Multiplicative properties of a quantum Caldero-Chapoton map associated to valued quivers

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Abstract We prove a multiplication theorem of a quantum Caldero-Chapoton map associated to valued quivers which extends the results in [8][6]. As an application, when Q is a valued quiver of finite type or rank 2, we obtain that the algebra $\mathscr{A}\mathscr{H}_{|k|}(Q)$ generated by all cluster characters (see Definition 1) is exactly the quantum cluster algebra $\mathscr{E}\mathscr{H}_{|k|}(Q)$ and various bases of the quantum cluster algebras of rank 2 can naturally be deduced.

Keywords cluster variable, quantum cluster algebra
MSC 16G20

1 Introduction

Ever since the emergency of cluster algebras, the close relation between it and quiver representations has always been emphasized. One interesting viewpoint is to consider cluster algebras as some kind of Hall algebras of quiver representations, that was particularly enhanced by [4] in which the so-called Caldero-Chapoton formula (or map, or character) was invented. Then the multiplication formulas ([5],[7],[16]) of Caldero-Chapoton characters become important, especially in the construction of integral bases of cluster algebras (e.g. see [8, 9]).

In [19], D. Rupel obtained a quantum analogue of the Caldero-Chapoton formula, which is crucial for the study of quantum cluster algebras. Unlike in the cluster algebras, it does not generally hold that $X_N X_M = |k|^{\pm\frac{1}{2}d_{N\oplus M}} X_{N\oplus M}$ for any $d_{N\oplus M} \in \mathbb{Z}$. A natural question to ask is whether the quantized Caldero-Chapoton formula could be extended to the cluster category. In [6], this aim was achieved for equally-valued quivers and a multiplication formula was verified, which implies that for finite type the algebra $\mathscr{A}\mathscr{H}_{|k|}(Q)$ generated by all cluster characters (see Definition 1) is exactly the quantum cluster algebra $\mathscr{E}\mathscr{H}_{|k|}(Q)$.

In this paper, we will extend the results in [6] to valued quivers and prove

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two multiplication formulas therein. As an application, when Q is a valued quiver of finite type or rank 2, we prove that the algebra $\mathscr{AH}_{|k|}(Q)$ is exactly the quantum cluster algebra $\mathscr{EH}_{|k|}(Q)$. In particular, we obtain various bases of the quantum cluster algebras of rank 2 by using the standard monomials in [3].

2 Preliminaries and statement of the main result

2.1 Definition of quantum cluster algebras Let L be a lattice of rank m and $\Lambda : L \times L \to \mathbb{Z}$ a skew-symmetric bilinear form. Note that Λ can be identified with an $m \times m$ skew-symmetric matrix which still denoted by Λ if there is no confusion. Set a formal variable q and the ring of integer Laurent polynomials $\mathbb{Z}[q^{\pm 1/2}]$. Define the *based quantum torus* associated to the pair (L, Λ) to be the $\mathbb{Z}[q^{\pm 1/2}]$-algebra $\mathscr{T}$ with a distinguished $\mathbb{Z}[q^{\pm 1/2}]$-basis $\{X^e : e \in L\}$ and the multiplication

$$X^e X^f = q^{\Lambda(e,f)/2} X^{e+f}.$$

It is known that $\mathscr{T}$ is contained in its skew-field of fractions $\mathscr{F}$. A *toric frame* in $\mathscr{F}$ is a map $M : \mathbb{Z}^m \to \mathscr{F} \setminus \{0\}$ given by

$$M(\mathbf{c}) = \varphi(X^{\eta(\mathbf{c})})$$

where φ is an automorphism of $\mathscr{F}$ and $\eta : \mathbb{Z}^m \to L$ is an isomorphism of lattices. By the definition, the elements $M(\mathbf{c})$ form a $\mathbb{Z}[q^{\pm 1/2}]$-basis of the based quantum torus $\mathscr{T}_M := \varphi(\mathscr{T})$ and satisfy the following relations:

$$M(\mathbf{c})M(\mathbf{d}) = q^{\Lambda_M(\mathbf{c},\mathbf{d})/2} M(\mathbf{c}+\mathbf{d}),\ M(\mathbf{c})M(\mathbf{d}) = q^{\Lambda_M(\mathbf{c},\mathbf{d})} M(\mathbf{d})M(\mathbf{c}),$$
$$M(\mathbf{0}) = 1,\ M(\mathbf{c})^{-1} = M(-\mathbf{c}),$$

where Λ_M is the skew-symmetric bilinear form on $\mathbb{Z}^m$ obtained from the lattice isomorphism η. Let Λ_M be the skew-symmetric $m \times m$ matrix defined by $\lambda_{ij} = \Lambda_M(e_i, e_j)$ where $\{e_1, \ldots, e_m\}$ is the standard basis of $\mathbb{Z}^m$. Given a toric frame M, let $X_i = M(e_i)$. Then we have

$$\mathscr{T}_M = \mathbb{Z}[q^{\pm 1/2}]\langle X_1^{\pm 1}, \ldots, X_m^{\pm 1} : X_i X_j = q^{\lambda_{ij}} X_j X_i \rangle.$$

