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The Structure of AS-Gorenstein Algebras

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Abstract

In this paper, we define a notion of AS-Gorenstein algebra for $\mathbb{N}$-graded algebras, and show that symmetric AS-regular algebras of Gorenstein parameter 1 are exactly preprojective algebras of quasi-Fano algebras. This result can be compared with the fact that symmetric graded Frobenius algebras of Gorenstein parameter -1 are exactly trivial extensions of finite dimensional algebras. The results of this paper suggest that there is a strong interaction between classification problems in noncommutative algebraic geometry and those in representation theory of finite dimensional algebras.

Keywords: AS-regular algebras, Fano algebras, preprojective algebras, graded Frobenius algebras, trivial extensions

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1 Introduction

In noncommutative algebraic geometry, classifying AS-regular algebras is one of the major projects, while, in representation theory of finite dimensional algebras, classifying finite dimensional algebras of finite global dimension is one of the major projects. The purpose of this paper is to connect these classification problems. The first author of this paper recently introduced a notion of Fano algebra in [15], which is a nice class of finite dimensional algebras of finite global dimension. For example, every path algebra of finite acyclic quiver of infinite representation type is Fano [15]. In this paper, by extending the notion of AS-Gorenstein algebra to $\mathbb{N}$-graded algebras, we will give a nice correspondence between these algebras, namely, for an AS-regular algebra A of global dimension $n \geq 1$, we define the Beilinson algebra ∇A, which is a quasi-Fano algebra of global dimension $n-1$, and, for a quasi-Fano algebra R of global dimension n, we define the preprojective algebra ΠR, which is exactly a symmetric AS-regular algebra of global dimension $n+1$ and of Gorenstein parameter 1. This correspondence is nice in a sense that $\nabla(\Pi R)$ is isomorphic to R as algebras

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for every quasi-Fano algebra R, and $\Pi(\nabla A)$ is graded Morita equivalent to A for every AS-regular algebra A. These results suggest that classifying AS-regular algebras of dimension $n \geq 1$ up to graded Morita equivalence is strongly related to classifying quasi-Fano algebras of dimension $n-1$ up to isomorphism. In fact, classifying quantum projective spaces of dimension n up to derived equivalence is the same as classifying quasi-Fano algebras of dimension n up to derived equivalence. We will also show that the analogous results hold for graded Frobenius algebras, which are exactly AS-Gorenstein algebras of dimension 0 in our sense.

For the rest of this section, we fix basic definitions and notations used in this paper. Throughout, let k be a field. In this paper, an algebra means an algebra over k unless otherwise stated. For a ring R, we denote by $\operatorname{Mod} R$ the category of right R-modules, and by $\operatorname{mod} R$ the full subcategory consisting of finitely presented modules. In this paper, R is often a finite dimensional algebra over k. In this case, $\operatorname{mod} R$ is simply the full subcategory consisting of modules finite dimensional over k. We denote by R^o the opposite ring of R and define $R^e := R^o \otimes R$. For rings R, S, $\operatorname{Mod} R^o$ is identified with the category of left R-modules, and $\operatorname{Mod}(R^o \otimes S)$ is identified with the category of R-S bimodules, so that $\operatorname{Mod} R^e$ is identified with the category of R-R bimodules. We denote by $DV := \operatorname{Hom}_k(V, k)$ the vector space dual of a vector space V. If R is a finite dimensional algebra, then $D(-) : \operatorname{mod} R \leftrightarrow \operatorname{mod} R^o : D(-)$ gives a duality.

For an $\mathbb{N}$-graded ring $A = \oplus_{i\in\mathbb{N}} A_i$, we denote by $\operatorname{GrMod} A$ the category of graded right A-modules, and by $\operatorname{grmod} A$ the full subcategory consisting of finitely presented modules. Morphisms in $\operatorname{GrMod} A$ are A-module homomorphisms preserving degrees. Note that A is graded right coherent if and only if $\operatorname{grmod} A$ is an abelian category. We say that A is connected over R if $A_0 = R$. A graded algebra A is called locally finite if $\dim_k A_i < \infty$ for all $i \in \mathbb{N}$.

For a graded module $M \in \operatorname{GrMod} A$ and an integer $m \in \mathbb{Z}$, we define the truncation $M_{\geq m} := \oplus_{i \geq m} M_i \in \operatorname{GrMod} A$ and the shift $M(m) \in \operatorname{GrMod} A$ by $M(m)_i := M_{m+i}$ for $i \in \mathbb{Z}$. We say that M is bounded below if $M = M_{\geq m}$ for some $m \in \mathbb{Z}$, and bounded above if $M_{\geq m} = 0$ for some $m \in \mathbb{Z}$. Note that the rule $M \mapsto M(m)$ is an autoequivalence for both $\operatorname{GrMod} A$ and $\operatorname{grmod} A$, called a shift functor. For $M, N \in \operatorname{GrMod} A$, we write $\operatorname{Ext}^i_A(M, N) = \operatorname{Ext}^i_{\operatorname{GrMod} A}(M, N)$ and

$$\underline{\operatorname{Ext}}^i_A(M, N) := \bigoplus_{m\in\mathbb{Z}} \operatorname{Ext}^i_A(M, N(m)).$$

Let A be a graded algebra. We denote by $\operatorname{tors} A$ the full subcategory of $\operatorname{GrMod} A$ consisting of modules finite dimensional over k, and by $\operatorname{Tors} A$ the full subcategory of $\operatorname{GrMod} A$ consisting of direct limits of modules in $\operatorname{tors} A$. By abuse of notation, we denote by $DV := \underline{\operatorname{Hom}}_k(V, k)$ the graded vector space dual of a graded vector space V, i.e. $(DV)_i := D(V_{-i})$. If A is a locally finite $\mathbb{N}$-graded algebra, then $D(-) : \operatorname{tors} A \leftrightarrow \operatorname{tors} A^o : D(-)$ gives a duality.

AS-regular algebras defined below are the most important class of algebras in noncommutative algebraic geometry.

Definition 1.1. A graded algebra A connected over k is called AS-Gorenstein (resp. AS-regular) of dimension n and of Gorenstein parameter ℓ if $\operatorname{id} A = n$ (resp. $\operatorname{gldim} A = n$) and A satisfies the Gorenstein condition, that is,

$$\underline{\operatorname{Ext}}^i_A(k, A) \cong \begin{cases} k(\ell) & \text{if } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

We denote by $\mathcal{D}(\mathcal{C})$ the derived category of an abelian category $\mathcal{C}$, and by $\mathcal{D}^b(\mathcal{C})$ the bounded derived category of $\mathcal{C}$. Then the Gorenstein condition above is equivalent to $\mathbf{R}\underline{\operatorname{Hom}}_A(k, A) \cong k(\ell)[-n]$ in $\mathcal{D}(\operatorname{GrMod} k)$. The purpose of this paper is to extend the notion of AS-Gorenstein algebra to $\mathbb{N}$-graded algebras, and study their structures. In particular, we present some interactions between noncommutative algebraic geometry and representation theory of finite dimensional algebras.

A complex $M \in \mathcal{D}(\mathcal{C})$ is called pure if it is quasi-isomorphic to a complex concentrated in degree 0, i.e., $h^i(M) = 0$ for $i \neq 0$. Let R be a ring. A two-sided titling complex of R is a complex of R-R bimodules $L \in \mathcal{D}(\operatorname{Mod} R^e)$ such that $L \otimes^{\mathbf{L}}_R L^{-1} \cong R$ and $L^{-1} \otimes^{\mathbf{L}}_R L \cong R$ in $\mathcal{D}(\operatorname{Mod} R^e)$ where $L^{-1} := \mathbf{R}\operatorname{Hom}_R(L, R)$. For a two-sided tilting complex L of R, we define the following full subcategories of $\mathcal{D}^b(\operatorname{mod} R)$:

$$\mathcal{D}^{L,\geq 0} := \{M \in \mathcal{D}^b(\operatorname{mod} R) \mid h^q(M \otimes^{\mathbf{L}}_R L^{\otimes^{\mathbf{L}}_R i}) = 0 \text{ for all } q < 0, i \gg 0\},$$
$$\mathcal{D}^{L,\leq 0} := \{M \in \mathcal{D}^b(\operatorname{mod} R) \mid h^q(M \otimes^{\mathbf{L}}_R L^{\otimes^{\mathbf{L}}_R i}) = 0 \text{ for all } q > 0, i \gg 0\}.$$

Fano algebras and their preprojective algebras introduced in [15] and [16] will play an essential role to connect noncommutative algebraic geometry and representation theory of finite dimensional algebras.

Definition 1.2. Let R be a ring. A two-sided tilting complex L of R is called quasi-ample if $L^{\otimes^{\mathbf{L}}_R i}$ are pure for all $i \geq 0$. We say that a finite dimensional algebra R is quasi-Fano of dimension n if $\operatorname{gldim} R = n$ and ω_R^{-1} is a quasi-ample two-sided tilting complex where $\omega_R := DR[-n]$. A quasi-Fano algebra is called extremely Fano if $(\mathcal{D}^{\omega_R^{-1},\geq 0}, \mathcal{D}^{\omega_R^{-1},\leq 0})$ is a t-structure in $\mathcal{D}^b(\operatorname{mod} R)$. The preprojective algebra of a quasi-Fano algebra R is defined by the tensor algebra $\Pi R := T_R(\omega_R^{-1})$ of ω_R^{-1} over R.

Remark 1.3. Let R be a finite dimensional algebra, and $\omega_R := DR[-n]$ for some $n \in \mathbb{Z}$. If $\operatorname{gldim} R < \infty$, then ω_R is a two-sided tilting complex of R. Moreover, if ω_R^{-1} is quasi-ample, then $\operatorname{gldim} R = n$ so that R is a quasi-Fano algebra of dimension n.

Remark 1.4. The preprojective algebra of a quasi-Fano algebra of dimension n was called the $(n+1)$-Calabi-Yau completion or the derived $(n+1)$-preprojective algebra in [10]. One of the conditions for an algebra R to be quasi-Fano of dimension n can be thought of as the condition that the derived m-preprojective algebra of R is a usual algebra (not only a dg-algebra) if and only if $m = n+1$, so we drop the notion "derived $(n+1)$-" in our definition of the preprojective algebra.

Every finite dimensional algebra R of $\operatorname{gldim} R = 0$ is extremely Fano. In this case, $\Pi R \cong R[x]$ the polynomial algebra over R. By [15, Theorem 3.6], every quasi-Fano algebra R of dimension 1 is extremely Fano. The path algebra kQ of a finite acyclic quiver Q of infinite representation type is a concrete example of an extremely Fano algebra of dimension 1. In this case, $\Pi(kQ)$ is isomorphic to the usual preprojective algebra of kQ.(See [15, 4.1])

2 Preliminaries

Before extending the notion of AS-Gorenstein algebra, we will introduce the notion of minimal projective resolution for $\mathbb{N}$-graded algebras, and review modules and graded rings twisted by automorphisms.

2.1 Minimal projective resolutions

In this subsection, we fix a finite dimensional algebra R, and an $\mathbb{N}$-graded algebra A connected over R. We denote by J_R the Jacobson radical of R, and define the Jacobson radical of A by the homogenous ideal $J_A := J_R \oplus A_{\geq 1}$ of A. In this paper, we sometimes view R as a graded algebra concentrated in degree 0. Since R is finite dimensional, every R-module M has a projective cover $\bar{f} : P \to M$, i.e., P is a projective R-module and $\operatorname{Ker}(\bar{f}) \subset PJ_R$ (cf. [1, Proposition 28.13]). Therefore it is easy to see that every graded right R-module M has a graded projective cover $\bar{f} : P \to M$, i.e., P is a projective graded right R-module and $\operatorname{Ker}(\bar{f}) \subset PJ_R$.

We denote by $\underline{\otimes}$ the graded tensor product. Then there is the following adjoint pair of functors

$$- \underline{\otimes}_R A : \operatorname{GrMod} R \rightleftarrows \operatorname{GrMod} A : \underline{\operatorname{Hom}}_A(A, -)$$

where we view A as a graded R-A bimodule. We identify the base algebra R with $A/A_{\geq 1}$ by the canonical morphism $R \hookrightarrow A \twoheadrightarrow A/A_{\geq 1}$. Note that, for a projective graded right R-module P, the tensor product $P \underline{\otimes}_R A$ with A is a projective graded right A-module. Note that, for each graded right R-module M, the tensor product $M \underline{\otimes}_A R$ is naturally isomorphic to M as graded right R-modules.

The next lemma is called the Nakayama lemma.

Lemma 2.1. *For $M \in \operatorname{Mod} R$, $M \subset MJ_R$ if and only if $M = 0$. Moreover, for $M \in \operatorname{GrMod} A$ bounded below, $M \subset MJ_A$ if and only if $M = 0$.*

Proof. Since R is finite dimensional, if $M \subset MJ_R$, then $M = MJ_R = MJ_R^2 = \cdots = 0$.

Moreover, let $M \in \operatorname{GrMod} A$ be bounded below. If $M \neq 0$, then there is an integer $p \in \mathbb{Z}$ such that $M_p \neq 0$ and $M_q = 0$ for $q < p$. If $M \subset MJ_A$, then $M_p \subset (MJ_A)_p = M_pJ_R$, so $M_p = 0$, which is a contradiction. □

The following version of Nakayama lemma is well-known, and easily follows from the above lemma.

Lemma 2.2. *For $M \in \operatorname{GrMod} A$ bounded below, $M \underline{\otimes}_A R = 0$ if and only if $M = 0$.*

Lemma 2.3. *A projective R-module Q is isomorphic to a direct sum of indecomposable direct summands of R.*

Proof. By [1, Proposition 28.13], there is a projective cover $\bar{f} : P \to Q$ so that $\operatorname{Ker}(\bar{f}) \subset PJ_R$ where P is a direct sum of direct summands of R. Since Q is projective, there is a projective module Q' and an isomorphism $\alpha : Q \oplus Q' \xrightarrow{\cong} P$ such that the composition $\bar{f} \circ \alpha$ is equal to the first projection $Q \oplus Q' \to Q$. Under the isomorphism α, Q' is isomorphic to $\operatorname{Ker}(\bar{f})$. Hence $Q' \subset Q \oplus Q'$ is contained in $(Q \oplus Q')J_R = QJ_R \oplus Q'J_R$. Therefore we have $Q' \subset Q'J_R$, so $Q' = 0$ by Lemma 2.1. □

Lemma 2.4. *Let P be a projective graded right R-module. If $g : P \underline{\otimes}_R A \to P \to P/PJ_R$ is the canonical projection, then $\operatorname{Ker}(g) = (P \underline{\otimes}_R A)J_A$.*

Proof. By the above lemma, P is of the form $\oplus_{i \in I} P^i(s^i)$ where P^i is an indecomposable direct summand of R and $s^i \in \mathbb{Z}$ for each i. Therefore, we reduce to the case when P is a direct summand of R as a right R-module. In this case, there is a primitive idempotent element e of R such that $P \cong eR$. Then we have $P \underline{\otimes}_R A \cong eA$ and $P/PJ_R \cong e(R/J_R)$. Now it is easy to see the claim. □

Lemma 2.5. *Let M be a graded right A-module bounded below. Then there is a graded projective right R-module P bounded below and a surjective homomorphism $f : P \underline{\otimes}_R A \to M$ in $\operatorname{GrMod} A$ such that $\operatorname{Ker}(f) \subset (P \underline{\otimes}_R A)J_A$.*

We call a surjective homomorphism $f : P \underline{\otimes}_R A \to M$ in $\operatorname{GrMod} A$ satisfying this property *a projective cover* of M.

Proof. By [1, Proposition 28.13], there is a projective cover $\bar{f} : P \to M \underline{\otimes}_A R$ of a graded right R-module $M \underline{\otimes}_A R$ so that $\operatorname{Ker}(\bar{f}) \subset PJ_R$. Since $P \underline{\otimes}_R A$ is a graded projective right A-module, there is a homomorphism $f : P \underline{\otimes}_R A \to M$ in $\operatorname{GrMod} A$ which completes the following commutative diagram

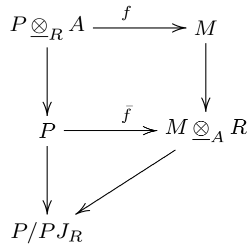
$$\begin{CD}
P \underline{\otimes}_R A @>{f}>> M \\
@VVV @VVV \\
P @>{\bar{f}}>> M \underline{\otimes}_A R
\end{CD}$$

in $\operatorname{GrMod} A$. We claim that f is surjective. Indeed, applying $- \underline{\otimes}_A R$ to the exact sequence $P \underline{\otimes}_R A \to M \to \operatorname{Coker}(f) \to 0$, we obtain the following commutative diagram

$$\begin{CD}
(P \underline{\otimes}_R A) \underline{\otimes}_A R @>{f \underline{\otimes}_A R}>> M \underline{\otimes}_A R @>>> \operatorname{Coker}(f) \underline{\otimes}_A R @>>> 0 \\
@V{\cong}VV @| \\
P @>{\bar{f}}>> M \underline{\otimes}_A R
\end{CD}$$

in GrMod R. Since vertical arrows are isomorphisms and $\bar{f}$ is surjective, we see $\operatorname{Coker}(f) \underline{\otimes}_A R = 0$. Since $\operatorname{Coker}(f)$ is bounded below, we conclude that $\operatorname{Coker}(f) = 0$ by Lemma 2.2. This proves the claim.

