalent, where $\mathscr{X}_{\mathcal{F}}^{\Sigma_3^{n-2},\Phi}(X), \mathscr{Y}_{\mathcal{F}}^{\Sigma_3^{n-2},\Phi}(X), I$ and $J$ are defined as in Theorem* 1.1.

**Proof.** This follows from [11, Theorem 3.1] and Theorem 1.1. $\square$

In [20], Iyama and Yoshino introduced Auslander-Reiten $n$-angles in $(n-2)$-cluster tilting subcategories of triangulated $k$-categories and proved that they always exist. Let $\mathcal{T}$ be a Krull-Schmidt triangulated category with shift functor $\Sigma_3$, and let $\mathcal{S}$ be an $n$-cluster tilting subcategory of $\mathcal{T}$.

$$X_{i+1} \overset{b_{i+1}}{\to} C_i \overset{a_i}{\to} X_i \to \Sigma_3 X_{i+1} \quad (0 \le i < n).$$

are triangles in $\mathcal{T}$. A complex

$$X_n \overset{b_n}{\to} C_{n-1} \overset{a_{n-1}b_{n-1}}{\to} C_{n-2} \overset{a_{n-2}b_{n-2}}{\to} \cdots \overset{a_2b_2}{\to} C_1 \overset{a_1b_1}{\to} C_0 \overset{a_0}{\to} X_0$$

is called an *Auslander-Reiten* $(n+2)$*-angle* if the following conditions are satisfied.

(1) $X_n, X_0$ and $C_i (0 \le i < n)$ belong to $\mathcal{S}$.

(2) $a_0$ is a sink map of $X_0$ in $\mathcal{S}$ and $b_n$ is a source map of $X_n$ in $\mathcal{S}$.

(3) $a_i$ is a minimal right $\mathcal{S}$-approximation of $X_i$ for $0 < i < n$.

(4) $b_i$ is a minimal left $\mathcal{S}$-approximation of $X_i$ for $0 < i < n$.

As a corollary of Corollary 3.3, we can establish a relationship between Auslander-Reiten $n$-angle and derived equivalences.

**Corollary 3.4.** *Let $\mathcal{T}$ be a Krull- Schmidt triangulated k-category with shift functor $\Sigma_3$, and let $\mathcal{S}$ be an $(n-2)$-cluster tilting subcategory of $\mathcal{T}$, which is closed under $\Sigma_3^{n-2}$. Suppose that*

$$X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} X_3 \to \cdots \to X_n$$

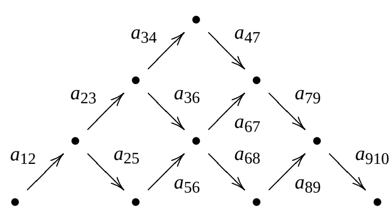
*is an Auslander-Reiten n-angle in $\mathcal{S}$ and $X_1, X_n \notin \mathrm{add}(\oplus_{i=2}^{n-1} X_i)$. Then the two rings $\mathrm{End}_{\mathcal{S}}(\oplus_{i=1}^{n-1} X_i)/I$ and $\mathrm{End}_{\mathcal{S}}(\oplus_{i=2}^{n} X_i)/J$ are derived equivalent, where $I, J$ are defined as in Theorem 1.1.*

**Proof.** By [20, Proposition 3.9] and Corollary 3.3, we can get the conclusion. $\square$

# 4 Examples

In this part, we give an example to illustrate the main result of this paper.