An easy computation shows that:

$$M(\mathbf{c}) = q^{\frac{1}{2}\sum_{i<j} c_i c_j \lambda_{ji}} X_1^{c_1} X_2^{c_2} \cdots X_m^{c_m} =: X^{(\mathbf{c})} \quad (\mathbf{c} \in \mathbb{Z}^m).$$

Let Λ be an $m \times m$ skew-symmetric matrix and $\tilde{B}$ an $m \times n$ matrix with $n \leq m$. We call the pair $(\Lambda, \tilde{B})$ *compatible* if up to permuting rows and columns $\tilde{B}^T \Lambda = (D|0)$ with $D = diag(d_1, \cdots, d_n)$ where $d_i \in \mathbb{N}$ for $1 \leq i \leq n$. The pair $(M, \tilde{B})$ is called a *quantum seed* if the pair $(\Lambda_M, \tilde{B})$ is compatible. Define the $m \times m$ matrix $E = (e_{ij})$ as follows

$$e_{ij} = \begin{cases} \delta_{ij} & \text{if } j \neq k; \\ -1 & \text{if } i = j = k; \\ max(0, -b_{ik}) & \text{if } i \neq j = k. \end{cases}$$

For $n, k \in \mathbb{Z}$, $k \geq 0$, denote $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n - q^{-n})\cdots(q^{n-k+1} - q^{-n+k-1})}{(q^k - q^{-k})\cdots(q - q^{-1})}$. Let $k \in [1, n]$ where $[1, n] = \{1, \cdots, n\}$ and $\mathbf{c} = (c_1, \ldots, c_m) \in \mathbb{Z}^m$ with $c_k \geq 0$. Define the toric frame $M' : \mathbb{Z}^m \to \mathscr{F} \setminus \{0\}$ as follows

$$M'(\mathbf{c}) = \sum_{p=0}^{c_k} \begin{bmatrix} c_k \\ p \end{bmatrix}_{q^{d_k/2}} M(E\mathbf{c} + p\mathbf{b}^k), \quad M'(-\mathbf{c}) = M'(\mathbf{c})^{-1}. \tag{1}$$

where the vector $\mathbf{b}^k \in \mathbb{Z}^m$ is the kth column of $\tilde{B}$. Following [10], we say a real $m \times n$ matrix $\tilde{B}'$ is obtained from $\tilde{B}$ by matrix mutation in direction k if the entries of $\tilde{B}'$ are given by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k; \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{otherwise.} \end{cases}$$

Then the quantum seed $(M', \tilde{B}')$ is defined to be the mutation of $(M, \tilde{B})$ in direction k. Two quantum seeds are called mutation-equivalent if they can be obtained from each other by a sequence of mutations. Let $\mathscr{C} = \{M'(e_i) : i \in [1, n]\}$ where $(M', \tilde{B}')$ is mutation-equivalent to $(M, \tilde{B})$. The elements of $\mathscr{C}$ are called the *cluster variables*. Let $\mathbb{P} = \{M(e_i) : i \in [n+1, m]\}$ and the elements of $\mathbb{P}$ are called *coefficients*. Denote by $\mathbb{ZP}$ the ring of Laurent polynomials generated by $q^{\frac{1}{2}}, \mathbb{P}$ and their inverses. Then the *quantum cluster algebra* $\mathscr{A}_q(\Lambda_M, \tilde{B})$ is defined to be the $\mathbb{ZP}$-subalgebra of $\mathscr{F}$ generated by $\mathscr{C}$.

2.2 The quantum Caldero-Chapoton map and main result Let k be a finite field with cardinality $|k| = q$ and $m \geq n$ be two positive integers. Let Δ be a valued graph without vertex loops and with vertex set $\{1, \ldots, m\}$. The edges of Δ are of the form $i \overset{(a_{ij}, a_{ji})}{\text{———}} j$, in which the positive integers a_{ij} form a symmetrizable matrix.

Let $\widetilde{Q}$ be an orientation of Δ containing no oriented cycles: that is, we replace each valued edge by a valued arrow. Thus $\widetilde{Q}$ is called a valued quiver. Note that any finite dimensional basic hereditary k-algebra can be obtained by taking the tensor algebra of the k-species associated to $\widetilde{Q}$. In what follows we will denote by $\widetilde{\mathfrak{S}}$ the k-species of type $\widetilde{Q}$ in the sense of [16], which identified a k-species with its corresponding tensor algebra.

The full subquiver Q on the vertices $1, \ldots, n$ is called the principal part of $\widetilde{Q}$, with the corresponding k-species denoted by $\mathfrak{S}$. For $1 \leq i \leq m$, let S_i be the i-th simple module for $\widetilde{\mathfrak{S}}$.

Let $\widetilde{B}$ be the $m \times n$ matrix associated to the quiver $\widetilde{Q}$ whose entry in position (i, j) given by

$$b_{ij} = \dim_{\mathrm{End}_{\widetilde{\mathfrak{S}}}(S_i)^{op}} \mathrm{Ext}^1_{\widetilde{\mathfrak{S}}}(S_i, S_j) - \dim_{\mathrm{End}_{\widetilde{\mathfrak{S}}}(S_i)} \mathrm{Ext}^1_{\widetilde{\mathfrak{S}}}(S_j, S_i)$$

for $1 \leq i \leq m$, $1 \leq j \leq n$. Denote by $\widetilde{I}$ the left $m \times n$ submatrix of the identity matrix of size $m \times m$.