Since $\bar{f} : P \to M \underline{\otimes}_A R$ is a projective cover, $\operatorname{Ker} \bar{f} \subset PJ_R$, so there is a homomorphism $M \underline{\otimes}_R A \to P/PJ_R$ in GrMod A which completes the following commutative diagram



in GrMod A. If $g : P \underline{\otimes}_R A \to P \to P/PJ_R$ is the canonical projection, then $\operatorname{Ker}(f) \subset \operatorname{Ker}(g) = (P \underline{\otimes}_R A)J_A$ by Lemma 2.4. □

Lemma 2.6. *For a graded projective right A-module Q bounded below, there is a graded projective right R-module P bounded below such that $Q \cong P \underline{\otimes}_R A$.*

Proof. Let $f : P \underline{\otimes}_R A \to Q$ be a projective cover of Q. Since Q is projective, there is a projective graded right A-module Q' bounded below and an isomorphism $\alpha : Q \oplus Q' \cong P \underline{\otimes}_R A$ such that the composition $p \circ \alpha$ is the first projection. Since $\alpha(Q') \subset \operatorname{Ker}(f) \subset (P \underline{\otimes}_R A)J_A$, it follows that $Q' \subset (Q \oplus Q')J_A = QJ_A \oplus Q'J_A$, so $Q' \subset Q'J_A$, hence $Q' = 0$ by Lemma 2.1. □

By the above lemma, we often write a graded projective right A-module bounded below in the form $P \underline{\otimes}_R A$ where P is a graded projective right R-module bounded below in order to avoid some potential confusion. Let M be a graded right A-module bounded below. By Lemma 2.5, we can construct a projective resolution of M

$$\cdots \to P^{-2} \underline{\otimes}_R A \xrightarrow{d^{-2}} P^{-1} \underline{\otimes}_R A \xrightarrow{d^{-1}} P^0 \underline{\otimes}_R A \to M \to 0$$

such that $\operatorname{Im}(d^i) \subset (P^{i+1} \underline{\otimes}_R A)J_A$ for each $i < 0$. We call such a resolution *a minimal projective resolution* of M. Since a simple module in GrMod A^o is naturally viewed as a shift of simple module in Mod R^o, the differential d^i of a minimal projective resolution vanishes when d^i is tensored by a graded simple A^o-module S, i.e., $d^i \underline{\otimes}_A S = 0$.

We will define the graded small global dimension of A by $\operatorname{sgldim} A := \sup\{\operatorname{pd}_A S \mid S \in \operatorname{GrMod} A \text{ simple }\}$.

Proposition 2.7. *We have*

$$\operatorname{gldim} A = \operatorname{sgldim} A = \operatorname{sgldim} A^o = \operatorname{gldim} A^o.$$

Proof. Clearly, $\operatorname{sgldim} A \leq \operatorname{gldim} A$, so it is enough to show that $\operatorname{gldim} A \leq \operatorname{sgldim} A^o$ when the right hand side is finite. Assume that $n = \operatorname{sgldim} A < \infty$.

By [23, Theorem 4.1.2], it is enough to show that $\operatorname{pd}_A A/I \leq n$ for every graded right ideal I of A. Therefore it is enough to show that $\operatorname{pd}_A M \leq n$ for every graded right A-module M bounded below. Let

$$\cdots \to P^{-2} \underline{\otimes}_R A \xrightarrow{d^{-2}} P^{-1} \underline{\otimes}_R A \xrightarrow{d^{-1}} P^0 \underline{\otimes}_R A \to M \to 0$$

be a minimal projective resolution of M. By the above remark, for any simple module $S \in \operatorname{GrMod} A^o$, we have $(P^{-p} \underline{\otimes}_R A) \underline{\otimes}_A S \cong \underline{\operatorname{Tor}}_p^A(M, S) = 0$ for all $p > n$. Since R is finite dimensional, it is a finite extension of graded simple right A-modules, so $(P^{-p} \underline{\otimes}_R A) \underline{\otimes}_A R = 0$, hence $P^{-p} = (P^{-p} \underline{\otimes}_R A) \underline{\otimes}_A R = 0$ for all $p > n$. □

2.2 Twist by an Automorphism

First, we define a module twisted by a ring automorphism. Let R be a ring and $\sigma \in \operatorname{Aut} R$ a ring automorphism. For $M \in \operatorname{Mod} R$, we define $M_\sigma \in \operatorname{Mod} R$ by $M_\sigma = M$ as an abelian group with the new right action $m * a = m\sigma(a)$, which induces an equivalence functor $(-)_\sigma \cong - \otimes_R R_\sigma : \operatorname{Mod} R \to \operatorname{Mod} R$. Similarly, for $M \in \operatorname{Mod} R^o$, we define ${}_\sigma M \in \operatorname{Mod} R^o$ by ${}_\sigma M = M$ as an abelian group with the new left action $a * m = \sigma(a)m$, which induces an equivalence functor ${}_\sigma(-) \cong {}_\sigma R \otimes_R - : \operatorname{Mod} R^o \to \operatorname{Mod} R^o$. The map $\sigma : R \to R$ gives an isomorphism ${}_{\sigma^{-1}}R \to R_\sigma$ in $\operatorname{Mod} R^e$. Note that ${}_{\sigma^{-1}}M$ is not necessarily isomorphic to M_σ in $\operatorname{Mod} R^e$ for a general $M \in \operatorname{Mod} R^e$, however, we have the following result.

Lemma 2.8. *Let R be a ring and $M \in \operatorname{Mod} R$.*
(1) *For $\sigma \in \operatorname{Aut} R$, $\operatorname{Hom}_R(M_\sigma, R) \cong {}_\sigma \operatorname{Hom}_R(M, R)$ in $\operatorname{Mod} R^o$.*
(2) *If R is an algebra and $\sigma \in \operatorname{Aut}_k R$, then $D(M_\sigma) \cong {}_\sigma(DM)$ in $\operatorname{Mod} R^o$.*

Proof. Left to the reader. □

A graded module twisted by a graded ring automorphism can be defined in a similar way. The following lemma is standard.

Lemma 2.9. *Let R be a finite dimensional algebra and M an R-R bimodule. If $M \cong R$ in $\operatorname{Mod} R$ and $\operatorname{Mod} R^o$, then there exists an algebra automorphism $\sigma \in \operatorname{Aut}_k R$ such that $M \cong R_\sigma \cong {}_{\sigma^{-1}}R$ in $\operatorname{Mod} R^e$.*

Similarly, let A be a locally finite $\mathbb{N}$-graded algebra and M a graded A-A bimodule. If $M \cong A$ in $\operatorname{GrMod} A$ and in $\operatorname{GrMod} A^o$, then there exists a graded algebra automorphism $\tau \in \underline{\operatorname{Aut}}_k A$ such that $M \cong A_\tau \cong {}_{\tau^{-1}}A$ in $\operatorname{GrMod} A^e$.

Proof. The first claim is a special case of the second claim for a graded algebra concentrated in degree 0.

Fix a graded left A-module isomorphism $\phi : M \to A$ and define a graded vector space map $\tau : A \to A$ by $\tau(a) := \phi(\phi^{-1}(1)a)$. Since an element $\tau(a)$ of the LHS corresponds to $\phi \circ (- \times a) \circ \phi^{-1}$ under the anti-isomorphism $A \cong$

$\underline{\mathrm{Hom}}_{A^o}(A, A)$ of graded algebras where $- \times a$ is the right multiplication by a, it is easy to see that τ is a graded algebra homomorphism and ϕ induces an isomorphism $M \to A_\phi$ of graded A-A-bimodules. Since M is isomorphic to A as graded right A-modules, the right action on M of A is faithful. Therefore we see that τ is injective. Since A is locally finite, τ is surjective. Hence τ is a graded algebra automorphism of A. □

Second, we define a graded ring twisted by a graded ring automorphism. Let A be a graded ring and $\tau \in \underline{\mathrm{Aut}} A$ a graded ring automorphism. We define a new graded ring A^τ by $A^\tau = A$ as a graded abelian group with new multiplication defined by $a * b = a\tau^i(b)$ where $a \in A_i, b \in A_j$. Similarly, we define a new graded ring ${}^\tau A$ by ${}^\tau A = A$ as a graded abelian group with new multiplication defined by $a * b = \tau^j(a)b$ where $a \in A_i, b \in A_j$. The map $A \to A$ defined by $a \mapsto \tau^i(a)$ for $a \in A_i$ gives an isomorphism ${}^{\tau^{-1}}A \to A^\tau$ as graded rings. By [25], $\mathrm{GrMod}\, {}^\tau A \cong \mathrm{GrMod}\, A^\tau \cong \mathrm{GrMod}\, A$ for any $\tau \in \underline{\mathrm{Aut}} A$.

We will give some relationships between these two notions of twist.

Lemma 2.10. *Let A be an $\mathbb{N}$-graded ring connected over R, $\tau \in \underline{\mathrm{Aut}} A$ a graded ring automorphism, and $\sigma = \tau|_R \in \mathrm{Aut}\, R$ the restriction.*

(1) *For $M \in \mathrm{GrMod}\, A$, $(M_i)_\sigma = (M_\tau)_i$ in $\mathrm{Mod}\, R$.*

(2) *$(A^\tau)_i = (A_i)_{\sigma^i}$ in $\mathrm{Mod}\, R^e$.*

Proof. (1) For $m \in M_i$ and $a \in R \subset A$, the right action of a in $(M_i)_\sigma$ is $m * a = m\sigma(a) = m\tau(a)$, and the right action of a in $(M_\tau)_i$ is $m * a = m\tau(a)$, hence the result.

(2) For $b \in A_i$ and $a \in R$, the right action of a in $(A^\tau)_i$ is $b * a = b\tau^i(a)$, and the right action of a in $(A_i)_{\sigma^i}$ is $b * a = b\sigma^i(a) = b\tau^i(a)$, hence the result. □

Lemma 2.11. *Let R be a ring and M an R-R bimodule. If $\tau \in \underline{\mathrm{Aut}} T_R(M)$ is a graded ring automorphism, and $\sigma = \tau|_R \in \mathrm{Aut}\, R$ is the restriction, then $T_R({}_\sigma M) \cong {}^\tau T_R(M)$ as graded algebras over R.*

Proof. We denote by $\tau_i = \tau|_{M^{\otimes_R i}} : M^{\otimes_R i} \to M^{\otimes_R i}$ the degree i component of τ so that $\sigma = \tau_0$. The map $\tau_1^i : {}_\sigma M_{\sigma^{-i}} \to {}_{\sigma^{i+1}} M$ is an isomorphism in $\mathrm{Mod}\, R^e$ because, for $a, b \in R, m \in M$,

$$\begin{aligned} \tau_1^i(a * m * b) &= \tau_1^i(\sigma(a) m \sigma^{-i}(b)) = \tau_0^i(\sigma(a))\tau_1^i(m)\tau_0^i(\sigma^{-i}(b)) \\ &= \sigma^{i+1}(a)\tau_1^i(m)b = a * \tau_1^i(m) * b, \end{aligned}$$

so the map

$$\tau_1^i \otimes \mathrm{id}_M : {}_\sigma M \otimes {}_{\sigma^i} M \cong {}_\sigma M \otimes {}_{\sigma^i} R \otimes M \cong {}_\sigma M \otimes R_{\sigma^{-i}} \otimes M \cong {}_\sigma M_{\sigma^{-i}} \otimes M \to {}_{\sigma^{i+1}} M \otimes M$$

is an isomorphism in $\mathrm{Mod}\, R^e$ for each $i \in \mathbb{N}^+$ where $\otimes = \otimes_R$. We define an isomorphism $\alpha_i : ({}_\sigma M)^{\otimes_R i} \to {}_{\sigma^i} M^{\otimes_R i}$ in $\operatorname{Mod} R^e$ by the composition

$$\begin{array}{c}
{}_\sigma M \otimes {}_\sigma M \otimes \cdots \otimes {}_\sigma M \otimes {}_\sigma M \otimes {}_\sigma M \\
\big\downarrow \scriptstyle \mathrm{id}_M \otimes \mathrm{id}_M \otimes \cdots \otimes \mathrm{id}_M \otimes \tau_1 \otimes \mathrm{id}_M \\
{}_\sigma M \otimes {}_\sigma M \otimes \cdots \otimes {}_\sigma M \otimes {}_{\sigma^2} M \otimes M \\
\big\downarrow \scriptstyle \mathrm{id}_M \otimes \mathrm{id}_M \otimes \cdots \otimes \tau_1^2 \otimes \mathrm{id}_M \otimes \mathrm{id}_M \\
{}_\sigma M \otimes {}_\sigma M \otimes \cdots \otimes {}_{\sigma^3} M \otimes M \otimes M \\
\big\downarrow \scriptstyle \mathrm{id}_M \otimes \mathrm{id}_M \otimes \cdots \otimes \tau^3 \otimes \mathrm{id}_M \otimes \mathrm{id}_M \\
\vdots \\
\big\downarrow \scriptstyle \tau_1^{i-1} \otimes \mathrm{id}_M \otimes \cdots \otimes \mathrm{id}_M \otimes \mathrm{id}_M \otimes \mathrm{id}_M \\
{}_{\sigma^i} M \otimes M \otimes \cdots \otimes M \otimes M \otimes M,
\end{array}$$

i.e.,

$$\alpha_i(m_1 \otimes m_2 \otimes \cdots \otimes m_i) := \tau_1^{i-1}(m_1) \otimes \tau_1^{i-2}(m_2) \otimes \tau_1^{i-3}(m_3) \otimes \cdots \otimes m_i$$

where $m_p \in M$ for each $p = 1, \dots, i$. We set $\alpha_0 := \mathrm{id}_R : R \to R$. It is now routine to check that $\alpha := \oplus_{i \geq 0} \alpha_i : T_R({}_\sigma M) \to {}^\tau T_R(M)$ is an isomorphism of graded algebras over R. We leave the details to the reader. □

3 Generalizations of AS-Gorenstein Algebras

Throughout this section, let R be a finite dimensional algebra, and A a locally finite $\mathbb{N}$-graded algebra connected over R. We give a few possible generalizations of the notion of AS-Gorenstein algebra for such an A, and compare these generalizations.

3.1 AS-Gorenstein Algebras over R

The following is the most natural generalization of AS-Gorenstein algebra for $\mathbb{N}$-graded algebras.

Definition 3.1. A locally finite $\mathbb{N}$-graded algebra A connected over R is called AS-Gorenstein over R (resp. AS-regular over R) of dimension n and of Gorenstein parameter ℓ if $\operatorname{id} A = n$ (resp. $\operatorname{gldim} A = n$ and $\operatorname{gldim} R < \infty$) and A satisfies the Gorenstein condition over R, that is,

$$\underline{\mathrm{Ext}}^i_A(R, A) \cong \begin{cases} (DR)(\ell) & \text{if } i = n, \\ 0 & \text{otherwise} \end{cases}$$

in $\operatorname{GrMod} R$ and in $\operatorname{GrMod} R^o$, or equivalently, $\mathbf{R}\underline{\mathrm{Hom}}_A(R, A) \cong (DR)(\ell)[-n]$ in $\mathcal{D}(\operatorname{GrMod} R)$ and in $\mathcal{D}(\operatorname{GrMod} R^o)$.

First note that the usual AS-Gorenstein algebras are exactly AS-Gorenstein algebras over k in the above definition. If A satisfies the Gorenstein condition over R as above, then $D\,\underline{\mathrm{Ext}}^n_A(R,A) \cong R(-\ell)$ in $\mathrm{GrMod}\,R$ and in $\mathrm{GrMod}\,R^o$, so $D\,\underline{\mathrm{Ext}}^n_A(R,A) \cong R_\sigma(-\ell)$ in $\mathrm{GrMod}\,R^e$ for some algebra automorphism $\sigma \in \mathrm{Aut}_k\,R$ by Lemma 2.9. It follows that the Gorenstein condition over R is equivalent to the condition

$$\underline{\mathrm{Ext}}^i_A(R,A) \cong \begin{cases} {}_\sigma(DR)(\ell) & \text{if } i = n, \\ 0 & \text{otherwise} \end{cases}$$

in $\mathrm{GrMod}\,R^e$ for some algebra automorphism $\sigma \in \mathrm{Aut}_k\,R$ by Lemma 2.8 (2). In this case, we say that A satisfies the Gorenstein condition over R with respect to (n,ℓ,σ). By convention, we say that A^o satisfies the Gorenstein condition over R with respect to (n,ℓ,σ) if

$$\underline{\mathrm{Ext}}^i_{A^o}(R,A) \cong \begin{cases} (DR)_\sigma(\ell) & \text{if } i = n, \\ 0 & \text{otherwise} \end{cases}$$

in $\mathrm{GrMod}\,R^e$.

In the sequel, we use the following notations

$$\begin{array}{lll} (-)^\vee & := \underline{\mathrm{Hom}}_A(-,A) & : \mathrm{GrMod}\,A \to \mathrm{GrMod}\,A^o \\ (-)^\vee & := \underline{\mathrm{Hom}}_{A^o}(-,A) & : \mathrm{GrMod}\,A^o \to \mathrm{GrMod}\,A \\ (-)^* & := \underline{\mathrm{Hom}}_R(-,R) & : \mathrm{GrMod}\,R \to \mathrm{GrMod}\,R^o \\ (-)^* & := \underline{\mathrm{Hom}}_{R^o}(-,R) & : \mathrm{GrMod}\,R^o \to \mathrm{GrMod}\,R. \end{array}$$

For the graded projective right R-module P, we have a natural isomorphism $(P \underline{\otimes}_R A)^\vee \cong A \underline{\otimes}_R P^*$ in $\mathrm{GrMod}\,A^o$.

Lemma 3.2. *Let $0 \neq M \in \mathrm{GrMod}\,A$ such that $\mathrm{pd}_A M < \infty$. If, for some $n \in \mathbb{N}$, $\underline{\mathrm{Ext}}^i_A(M,A) = 0$ for $i \neq n$, then $\mathrm{pd}_A M = n$.*

Proof. Left to the reader. □

The *Jacobson radical* $\mathrm{rad}\,M$ of a graded right A-module M is the intersection of the maximal graded submodules.