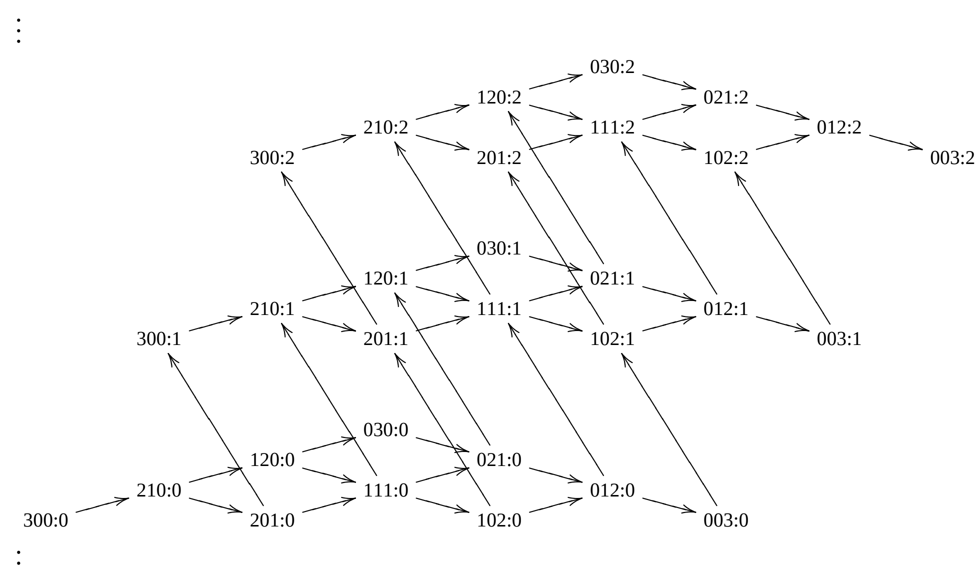
Consider the 2-representation finite algebra $A$ of type ‘A’. The quiver with relation of $A$ is given by the following diagram.




with relations $\{a_{23}a_{36} - a_{25}a_{56}, a_{34}a_{47} - a_{36}a_{67}, a_{67}a_{79} - a_{68}a_{89}, a_{12}a_{25}, a_{56}a_{68}, a_{89}a_{910}\}$.

Assume that $\nu := DA \otimes_A^L - : D(A) \to D(A)$ is the derived functor of Nakayama functor and $\nu_n = \nu[-n]$. By [11, Theorem 1], The 2-cluster tilting subcategory $\mathcal{U} = \mathrm{add}\{\nu_2^i A \mid i \in \mathbb{Z}\}$ of $D(A)$ is a 4-angulated category with suspension functor $\Sigma_4$. And the Auslander-Reiten quiver of $\mathcal{U}$ is given as follows. (see [18, 19])

$\widetilde{Q^{(2,4)}}$ :



Note that the functor $\nu_2$ can be viewed as the automorphism of $\widetilde{Q^{(2,4)}}$ which send $(l_1, l_2, l_3 : i)$ to $(l_1, l_2, l_3 : i-1)$. Select a source map $f_1 : 111:0 \to 210:1 \oplus 021:0 \oplus 102:0$. There is a 4-angle

$$111:0 \xrightarrow{f_1} 210:1 \oplus 021:0 \oplus 102:0 \to X_3 \xrightarrow{g} X_4 \to \Sigma_4 111:0 \qquad (*)$$

in $\mathcal{U}$. By [20, Proposition 3.9], $(*)$ is an Auslander-Reiten 4-angle in $\mathcal{U}$ and $g$ is a sink map. By [20, Theorem 3.10], we have $111:0 = \nu_2 X_4$. Thus,

$$111:0 \xrightarrow{f_1} 210:1 \oplus 021:0 \oplus 102:0 \xrightarrow{f_2} 120:1 \oplus 201:1 \oplus 012:0 \xrightarrow{f_3} 111:1 \xrightarrow{f_4} \Sigma_4 111:0$$

is an Auslander-Reiten 4-angle in $\mathcal{U}$.

We denote $210:1 \oplus 021:0 \oplus 120:1 \oplus 102:0 \oplus 201:1 \oplus 012:0$ by $M$. Clearly, the morphism $f_1 : 111:0 \to 210:1 \oplus 021:0 \oplus 102:0$ is a left add$(M)$-approximation of $111:0$ and the morphism $f_3 : 120:1 \oplus 201:1 \oplus 012:0 \to 111:1$ is a right add$(M)$-approximation of $111:1$. By Corollary 3.4, we can get that the two rings $\mathrm{End}_{D(A\text{-mod})}(111:0 \oplus M)/I$ and $\mathrm{End}_{D(A\text{-mod})}(M \oplus 111:1)/J$ are derived equivalent where $I, J$ are defined as in Theorem 1.1.

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