By [19], we can assume that there exists some antisymmetric $m \times m$ integer matrix Λ such that

$$\Lambda(-\widetilde{B}) = \begin{bmatrix} D_n \\ 0 \end{bmatrix},$$

where $D_n = diag(d_1, \cdots, d_n)$ where $d_i \in \mathbb{N}$ for $1 \leq i \leq n$. Let $\widetilde{R} = \widetilde{R}_{\widetilde{Q}}$ be the $m \times n$ matrix with its entry in position (i, j) is

$$\widetilde{r}_{ij} := \dim_{\mathrm{End}_{\widetilde{\mathfrak{S}}}(S_i)} \mathrm{Ext}^1_{\widetilde{\mathfrak{S}}}(S_j, S_i)$$

for $1 \leq i \leq m$, $1 \leq j \leq n$ respectively. And define $\widetilde{R}' := \widetilde{R}_{\widetilde{Q}^{op}}$. Denote the principal parts of the matrices $\widetilde{B}$ and $\widetilde{R}$ by B and R respectively. Note that $\widetilde{B} = \widetilde{R}' - \widetilde{R}$ and $B = R' - R$.

Let $\mathscr{C}_Q$ be the cluster category (see [2]) of the valued quiver Q, i.e., the orbit category of the derived category $\mathscr{D}^b(\mathfrak{S})$ by the functor $F = \tau \circ [-1]$. We note that the indecomposable $\mathfrak{S}$-modules and $P_i[1]$ for $1 \leq i \leq n$ exhaust the indecomposable objects of the cluster category $\mathscr{C}_Q$:

$$\mathrm{ind}\, \mathscr{C}_Q = \mathrm{ind}\, \mathrm{mod}\mathfrak{S} \sqcup \{P_i[1] : 1 \leq i \leq n\}$$

where P_i is the indecomposable projective $\mathfrak{S}$-module at i for $i = 1, \cdots, n$. Each object M in $\mathscr{C}_Q$ can be uniquely decomposed in the following way:

$$M = M_0 \oplus P_M[1]$$

where M_0 is a $\mathfrak{S}$-module and P_M is a projective module. Let $P_M = \bigoplus_{1 \leq i \leq n} m_i P_i$. We extend the definition of the dimension vector $\underline{\dim}$ on modules in $\mathrm{mod}\mathfrak{S}$ to objects in $\mathscr{C}_Q$ by setting

$$\underline{\dim} M = \underline{\dim} M_0 - (m_i)_{1 \leq i \leq n}.$$

The Euler form on $\mathfrak{S}$-modules M and N is given by

$$\langle M, N \rangle = \dim_k \mathrm{Hom}_{\mathfrak{S}}(M, N) - \dim_k \mathrm{Ext}^1_{\mathfrak{S}}(M, N).$$

Note that the Euler form only depends on the dimension vectors of M and N and the matrix representing this form is $(I_n - R^{tr})D_n = D_n(I_n - R')$.

The quantum Caldero-Chapoton map of an acyclic quiver Q has been defined in [19][18][6]. The quantum Caldero-Chapoton map was defined in [19] for $\mathfrak{S}$-modules, in [18] for coefficient-free rigid object in $\mathscr{C}_{\widetilde{Q}}$. Later it was extended in [6] to the cluster category for equally-valued quivers. For valued quivers, we also have

$$X_? : \mathrm{Obj}\, \mathscr{C}_{\widetilde{Q}} \longrightarrow \mathscr{T}$$

defined by the following rules:

(1) If M is a $\mathfrak{S}$-module, then

$$X_M = \sum_{\underline{e}} |\mathrm{Gr}_{\underline{e}} M| q^{-\frac{1}{2}\langle \underline{e}, \underline{m} - \underline{e} \rangle} X^{-\widetilde{B}\underline{e} - (\widetilde{I} - \widetilde{R}')\underline{m}};$$

(2) If M is a $\mathfrak{S}$-module and I is an injective $\widetilde{\mathfrak{S}}$-module, then

$$X_{M\oplus I[-1]}=\sum_{\underline{e}}|\mathrm{Gr}_{\underline{e}}M|q^{-\frac{1}{2}\langle\underline{e},\underline{m}-\underline{e}-\underline{i}\rangle}X^{-\widetilde{B}\underline{e}-(\widetilde{I}-\widetilde{R}')\underline{m}+\underline{\dim}\mathrm{Soc}I},$$

where $\underline{\dim}I=\underline{i}, \underline{\dim}M=\underline{m}$ and $\mathrm{Gr}_{\underline{e}}M$ denotes the set of all submodules V of M with $\underline{\dim}V=\underline{e}$. We note that

$$X_{P[1]}=X_{\tau P}=X^{\underline{\dim}P/\mathrm{rad}P}=X^{\underline{\dim}\mathrm{Soc}I}=X_{I[-1]}=X_{\tau^{-1}I}.$$

for any projective $\widetilde{\mathfrak{S}}$-module P and injective $\widetilde{\mathfrak{S}}$-module I with $\mathrm{Soc}I=P/\mathrm{rad}P$. In the following, we denote by the corresponding underlined lower case letter $\underline{x}$ the dimension vector of a $\mathfrak{S}$-module X and view $\underline{x}$ as a column vector in $\mathbb{Z}^n$.