Lemma 3.3. *Let P and Q be graded projective right R-modules and let $f : P \underline{\otimes}_R A \to Q \underline{\otimes}_R A$ be a homomorphism in $\mathrm{GrMod}\,A$. If $\mathrm{Im}(f) \subset (Q \underline{\otimes}_R A)J_A$, then $\mathrm{Im}(f^\vee) \subset \mathrm{rad}((P \underline{\otimes}_R A)^\vee)$.*

Proof. Let $P = \oplus_{\lambda\in\Lambda} P^\lambda$ be a decomposition into indecomposable projective graded right R-modules P^λ. Then we have $(P \underline{\otimes}_R A)^\vee \cong \Pi_{\lambda\in\Lambda}(P^\lambda \underline{\otimes}_R A)^\vee$ and $\mathrm{rad}((P \underline{\otimes}_R A)^\vee) \cong \Pi_{\lambda\in\Lambda} J_A (P^\lambda \underline{\otimes}_R A)^\vee$ in $\mathrm{GrMod}\,A^o$. Let $\iota^\lambda : P^\lambda \subset P$ be the canonical inclusion. It is enough to show that the image of the composition $(\iota^\lambda \underline{\otimes}_R A)^\vee \circ f^\vee : (Q \underline{\otimes}_R A)^\vee \to (P^\lambda \underline{\otimes}_R A)^\vee$ is contained in $J_A(P^\lambda \underline{\otimes}_R A)^\vee$ for each $\lambda \in \Lambda$. Since P^λ is finitely generated by Lemma 2.3, there is a finite direct summand $Q' \subset Q$ such that f is factored by $f' : P^\lambda \underline{\otimes}_R A \to Q' \underline{\otimes}_R A$. Therefore it is enough to show that $\operatorname{Im}(f'^{\vee}) \subset J_A(P^{\lambda} \underline{\otimes}_R A)^{\vee}$. Let $Q' = \oplus_{\lambda' \in \Lambda'} Q'^{\lambda'}$ be a decomposition into indecomposable projective graded right R-modules $Q'^{\lambda'}$. Since the index set Λ' is finite, $(Q' \underline{\otimes}_R A)^{\vee} \cong \oplus_{\lambda' \in \Lambda'} (Q'^{\lambda'} \underline{\otimes}_R A)^{\vee}$ in $\operatorname{GrMod} A^o$. Therefore it is enough to show that $\operatorname{Im}(f'^{\vee}|_{(Q'^{\lambda'} \underline{\otimes}_R A)^{\vee}}) \subset J_A(P^{\lambda} \underline{\otimes}_R A)^{\vee}$. Let $p^{\lambda'} : Q' \twoheadrightarrow Q'^{\lambda'}$ be the canonical projection. Since $f'^{\vee}|_{(Q'^{\lambda'} \underline{\otimes}_R A)^{\vee}} = (p^{\lambda'} \circ f')^{\vee}$, we may assume that Q' is indecomposable by replacing $Q'^{\lambda'}$ by Q'. Note that if P is finitely generated then $\operatorname{rad}((P \underline{\otimes}_R A)^{\vee}) = J_A(P \underline{\otimes}_R A)^{\vee}$. Replacing P^{λ} and Q' by P and Q respectively, we may assume that P and Q are indecomposable projective graded right R-modules.

By shifting the grading, we may assume that P is concentrated in degree 0. If Q is concentrated in degree greater than 0, then $\underline{\operatorname{Hom}}_A(P \underline{\otimes}_R A, Q \underline{\otimes}_R A) = 0$, hence $f = 0$. If Q is concentrated in degree less than 0, then it is easy to see that $\operatorname{Im}(f^{\vee}) \subset (P \underline{\otimes}_R A)^{\vee}_{\geq 0} \subset J_A(P \underline{\otimes}_R A)^{\vee}$. Therefore we may assume that Q is concentrated in degree 0.

Note that, for $f : P \underline{\otimes}_R A \to Q \underline{\otimes}_R A$, there is a homomorphism $g : P \to Q$ in $\operatorname{Mod} R$ such that $f = g \underline{\otimes}_R A$. Under the isomorphisms $(P \underline{\otimes}_R A)^{\vee} \cong A \underline{\otimes}_R P^*$ and $(Q \underline{\otimes}_R A)^{\vee} \cong A \underline{\otimes}_R Q^*$, $(g \underline{\otimes}_R A)^{\vee}$ corresponds to $A \underline{\otimes}_R g^*$. It follows that $\operatorname{Im}(f^{\vee}) = \operatorname{Im}((g \underline{\otimes}_R A)^{\vee}) \subset J_A(P \underline{\otimes}_R A)^{\vee}$ if and only if $\operatorname{Im}(A \underline{\otimes}_R g^*) \subset J_A(A \underline{\otimes}_R P^*)$ which is equivalent to $\operatorname{Im}(g^*) \subset J_R P^*$ because P^*, Q^* are concentrated in degree 0. Since $\operatorname{Im}(g) \subset Q J_R$ if and only if g is not an isomorphism and $\operatorname{Im}(g^*) \subset J_R P$ if and only if g^* is not an isomorphism, now it is easy to see the claim.

□

Proposition 3.4 ([20, Proposition 3.1]). *Let*

$$0 \to P^{-n} \underline{\otimes}_R A \xrightarrow{d^{-n}} P^{-n+1} \underline{\otimes}_R A \xrightarrow{d^{-n+1}} \cdots \xrightarrow{d^{-2}} P^{-1} \underline{\otimes}_R A \xrightarrow{d^{-1}} P^0 \underline{\otimes}_R A \xrightarrow{\epsilon} M \to 0$$

be a minimal projective resolution of $M \in \operatorname{GrMod} A$ *so that* $\operatorname{pd}_A M = n$. *If* $\underline{\operatorname{Ext}}^i_A(M, A) = 0$ *for* $i \neq n$ *and* $\dim_k \underline{\operatorname{Ext}}^n_A(M, A) < \infty$, *then the following hold:*

(1) *For each* i, P^{-i} *is finitely generated as a graded right* R-*module.*

(2) *For each* i, $\min\{s \mid P^{-i}_s \neq 0\} \leq \min\{s \mid P^{-(i+1)}_s \neq 0\}$ *and* $\max\{s \mid P^{-i}_s \neq 0\} \leq \max\{s \mid P^{-(i+1)}_s \neq 0\}$.

(3) $\{s \mid P^{-n}_s \neq 0\} \subset \{s \in \mathbb{Z} \mid \underline{\operatorname{Ext}}^n_A(M, A)_{-s} \neq 0\}$.

Proof. (1) Applying $(-)^{\vee}$ to the minimal projective resolution of M, we obtain the following exact sequence:

$$0 \leftarrow \underline{\operatorname{Ext}}^n_A(M, A) \xleftarrow{\epsilon'} A \underline{\otimes}_R (P^{-n})^* \xleftarrow{\partial^{-n}} A \underline{\otimes}_R (P^{-n+1})^* \xleftarrow{\partial^{-n+1}} \cdots \xleftarrow{\partial^{-1}} A \underline{\otimes}_R (P^0)^* \leftarrow 0$$

in $\operatorname{GrMod} A^o$ where $\partial^i := (d^i)^{\vee}$. Since $\operatorname{Ker} \epsilon' = \operatorname{Im}(\partial^{-n}) \subset \operatorname{rad}(A \underline{\otimes}_R (P^{-n})^*)$ by Lemma 3.3, we have a surjective homomorphism

$$\underline{\operatorname{Ext}}^n_A(M, A) \twoheadrightarrow T \tag{3-1}$$

where we set $T := A \underline{\otimes}_R (P^{-n})^* / \operatorname{rad}(A \underline{\otimes}_R (P^{-n})^*)$. Let $P^{-n} = \oplus_{\lambda \in \Lambda} P^{\lambda}$ be a decomposition into indecomposable projective graded R-modules. We denote by S^{λ} the simple graded R^o-module corresponding to P^{λ}, i.e. if $P^{\lambda} \cong eR(-l)$

for some idempotent element $e \in R$ and some $l \in \mathbb{Z}$, then $S^\lambda := (Re/J_R Re)(l)$. Then we have

$$T \cong \Pi_{\lambda\in\Lambda} S^\lambda. \tag{3-2}$$

Since $\underline{\mathrm{Ext}}^n_A(M, A)$ is finite dimensional, we see that the index set Λ is finite by the surjection (3-1). Hence P^{-n} is finitely generated.

We prove P^{-i} is finitely generated by descending induction on i. Suppose that $P^{-(i+1)}$ is finitely generated. Then P^{-i} has a finite direct summand P'^{-i} such that the image of $d^{-(i+1)}$ is contained in $P'^{-i} \underline{\otimes}_R A$. Let P''^{-i} be the complement of P'^{-i} in P^{-i}, i.e., $P^{-i} = P'^{-i} \oplus P''^{-i}$. Then the restriction of $\partial^{-(i+1)}$ to $A \underline{\otimes}_R (P''^{-i})^*$ vanishes. Since $0 = \underline{\mathrm{Ext}}^i(M, A) = \mathrm{Ker}\,\partial^{-(i+1)}/\,\mathrm{Im}\,\partial^{-i}$, we see that $\mathrm{Im}\,\partial^{-i} = \mathrm{Ker}\,\partial^{-(i+1)}$ and that $A \underline{\otimes}_R (P''^{-i})^*$ is contained in $\mathrm{Im}\,\partial^{-i}$. By Lemma 3.3, $\mathrm{Im}\,\partial^{-i} \subset \mathrm{rad}(A \underline{\otimes}_R (P^{-i})^*)$. By the graded version of [1, 9.19], $\mathrm{rad}(A \underline{\otimes}_R (P''^{-i})^*) = \mathrm{rad}(A \underline{\otimes}_R (P'^{-i})^*) \oplus \mathrm{rad}(A \underline{\otimes}_R (P''^{-i})^*)$. Therefore we see that $A \underline{\otimes}_R (P''^{-i})^* \subset \mathrm{rad}(A \underline{\otimes}_R (P''^{-i})^*)$. Hence $A \underline{\otimes}_R (P''^{-i})^* = 0$ and $P''^{-i} = 0$. Therefore $P^{-i} = P'^{-i}$ is finitely generated.

(2) The first part of (2) is by the definition of a minimal projective resolution. Set $s_0 := \max\{s \mid P_s^{-(i+1)} \neq 0\}$. We take the decomposition $P^{-i} = P^{-i}_{\leq s_0} \oplus P^{-i}_{>s_0}$. Since $(P^{-i})^*_{<-s_0} = (P^{-i}_{>s_0})^*$ and $-s_0 = \min\{s \mid (P^{-(i+1)})^*_s \neq 0\}$, we have $\mathrm{Hom}_{A^o}(A \underline{\otimes}_R (P^{-i}_{>s_0})^*, A \underline{\otimes}_R (P^{-(i+1)})^*) = 0$. Therefore the restriction of $\partial^{-(i+1)}$ to $A \underline{\otimes}_R (P^{-i}_{>s_0})^*$ vanishes. Hence $A \underline{\otimes}_R (P^{-i}_{>s_0})^*$ is contained in $\mathrm{Ker}\,\partial^{-(i+1)}$. By the same argument in the proof of (1), we see that $P^{-i}_{>s_0} = 0$. This finishes the proof of (2).

(3) By the isomorphism (3-2), we see that if $P^{-n}_s \neq 0$ then $T_{-s} \neq 0$. Therefore by looking at the degree $-s$ part of the surjection (3-1), we see the claim. □

In the sequel, we use the following notations

$$\begin{aligned} \mathbb{E}^i(-) &:= \underline{\mathrm{Ext}}^i_A(-, A) &&: \mathrm{GrMod}\,A \to \mathrm{GrMod}\,A^o \\ \mathbb{E}'^i(-) &:= \underline{\mathrm{Ext}}^i_{A^o}(-, A) &&: \mathrm{GrMod}\,A^o \to \mathrm{GrMod}\,A. \end{aligned}$$

If A satisfies the Gorenstein condition over R with respect to (n, ℓ, σ), then we also use the following notations

$$\begin{aligned} F(-) &:= {}_\sigma D(-)(\ell) &&: \mathrm{GrMod}\,R \to \mathrm{GrMod}\,R^o \\ F'(-) &:= D(-)_{\sigma^{-1}}(\ell) &&: \mathrm{GrMod}\,R^o \to \mathrm{GrMod}\,R. \end{aligned}$$

Clearly, F and F' are exact functors. For $N \in \mathrm{grmod}\,R^o$,

$$F(F'N)) \cong {}_\sigma D((DN)_{\sigma^{-1}}(\ell))(\ell) \cong {}_{\sigma^{-1}}({}_\sigma DDN) \cong N$$

by Lemma 2.8, so it is easy to see that $F : \mathrm{grmod}\,R \leftrightarrow \mathrm{grmod}\,R^o : F'$ give a duality. By abusing notation, we denote by $\mathbb{E}^i$ the functor $U \circ \mathbb{E}^i \circ I : \mathrm{GrMod}\,R \to \mathrm{GrMod}\,R^o$ where $I := - \underline{\otimes}_R A : \mathrm{GrMod}\,R \hookrightarrow \mathrm{GrMod}\,A$ is the embedding functor and $U := \underline{\mathrm{Hom}}_{A^o}(A, -) : \mathrm{GrMod}\,A^o \twoheadrightarrow \mathrm{GrMod}\,R^o$ is the forgetful functor. In the same way, we denote by $\mathbb{E}'^i$ the functor $\mathrm{GrMod}\,R^o \to \mathrm{GrMod}\,R$ induced by $\mathbb{E}'^i$.

Proposition 3.5. *Let R be a finite dimensional algebra of finite global dimension. If A is an $\mathbb{N}$-graded algebra satisfying the Gorenstein condition over R with respect to (n, ℓ, σ), then* $\mathbb{E}^i \cong \begin{cases} F & i = n \\ 0 & \text{otherwise} \end{cases}$ *as functors* $\operatorname{GrMod} R \to \operatorname{GrMod} R^o$.

Proof. Since every module in $\operatorname{GrMod} R$ is a direct sum of shifts of modules in $\operatorname{Mod} R$, and both $\mathbb{E}^i$ and F commute with direct sum and shifts, it is enough to show that $\mathbb{E}^i \cong \begin{cases} F & i = n \\ 0 & \text{otherwise} \end{cases}$ as functors $\operatorname{Mod} R \to \operatorname{GrMod} R^o$. Let $\mathcal{C}_p$ be the full subcategory of $\operatorname{Mod} R$ consisting of modules of projective dimension less than or equal to p. We prove the functor $\mathbb{E}^n$ is naturally isomorphic to the functor F and $\mathbb{E}^i = 0$ for $i \neq n$ on $\mathcal{C}_p$ by induction on p.

We fix an isomorphism $\chi_R : \mathbb{E}^n(R) := \underline{\operatorname{Ext}}^n_A(R, A) \xrightarrow{\cong} {}_\sigma(DR)(\ell) =: F(R)$ in $\operatorname{Mod} R^e$. For any index set I, χ_R induces an isomorphism $\chi_{R^{\oplus I}} : \mathbb{E}^n(R^{\oplus I}) \xrightarrow{\cong} F(R^{\oplus I})$ in $\operatorname{Mod} R^e$. The right R-module structure on R induces the left R-module structure on $\mathbb{E}^n(R)$ and on $F(R)$. Therefore any homomorphism $\phi : R^{\oplus I} \to R^{\oplus J}$ in $\operatorname{Mod} R$ induces the following commutative diagram

$$\begin{array}{ccc} \mathbb{E}^n(R^{\oplus J}) & \xrightarrow{\mathbb{E}^n(\phi)} & \mathbb{E}^n(R^{\oplus I}) \\ \cong \downarrow \chi_{R^{\oplus J}} & & \cong \downarrow \chi_{R^{\oplus I}} \\ F(R^{\oplus J}) & \xrightarrow[F(\phi)]{} & F(R^{\oplus I}) \end{array}$$

in $\operatorname{GrMod} R^o$.

The case $p = 0$. Let P be a projective right R-module. It is easy to see that $\mathbb{E}^i(P) = 0$ for $i \neq n$. Let $R^{\oplus J} \xrightarrow{f} R^{\oplus I} \xrightarrow{g} P \to 0$ be an exact sequence in $\operatorname{Mod} R$. Since the kernel of the first morphism $g : R^{\oplus I} \to P$ is projective, we obtain the following commutative diagram

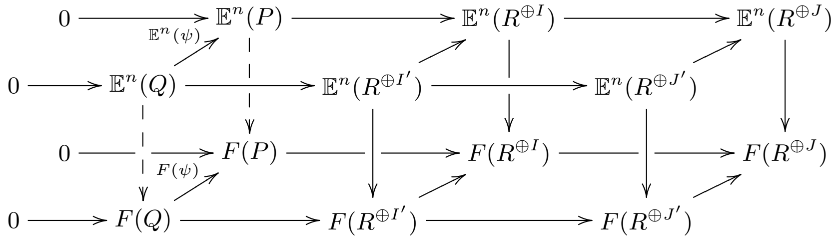
$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{E}^n(P) & \xrightarrow{\mathbb{E}^n(g)} & \mathbb{E}^n(R^{\oplus I}) & \xrightarrow{\mathbb{E}^n(f)} & \mathbb{E}^n(R^{\oplus J}) \\ & & \chi_P \downarrow & & \cong \downarrow \chi_{R^{\oplus I}} & & \cong \downarrow \chi_{R^{\oplus J}} \\ 0 & \longrightarrow & F(P) & \xrightarrow[F(g)]{} & F(R^{\oplus I}) & \xrightarrow[F(f)]{} & F(R^{\oplus J}). \end{array}$$

in $\operatorname{GrMod} R^o$, hence we obtain an isomorphism $\chi_P : \mathbb{E}^n(P) \xrightarrow{\cong} F(P)$ in $\operatorname{GrMod} R^o$.

Let Q be another projective right R-module, and $R^{\oplus J'} \xrightarrow{f'} R^{\oplus I'} \xrightarrow{g'} Q \to 0$ an exact sequence in $\operatorname{Mod} R$. For any homomorphism $\psi : P \to Q$ in $\operatorname{Mod} R$, we obtain the following commutative diagram

$$\begin{array}{ccccccc} R^{\oplus J} & \xrightarrow{f} & R^{\oplus I} & \xrightarrow{g} & P & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \psi} & & \\ R^{\oplus J'} & \xrightarrow{f'} & R^{\oplus I'} & \xrightarrow{g'} & Q & \longrightarrow & 0. \end{array}$$

in $\operatorname{Mod} R$. Applying the functors $\mathbb{E}^n(-)$ and $F(-)$ to this diagram, we obtain the following commutative diagram



in $\operatorname{GrMod} R^o$. From this diagram, we see that the assignment χ_P is functorial in P. Note that this also shows that χ_P dose not depend on the choice of a presentation $R^{\oplus J} \xrightarrow{f} R^{\oplus I} \xrightarrow{g} P \to 0$. Therefore χ gives a natural isomorphism between $\mathbb{E}^n$ and F on $\mathcal{C}_0$.