Now we need to recall some notations. For any $\widetilde{\mathfrak{S}}$-modules M, N and E, denote by ε_{MN}^{E} the cardinality of the set $\mathrm{Ext}^1_{\widetilde{\mathfrak{S}}}(M,N)_E$ which is the subset of $\mathrm{Ext}^1_{\widetilde{\mathfrak{S}}}(M,N)$ consisting of those equivalence classes of short exact sequences with middle term isomorphic to E ([16, Section 4]). Let F_{AB}^{M} be the number of submodules U of M such that U is isomorphic to B and M/U is isomorphic to A. Then by definition, we have

$$|\mathrm{Gr}_{\underline{e}}(M)|=\sum_{A,B;\underline{\dim}B=\underline{e}}F_{AB}^{M}.$$

Denote by $[M,N]^1=\dim_k\mathrm{Ext}^1_{\widetilde{\mathfrak{S}}}(M,N)$ and $[M,N]=\dim_k\mathrm{Hom}_{\widetilde{\mathfrak{S}}}(M,N)$.

Let M, N be any $\mathfrak{S}-$modules and I any injective $\widetilde{\mathfrak{S}}-$module. Define

$$\mathrm{Hom}_{\widetilde{\mathfrak{S}}}(M,I)_{BI'}:=\{f:M\longrightarrow I|\mathrm{Ker}f\cong B,\mathrm{Coker}f\cong I'\}.$$

Note that I' is an injective $\widetilde{\mathfrak{S}}-$module.

The main result of this article is the following theorem:

Theorem 1.

$$(1)\ q^{[M,N]^1}X_MX_N=q^{\frac{1}{2}\Lambda((\widetilde{I}-\widetilde{R}')\underline{m},(\widetilde{I}-\widetilde{R}')\underline{n})}\sum_{E}\varepsilon_{MN}^{E}X_E,$$

$$(2)\ q^{[M,I]}X_MX_{I[-1]}=q^{\frac{1}{2}\Lambda((\widetilde{I}-\widetilde{R}')\underline{m},-\underline{\dim}SocI)}\sum_{B,I'}|\mathrm{Hom}_{\widetilde{\mathfrak{S}}}(M,I)_{BI'}|X_{B\oplus I'[-1]}.$$

Definition 1. X_L *is called* the corresponding cluster character*, if* L *is a* $\mathfrak{S}$*-module or* $L=M\oplus I[-1]\in\mathscr{C}_{\widetilde{Q}}$ *satisfying that* M *is a* $\mathfrak{S}$*-module and* I *is an injective* $\widetilde{\mathfrak{S}}$*-module.*

For a valued quiver Q, denote by $\mathscr{AH}_{|k|}(Q)$ the $\mathbb{ZP}$-subalgebra of $\mathscr{F}$ generated by all the cluster characters and by $\mathscr{EH}_{|k|}(Q)$ the corresponding quantum cluster algebra, i.e, the $\mathbb{ZP}$-subalgebra of $\mathscr{F}$ generated by all the cluster variables. Then we have the following corollary:

Corollary 1. *For any valued quiver Q of finite type or rank 2, we have $\mathscr{E}\mathscr{H}_{|k|}(Q) = \mathscr{A}\mathscr{H}_{|k|}(Q)$.*

3 Proof of the main theorem

In this section, we fix a valued quiver Q with n vertices.

Lemma 1. *For any dimension vector $\underline{m}, \underline{e}, \underline{f} \in \mathbb{Z}^n_{\geq 0}$, we have*

$$(1)\ \Lambda((\widetilde{I} - \widetilde{R}')\underline{m}, \widetilde{B}\underline{e}) = -\langle \underline{e}, \underline{m} \rangle;$$

$$(2)\ \Lambda(\widetilde{B}\underline{e}, \widetilde{B}\underline{f}) = \langle \underline{f}, \underline{e} \rangle - \langle \underline{e}, \underline{f} \rangle.$$

Proof. By definition, we have

$$\begin{aligned}
& \Lambda((\widetilde{I} - \widetilde{R}')\underline{m}, \widetilde{B}\underline{e}) \\
= \ & \underline{m}^{tr}(\widetilde{I} - \widetilde{R}')^{tr}\Lambda\widetilde{B}\underline{e} = -\underline{m}^{tr}(\widetilde{I} - \widetilde{R}')^{tr}\begin{bmatrix} D_n \\ 0 \end{bmatrix}\underline{e} \\
= \ & -\underline{m}^{tr}(I_n - (R')^{tr})D_n\underline{e} = -\underline{e}^{tr}D_n(I_n - R')\underline{m} \\
= \ & -\langle \underline{e}, \underline{m} \rangle.
\end{aligned}$$

As for (2), the left side of the desired equation is equal to

$$\underline{e}^{tr}\widetilde{B}^{tr}\Lambda\widetilde{B}\underline{f} = -\underline{e}^{tr}\widetilde{B}^{tr}\begin{bmatrix} D_n \\ 0 \end{bmatrix}\underline{f} = -\underline{e}^{tr}B^{tr}D_n\underline{f}.$$

The right side is

$$\begin{aligned}
& \langle \underline{f}, \underline{e} \rangle - \langle \underline{e}, \underline{f} \rangle \\
= \ & \underline{f}^{tr}D_n(I_n - R')\underline{e} - \underline{e}^{tr}(I_n - R^{tr})D_n\underline{f} \\
= \ & \underline{e}^{tr}(I_n - R')^{tr}D_n\underline{f} - \underline{e}^{tr}(I_n - R^{tr})D_n\underline{f} \\
= \ & \underline{e}^{tr}(R^{tr} - (R')^{tr})D_n\underline{f} = -\underline{e}^{tr}B^{tr}D_n\underline{f}.
\end{aligned}$$

Thus we prove the lemma. □

Corollary 2. *For any dimension vector $\underline{m}, \underline{l}, \underline{e}, \underline{f} \in \mathbb{Z}^n_{\geq 0}$, we have*

$$\begin{aligned}
& \Lambda(-\widetilde{B}\underline{e} - (\widetilde{I} - \widetilde{R}')\underline{m}, -\widetilde{B}\underline{f} - (\widetilde{I} - \widetilde{R}')\underline{l}) \\
= \ & \Lambda((\widetilde{I} - \widetilde{R}')\underline{m}, (\widetilde{I} - \widetilde{R}')\underline{l}) + \langle \underline{f}, \underline{e} \rangle - \langle \underline{e}, \underline{f} \rangle + \langle \underline{e}, \underline{l} \rangle - \langle \underline{f}, \underline{m} \rangle.
\end{aligned}$$

Proof. It follows from Lemma 1. □

Proof of Theorem 1(1): By Green's formula [15], we have

$$\sum_E \varepsilon^E_{MN} F^E_{XY} = \sum_{A,B,C,D} q^{[M,N]-[A,C]-[B,D]-\langle A,D\rangle} F^M_{AB} F^N_{CD} \varepsilon^X_{AC} \varepsilon^Y_{BD}.$$

Then

$$\begin{aligned}
& \sum_E \varepsilon^E_{MN} X_E \\
= & \sum_{E,X,Y} \varepsilon^E_{MN} q^{-\frac{1}{2}\langle Y,X\rangle} F^E_{XY} X^{-\widetilde{B}\underline{y}-(\widetilde{I}-\widetilde{R}')\underline{e}} \\
= & \sum_{A,B,C,D,X,Y} q^{[M,N]-[A,C]-[B,D]-\langle A,D\rangle-\frac{1}{2}\langle B+D,A+C\rangle} F^M_{AB} F^N_{CD} \varepsilon^X_{AC} \varepsilon^Y_{BD} X^{-\widetilde{B}\underline{y}-(\widetilde{I}-\widetilde{R}')\underline{e}}.
\end{aligned}$$

By Corollary 2, we have

$$\begin{aligned}
& X^{-\widetilde{B}\underline{y}-(\widetilde{I}-\widetilde{R}')\underline{e}} \\
= & X^{-\widetilde{B}(\underline{b}+\underline{d})-(\widetilde{I}-\widetilde{R}')(\underline{m}+\underline{n})} \\
= & q^{\frac{1}{2}\Lambda(-\widetilde{B}\underline{d}-(\widetilde{I}-\widetilde{R}')\underline{n},-\widetilde{B}\underline{b}-(\widetilde{I}-\widetilde{R}')\underline{m})} X^{-\widetilde{B}\underline{b}-(\widetilde{I}-\widetilde{R}')\underline{m}} X^{-\widetilde{B}\underline{d}-(\widetilde{I}-\widetilde{R}')\underline{n}} \\
= & q^{\frac{1}{2}\Lambda((\widetilde{I}-\widetilde{R}')\underline{n},(\widetilde{I}-\widetilde{R}')\underline{m})+\frac{1}{2}[-\langle D,B\rangle+\langle B,D\rangle+\langle D,M\rangle-\langle B,N\rangle]} X^{-\widetilde{B}\underline{b}-(\widetilde{I}-\widetilde{R}')\underline{m}} X^{-\widetilde{B}\underline{d}-(\widetilde{I}-\widetilde{R}')\underline{n}} \\
= & q^{\frac{1}{2}\Lambda((\widetilde{I}-\widetilde{R}')\underline{n},(\widetilde{I}-\widetilde{R}')\underline{m})} q^{\frac{1}{2}\langle D,A\rangle-\frac{1}{2}\langle B,C\rangle} X^{-\widetilde{B}\underline{b}-(\widetilde{I}-\widetilde{R}')\underline{m}} X^{-\widetilde{B}\underline{d}-(\widetilde{I}-\widetilde{R}')\underline{n}}.
\end{aligned}$$

Thus

$$\begin{aligned}
& \sum_E \varepsilon^E_{MN} X_E \\
= & q^{-\frac{1}{2}\Lambda((\widetilde{I}-\widetilde{R}')\underline{m},(\widetilde{I}-\widetilde{R}')\underline{n})} \sum_{A,B,C,D} q^{[M,N]-[A,C]-[B,D]-\langle A,D\rangle-\frac{1}{2}\langle B+D,A+C\rangle+[A,C]^1+[B,D]^1} \cdot \\
& q^{\frac{1}{2}\langle D,A\rangle-\frac{1}{2}\langle B,C\rangle} F^M_{AB} F^N_{CD} X^{-\widetilde{B}\underline{b}-(\widetilde{I}-\widetilde{R}')\underline{m}} X^{-\widetilde{B}\underline{d}-(\widetilde{I}-\widetilde{R}')\underline{n}}.
\end{aligned}$$