Now assume that we have a natural isomorphism $\chi : \mathbb{E}^n \Rightarrow F$ and $\mathbb{E}^i(-) = 0$ for $i \neq n$ on $\mathcal{C}_{p-1}$. Let M be a right R-module of projective dimension p. Let $0 \to M' \xrightarrow{f} P \xrightarrow{g} M \to 0$ be an exact sequence in $\operatorname{Mod} R$ such that P is a projective right R-module. Then $f : M' \to P$ is a morphism in $\mathcal{C}_{p-1}$. Therefore there is the following commutative diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathbb{E}^n(M) & \xrightarrow{\mathbb{E}^n(g)} & \mathbb{E}^n(P) & \xrightarrow{\mathbb{E}^n(f)} & \mathbb{E}^n(M') & \longrightarrow & \mathbb{E}^{n+1}(M) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \chi_M} & & \downarrow{\scriptstyle \chi_P} & & \downarrow{\scriptstyle \chi_{M'}} & & & & \\ 0 & \longrightarrow & F(M) & \xrightarrow{F(g)} & F(P) & \xrightarrow{F(f)} & F(M') & \longrightarrow & 0 & & \end{array}$$

in $\operatorname{GrMod} R^o$ where the top and bottom lows are exact. Therefore we obtain an isomorphism $\chi_M : \mathbb{E}^n(M) \xrightarrow{\cong} F(M)$ and see that $\mathbb{E}^i(M) = 0$ for $i \neq n$. By the same method as in the case $p = 0$, we see that χ_M dose not depend on the choice of the surjection $g : P \twoheadrightarrow M$ and that the assignment χ_M is functorial in M. $\square$

Proposition 3.6. *Let R be a finite dimensional algebra of finite global dimension, and A an $\mathbb{N}$-graded algebra satisfying the Gorenstein condition over R with respect to (n, ℓ, σ). If $\operatorname{gldim} A < \infty$, then the following hold:*

(1) *A^o satisfies the Gorenstein condition over R with respect to (n, ℓ, σ^{-1}), so that* $\mathbb{E}'^i \cong \begin{cases} F' & i = n \\ 0 & \text{otherwise} \end{cases}$ *as functors* $\operatorname{GrMod} R^o \to \operatorname{GrMod} R$.

(2) *For every* $0 \neq M \in \operatorname{Mod} R$*,* $\operatorname{pd}_A M = n$*. Dually, for every* $0 \neq N \in \operatorname{Mod} R^o$*,* $\operatorname{pd}_{A^o} N = n$*.*

(3) $\operatorname{gldim} A = \operatorname{gldim} A^o = n$*.*

Proof. (1) Since $\operatorname{gldim} A < \infty$, if $M \in \operatorname{mod} R$, then $\operatorname{pd}_A M = n$ by Proposition 3.5 and Lemma 3.2. Let $0 \to P^{-n} \underline{\otimes}_R A \to \cdots \to P^0 \underline{\otimes}_R A \to M \to 0$ be a minimal projective resolution of M in $\operatorname{GrMod} A$. Applying $(-)^\vee = \underline{\operatorname{Hom}}_A(-, A)$ to this resolution, we obtain a resolution $0 \to (P^0 \underline{\otimes}_R A)^\vee \to \cdots \to (P^{-n} \underline{\otimes}_R A)^\vee \to \mathbb{E}^n(M) \to 0$ of $\mathbb{E}^n(M)$ in $\operatorname{GrMod} A^o$ by Proposition 3.5. Since $P^{-i} \in \operatorname{grmod} R$ by Proposition 3.4, we have a natural isomorphism $(P^{-i} \underline{\otimes}_R A)^{\vee\vee} \cong P^{-i} \underline{\otimes}_R A$ for each i. Therefore, applying $(-)^\vee = \underline{\operatorname{Hom}}_{A^o}(-, A)$ to the above resolution of $\mathbb{E}^n(M)$, we see that $\mathbb{E}'^i \mathbb{E}^n \cong 0$ for $i \neq n$ and $\mathbb{E}'^n \circ \mathbb{E}^n \cong \operatorname{id}$ on $\operatorname{mod} R$.

For $N \in \operatorname{mod} R^o$, we have a natural isomorphism $\mathbb{E}^n(F'(N)) \cong F(F'(N)) \cong N$ in $\operatorname{GrMod} R^o$, so we have a natural isomorphism $\mathbb{E}^n \circ F' \cong \operatorname{id}_{\operatorname{mod} R^o}$. Since the right quasi-inverse of $\mathbb{E}^n$ and the left quasi-inverse of $\mathbb{E}^n$ agree, $\mathbb{E}'^n$ is naturally isomorphic to F'. Since this isomorphism is natural in N, the isomorphism $\mathbb{E}'^n(R) \cong F'(R) := (DR)_{\sigma^{-1}}(\ell)$ is an isomorphism in $\operatorname{GrMod} R^e$. Since $\mathbb{E}'^i(R) \cong \mathbb{E}'^i \mathbb{E}^n F'(R) \cong 0$, A^o satisfies the Gorenstein condition over R with respect to (n, ℓ, σ^{-1}), so we finish the proof of (1) by the dual of Proposition 3.5.

(2) follows from Lemma 3.2.

(3) follows from (2) and Proposition 2.7. □

If R is a finite dimensional algebra, then $\operatorname{gldim} R = \operatorname{gldim} R^o$, hence the following result.

Corollary 3.7. *A locally finite* $\mathbb{N}$*-graded algebra* A *is AS-regular over* R *of dimension* n *and of Gorenstein parameter* ℓ *if and only if* A^o *is AS-regular over* R^o *of dimension* n *and of Gorenstein parameter* ℓ*.*

Corollary 3.8. *If* A *is an AS-regular algebra over* R *of dimension* n*, then the functors* $\mathbb{E}^n(-) = \underline{\operatorname{Ext}}^n_A(-, A) : \operatorname{tors} A \leftrightarrow \operatorname{tors} A^o : \mathbb{E}'^n(-) = \underline{\operatorname{Ext}}^n_{A^o}(-, A)$ *give a duality.*

Proof. Since every $M \in \operatorname{tors} A$ is a finite extension of modules in $\operatorname{grmod} R$, we see that $\mathbb{E}^i(M) = \underline{\operatorname{Ext}}^i_A(M, A) = 0$ for $i \neq n$ and $\mathbb{E}^n(M) := \underline{\operatorname{Ext}}^n_A(M, A) \in \operatorname{tors} A^o$ by Proposition 3.5. It follows that each term $P^{-i} \underline{\otimes}_R A$ of a minimal projective resolution $P^\bullet \underline{\otimes}_R A$ of M in $\operatorname{GrMod} A$ is finitely generated as graded right A-modules by Lemma 3.4. Therefore, by the same argument in the proof of Proposition 3.5, we see that $\mathbb{E}'^i(\mathbb{E}^n(M)) = 0$ for $i \neq n$ and $\mathbb{E}'^n(\mathbb{E}^n(M)) \cong M$ in $\operatorname{tors} A$. Since A^o is also AS-regular of dimension n by Corollary 3.7, we see that $\mathbb{E}^n(\mathbb{E}'^n(N)) \cong N$ in $\operatorname{tors} A^o$ for $N \in \operatorname{tors} A^o$. □

3.2 ASF-Gorenstein Algebras

We will give the second possible generalization of an AS-Gorenstein algebra, which is also a generalization of a graded Frobenius algebra. In the definition below, ASF stands for Artin-Schelter-Frobenius.

Definition 3.9. A locally finite $\mathbb{N}$-graded algebra A is called ASF-Gorenstein (resp. ASF-regular) of dimension n and of Gorenstein parameter ℓ if $\operatorname{id} A = n$ (resp. $\operatorname{gldim} A = n$) and

$$\mathbf{R}\underline{\Gamma}_{\mathfrak{m}}(A) \cong (DA)(\ell)[-n]$$

in $\mathcal{D}(\operatorname{GrMod} A)$ and in $\mathcal{D}(\operatorname{GrMod} A^o)$ where

$$\underline{\Gamma}_{\mathfrak{m}}(M) := \lim_{m\to\infty} \underline{\operatorname{Hom}}_A(A/A_{\geq m}, M),$$

or equivalently,

$$\underline{\operatorname{H}}^i_{\mathfrak{m}}(A) := h^i(\mathbf{R}\underline{\Gamma}_{\mathfrak{m}}(A)) \cong \begin{cases} (DA)(\ell) & \text{if } i = n, \\ 0 & \text{otherwise} \end{cases}$$

in $\operatorname{GrMod} A$ and in $\operatorname{GrMod} A^o$. A graded A-A bimodule $\omega_A := D\,\underline{\operatorname{H}}^n_{\mathfrak{m}}(A)$ is called the canonical module of A. The generalized Nakayama automorphism of A is a graded algebra automorphism $\nu \in \underline{\operatorname{Aut}}_k A$ such that $\omega_A \cong A_\nu(-\ell)$ in $\operatorname{GrMod} A^e$. An ASF-Gorenstein (ASF-regular) algebra is called symmetric if $\omega_A \cong A(-\ell)$ in $\operatorname{GrMod} A^e$, i.e., ν is the identity (up to inner automorphisms).

Note that, for any ASF-Gorenstein algebra A of Gorenstein parameter ℓ, $\omega_A \cong A(-\ell)$ in $\operatorname{GrMod} A$ and in $\operatorname{GrMod} A^o$, so the generalized Nakayama automorphism of A exists by Lemma 2.9.

Remark 3.10. Let A be a graded algebra connected over k. If A is a coherent AS-Gorenstein algebra with the usual Gorenstein condition $\mathbf{R}\underline{\operatorname{Hom}}_A(k, A) \cong k(\ell)[-n]$, then $\mathbf{R}\underline{\Gamma}_{\mathfrak{m}}(A) \cong (DA)(\ell)[-n]$ in $\mathcal{D}(\operatorname{GrMod} A)$ by [24, Proposition 4.4]. Conversely, suppose that A is an Ext-finite graded algebra connected over k such that $\underline{\Gamma}_{\mathfrak{m}}$ has finite cohomological dimension. (Note that every noetherian AS-Gorenstein algebra over k satisfies these conditions.) If $\mathbf{R}\underline{\Gamma}_{\mathfrak{m}}(A) \cong (DA)(\ell)[-n]$ in $\mathcal{D}(\operatorname{GrMod} A)$, then

$$\begin{aligned} \mathbf{R}\underline{\operatorname{Hom}}_A(k, A) &\cong \mathbf{R}\underline{\operatorname{Hom}}_A(k, D\,\mathrm{R}\,\underline{\Gamma}_{\mathfrak{m}}(A)(\ell)[-n]) \\ &\cong \mathbf{R}\underline{\operatorname{Hom}}_A(k, D\,\mathrm{R}\,\underline{\Gamma}_{\mathfrak{m}}(A))(\ell)[-n] \\ &\cong D\mathbf{R}\underline{\Gamma}_{\mathfrak{m}}(k)(\ell)[-n] \cong k(\ell)[-n] \end{aligned}$$

by local duality theorem [21, Theorem 5.1]. These facts justify the above definition.

Remark 3.11. If A is an $\mathbb{N}$-graded algebra finite dimensional over k, then $\mathbf{R}\underline{\Gamma}_{\mathfrak{m}}(A) \cong A$ in $\mathcal{D}(\operatorname{GrMod} A^e)$. It follows that every graded Frobenius algebra is ASF-Gorenstein of dimension 0, and in this case, the generalized Nakayama automorphism is the usual Nakayama automorphism. Conversely, suppose that A is an ASF-Gorenstein algebra of dimension 0. Since $\underline{\Gamma}_{\mathfrak{m}}(A) \subset A$ is bounded below and $A \cong D\underline{\Gamma}_{\mathfrak{m}}(A)(\ell)$ as graded k-vector spaces, A is bounded. Since A is locally finite, A is finite dimensional over k, so $\mathbf{R}\underline{\Gamma}_{\mathfrak{m}}(A) \cong A$ in $\mathcal{D}(\operatorname{GrMod} A^e)$, hence A is a graded Frobenius algebra.

We now compare these two notions of AS-regular algebras.

Theorem 3.12. *If A is an AS-regular algebra over R of dimension n and of Gorenstein parameter ℓ, then A is an ASF-regular algebra of dimension n and of Gorenstein parameter ℓ.*

Proof. Since $A/A_{\geq m}$ is a finite extension of modules in $\operatorname{GrMod} R$,

$$\begin{aligned}\underline{\mathrm{H}}^i_{\mathfrak{m}}(A) &= \lim_{m\to\infty} \underline{\mathrm{Ext}}^i_A(A/A_{\geq m}, A) =: \lim_{m\to\infty} \mathbb{E}^i(A/A_{\geq m}) \\ &\cong \begin{cases} \lim_{m\to\infty} F(A/A_{\geq m}) := \lim_{m\to\infty} {}_\sigma D(A/A_{\geq m})(\ell) & \text{if } i = n \\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

in $\operatorname{GrMod} R^o$ by Proposition 3.5. First, we will show that $\underline{\mathrm{H}}^n_{\mathfrak{m}}(A) \cong (DA)(\ell)$ in $\operatorname{GrMod}(R^o \underline{\otimes}_k A)$. Since we have an isomorphism

$$\lim_{m\to\infty} {}_\sigma D(A/A_{\geq m}) \cong {}_\sigma(\lim_{m\to\infty} D(A/A_{\geq m})) \cong {}_\sigma D(A)$$

in $\operatorname{GrMod}(R^o \underline{\otimes}_k A)$, it is enough to show that

$$\mathbb{E}^n(A/A_{\geq m}) \cong F(A/A_{\geq m}) \tag{3-3}$$

in $\operatorname{GrMod}(R^o \underline{\otimes}_k A)$. The left multiplication $b \times - : A/A_{\geq m} \to A/A_{\geq m}(q)$ of a homogeneous element $b \in A_q$ of degree q is a homomorphism of graded right R-modules. This induces the graded right A-module structures on $\mathbb{E}^n(A/A_{\geq m})$ and on $F(A/A_{\geq m})$. Therefore we see that $\mathbb{E}^n(A/A_{\geq m}) \cong F(A/A_{\geq m})$ is an isomorphism in $\operatorname{GrMod}(R^o \otimes A)$. Hence $\underline{\mathrm{H}}^n_{\mathfrak{m}}(A) \cong (DA)(\ell)$ in $\operatorname{GrMod}(R^o \underline{\otimes}_k A)$.

Second, we will show that $\underline{\mathrm{H}}^n_{\mathfrak{m}}(A) \cong (DA)(\ell)$ in $\operatorname{GrMod} A^o$, or equivalently, $D\,\underline{\mathrm{H}}^n_{\mathfrak{m}}(A) \cong A(-\ell)$ in $\operatorname{GrMod} A$. We claim that $D\,\underline{\mathrm{H}}^n_{\mathfrak{m}}(A) \underline{\otimes}_A R \cong R(-\ell)$ in $\operatorname{GrMod} A$. Since $A/A_{\geq m}, DR \in \operatorname{tors} A$ and $\mathbb{E}^n({}_\sigma DR) \cong {}_\sigma D\left({}_\sigma D(R)\right)(\ell) \cong {}_{\sigma\sigma^{-1}}R(\ell) \cong R(\ell)$ in $\operatorname{GrMod} A^e$, we have an isomorphism

$$\begin{aligned}DR \cong {}_\sigma DR \cong \underline{\mathrm{Hom}}_A(A/A_{\geq m}, {}_\sigma DR) &\cong \underline{\mathrm{Hom}}_{A^o}(\mathbb{E}^n({}_\sigma DR), \mathbb{E}^n(A/A_{\geq m})) \\ &\cong \underline{\mathrm{Hom}}_{A^o}(R(\ell), \mathbb{E}^n(A/A_{\geq m}))\end{aligned}$$

in $\operatorname{GrMod} A^o$ for $m \geq 1$ by Corollary 3.8. Since $R(\ell)$ is a finitely presented graded left A-module by Proposition 3.4, the functor $\underline{\mathrm{Hom}}_{A^o}(R(\ell), -)$ commutes with direct limits. Therefore we have an isomorphism

$$\begin{aligned}DR &\cong \lim_{m\to\infty} \underline{\mathrm{Hom}}_{A^o}(R(\ell), \mathbb{E}^n(A/A_{\geq m})) \\ &\cong \underline{\mathrm{Hom}}_{A^o}(R(\ell), \lim_{m\to\infty} \mathbb{E}^n(A/A_{\geq m})) \\ &\cong \underline{\mathrm{Hom}}_{A^o}(R(\ell), \underline{\mathrm{H}}^n_{\mathfrak{m}}(A)) \\ &\overset{*}{\cong} \underline{\mathrm{Hom}}_A(D\,\underline{\mathrm{H}}^n_{\mathfrak{m}}(A), (DR)(-\ell)) \\ &\cong \underline{\mathrm{Hom}}_R((D\,\underline{\mathrm{H}}^n_{\mathfrak{m}}(A)) \underline{\otimes}_A R, (DR)(-\ell)) \\ &\cong \underline{\mathrm{Hom}}_{R^o}(R, D\left((D\,\underline{\mathrm{H}}^n_{\mathfrak{m}}(A)) \underline{\otimes}_A R\right)(-\ell)) \\ &\cong D\left((D\,\underline{\mathrm{H}}^n_{\mathfrak{m}}(A)) \underline{\otimes}_A R\right)(-\ell)\end{aligned}$$

in $\operatorname{GrMod} R^o$ where, for the isomorphism $\overset{*}{\cong}$, we use the fact that $R(\ell)$ and $\underline{\mathrm{H}}^n_{\mathfrak{m}}(A)$ are locally finite. Therefore we have an isomorphism $\bar{f} : R(-\ell) \xrightarrow{\cong} D\,\underline{\mathrm{H}}^n_{\mathfrak{m}}(A) \underline{\otimes}_A R$ in $\operatorname{GrMod} R$. The isomorphism $\bar{f}$ of graded right R-modules lift to the projective cover

$$f : A(-\ell) \cong R(-\ell) \underline{\otimes}_R A \twoheadrightarrow D\,\underline{\mathrm{H}}^n_{\mathfrak{m}}(A)$$

of $D\,\underline{\mathrm{H}}^n_{\mathfrak{m}}(A)$ in $\operatorname{GrMod} A$ by the proof of Lemma 2.5. Since $D\,\underline{\mathrm{H}}^n_{\mathfrak{m}}(A)$ is isomorphic to $A(-\ell)$ in $\operatorname{GrMod} A^o$ and A is locally finite, we see that f is an isomorphism in $\operatorname{GrMod} A$. Hence $D\,\underline{\mathrm{H}}^n_{\mathfrak{m}}(A) \cong A(-\ell)$ in $\operatorname{GrMod} A$.