Here we use the following fact

$$\sum_X \varepsilon^X_{AC} = q^{[A,C]^1}, \sum_Y \varepsilon^Y_{BD} = q^{[B,D]^1}$$

An easy calculation shows that

$$[M,N]-[A,C]-[B,D]-\langle A,D\rangle+[A,C]^1+[B,D]^1 = [M,N]^1+\langle B,C\rangle.$$

Hence

$$\begin{aligned}
& \sum_E \varepsilon^E_{MN} X_E \\
= & q^{-\frac{1}{2}\Lambda((\widetilde{I}-\widetilde{R}')\underline{m},(\widetilde{I}-\widetilde{R}')\underline{n})} q^{[M,N]^1} \sum_{A,B,C,D} F^M_{AB} q^{-\frac{1}{2}\langle B,A\rangle} X^{-\widetilde{B}\underline{b}-(\widetilde{I}-\widetilde{R}')\underline{m}} F^N_{CD} q^{-\frac{1}{2}\langle D,C\rangle} X^{-\widetilde{B}\underline{d}-(\widetilde{I}-\widetilde{R}')\underline{n}} \\
= & q^{-\frac{1}{2}\Lambda((\widetilde{I}-\widetilde{R}')\underline{m},(\widetilde{I}-\widetilde{R}')\underline{n})} q^{[M,N]^1} X_M X_N.
\end{aligned}$$

This finishes the proof.

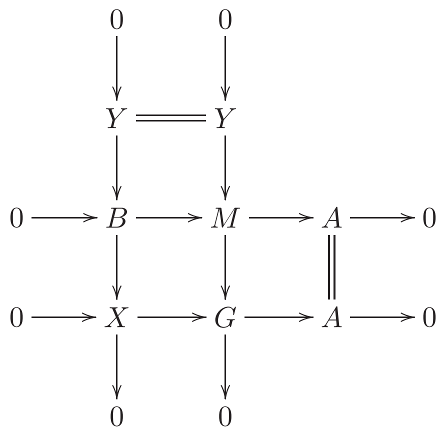
Proof of Theorem 1(2): We calculate

$$
\begin{aligned}
& X_M X_{I[-1]} \\
= & \sum_{G,H} q^{-\frac{1}{2}\langle H,G\rangle} F_{GH}^{M} X^{-\widetilde{B}\underline{h}-(\widetilde{I}-\widetilde{R}')\underline{m}} X^{\underline{\dim} socI} \\
= & \sum_{G,H} q^{-\frac{1}{2}\langle H,G\rangle} F_{GH}^{M} q^{\frac{1}{2}\Lambda(-\widetilde{B}\underline{h}-(\widetilde{I}-\widetilde{R}')\underline{m},\underline{\dim} socI)} X^{-\widetilde{B}\underline{h}-(\widetilde{I}-\widetilde{R}')\underline{m}+\underline{\dim} socI} \\
= & q^{\frac{1}{2}\Lambda(-(\widetilde{I}-\widetilde{R}')\underline{m},\underline{\dim} socI)} \sum_{G,H} q^{-\frac{1}{2}\langle H,G\rangle} q^{\frac{1}{2}\Lambda(-\widetilde{B}\underline{h},\underline{\dim} socI)} F_{GH}^{M} X^{-\widetilde{B}\underline{h}-(\widetilde{I}-\widetilde{R}')\underline{m}+\underline{\dim} socI} \\
= & q^{\frac{1}{2}\Lambda((\widetilde{I}-\widetilde{R}')\underline{m},-\underline{\dim} socI)} \sum_{G,H} q^{-\frac{1}{2}\langle H,G\rangle} q^{-\frac{1}{2}[H,I]} F_{GH}^{M} X^{-\widetilde{B}\underline{h}-(\widetilde{I}-\widetilde{R}')\underline{m}+\underline{\dim} socI}.
\end{aligned}
$$

Here we use the fact that

$$\Lambda(-\widetilde{B}\underline{h},\underline{\dim} socI) = -\underline{h}^{tr}\widetilde{B}^{tr}\Lambda(\underline{\dim} socI) = -[H,I].$$

Note that we have the following commutative diagram



and short exact sequence

$$0 \longrightarrow A \longrightarrow I \longrightarrow I' \longrightarrow 0,$$

Thus by [16] we have

$$\sum_{B} F_{XY}^{B} F_{AB}^{M} = \sum_{G} F_{AX}^{G} F_{GY}^{M},\ |\mathrm{Hom}_{\widetilde{\mathfrak{S}}}(M,I)_{BI'}| = \sum_{A} |\mathrm{Aut}(A)| F_{AB}^{M} F_{I'A}^{I}$$

and

$$\sum_{A,I',X} |\mathrm{Aut}(A)| F_{I'A}^{I} F_{AX}^{G} = \sum_{I',X} |\mathrm{Hom}_{\widetilde{\mathfrak{S}}}(G,I)_{XI'}| = q^{[G,I]} = q^{\langle G,I\rangle}.$$