□

Unfortunately, we do not not know if the converse of the above theorem holds. However, we can define the notion of generalized Nakayama automorphism for an AS-regular algebra over R by the above theorem.

Corollary 3.13. *Let A be an AS-regular algebra over R of dimension n and of Gorenstein parameter ℓ, and $\nu \in \underline{\mathrm{Aut}}_k A$ the generalized Nakayama automorphism of A. For $M \in \operatorname{tors} A$, we have a natural isomorphism $\underline{\mathrm{Ext}}^n_A(M, A) \cong {}_\nu(DM)(\ell)$ in* $\operatorname{GrMod} A^o$.

Proof. For each $m \in \mathbb{N}$, we have an isomorphism $\mathbb{E}^n(A/A_{\geq m}) \cong {}_\nu D(A/A_{\geq m})(\ell)$ in $\operatorname{GrMod} A^e$. Therefore, for a projective graded right R-module P, we have an isomorphism

$$\chi'_{P,m} : \mathbb{E}^n\left((P \underline{\otimes}_R A)/(P \underline{\otimes}_R A)_{\geq m}\right) \xrightarrow{\cong} {}_\nu D\left((P \underline{\otimes}_R A)/(P \underline{\otimes}_R A)_{\geq m}\right)(\ell)$$

in $\operatorname{GrMod} A^o$ which is functorial in P. Let

$$0 \to P^{-n} \underline{\otimes}_R A \to P^{-(n-1)} \underline{\otimes}_R A \to \cdots \to P^0 \underline{\otimes}_R A \to M \to 0$$

be the minimal projective resolution of M in $\operatorname{GrMod} A$. Let m be an integer such that $M_{\geq m} = 0$. We set $X^{-i} := (P^{-i} \underline{\otimes}_R A)/(P^{-i} \underline{\otimes}_R A)_{\geq m}$ for each $i = 0, \dots, n$. Note that $X^{-i} \in \operatorname{tors} A$ by Proposition 3.4. Then we obtain an exact sequence $0 \to X^{-n} \to \cdots \to X^0 \to M \to 0$ in $\operatorname{tors} A$, which induces the following commutative diagram in $\operatorname{tors} A^o$

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathbb{E}^n(M) & \longrightarrow & \mathbb{E}^n\left(X^0\right) & \longrightarrow \cdots \longrightarrow & \mathbb{E}^n\left(X^{-n}\right) & \longrightarrow 0 \\
& \big\downarrow \chi_M & & \cong \big\downarrow \chi'_{P^0,m} & & \cong \big\downarrow \chi'_{P^{-n},m} & \\
0 \longrightarrow & {}_\nu D(M)(\ell) & \longrightarrow & {}_\nu D\left(X^0\right)(\ell) & \longrightarrow \cdots \longrightarrow & {}_\nu D\left(X^{-n}\right)(\ell) & \longrightarrow 0.
\end{array}$$

Since both functors $\mathbb{E}^n : \operatorname{tors} A \to \operatorname{tors} A^o$ and ${}_\nu D(-)(\ell) : \operatorname{tors} A \to \operatorname{tors} A^o$ give dualities, the top and bottom rows are exact. Hence we obtain the isomorphism $\chi_M : \mathbb{E}^n(M) \xrightarrow{\cong} {}_\nu D(M)$. It is easy to see that χ_M does not depend on the choice of m. By the same method in the proof of Proposition 3.5, we see that the assignment χ_M is functorial in M. □

For a graded A^o-module N, we set $\underline{\Gamma}_{\mathfrak{m}^o}(N) := \lim_{m\to\infty} \underline{\mathrm{Hom}}_{A^o}(A/A_{\geq m}, N)$.

Corollary 3.14. *Let A be an AS-regular algebra over R of dimension n and of Gorenstein parameter ℓ. If $\nu \in \underline{\mathrm{Aut}}_k A$ is the generalized Nakayama automorphism, then $\nu^{-1} \in \underline{\mathrm{Aut}}_k A^o$ is the generalized Nakayama automorphism of A^o so that $\omega_A \cong A_\nu(-\ell) \cong {}_{\nu^{-1}}A(-\ell) \cong \omega_{A^o}$ in $\mathrm{GrMod}\, A^e$. Moreover, $\omega_A[n] \cong A_\nu(-\ell)[n] \in \mathcal{D}(\mathrm{GrMod}\, A^e)$ is the balanced dualizing complex of A, i.e., we have isomorphisms $\mathbf{R}\underline{\Gamma}_{\mathfrak{m}}(\omega_A[n]) \cong DA$ and $\mathbf{R}\underline{\Gamma}_{\mathfrak{m}^o}(\omega_A[n]) \cong DA$ in $\mathcal{D}(\mathrm{GrMod}\, A^e)$.*

Proof. By Corollary 3.7, A^o is AS-regular, so A^o has the generalized Nakayama automorphism $\nu^o \in \underline{\mathrm{Aut}}_k A^o = \underline{\mathrm{Aut}}_k A$. For $N \in \mathrm{tors}\, A^o$, we have a natural isomorphism $\underline{\mathrm{Ext}}^n_{A^o}(N, A) \cong (DN)_{\nu^o}(\ell)$ in $\mathrm{GrMod}\, A$ by the above corollary. Note that since this isomorphism is natural in N, if N is a graded A^o-A^o-bimodule then this is an isomorphism of graded A-A-bimodules. By Corollary 3.8 for $m \geq 1$, we have an isomorphism

$$\begin{aligned} D(A/A_{\geq m}) \cong \mathbb{E}'^n\mathbb{E}^n(D(A/A_{\geq m})) &\cong D\left({}_\nu D(D(A/A_{\geq m}))(\ell)\right)_{\nu^o}(\ell) \\ &\cong D({}_{\nu\nu^o}(A/A_{\geq m})) \end{aligned}$$

in $\mathrm{GrMod}\, A^e$. Taking colimit by m in the above isomorphism and then taking the k-dual, we obtain an isomorphism $A \cong {}_{\nu\nu^o}A$ in $\mathrm{GrMod}\, A^e$. Therefore $\nu\nu^o$ is the identity id_A up to inner automorphisms. Hence we can adjust ν^o by inner automorphisms in such a way that $\nu^o = \nu^{-1}$.

Since $D\mathbf{R}\underline{\Gamma}_{\mathfrak{m}}(A) \cong \omega_A[n] \cong A_\nu(-\ell)[n]$ in $\mathcal{D}(\mathrm{GrMod}\, A^e)$,

$$\begin{aligned} D\mathbf{R}\underline{\Gamma}_{\mathfrak{m}}(\omega_A[n]) &\cong D\mathbf{R}\underline{\Gamma}_{\mathfrak{m}}(A_\nu(-\ell)[n]) \cong D(\mathbf{R}\underline{\Gamma}_{\mathfrak{m}}(A)_\nu(-\ell)[n]) \\ &\cong {}_\nu D\mathbf{R}\underline{\Gamma}_{\mathfrak{m}}(A)(\ell)[-n] \cong {}_\nu(A_\nu(-\ell)[n])(\ell)[-n] \\ &\cong (A_\nu)_{\nu^{-1}} \cong A \end{aligned}$$

in $\mathcal{D}(\mathrm{GrMod}\, A^e)$ by a graded version of Lemma 2.8 (2), hence the second claim. □

3.3 Generalized AS-regular Algebras due to Martinez-Villa.

In [12], Martinez-Villa gave another generalization of AS-regular algebra for $\mathbb{N}$-graded algebras.

Definition 3.15. A locally finite $\mathbb{N}$-graded algebra A is called generalized AS-regular of dimension n if the following conditions are satisfied: (1) $\mathrm{gldim}\, A = n$, (2) for any simple graded right A-module S, we have $\mathbb{E}^i(S) := \underline{\mathrm{Ext}}^i_A(S, A) = 0$ for $i \neq n$, and (3) the functors $\mathbb{E}^n(-) := \underline{\mathrm{Ext}}^n_A(-, A) : \mathrm{GrMod}\, A \leftrightarrow \mathrm{GrMod}\, A^o : \mathbb{E}'^n(-) := \underline{\mathrm{Ext}}^n_{A^o}(-, A)$ induce a bijection between the set of all simple graded right A-modules and the set of all simple graded left A-modules.

Remark 3.16. In [12], Martinez-Villa called the algebra defined as above generalized Auslander regular. His definition has one more condition (4) every simple graded right A-module has projective resolution by finitely generated projectives. By Proposition 3.4, this condition follows from the other three conditions.

By Corollary 3.8, we immediately see the following theorem.

Theorem 3.17. *Every AS-regular algebra over R is generalized AS-regular.*

In general, the converse of the above proposition dose not hold as the following examples show. The point is that, for a generalized AS-regular algebra, the Gorenstein parameter may not be uniquely determined.

Example 3.18. (1) If A and A' are AS-regular algebras over A_0 and A'_0 of the same dimension and of Gorenstein parameters ℓ and ℓ', then the direct sum $A \oplus A'$ is a generalized AS-regular algebra, but not an AS-regular algebra over $A_0 \oplus A'_0$ unless $\ell = \ell'$.

(2) Let $A = kQ$ be the path algebra of the following graded quiver

$$a \underset{y}{\overset{x}{\rightleftarrows}} b \qquad \deg a = \deg b = 0,\ \deg x = \ell, \text{ and } \deg y = \ell'.$$

If ${}_aS, {}_bS$ (resp. S_a, S_b) are the simple right R-modules (resp. left R-modules) corresponding to the vertices a, b, then we have $\underline{\mathrm{Hom}}_A({}_aS, A) = \underline{\mathrm{Hom}}_A({}_bS, A) = 0$, $\underline{\mathrm{Ext}}^1_A({}_aS, A) \cong S_b(\ell)$ and $\underline{\mathrm{Ext}}^1_A({}_bS, A) \cong S_a(\ell')$. It follows that A is a generalized AS-regular algebra, but A not an AS-regular algebra over A_0 unless $\ell = \ell'$. Note that if either $\ell > 1$ or $\ell' > 1$, then A is not generated in degree 1 over A_0, and if either $\ell = 0$ or $\ell' = 0$, then A_0 is not semi-simple.

Martinez-Villa defined a generalized AS-regular algebra only when A is a quiver algebra kQ/I for a finite quiver Q such that every arrow has degree 1. In this case, R is basic and semi-simple, and A is generated in degree 1 over R, so we see that these two notions of AS-regular algebras agree.

Theorem 3.19. *Let R be a basic semi-simple finite dimensional algebra, and A an $\mathbb{N}$-graded algebra connected over R. Assume that A is generated in degree 1 over R and not a direct sum of two non-trivial graded algebras. Then A is AS-regular over R if and only if A is generalized AS-regular.*

We need the following lemma.

Lemma 3.20. *Let R and A be as above. Then, for simple right R-modules S, S', we have $\underline{\mathrm{Ext}}^1_A(S, S')_i = 0$ for $i \neq -1$. Dually, for simple left R-modules T, T', we have $\underline{\mathrm{Ext}}^1_{A^o}(T, T')_i = 0$ for $i \neq -1$.*

Proof. Recall that an i-th homogeneous element $\xi \in \underline{\mathrm{Ext}}^1_A(S, S')_i$ corresponds to an (isomorphism class of) extension $[\xi] : 0 \to S'(i) \to M \to S \to 0$ in $\mathrm{GrMod}\, A$ and that $\xi = 0$ if and only if the corresponding extension $[\xi]$ splits. Therefore $\underline{\mathrm{Ext}}^1_A(S, S')_i = 0$ if and only if the graded vector space $S \oplus S'(i)$ has no non-trivial graded right A-module structure having $S'(i)$ as a graded submodule and S as a graded factor module. Since A is $\mathbb{N}$-graded, we see that

$\underline{\mathrm{Ext}}^1_A(S,S')_i = 0$ for $i > 0$. Since A is generated in degree 1 over $A_0 = R$, we see that $\underline{\mathrm{Ext}}^1_A(S,S')_i = 0$ for $i < -1$. Since $S \oplus S'$ is concentrated in degree 0, giving a graded right A-module structure on $S \oplus S'$ with desired properties is equivalent to giving a right R-module structure on $S \oplus S'$ with similar properties. Therefore $\underline{\mathrm{Ext}}^1_A(S,S')_0 = \mathrm{Ext}^1_R(S,S')$. Since R is semi-simple, we see that $\underline{\mathrm{Ext}}^1_A(S,S')_0 = \mathrm{Ext}^1_R(S,S') = 0$. This proves the first statement. The second statement is proved in the same way. □

Proof of Theorem 3.19. Only if part is already proved. We assume that A is generalized AS-regular of dimension n. Since every $M \in \operatorname{tors} A$ is a finite extension of simple graded right A-modules, we see that $\mathbb{E}^n(M) \in \operatorname{tors} A^o$ and $\mathbb{E}^i(M) = 0$ for $i \neq n$. Then by the similar argument in the proof of Proposition 3.6 (1), we see that $\mathbb{E}'^n\mathbb{E}^n(M) \cong M$. In a similar way, we see that $\mathbb{E}^n\mathbb{E}'^n(N) \cong N$ for $N \in \operatorname{tors} A^o$. Hence $\mathbb{E}^n : \operatorname{tors} A \leftrightarrow \operatorname{tors} A^o : \mathbb{E}'^n$ give a duality.

Let $S_1, \dots, S_p$ (resp. $T_1, \dots, T_p$) be the set of complete representatives of isomorphism classes of simple right R-modules (resp. left R-modules), which is the set of complete representatives of simple graded right A-modules (resp. left A-modules) concentrated in degree 0. By definition, there is a permutation $\sigma \in \mathcal{S}_p$ such that $\mathbb{E}^n(S_i) \cong T_{\sigma(i)}(\ell_i)$ for some $\ell_i \in \mathbb{Z}$. We claim that $\ell_i = \ell_j$ for all i, j. Indeed, since A is not a direct sum of two non-trivial graded algebras, for any S_i, S_j, there is a finite sequence of simple right R-modules $S_i = S_{i_1}, S_{i_2}, \dots, S_{i_q} = S_j$ such that $\underline{\mathrm{Ext}}^1_A(S_l, S_{l+1}) \neq 0$ or $\underline{\mathrm{Ext}}^1_A(S_{l+1}, S_l) \neq 0$ for $l = 1, 2, \dots, q-1$ by the graded version of [3, III.1.14]. Therefore it is enough to show that $\ell_i = \ell_j$ whenever $\underline{\mathrm{Ext}}^1_A(S_i, S_j) \neq 0$. Since $\operatorname{tors} A$ is a full subcategory of $\operatorname{GrMod} A$ closed under extensions, $\underline{\mathrm{Ext}}^1_A(M,N) \cong \underline{\mathrm{Ext}}^1_{\operatorname{tors} A}(M,N)$ for all $M, N \in \operatorname{tors} A$. Since $\mathbb{E}^n : \operatorname{tors} A \leftrightarrow \operatorname{tors} A^o : \mathbb{E}'^n$ give a duality, we have

$$0 \neq \underline{\mathrm{Ext}}^1_A(S_i, S_j) \cong \underline{\mathrm{Ext}}^1_{A^o}(\mathbb{E}^n(S_j), \mathbb{E}^n(S_i)) \cong \underline{\mathrm{Ext}}^1_{A^o}(T_{\sigma(j)}, T_{\sigma(i)})(\ell_i - \ell_j).$$

By Lemma 3.20, we conclude that $\ell_i = \ell_j$.

We set $\ell = \ell_i$. Then, for $M \in \operatorname{mod} R$, we see that $\mathbb{E}^n(M)$ is concentrated in degree $-\ell$. Hence $G(-) := D(\mathbb{E}^n(-)(-\ell))$ gives an autoequivalence of $\operatorname{mod} R$. Since R is basic, by Morita theory, there exists an algebra automorphism $\sigma \in \operatorname{Aut}_k R$ such that $G(R) \cong R_\sigma$ in $\operatorname{Mod} R^e$, so $\underline{\mathrm{Ext}}^n_A(R, A) = \mathbb{E}^n(R) \cong DG(R)(\ell) \cong D(R_\sigma)(\ell) \cong {}_\sigma(DR)(\ell)$ in $\operatorname{Mod} R^e$ by Lemma 2.8. □

4 The Structure of AS-Gorenstein Algebras

In this section, we study the structure of AS-regular algebras. We also see that there are some analogies to the structure of graded Frobenius algebras, that is, ASF-Gorenstein algebras of dimension 0.

4.1 The Structure of AS-regular Algebras

First we will show that the preprojective algebra of a quasi-Fano algebra R is an AS-regular algebra over R of Gorenstein parameter 1.