By [16, Lemma 1], we have $(\widetilde{I}-\widetilde{R}')\underline{i}=\underline{\dim}socI$. Now we can calculate the term

$$\begin{aligned}
&\sum_{B,I'}|\mathrm{Hom}_{\widetilde{\mathfrak{C}}}(M,I)_{BI'}|X_{B\oplus I'[-1]}\\
=&\sum_{A,B,I',X,Y}|\mathrm{Aut}(A)|F^{M}_{AB}F^{I}_{I'A}q^{-\frac{1}{2}\langle Y,X-I'\rangle}F^{B}_{XY}X^{-\widetilde{B}\underline{y}-(\widetilde{I}-\widetilde{R}')\underline{b}+\underline{\dim}socI'}\\
=&\sum_{A,G,I',X,Y}q^{-\frac{1}{2}\langle Y,X-I'\rangle}|\mathrm{Aut}(A)|F^{I}_{I'A}F^{G}_{AX}F^{M}_{GY}X^{-\widetilde{B}\underline{y}-(\widetilde{I}-\widetilde{R}')\underline{b}+\underline{\dim}socI'}.
\end{aligned}$$

Note that we have the following facts

$$\underline{i'}+\underline{a}=\underline{i},\ \underline{x}+\underline{a}=\underline{g}\Longrightarrow\underline{x}-\underline{i'}=\underline{g}-\underline{i},$$

and

$$\begin{aligned}
&-\widetilde{B}\underline{y}-(\widetilde{I}-\widetilde{R}')\underline{b}+\underline{\dim}socI'\\
=&-\widetilde{B}\underline{h}-(\widetilde{I}-\widetilde{R}')(\underline{m}-\underline{i}+\underline{i'})+\underline{\dim}socI'\\
=&-\widetilde{B}\underline{h}-(\widetilde{I}-\widetilde{R}')\underline{m}+(\widetilde{I}-\widetilde{R}')(\underline{i}-\underline{i'})+\underline{\dim}socI'\\
=&-\widetilde{B}\underline{h}-(\widetilde{I}-\widetilde{R}')\underline{m}+(\widetilde{I}-\widetilde{R}')\underline{i}\\
=&-\widetilde{B}\underline{h}-(\widetilde{I}-\widetilde{R}')\underline{m}+\underline{\dim}socI.
\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{B,I'}|\mathrm{Hom}_{\widetilde{\mathfrak{C}}}(M,I)_{BI'}|X_{B\oplus I'[-1]}\\
=&\sum_{G,H}q^{\langle G,I\rangle}q^{-\frac{1}{2}\langle H,G-I\rangle}F^{M}_{GH}X^{-\widetilde{B}\underline{h}-(\widetilde{I}-\widetilde{R}')\underline{m}+\underline{\dim}socI}\\
=&\sum_{G,H}q^{\langle M,I\rangle}q^{-\frac{1}{2}\langle H,I\rangle}q^{-\frac{1}{2}\langle H,G\rangle}F^{M}_{GH}X^{-\widetilde{B}\underline{h}-(\widetilde{I}-\widetilde{R}')\underline{m}+\underline{\dim}socI}\\
=&\ q^{[M,I]}\sum_{G,H}q^{-\frac{1}{2}[H,I]}q^{-\frac{1}{2}\langle H,G\rangle}F^{M}_{GH}X^{-\widetilde{B}\underline{h}-(\widetilde{I}-\widetilde{R}')\underline{m}+\underline{\dim}socI}.
\end{aligned}$$

This finishes the proof.

To prove Corollary 1, we recall the following lemma which can be found in [5][6].

Lemma 2. *Let*

$$M\longrightarrow E\longrightarrow N\xrightarrow{\epsilon}M[1]$$

be a non-split triangle in $\mathscr{C}_{\widetilde{Q}}$. *Then*

$$\dim_k\mathrm{Ext}^1_{\mathscr{C}_{\widetilde{Q}}}(E,E)<\dim_k\mathrm{Ext}^1_{\mathscr{C}_{\widetilde{Q}}}(M\oplus N,M\oplus N).$$

Proof of Corollary 1: Firstly, we prove that for any indecomposable object $M \in \mathscr{C}_{\widetilde{Q}}$, X_M is in the quantum cluster algebra $\mathscr{EH}_{|k|}(Q)$.

When Q is a valued quiver of finite type, it follows that X_M is a cluster variable for any indecomposable object $M \in \mathscr{C}_{\widetilde{Q}}$ by [19].

When Q is a valued quiver of rank 2. Denoted by

$$\Phi_i : \mathscr{A}_{|k|}(Q) \to \mathscr{A}_{|k|}(Q')$$

the canonical isomorphism of quantum cluster algebras associated to sink or source $1 \leq i \leq 2$. Let $\Sigma_i :\ \mathrm{mod}Q \longrightarrow\ \mathrm{mod}Q'$ be the standard BGP-reflection functor. It follows from [19, Theorem 2.4] that $\Phi_i(X_M^Q) = X_{\Sigma_i M}^{Q'}$ for any regular module M of Q. Note also the fact Q is an acyclic valued quiver of rank 2, we have that X_M is in the upper quantum cluster algebra associated to Q which coincides with the quantum cluster algebra $\mathscr{EH}_{|k|}(Q)$ by the acyclicity of Q [3]. When M is an indecomposable preprojective or preinjective module, it follows from [19] X_M is a cluster variable, hence in the quantum cluster algebra $\mathscr{EH}_{|k|}(Q)$.