Lemma 4.1. *If R is a ring, L is a quasi-ample two-sided tilting complex of R, and $A = T_R(L)$ is the tensor algebra, then $\mathbf{R}\,\underline{\mathrm{Hom}}_A(R, A) \cong L^{-1}(1)[-1]$ in $\mathcal{D}(\mathrm{GrMod}\, R^e)$.*

Proof. Since $A_{\geq 1} = L \underline{\otimes}_R A(-1)$ in $\mathrm{GrMod}(R^o \underline{\otimes}_k A)$, we have an exact sequence

$$0 \to L \underline{\otimes}_R A(-1) \xrightarrow{\iota} A \to R \to 0$$

in $\mathrm{GrMod}(R^o \underline{\otimes}_k A)$ where ι is the inclusion. Applying $\mathbf{R}\underline{\mathrm{Hom}}_A(-, A)$ to this exact sequence, we obtain an exact triangle

$$\mathbf{R}\underline{\mathrm{Hom}}_A(R, A) \to \mathbf{R}\underline{\mathrm{Hom}}_A(A, A) \xrightarrow{\mathbf{R}\underline{\mathrm{Hom}}_A(\iota, A)} \mathbf{R}\underline{\mathrm{Hom}}_A(L \underline{\otimes}_R A(-1), A) \xrightarrow{[1]}$$

in $\mathcal{D}(\mathrm{GrMod}(A^o \underline{\otimes}_k R))$, hence in $\mathcal{D}(\mathrm{GrMod}\, R^e)$. Therefore it is enough to prove that the cone of $\mathbf{R}\underline{\mathrm{Hom}}_A(\iota, A)$ is isomorphic to $L^{-1}(1)$ in $\mathcal{D}(\mathrm{GrMod}\, R^e)$.

Since L is quasi-ample, $L \underline{\otimes}_R A \cong L \underline{\otimes}^{\mathbf{L}}_R A$ in $\mathcal{D}(\mathrm{GrMod}(R^o \underline{\otimes}_k A))$. Therefore, by adjunction, we have an isomorphism

$$\begin{aligned} \mathbf{R}\underline{\mathrm{Hom}}_A(L \underline{\otimes}_R A(-1), A) &\cong \mathbf{R}\underline{\mathrm{Hom}}_A(L \underline{\otimes}^{\mathbf{L}}_R A(-1), A) \cong \mathbf{R}\underline{\mathrm{Hom}}_R(L(-1), A) \\ &\cong \bigoplus_{i \geq -1} \mathbf{R}\,\mathrm{Hom}_R(L, L^{\otimes^{\mathbf{L}}_R i+1})(-i) \cong L^{-1}(1) \oplus \bigoplus_{i \geq 0} \mathrm{Hom}_R(L, L^{\otimes_R i+1})(-i) \end{aligned} \tag{4-4}$$

in $\mathcal{D}(\mathrm{GrMod}\, R^e)$. Since $\mathbf{R}\underline{\mathrm{Hom}}_A(L \underline{\otimes}_R A(-1), A)_{-1} \cong L^{-1}$ in $\mathcal{D}(\mathrm{Mod}\, R^e)$, it is enough to show that

$$\mathbf{R}\underline{\mathrm{Hom}}(\iota, A)_{\geq 0} : \mathbf{R}\underline{\mathrm{Hom}}_A(A, A)_{\geq 0} \to \mathbf{R}\underline{\mathrm{Hom}}_A(L \underline{\otimes}_R A(-1), A)_{\geq 0}$$

is an isomorphism in $\mathcal{D}(\mathrm{GrMod}\, R^e)$. Since we have canonical isomorphisms

$$\begin{aligned} &\mathbf{R}\underline{\mathrm{Hom}}_A(A, A) \cong \underline{\mathrm{Hom}}_A(A, A), \\ &\mathbf{R}\underline{\mathrm{Hom}}_A(L \underline{\otimes}_R A(-1), A)_{\geq 0} \cong \underline{\mathrm{Hom}}_A(L \underline{\otimes}_R A(-1), A)_{\geq 0} \end{aligned}$$

in $\mathcal{D}(\mathrm{GrMod}\, R^e)$, it is enough to show that

$$\underline{\mathrm{Hom}}_A(\iota, A)_i : \underline{\mathrm{Hom}}_A(A, A)_i \to \underline{\mathrm{Hom}}_A(L \underline{\otimes}_R A(-1), A)_i$$

is an isomorphism in $\mathrm{Mod}\, R^e$ for each $i \geq 0$.

We have an isomorphism

$$\underline{\mathrm{Hom}}_A(A, A) \cong \underline{\mathrm{Hom}}_R(R, A) = \bigoplus_{i \geq 0} \mathrm{Hom}_R(R, L^{\otimes_R i})(-i) \cong \bigoplus_{i \geq 0} L^{\otimes i}(-i)$$

in $\mathrm{GrMod}\, R^e$. Under this isomorphism, a homogeneous element $a \in L^{\otimes_R i}$ in the RHS corresponds to the left multiplication $a \times - : A \to A(i)$. Since L is a quasi-ample two-sided tilting complex, we have an isomorphism $\mathrm{Hom}_R(L, L^{\otimes_R i+1}) \cong \mathrm{Hom}_R(R, L^{\otimes_R i}) \cong L^{\otimes_R i}$ for each $i \geq 0$. Under this isomorphism, an element $a \in L^{\otimes_R i}$ in the RHS corresponds to the left multiplication $a \otimes - : L \to L^{\otimes_R i+1}$. Therefore, by (4-4), we have an isomorphism

$$\underline{\mathrm{Hom}}_A(L \underline{\otimes}_R A(-1), A)_{\geq 0} \cong \bigoplus_{i \geq 0} \mathrm{Hom}_R(L, L^{\otimes_R i+1})(-i) \cong \bigoplus_{i \geq 0} L^{\otimes i}(-i)$$

in $\operatorname{GrMod} R^e$. Under this isomorphism, a homogeneous element $a \in L^{\otimes_R i}$ in the RHS corresponds to the left multiplication $a \times - : A_{\geq 1} = L \underline{\otimes}_R A(-1) \to A(i)$. Since $\iota : L \underline{\otimes}_R A(-1) = A_{\geq 1} \hookrightarrow A$ is the inclusion, $\underline{\operatorname{Hom}}(\iota, A)$ sends the left multiplication $a \times - : A \to A(i)$ to the left multiplication $a \times - : A_{\geq 1} = L \underline{\otimes}_R A(-1) \to A(i)$. Therefore $\operatorname{Hom}_A(\iota, A)_i$ is an isomorphism for each $i \geq 0$. This finishes the proof. □

Theorem 4.2. *If R is a quasi-Fano algebra of dimension n, then the preprojective algebra $A = \Pi R$ is an AS-regular algebra over R of dimension $n+1$ and of Gorenstein parameter 1.*

Proof. By Lemma 4.1,

$$\mathbf{R}\underline{\operatorname{Hom}}_A(R, A) \cong \omega_R(1)[-1] \cong DR(1)[-n-1]$$

in $\mathcal{D}(\operatorname{GrMod} R^e)$. By [15, Theorem A.1], $\operatorname{gldim} A < \infty$, so $\operatorname{gldim} A = n+1$ by Proposition 3.6, hence the result. □

Next we will show that every AS-regular algebra is a preprojective algebra of a quasi-Fano algebra up to graded Morita equivalence. Here, we say that two graded rings are *graded Morita equivalent* if their categories of graded right modules are equivalent. Note that not every graded Morita equivalence commutes with the shift functor. (This is different from the notion of graded Morita equivalent used in [25].) We will also show that every AS-regular algebra of Gorenstein parameter 1 is isomorphic to a preprojective algebra twisted by an automorphism. In particular, we will show that symmetric AS-regular algebras of Gorenstein parameter 1 are exactly preprojective algebras of quasi-Fano algebras.

The quotient category $\operatorname{Tails} A := \operatorname{GrMod} A / \operatorname{Tors} A$ is called the noncommutative projective scheme associated to A in [2]. Let $\pi : \operatorname{GrMod} A \to \operatorname{Tails} A$ be the quotient functor. We often denote by $\mathcal{M} := \pi M \in \operatorname{Tails} A$ for $M \in \operatorname{GrMod} A$. Note that the autoequivalence $M \mapsto M(m)$ preserves (direct limits of) modules finite dimensional over k, so it induces an autoequivalence $\mathcal{M} \mapsto \mathcal{M}(m)$ for $\operatorname{Tails} A$, also called a shift functor. The set of morphisms in $\operatorname{Tails} A$ is denoted by $\operatorname{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{N}) := \operatorname{Hom}_{\operatorname{Tails} A}(\mathcal{M}, \mathcal{N})$, and we write

$$\underline{\operatorname{Ext}}^i_{\mathcal{A}}(\mathcal{M}, \mathcal{N}) := \bigoplus_{m \in \mathbb{Z}} \operatorname{Ext}^i_{\mathcal{A}}(\mathcal{M}, \mathcal{N}(m))$$

as before.

Let A be an AS-regular algebra over R of dimension $n \geq 1$ and of Gorenstein parameter ℓ. Note that, by Proposition 3.4, $n \geq 1$ implies $\ell \geq 1$. Proposition 3.4 for $M = A/A_{\geq m}$ allows us to use results given in [5, Section 4]. Recall that an object $\mathcal{M}$ of $\mathcal{D}(\operatorname{Tails} A)$ is called compact if $\operatorname{Hom}_{\mathcal{D}(\operatorname{Tails} A)}(\mathcal{M}, -)$ commutes with direct sums. By [5], $\mathcal{A}(i)$ are compact in $\mathcal{D}(\operatorname{Tails} A)$ for $i \in \mathbb{Z}$, and $\{\mathcal{A}(i) \mid i \in \mathbb{Z}\}$ generates $\mathcal{D}(\operatorname{Tails} A)$, i.e., an object $\mathcal{M}$ of $\mathcal{D}(\operatorname{Tails} A)$ is zero if and only if $\operatorname{Hom}_{\mathcal{D}(\operatorname{Tails} A)}(\mathcal{A}(i), \mathcal{M}[p]) = 0$ for all $i \in \mathbb{Z}$ and all $p \in \mathbb{Z}$.

Proposition 4.3. *If A is an AS-regular algebra over R of dimension $n \geq 1$ and of Gorenstein parameter ℓ, then the set $\{\mathcal{A}(i) \mid 0 \leq i \leq \ell - 1\}$ compactly generates $\mathcal{D}(\mathrm{Tails}\, A)$.*

Proof. It is enough to show that, for each $i \in \mathbb{Z}$, $\mathcal{A}(i)$ belongs to the full subcategory $Z := \langle \mathcal{A}, \mathcal{A}(1), \dots, \mathcal{A}(\ell - 1)\rangle$, i.e., $\mathcal{A}(i)$ can be obtained from $\{\mathcal{A}, \dots, \mathcal{A}(\ell - 1)\}$ by taking cones, shifts of complexes, and direct summands finitely many times. Let

$$0 \to P^{-n} \underline{\otimes}_R A \to \cdots \to P^{-1} \underline{\otimes}_R A \to A(\ell) \to R(\ell) \to 0$$

be a minimal projective resolution of $R(\ell)$ in $\mathrm{GrMod}\, A$. Note that P^{-i} is a finitely generated projective graded right R-module for each i by Proposition 3.4. Since the kernel of the canonical projection $A(\ell) \to R(\ell)$ is $A_{\geq 1}(\ell)$ and $\underline{\mathrm{Ext}}^n_A(R(\ell), A) \cong {}_\sigma(DR)$ is concentrated in degree 0, we see that, for each i, P^{-i} is concentrated in degrees between 0 and $-(\ell - 1)$ by Proposition 3.4. Therefore $\pi(P^{-i} \underline{\otimes}_R A)$ belongs to Z. Since the quotient functor π is exact, we have an exact sequence

$$0 \to \pi(P^{-n} \underline{\otimes}_R A) \to \cdots \to \pi(P^{-1} \underline{\otimes}_R A) \to \mathcal{A}(\ell) \to 0.$$

in $\mathrm{Tails}\, A$, so $\mathcal{A}(\ell)$ belongs to Z. By induction, we see that $\mathcal{A}(j)$ belongs to Z for $j \geq \ell$.

Let

$$0 \to A \underline{\otimes}_R Q^{-n} \to \cdots \to A \underline{\otimes}_R Q^{-1} \to A(1) \to R(1) \to 0$$

be a minimal projective resolution of $R(1)$ in $\mathrm{GrMod}\, A^o$. Applying $(-)^\vee$ to this exact sequence, we obtain an exact sequence

$$0 \to A(-1) \to (Q^{-1})^* \underline{\otimes}_R A \to \cdots \to (Q^{-n})^* \underline{\otimes}_R A \to \underline{\mathrm{Ext}}^n_{A^o}(R, A)(-1) \to 0$$

in $\mathrm{GrMod}\, A$. By Proposition 3.6 (1), $\pi\, \underline{\mathrm{Ext}}^n_{A^o}(R, A) = 0$, so we see that $\mathcal{A}(-1)$ belongs to Z by the same argument. By induction, we see that $\mathcal{A}(j)$ belongs to Z for $j < 0$. This finishes the proof. $\square$

Proposition 4.4. *If A is an AS-regular algebra over R of dimension $n \geq 1$ and of Gorenstein parameter ℓ, then there are following isomorphisms of k-vector spaces: In the case $n = 1$,*

$$\mathrm{Ext}^q_{\mathcal{A}}(\mathcal{A}(i), \mathcal{A}(j)) \cong \begin{cases} A_{j-i} & q = 0 \quad \text{and } j \geq i \\ D(A_{i-j-\ell}) & q = 0 \quad \text{and } j < i \\ 0 & \text{otherwise.} \end{cases}$$

In the case $n \geq 2$,

$$\mathrm{Ext}^q_{\mathcal{A}}(\mathcal{A}(i), \mathcal{A}(j)) \cong \begin{cases} A_{j-i} & q = 0 \\ D(A_{i-j-\ell}) & q = n - 1 \\ 0 & \text{otherwise.} \end{cases}$$

In particular, for any $n \geq 1$, we have

$$\mathrm{Ext}^q_{\mathrm{Tails}\,A}(\mathcal{A}(i), \mathcal{A}(j + m\ell)) = 0$$

for $q \neq 0$, $0 \leq i, j \leq \ell - 1$ and $m \geq 0$.

Proof. This is nothing but [2, Theorem 8.1] without noetherian hypothesis. Even without noetherian hypothesis, the same proof works using the exact sequence and the isomorphisms obtained in [5] for $M \in \mathrm{GrMod}\,A$:

$$\begin{gathered} 0 \to \underline{\mathrm{H}}^0_{\mathfrak{m}}(M) \to M \to \underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{A}, \mathcal{M}) \to \underline{\mathrm{H}}^1_{\mathfrak{m}}(M) \to 0 \\ \underline{\mathrm{Ext}}^{i-1}_{\mathcal{A}}(\mathcal{A}, \mathcal{M}) \cong \underline{\mathrm{H}}^i_{\mathfrak{m}}(M) \quad \text{for } i \geq 2. \end{gathered} \tag{4-5}$$

□

Remark 4.5. The isomorphisms in the above proposition are natural in the following sense: We have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(\mathcal{A}(j), \mathcal{A}(k)) \times \mathrm{Hom}(\mathcal{A}(i), \mathcal{A}(j)) & \longrightarrow & \mathrm{Hom}(\mathcal{A}(i), \mathcal{A}(k)) \\ \cong \downarrow & & \downarrow \cong \\ A_{k-j} \times A_{j-i} & \longrightarrow & A_{k-i} \end{array}$$

where the vertical arrows are isomorphisms in Proposition 4.4, the top horizontal arrow is the composition and the bottom horizontal arrow is the multiplication.

Quasi-Veronese algebras and Beilinson algebras introduced in [16] and [6] will play an important role for the rest of this paper.

Definition 4.6. Let $A = \oplus_{i \in \mathbb{Z}} A_i$ be a $\mathbb{Z}$-graded algebra and $r \in \mathbb{N}^+$.

1. The r-th Veronese algebra of A is a $\mathbb{Z}$-graded algebra defined by

$$A^{(r)} := \bigoplus_{i \in \mathbb{Z}} A_{ri}.$$

2. The r-th quasi-Veronese algebra of A is a $\mathbb{Z}$-graded algebra defined by

$$A^{[r]} := \bigoplus_{i \in \mathbb{Z}} \begin{pmatrix} A_{ri} & A_{ri+1} & \cdots & A_{ri+r-1} \\ A_{ri-1} & A_{ri} & \cdots & A_{ri+r-2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{ri-r+1} & A_{ri-r+2} & \cdots & A_{ri} \end{pmatrix}$$

with the multiplication defined as follows: for $(a_{ij}) \in (A^{[r]})_p, (b_{ij}) \in (A^{[r]})_q$ where $a_{ij} \in A_{rp+j-i}, b_{ij} \in A_{rq+j-i}$,

$$(a_{ij})(b_{ij}) = \left(\sum_{k=0}^{r-1} a_{kj} b_{ik} \right) \in (A^{[r]})_{p+q}.$$

Definition 4.7. If A is an AS-Gorenstein algebra over R of Gorenstein parameter $\ell \neq 0$, then we define the Beilinson algebra associated to A by

$$\nabla A := (A^{[r]})_0 = \begin{pmatrix} A_0 & A_1 & \cdots & A_{r-1} \\ 0 & A_0 & \cdots & A_{r-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_0 \end{pmatrix}$$

where $r = |\ell| \in \mathbb{N}^+$.

Remark 4.8. A graded algebra automorphism $\tau \in \underline{\mathrm{Aut}}_k A$ induces a graded algebra automorphism $\tau^{(r)} \in \underline{\mathrm{Aut}}_k A^{(r)}$ by $\tau^{(r)}(a) = \tau(a)$ for $a \in (A^{(r)})_p = A_{rp}$, and a graded algebra automorphism $\tau^{[r]} \in \underline{\mathrm{Aut}}_k A^{[r]}$ by $\tau^{[r]}((a_{ij})) = (\tau(a_{ij}))$ for $(a_{ij}) \in (A^{[r]})_p$.

Remark 4.9. It is hardly ever the case that $\mathrm{GrMod}\, A^{(r)} \cong \mathrm{GrMod}\, A$ for $r \geq 2$, however, the functor $Q : \mathrm{GrMod}\, A \to \mathrm{GrMod}\, A^{[r]}$ defined by

$$Q(M) := \oplus_{i\in\mathbb{Z}}(M_{ri} \oplus \cdots \oplus M_{ri-r+1}) = \oplus_{i\in\mathbb{Z}} \begin{pmatrix} M_{ri} \\ M_{ri-1} \\ \vdots \\ M_{ri-r+1} \end{pmatrix}$$

gives an equivalence of categories for any $r \in \mathbb{N}^+$ by [16]. It is easy to see that the functor Q induces an equivalence $Q : \mathrm{grmod}\, A \to \mathrm{grmod}\, A^{[r]}$. Note that $Q(\oplus_{i=0}^{r-1} A(i)) \cong A^{[r]}$ in $\mathrm{GrMod}\, A^{[r]}$.

The B-construction defined below is one of the basic tools in noncommutative algebraic geometry.

Definition 4.10. For a k-linear category $\mathcal{C}$, an object $\mathcal{O} \in \mathcal{C}$ and a k-linear autoequivalence $s \in \mathrm{Aut}_k\, \mathcal{C}$, we define a graded algebra over k by

$$B(\mathcal{C}, \mathcal{O}, s) := \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(\mathcal{O}, s^i\mathcal{O})$$

where the multiplication is given by the following rule: for $a \in B(\mathcal{C}, \mathcal{O}, s)_i = \mathrm{Hom}_{\mathcal{C}}(\mathcal{O}, s^i\mathcal{O})$ and $b \in B(\mathcal{C}, \mathcal{O}, s)_j = \mathrm{Hom}_{\mathcal{C}}(\mathcal{O}, s^j\mathcal{O})$, we define $ab := s^j(a) \circ b \in B(\mathcal{C}, \mathcal{O}, s)_{i+j} = \mathrm{Hom}_{\mathcal{C}}(\mathcal{O}, s^{i+j}\mathcal{O})$.

Remark 4.5 says that $B(\mathrm{Tails}\, A, \mathcal{A}, (1)) \cong A$ as graded algebras. Moreover, we have the following Lemma (cf. [16]).

Lemma 4.11. *If A is an AS-regular algebra of Gorenstein parameter $\ell \geq 1$, and $\mathcal{E} := \mathcal{A} \oplus \mathcal{A}(1) \oplus \cdots \oplus \mathcal{A}(\ell - 1) \in \mathrm{Tails}\, A$, then $B(\mathcal{D}(\mathrm{Tails}\, A), \mathcal{E}, (\ell)) \cong A^{[\ell]}$ as graded algebras, and $\mathrm{End}_{\mathcal{A}}(\mathcal{E}) \cong \nabla A$ as algebras.*

The following theorem says that every AS-regular algebra over R is graded Morita equivalent to the preprojective algebra of a quasi-Fano algebra, which is in particular a tensor algebra. Recall that every AS-regular algebra A over R is ASF-regular by Theorem 3.12, so we can define the generalized Nakayama automorphism of A and the notion of symmetric AS-regular algebra A over R.

Theorem 4.12. *If A is an AS-regular algebra over R of dimension $n \geq 1$ and of Gorenstein parameter ℓ, then the following hold.*

(1) *The Beilinson algebra ∇A associated to A is a quasi-Fano algebra of dimension $n-1$.*

(2) ${}^{\nu^{[\ell]}}(A^{[\ell]}) \cong \Pi(\nabla A)$ *as graded algebras.*

(3) $\operatorname{GrMod} A \cong \operatorname{GrMod} \Pi(\nabla A)$.

(4) $\mathcal{D}(\operatorname{Tails} A) \cong \mathcal{D}(\operatorname{Mod} \nabla A)$ *as triangulated categories.*

Proof. We set $\mathcal{E} := \mathcal{A} \oplus \mathcal{A}(1) \oplus \cdots \oplus \mathcal{A}(\ell-1) \in \operatorname{Tails} A$. By Lemma 4.11, $B(\mathcal{D}(\operatorname{Tails} A), \mathcal{E}, (\ell)) \cong A^{[\ell]}$ as graded algebras, and $S := \operatorname{End}_{\mathcal{A}}(\mathcal{E}) \cong \nabla A$ as algebras.

Since $\mathcal{E}$ compactly generates $\mathcal{D}(\operatorname{Tails} A)$ and we have $\operatorname{Ext}^q_{\mathcal{A}}(\mathcal{E}, \mathcal{E}) = 0$ for $q \geq 1$ by Proposition 4.4, the functor

$$F(-) := \mathbf{R}\operatorname{Hom}_{\mathcal{A}}(\mathcal{E}, -) : \mathcal{D}(\operatorname{Tails} A) \to \mathcal{D}(\operatorname{Mod} S) \tag{4-6}$$

gives an equivalence of triangulated categories by Keller's theorem [8, Theorem 4.3], proving (4).

We denote by $\mathcal{D}^{\mathrm{cpt}}(\operatorname{Tails} A)$ the full subcategory of $\mathcal{D}(\operatorname{Tails} A)$ consisting of compact objects. Since $\operatorname{gldim} S < \infty$ by [14, Proposition 7.5.1], the full subcategory $\mathcal{D}^{\mathrm{cpt}}(\operatorname{Mod} S)$ of $\mathcal{D}(\operatorname{Mod} R)$ consisting of compact objects is $\mathcal{D}^b(\operatorname{mod} S)$. Therefore $\mathcal{D}^{\mathrm{cpt}}(\operatorname{Tails} A)$ is equivalent to $\mathcal{D}^b(\operatorname{mod} S)$ under the equivalence (4-6).

The generalized Nakayama automorphism $\nu \in \underline{\operatorname{Aut}}_k A$ induces the autoequivalence $(-)_\nu : \operatorname{Tails} A \to \operatorname{Tails} A$. Since $\omega_A[n] \cong A_\nu(-\ell)[n]$ is the balanced dualizing complex of A by Corollary 3.14 and $\operatorname{gldim} A < \infty$, we see that the functor $\mathbb{S}_{\mathcal{A}}(-) = (-)_\nu(-\ell)[n-1]$ is the Serre functor of $\mathcal{D}^{\mathrm{cpt}}(\operatorname{Tails} A)$ by the same argument of [17, Appendix]. We denote by $\mathbb{S}_S$ the Serre functor $- \otimes^{\mathbf{L}}_S DS$ of $\mathcal{D}^b(\operatorname{mod} S)$. By the uniqueness of the Serre functor, we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{D}^{\mathrm{cpt}}(\operatorname{Tails} A) & \xrightarrow[\cong]{F} & \mathcal{D}^b(\operatorname{mod} S) \\ {\scriptstyle \mathbb{S}_{\mathcal{A}}} \downarrow {\scriptstyle \cong} & & {\scriptstyle \mathbb{S}_S} \downarrow {\scriptstyle \cong} \\ \mathcal{D}^{\mathrm{cpt}}(\operatorname{Tails} A) & \xrightarrow[\cong]{F} & \mathcal{D}^b(\operatorname{mod} S). \end{array}$$

We set $\omega_{\mathcal{A}} := \mathbb{S}_{\mathcal{A}}[-(n-1)]$. Since the Serre functor commutes with shifts of complexes, we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{D}^{\mathrm{cpt}}(\operatorname{Tails} A) & \xrightarrow[\cong]{F} & \mathcal{D}^b(\operatorname{mod} S) \\ {\scriptstyle \omega_{\mathcal{A}}^{-1}} \downarrow {\scriptstyle \cong} & & {\scriptstyle -\otimes^{\mathbf{L}}_S \omega_S^{-1}} \downarrow {\scriptstyle \cong} \\ \mathcal{D}^{\mathrm{cpt}}(\operatorname{Tails} A) & \xrightarrow[\cong]{F} & \mathcal{D}^b(\operatorname{mod} S) \end{array}$$

where $\omega_S := DS[-(n-1)]$. The map $\nu : A \to A$ gives an isomorphism $\nu : A_{\nu^{-1}} \to A$ in $\operatorname{GrMod} A$, so ν induces an isomorphism $\overline{\nu} : \mathcal{A}(i)_{\nu^{-1}} \to \mathcal{A}(i)$ in

Tails A for each $i \in \mathbb{Z}$. Since $F(\mathcal{E}) \cong S$,

$$\begin{aligned}
h^q((\omega_S^{-1})^{\otimes^{\mathbf{L}}_S m}) &= \operatorname{Ext}^q_S(S, S \otimes^{\mathbf{L}}_S (\omega_S^{-1})^{\otimes^{\mathbf{L}}_S m}) \\
&\cong \operatorname{Ext}^q_{\mathcal{A}}(\mathcal{E}, \omega_{\mathcal{A}}^{-m}\mathcal{E}) \\
&\cong \bigoplus_{0 \le i,j \le \ell-1} \operatorname{Ext}^q_{\mathcal{A}}(\mathcal{A}(i), \mathcal{A}(j+m\ell)_{\nu^{-m}}) \\
&\cong \bigoplus_{0 \le i,j \le \ell-1} \operatorname{Ext}^q_{\mathcal{A}}(\mathcal{A}(i), \mathcal{A}(j+m\ell)) = 0
\end{aligned}$$

for $m \geq 0$ and $q \neq 0$ by Proposition 4.4. It follows that ω_S^{-1} is a quasi-ample two-sided tilting complex of S, so S is a quasi-Fano algebra of dimension $n-1$ by Remark 1.3, which proves (1).

The generalized Nakayama automorphism $\nu \in \underline{\operatorname{Aut}}_k A$ of A induces an algebra automorphism $\sigma = \nu^{[\ell]}|_S \in \operatorname{Aut}_k S$. We claim that the diagram

$$\begin{array}{ccc}
\mathcal{D}^{\mathrm{cpt}}(\operatorname{Tails} A) & \xrightarrow[\cong]{F} & \mathcal{D}^b(\operatorname{mod} S) \\
{\scriptstyle (-)_\nu}\downarrow{\scriptstyle \cong} & & {\scriptstyle (-)_\sigma}\downarrow{\scriptstyle \cong} \\
\mathcal{D}^{\mathrm{cpt}}(\operatorname{Tails} A) & \xrightarrow[\cong]{F} & \mathcal{D}^b(\operatorname{mod} S)
\end{array}$$

commutes. By abuse of notation, we denote by $\overline{\nu}$ the isomorphism $\mathcal{E}_{\nu^{-1}} \to \mathcal{E}$ in Tails A induced by $\overline{\nu} : \mathcal{A}(i)_{\nu^{-1}} \to \mathcal{A}(i)$. It is easy to check that, under the isomorphism $\operatorname{Hom}_{\mathcal{A}}(\mathcal{A}(i), \mathcal{A}(j)) \cong A_{j-i}$ of Proposition 4.4, the map $\nu|_{A_{j-i}} : A_{j-i} \to A_{j-i}$ corresponds to the map

$$\operatorname{Hom}_{\mathcal{A}}(\mathcal{A}(i), \mathcal{A}(j)) \xrightarrow{(-)_{\nu^{-1}}} \operatorname{Hom}_{\mathcal{A}}(\mathcal{A}(i)_{\nu^{-1}}, \mathcal{A}(j)_{\nu^{-1}}) \xrightarrow{\overline{\nu} \circ - \circ \overline{\nu}^{-1}} \operatorname{Hom}_{\mathcal{A}}(\mathcal{A}(i), \mathcal{A}(j))$$

Therefore, for $\phi \in S$, we have $\sigma(\phi) = \overline{\nu} \circ (\phi)_{\nu^{-1}} \circ \overline{\nu}^{-1}$. Now it is easy to check that, for any $\mathcal{M} \in \operatorname{Tails} A$, the map

$$\operatorname{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{M}_\nu) \xrightarrow{(-)_{\nu^{-1}}} \operatorname{Hom}_{\mathcal{A}}(\mathcal{E}_{\nu^{-1}}, \mathcal{M}) \xrightarrow{- \circ \overline{\nu}^{-1}} \operatorname{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{M})$$

gives an isomorphism $\operatorname{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{M}_\nu) \cong \operatorname{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{M})_\sigma$ of S-modules. Since this isomorphism is natural in $\mathcal{M}$, this proves the claim.

Set $\theta_S := - \otimes^{\mathbf{L}}_S \omega_S^{-1} \circ (-)_\sigma$. Since $\omega_{\mathcal{A}}^{-1} \circ (-)_\nu = (\ell)$, we have the following commutative diagram

$$\begin{array}{ccc}
\mathcal{D}^{\mathrm{cpt}}(\operatorname{Tails} A) & \xrightarrow[\cong]{F} & \mathcal{D}^b(\operatorname{mod} S) \\
{\scriptstyle (\ell)}\downarrow{\scriptstyle \cong} & & {\scriptstyle \theta_S}\downarrow{\scriptstyle \cong} \\
\mathcal{D}^{\mathrm{cpt}}(\operatorname{Tails} A) & \xrightarrow[\cong]{F} & \mathcal{D}^b(\operatorname{mod} S).
\end{array}$$

Therefore we have an isomorphism

$$A^{[\ell]} \cong B(\mathcal{D}(\operatorname{Tails} A), \mathcal{E}, (\ell)) \cong B(\mathcal{D}^{\mathrm{cpt}}(\operatorname{Tails} A), \mathcal{E}, (\ell)) \cong B(\mathcal{D}^b(\operatorname{mod} S), S, \theta_S)$$

of graded algebras by [16, Remark 2.5 (4)]. Since $S_\sigma \otimes^{\mathbf{L}}_S \omega_S^{-1} \cong {}_{\sigma^{-1}}(\omega_S^{-1}) \cong ({}_\sigma\omega_S)^{-1}$ in $\mathcal{D}(\operatorname{Mod} S^e)$, θ_S is naturally isomorphic to $- \otimes^{\mathbf{L}}_S ({}_\sigma\omega_S)^{-1}$. Therefore we have an isomorphism

$$B(\mathcal{D}^b(\operatorname{mod} S), S, \theta_S) = \bigoplus_{m\geq 0} \operatorname{Hom}_S(S, \theta_S^m S) \cong \bigoplus_{m \geq 0} h^0\left((({}_\sigma\omega_S)^{-1})^{\otimes^{\mathbf{L}}_S m}\right) \tag{4-7}$$

of graded algebras where the RHS is equipped with a natural graded algebra structure induced from tensor products. Since $(({}_\sigma\omega_S)^{-1})^{\otimes^{\mathbf{L}}_S m} \cong {}_{\sigma^{-m}}((\omega_S)^{-1})^{\otimes^{\mathbf{L}}_S m}$ for $m \geq 0$, $({}_\sigma\omega_S)^{-1}$ is quasi-ample by (1). Therefore $h^0\left((({}_\sigma\omega_S)^{-1})^{\otimes^{\mathbf{L}}_S m}\right) \cong (({}_\sigma\omega_S)^{-1})^{\otimes_S m}$ for $m \geq 0$. Hence the RHS of (4-7) is isomorphic to the tensor algebra $T_S(({}_\sigma\omega_S)^{-1})$. Combining the above isomorphisms, we have $A^{[\ell]} \cong T_S(({}_\sigma\omega_S)^{-1})$ as graded algebras.

Since $\sigma = \nu^{[\ell]}|_S \in \operatorname{Aut}_k S$,

$$\Pi S = T_S(\omega_S^{-1}) \cong T_S({}_\sigma({}_{\sigma^{-1}}(\omega_S^{-1}))) \cong {}^{\nu^{[\ell]}}T_S(({}_\sigma\omega_S)^{-1}) \cong {}^{\nu^{[\ell]}}(A^{[\ell]})$$

as graded algebras by Lemma 2.11, which proves (2).

Now (3) follows from the sequence of equivalences of graded module categories

$$\operatorname{GrMod} \Pi(\nabla A) \cong \operatorname{GrMod} {}^{\nu^{[\ell]}}(A^{[\ell]}) \cong \operatorname{GrMod} A^{[\ell]} \cong \operatorname{GrMod} A.$$

□

In the above proof, we see that there exists $\sigma \in \operatorname{Aut}_k S$ such that $A^{[\ell]} \cong T_S(({}_\sigma\omega_S)^{-1})$, so every AS-regular algebra is a tensor algebra up to taking quasi-Veronese.

Corollary 4.13. *Symmetric AS-regular algebras over R of dimension $n \geq 1$ of Gorenstein parameter 1 are exactly preprojective algebras of quasi-Fano algebras of dimension $n-1$.*

If A is an AS-regular algebra, then $\operatorname{tors} A \subset \operatorname{grmod} A$ by the proof of Corollary 3.8, so the quotient category $\operatorname{tails} A := \operatorname{grmod} A/\operatorname{tors} A$ makes sense.

Theorem 4.14. *Let A be an AS-regular algebra over R. Then A is graded right coherent if and only if ∇A is extremely Fano, and, in this case, we have an equivalence of triangulated categories*

$$\mathcal{D}^b(\operatorname{tails} A) \cong \mathcal{D}^b(\operatorname{mod} \nabla A).$$

Proof. The equivalence $Q : \operatorname{GrMod} A \to \operatorname{GrMod} A^{[\ell]}$ induces an equivalence $Q : \operatorname{grmod} A \to \operatorname{grmod} A^{[\ell]}$, so A is graded right coherent if and only if $A^{[\ell]}$ is graded right coherent. Therefore the theorem follows from [15, Theorem 2.7] □

By brevity, we call the noncommutative projective scheme associated to an AS-regular algebra of dimension $n+1$ *a quantum projective space* of dimension

n. Then the above results show that classifying quantum projective spaces of dimension n up to derived equivalence is the same as classifying quasi-Fano algebras of dimension n up to derived equivalence.

We now focus on AS-regular algebras of dimension 1. It is well-known that if A is an AS-regular algebra over k of dimension 1 and of Gorenstein parameter ℓ, then $A \cong k[X]$ with $\deg X = \ell \geq 1$ as graded algebras, so that $A^{(\ell)} \cong k[X]$ with $\deg X = 1$ as graded algebras. The following theorem says that a similar result holds for AS-regular algebras over R of dimension 1 up to twists.