Now we need to prove that for any cluster character $X_L \in \mathscr{AH}_{|k|}(Q)$, then $X_L \in \mathscr{EH}_{|k|}(Q)$. Let $L \cong \bigoplus_{i=1}^{l} L_i^{\oplus n_i}, n_i \in \mathbb{N}$ where L_i $(1 \leq i \leq l)$ are indecomposable objects in $\mathscr{C}_{\widetilde{Q}}$. By Theorem 1 and Lemma 2, we have that

$$X_{L_1}^{n_1} X_{L_2}^{n_2} \cdots X_{L_l}^{n_l} = q^{\frac{1}{2}n_L} X_L + \sum_{\dim_k \mathrm{Ext}^1_{\mathscr{C}_{\widetilde{Q}}}(E,E) < \dim_k \mathrm{Ext}^1_{\mathscr{C}_{\widetilde{Q}}}(L,L)} f_{n_E}(q^{\pm\frac{1}{2}}) X_E$$

where $n_L \in \mathbb{Z}$ and $f_{n_E}(q^{\pm\frac{1}{2}}) \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$. Note that the left side of the equation above is in $\mathscr{EH}_{|k|}(Q)$, thus by induction, it follows that $X_L \in \mathscr{EH}_{|k|}(Q)$ which finishes the proof.

3 Bases in the quantum cluster algebras of rank 2

In this section, we consider a valued quiver (see [19] for details) associated to a given compatible pair (Λ, B) where $\Lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}$ for any $b, c \in \mathbb{Z}_{>0}$. Let $\mathscr{T} = \mathbb{Z}[q^{\pm 1/2}]\langle X_1^{\pm 1}, X_2^{\pm 1} : X_1X_2 = qX_2X_1\rangle$ and $\mathscr{F}$ be the skew field of fractions of $\mathscr{T}$ and thus the quantum cluster algebra of the valued quiver of rank 2 (denoted by $\mathscr{A}_q(b,c)$ in the sequel) is the $\mathbb{Z}[q^{\pm 1/2}]$-subalgebra of $\mathscr{F}$ generated by the cluster variables X_k, $k \in \mathbb{Z}$, defined recursively by

$$X_{m-1}X_{m+1} = \begin{cases} q^{\frac{b}{2}} X_m^b + 1 & \text{if } m \text{ is odd;} \\ q^{\frac{c}{2}} X_m^c + 1 & \text{if } m \text{ is even.} \end{cases}$$

Definition 2. *For any (r_1, r_2) and $(s_1, s_2) \in \mathbb{Z}^2$, we write $(r_1, r_2) \preceq (s_1, s_2)$ if $r_i \leq s_i$ for $1 \leq i \leq 2$. Moreover, if there exists some i such that $r_i < s_i$, then we write $(r_1, r_2) \prec (s_1, s_2)$.*

For any $\underline{m} \in \mathbb{Z}^2$, define $\underline{m}^+ = (m_1^+, m_2^+)$ such that $m_i^+ = m_i$ if $m_i > 0$ and $m_i^+ = 0$ if $m_i \leq 0$ for any $1 \leq i \leq 2$. Dually, we set $\underline{m}^- = \underline{m}^+ - \underline{m}$. Denote by $\underline{dim}I[-1] = -\underline{dim}socI$ for any injective module I. For any $\underline{d} \in \mathbb{Z}^2$, we make the following assignment by

$$X_{\underline{d}} := X_M \quad \text{for any } M \in \mathscr{C}_Q \text{ with } \underline{dim}M = \underline{d}.$$

Note that this assignment is not unique.

Theorem 2. *The set* $\mathscr{B} = \{X_{\underline{d}} | \underline{d} \in \mathbb{Z}^2\}$ *is a* $\mathbb{Z}[q^{\pm\frac{1}{2}}]-$*basis of the quantum cluster algebra* $\mathscr{A}_q(b, c)$.

Proof. Note that for any $\underline{d} \in \mathbb{Z}^2$, $X_{\underline{d}} \in \mathscr{A}_q(b, c)$ by Corollary 1. According to the definition of the quantum Caldero-Chapoton map and the partial order in Definition 2, we obtain a minimal term $a_{\underline{d}}X^{\underline{d}}$ in the laurent expansion in $X_{\underline{d}}$, for some nonzero $a_{\underline{d}} \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$. Then by the standard monomials in [3], we have

$$X_{\underline{d}} = b_{\underline{d}} X_1^{d_1^-} X_2^{d_2^-} X_{S_1}^{d_1^+} X_{S_2}^{d_2^+} + \sum_{\underline{d} \succ \underline{l}} b_{\underline{l}} X_1^{l_1^-} X_2^{l_2^-} X_{S_1}^{l_1^+} X_{S_2}^{l_2^+}$$

where $b_{\underline{d}}, b_{\underline{l}} \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$. It is easy to see that $b_{\underline{d}}$ must be some nonzero monomial in $q^{\pm\frac{1}{2}}$. Thus we obtain that $\mathscr{B}$ is a $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-basis of $\mathscr{A}_q(b, c)$. □

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