Theorem 4.15. *If A is an AS-regular algebra over R of dimension 1 and of Gorenstein parameter ℓ with the generalized Nakayama automorphism $\nu \in \underline{\mathrm{Aut}}_k A$, then*

(1) *R is semi-simple, that is, R is quasi-Fano of dimension 0, and*

(2) *${}^{\nu^{(\ell)}}(A^{(\ell)}) \cong \Pi R \cong R[X]$ with $\deg X = 1$ as graded algebras.*

Proof. (1) Considering the exact sequence $0 \to A_{\geq 1} \to A \to R \to 0$ in $\mathrm{GrMod}\, A$, we see that $A_{\geq 1}$ is a graded projective right A-module. By Lemma 2.6 and Proposition 3.4, there is a projective right R-module P such that $P \underline{\otimes}_R A(-\ell) \cong A_{\geq 1}$ in $\mathrm{GrMod}\, A$. Applying $(-)^{\vee}$ to the above exact sequence, we obtain the exact sequence $0 \to A \to A \underline{\otimes}_R P^*(\ell) \to (DR)(\ell) \to 0$ in $\mathrm{GrMod}\, A^o$. By looking at the degree $-\ell$ part of this exact sequence, we have an isomorphism $P^* \cong DR$ in $\mathrm{Mod}\, R^o$. Therefore we see that every finitely generated injective left R-module is projective. Dually we see that every finitely generated injective right R-module is projective. Hence R is self-injective. Since R is assumed to be of finite global dimension, we conclude that $\mathrm{gldim}\, R = 0$. Hence R is semi-simple.

(2) From the exact sequence $0 \to P \underline{\otimes}_R A(-\ell) \to A \to R \to 0$ in $\mathrm{GrMod}\, A$, we see that $A_i = 0$ for $1 \leq i \leq \ell - 1$ and that $A_i \cong P \otimes_R A_{i-\ell}$ for $i \geq \ell$. Thus we conclude that $A_i = 0$ unless $i \in \ell\mathbb{N}$. Hence $A = A^{(\ell)}$ as (ungraded) algebras. The exact sequence $0 \to P \underline{\otimes}_R A(-\ell) \to A \to R \to 0$ in $\mathrm{GrMod}\, A$ gives the exact sequence $0 \to P \underline{\otimes}_R A^{(\ell)}(-1) \to A^{(\ell)} \to R \to 0$ in $\mathrm{GrMod}\, A^{(\ell)}$. It is now easy to see that $A^{(\ell)}$ is an AS-regular algebra of dimension 1 and of Gorenstein parameter 1 with the generalized Nakayama automorphism $\nu^{(\ell)} \in \underline{\mathrm{Aut}}_k A^{(\ell)}$. Since $\nabla(A^{(\ell)}) \cong R$, it follows that ${}^{\nu^{(\ell)}}(A^{(\ell)}) \cong \Pi R$ as graded algebras by Theorem 4.12. Since R is semi-simple, $\omega_R \cong DR \cong R$ in $\mathrm{Mod}\, R^e$, so $\Pi R = T_R(R) \cong R[X]$ with $\deg X = 1$ as graded algebras. □

We now consider AS-regular algebras of dimension 2. As an application of Theorem 4.12, we prove the following theorem. If A is connected over k, then this theorem was already proved by D. Piontkovski [18]. (See also [13, Theorem 7.2].)

Theorem 4.16. *Every AS-regular algebra A over R of dimension 2 is graded right coherent.*

Proof. The equivalence $Q : \mathrm{GrMod}\, A \to \mathrm{GrMod}\, A^{[\ell]}$ tells us that A is graded right coherent if and only if $A^{[\ell]}$ is graded right coherent. By [25], $A^{[\ell]}$ is graded right coherent if and only if ${}^{\nu^{[\ell]}}(A^{[\ell]})$ is graded right coherent. The Beilinson algebra $S = \nabla A$ associated to A is a hereditary finite dimensional algebra. By [15], the tensor algebra ${}^{\nu^{[\ell]}}(A^{[\ell]}) \cong \Pi S = T_S(\omega_S^{-1})$ of a quasi-ample two-sided tilting complex over a hereditary finite dimensional algebra S is graded right coherent. Thus we conclude that A is graded right coherent. □

To prove this theorem, Piontkovski used an explicit description of an AS-regular algebra over k of dimension 2 due to J.J. Zhang [26], namely, every AS-regular algebra over k of dimension 2 is given by $k\langle x_1, \dots, x_p\rangle / (\sum_{i=1}^{p} \sigma(x_i)x_{p-i})$ for some $\sigma \in \underline{\mathrm{Aut}}_k k\langle x_1, \dots, x_p\rangle$. Using representation theory of quivers, we can also reprove this theorem by Theorem 4.12. Note that the relation $\sum_{i=1}^{p} \sigma(x_i)x_{p-i}$ turns out the (twisted) mesh relation which is an important notion in representation theory of quivers. We leave the details to the reader.

Finally, we will prove that classifying AS-regular algebras over k up to graded Morita equivalence is the same as classifying Beilinson algebras up to isomorphism. Recall that a $\mathbb{Z}$-algebra A is an algebra (not necessarily with unity) endowed with the k-vector space decomposition $A = \oplus_{i,j\in\mathbb{Z}} A_{ij}$ such that $A_{ij}A_{kl} \subset \delta_{li}A_{kj}$. Two $\mathbb{Z}$-algebras A and B are isomorphic as $\mathbb{Z}$-algebras if there is an algebra isomorphism $\phi : A \to B$ such that $\phi(A_{ij}) = B_{ij}$ for all $i, j \in \mathbb{Z}$. For any $\mathbb{Z}$-graded algebra A, we define the $\mathbb{Z}$-algebra $\overline{A} = \oplus_{i,j\in\mathbb{Z}} A_{j-i}$. We refer to [19] for more details on $\mathbb{Z}$-algebras. As another application, we have the following result (cf. [22]).

Theorem 4.17. *Let A, A' be AS-regular algebras over k, and $S = \nabla A, S' = \nabla A'$ their Beilinson algebras. If $A_1 \neq 0$, then the following are equivalent:*

(1) $\mathrm{GrMod}\, A \cong \mathrm{GrMod}\, A'$.
(2) $S \cong S'$ *as algebras.*
(3) $\Pi S \cong \Pi S'$ *as graded algebras.*
(4) $\mathrm{GrMod}\, \Pi S \cong \mathrm{GrMod}\, \Pi S'$.

Proof. (1) $\Longrightarrow$ (2) Since $A_1 \neq 0$, if $\mathrm{GrMod}\, A \cong \mathrm{GrMod}\, A'$, then there exists an equivalence functor $F : \mathcal{D}(\mathrm{Tails}\, A) \to \mathcal{D}(\mathrm{Tails}\, A')$ such that $F(\mathcal{A}(i)) \cong \mathcal{A}'(i)$ for all $i \in \mathbb{Z}$ by [25]. Since the length of a full strong exceptional sequence is preserved by derived equivalence, A and A' have the same Gorenstein parameter ℓ. By [19], $\overline{A} \cong \overline{A'}$ as $\mathbb{Z}$-algebras, so $S = \oplus_{0\le i,j\le \ell-1} \overline{A}_{ij} \cong \oplus_{0\le i,j\le \ell-1} \overline{A'}_{ij} = S'$ as sub-algebras. Clearly, (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4). Since $\mathrm{GrMod}\, A \cong \mathrm{GrMod}\, \Pi S$ and $\mathrm{GrMod}\, A' \cong \mathrm{GrMod}\, \Pi S'$ by Theorem 4.12, (4) $\Longrightarrow$ (1). □

4.2 The Structure of Graded Frobenius Algebras

In this subsection, we study the structure of graded Frobenius algebras. Recall that graded Frobenius algebras are exactly ASF-Gorenstein algebras of dimension 0. First we will show that the trivial extension of a finite dimensional algebra R is a symmetric AS-Gorenstein algebra over R of dimension 0 and of Gorenstein parameter -1.

Theorem 4.18. *If A is a graded Frobenius algebra connected over R of Gorenstein parameter $-\ell$, then A is an AS-Gorenstein algebra over R of dimension 0 and of Gorenstein parameter $-\ell$.*

Proof. Since $DA \cong A(\ell)$ in $\operatorname{GrMod} A$ and in $\operatorname{GrMod} A^o$,

$$\begin{aligned}\mathbf{R}\underline{\operatorname{Hom}}_A(R, A) &\cong \mathbf{R}\underline{\operatorname{Hom}}_A(R, DA(-\ell)) \\ &\cong \mathbf{R}\underline{\operatorname{Hom}}_A(R, \mathbf{R}\underline{\operatorname{Hom}}_k(A, k))(-\ell) \\ &\cong \mathbf{R}\underline{\operatorname{Hom}}_k(R \underline{\otimes}_A^{\mathbf{L}} A, k)(-\ell) \\ &\cong \mathbf{R}\underline{\operatorname{Hom}}_k(R, k)(-\ell) \\ &\cong DR(-\ell)\end{aligned}$$

in $\mathcal{D}(\operatorname{GrMod} R^o)$ and in $\mathcal{D}(\operatorname{GrMod} R)$. □

Corollary 4.19. *If R is a finite dimensional algebra, then the trivial extension $A := \Delta R$ is an AS-Gorenstein algebra over R of dimension 0 and of Gorenstein parameter -1.*

One of the characterizations of an AS-regular algebra over k was given in [11], namely, a graded algebra A connected over k is AS-regular if and only if the Yoneda Ext-algebra $A^! := \oplus_{i\in\mathbb{N}} \underline{\operatorname{Ext}}^i_A(k, k)$ is graded Frobenius. This fact motivates the following result.

Proposition 4.20. *If R is a quasi-Fano algebra of dimension n and $A = \Pi R$ is the preprojective algebra of R, then the Yoneda* Ext*-algebra $A^! := \oplus_{i\in\mathbb{N}} \underline{\operatorname{Ext}}^i_A(R, R)$ is a graded Frobenius algebra of Gorenstein parameter $-n-1$. Moreover, $(A^!)^{(n+1)} \cong \Delta R$ as graded algebras.*

Proof. Since ω_R^{-1} is a quasi-ample two-sided tilting complex of R, and $A = \Pi R = T_R(\omega_R^{-1})$ is the tensor algebra, the exact sequence

$$0 \to \omega_R^{-1} \underline{\otimes}_R A(-1) \to A \to R \to 0$$

in $\operatorname{GrMod}(R^o \underline{\otimes}_k A)$ induces an exact triangle

$$\mathbf{R}\underline{\operatorname{Hom}}_A(R, R) \longrightarrow \mathbf{R}\underline{\operatorname{Hom}}_A(A, R) \xrightarrow{\phi} \mathbf{R}\underline{\operatorname{Hom}}_A(\omega_R^{-1} \underline{\otimes}_R A(-1), R)$$

in $\mathcal{D}(\operatorname{GrMod}(A^o \underline{\otimes}_k R))$. Since $\mathbf{R}\underline{\operatorname{Hom}}_A(A, R) \cong R$ is concentrated in degree 1 and $\mathbf{R}\underline{\operatorname{Hom}}_A(\omega_R^{-1} \underline{\otimes}_R A(-1), R) \cong \mathbf{R}\underline{\operatorname{Hom}}_R(\omega_R^{-1}, R)(1) \cong \omega_R(1) = (DR)(1)[-n]$ is concentrated in degree -1, we have $\phi = 0$. It follows that

$$\underline{\operatorname{Ext}}^i_A(R, R) \cong \begin{cases} R & \text{if } i = 0 \\ (DR)(1) & \text{if } i = n+1 \\ 0 & \text{otherwise} \end{cases}$$

in $\operatorname{Mod} R^e$, so $(A^!)^{(n+1)} \cong \Delta R$ as graded algebras. It is now easy to see that $A^!$ is a graded Frobenius algebra of Gorenstein parameter $-n-1$. □

Next, we will see that a graded Frobenius algebra is the trivial extension of a finite dimensional algebra up to graded Morita equivalence. This fact was first shown by [6].

Lemma 4.21. *Let A be an $\mathbb{N}$-graded algebra connected over R. For a graded A-A bimodule M, $(DM)_0 \cong D(M_0)$ as R-R bimodules.*

Proof. Note that

$$(DM)_0 = \underline{\mathrm{Hom}}_k(M,k)_0 = \mathrm{Hom}_k(M_0,k) = D(M_0)$$

as k-vector spaces. For $m \in M_0 \subset M$, $a, b \in R \subset A$, and $f \in \underline{\mathrm{Hom}}_k(M,k)_0 = \mathrm{Hom}_k(M_0,k)$, the bimodule structure on $\underline{\mathrm{Hom}}_k(M,k)_0$ is given by $(afb)(m) = f(bma)$, and the bimodule structure on $\underline{\mathrm{Hom}}_k(M_0,k)$ is given by $(afb)(m) = f(bma)$. □

The equivalence $\mathrm{GrMod}\, A \cong \mathrm{GrMod}\, \Delta(\nabla A)$ in the theorem below was proved in [6] in the same manner. We include our proof to show that the results below are analogous to those in Theorem 4.12.

Theorem 4.22. *If A is a graded Frobenius algebra connected over R of Gorenstein parameter $-\ell$ with the Nakayama automorphism $\nu \in \underline{\mathrm{Aut}}_k A$, then the following hold.*

(1) *$(A^{[\ell]})^{\nu^{[\ell]}} \cong \Delta(\nabla A)$ as graded algebras.*

(2) $\mathrm{GrMod}\, A \cong \mathrm{GrMod}\, \Delta(\nabla A)$.

(3) *Moreover, if* $\mathrm{gldim}\, \nabla A < \infty$, *then* $\underline{\mathrm{GrMod}} A \cong D(\mathrm{Mod}\, \nabla A)$ *as triangulated categories.*

Proof. Let $S = \nabla A$ be the Beilinson algebra associated to A, and $\sigma = \nu^{[\ell]}|_S \in \mathrm{Aut}_k S$ the restriction. Since $DA \cong A_\nu(\ell)$ as graded A-A bimodules,

$$(A^{[\ell]})_{\nu^{[\ell]}}(1) \cong Q(\oplus_{i=0}^{\ell-1} A(i))_{\nu^{[\ell]}}(1) \cong Q(\oplus_{i=0}^{\ell-1} A_\nu(\ell+i)) \cong \oplus_{i=0}^{\ell-1} Q((DA)(i))$$

as graded $A^{[\ell]}$-$A^{[\ell]}$ bimodules by [16, Lemma 5.2], so

$$\begin{aligned}
DS &= D((A^{[\ell]})_0) \\
&= D\begin{pmatrix} A_0 & A_1 & \cdots & A_{\ell-1} \\ 0 & A_0 & \cdots & A_{\ell-2} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_0 \end{pmatrix} \cong \begin{pmatrix} D(A_0) & 0 & \cdots & 0 \\ D(A_1) & D(A_0) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ D(A_{\ell-1}) & D(A_{\ell-2}) & \cdots & D(A_0) \end{pmatrix} \\
&= \begin{pmatrix} (DA)_0 & (DA(1))_0 & \cdots & (DA(\ell-1))_0 \\ (DA)_{-1} & (DA(1))_{-1} & \cdots & (DA(\ell-1))_{-1} \\ \vdots & \vdots & & \vdots \\ (DA)_{1-\ell} & (DA(1))_{1-\ell} & \cdots & (DA(\ell-1))_{1-\ell} \end{pmatrix} \\
&= \oplus_{i=0}^{\ell-1} Q((DA)(i))_0 \cong (A^{[\ell]})_{\nu^{[\ell]}}(1)_0 = ((A^{[\ell]})_1)_\sigma = ((A^{[\ell]})^{\nu^{[\ell]}})_1
\end{aligned}$$

as S-S bimodules by Lemma 2.10. It follows that

$$(A^{[\ell]})^{\nu^{[\ell]}} = ((A^{[\ell]})^{\nu^{[\ell]}})_0 \oplus ((A^{[\ell]})^{\nu^{[\ell]}})_1 \cong S \oplus DS = \Delta S$$

as graded algebras, so

$$\mathrm{GrMod}\, \Delta S \cong \mathrm{GrMod}(A^{[\ell]})^{\nu^{[\ell]}} \cong \mathrm{GrMod}\, A^{[\ell]} \cong \mathrm{GrMod}\, A.$$

Moreover, if $\mathrm{gldim}\, S < \infty$, then

$$\underline{\mathrm{GrMod}} A \cong \underline{\mathrm{GrMod}} \Delta S \cong \mathcal{D}(\mathrm{Mod}\, S)$$

as triangulated categories by [7, Chapter II] (See also [9, Theorem 8.6]). □

Corollary 4.23. *Symmetric graded Frobenius algebras of Gorenstein parameter -1 are exactly trivial extensions of finite dimensional algebras.*

Proof. If A is a symmetric graded Frobenius algebra connected over R of Gorenstein parameter -1, then $A \cong \Delta R$ by Theorem 4.22. Conversely if $A = \Delta R$ is a trivial extension of a finite dimensional algebra R, then $D\,\mathrm{R}\,\underline{\Gamma}_{\mathfrak{m}}(A) \cong DA \cong D(R \oplus DR) \cong DR \oplus R \cong A(1)$ in $\mathrm{GrMod}\, A^e$, so A is symmetric graded Frobenius algebra of Gorenstein parameter -1. □

As an application, we have the following result, which is analogous to Theorem 4.17.

Theorem 4.24. *Let A, A' be graded Frobenius algebras connected over k, and $S = \nabla A, S' = \nabla A'$ their Beilinson algebras. If $A_1 \neq 0$, then the following are equivalent:*
1. $\mathrm{GrMod}\, A \cong \mathrm{GrMod}\, A'$.
2. $S \cong S'$ *as algebras.*
3. $\Delta S \cong \Delta S'$ *as graded algebras.*
4. $\mathrm{GrMod}\, \Delta S \cong \mathrm{GrMod}\, \Delta S'$.

Proof. It is enough to show that (1) $\Longrightarrow$ (2). Suppose that $\mathrm{GrMod}\, A \cong \mathrm{GrMod}\, A'$. Since A, A' are finite dimensional and $A \cong A'$ as graded vector spaces, they have the same Gorenstein parameter ℓ. The rest of the proof is similar to that of Theorem 4.17.

□

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