ISOMORPHIC AND STRONGLY CONNECTED COMPONENTS

Miloš S. Kurilić[1]

Abstract

We study the partial orderings of the form $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$, where $\mathbb{X}$ is a binary relational structure with the connectivity components isomorphic to a strongly connected structure $\mathbb{Y}$ and $\mathbb{P}(\mathbb{X})$ is the set of (domains of) substructures of $\mathbb{X}$ isomorphic to $\mathbb{X}$. We show that, for example, for a countable $\mathbb{X}$, the poset $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ is either isomorphic to a finite power of $\mathbb{P}(\mathbb{Y})$ or forcing equivalent to a separative atomless σ-closed poset and, consistently, to $P(\omega)/\mathrm{Fin}$. In particular, this holds for each ultrahomogeneous structure $\mathbb{X}$ such that $\mathbb{X}$ or $\mathbb{X}^c$ is a disconnected structure and in this case $\mathbb{Y}$ can be replaced by an ultrahomogeneous connected digraph.

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1 Introduction

We consider the partial orderings of the form $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$, where $\mathbb{X}$ is a relational structure and $\mathbb{P}(\mathbb{X})$ the set of the domains of its isomorphic substructures. A rough classification of countable binary structures related to the properties of their posets of copies is obtained in [6], defining two structures to be equivalent if the corresponding posets of copies have isomorphic Boolean completions or, equivalently, are forcing equivalent. So, for example, for the structures from column D of Diagram 1 of [6] the corresponding posets are forcing equivalent to an atomless ω_1-closed poset and, consistently, to $P(\omega)/\mathrm{Fin}$. This class of structures includes all scattered linear orders [9] (in particular, all countable ordinals [8]), all structures with maximally embeddable components [7] (in particular, all countable equivalence relations and all disjoint unions of countable ordinals) and in this paper we show that it contains a large class of ultrahomogeneous structures.

In Theorem 3.2 of Section 3 we show that the poset of copies of a binary structure with κ-many isomorphic and strongly connected components is either isomorphic to a finite power of the poset of copies of one component, or forcing equivalent to something like $P(\kappa)/[\kappa]^{<\kappa}$ and, for countable structures, consistently, to $P(\omega)/\mathrm{Fin}$. The main result of Section 4 is that each ultrahomogeneous binary structure which is not biconnected is determined by an ultrahomogeneous digraph in a simple way and this fact is used in Section 5, where we apply Theorem 3.2 to countable ultrahomogeneous binary structures.

[1]Department of Mathematics and Informatics, University of Novi Sad, Trg Dositeja Obradovića 4, 21000 Novi Sad, Serbia. e-mail: milos@dmi.uns.ac.rs

2 Preliminaries

The aim of this section is to introduce notation and to give basic definitions and facts concerning relational structures and partial orders which will be used.

We observe *binary structures*, the relational structures of the form $\mathbb{X} = \langle X, \rho \rangle$, where ρ is a binary relation on the set X. If $\mathbb{Y} = \langle Y, \tau \rangle$ is a binary structure too, a mapping $f : X \to Y$ is an *embedding* (we write $f : \mathbb{X} \hookrightarrow \mathbb{Y}$) iff f is an injection and $x_1 \rho x_2 \Leftrightarrow f(x_1) \tau f(x_2)$, for each $x_1, x_2 \in X$. $\mathrm{Emb}(\mathbb{X}, \mathbb{Y})$ will denote the set of all embeddings of $\mathbb{X}$ into $\mathbb{Y}$ and, in particular, $\mathrm{Emb}(\mathbb{X}) = \mathrm{Emb}(\mathbb{X}, \mathbb{X})$. If, in addition, f is a surjection, f is an *isomorphism* and the structures $\mathbb{X}$ and $\mathbb{Y}$ are called *isomorphic*, in notation $\mathbb{X} \cong \mathbb{Y}$. If, in particular, $\mathbb{Y} = \mathbb{X}$, then f is called an *automorphism* of the structure $\mathbb{X}$ and $\mathrm{Aut}(\mathbb{X})$ will denote the set of all automorphisms of $\mathbb{X}$. If $\mathbb{X} = \langle X, \rho \rangle$ is a binary structure, $A \subset X$ and $\rho_A = \rho \cap (A \times A)$, then $\langle A, \rho_A \rangle$ is the corresponding *substructure* of $\mathbb{X}$. By $\mathbb{P}(\mathbb{X})$ we denote the set of domains of substructures of $\mathbb{X}$ which are isomorphic to $\mathbb{X}$, that is

$$\mathbb{P}(\mathbb{X}) = \{A \subset X : \langle A, \rho_A \rangle \cong \langle X, \rho \rangle\} = \{f[X] : f \in \mathrm{Emb}(\mathbb{X})\}.$$

More generally, if $\mathbb{X} = \langle X, \rho \rangle$ and $\mathbb{Y} = \langle Y, \tau \rangle$ are binary structures we define $\mathbb{P}(\mathbb{X}, \mathbb{Y}) = \{B \subset Y : \langle B, \tau_B \rangle \cong \langle X, \rho \rangle\} = \{f[X] : f \in \mathrm{Emb}(\mathbb{X}, \mathbb{Y})\}$. By $\mathrm{Pi}(\mathbb{X})$ we denote the set of all finite partial isomorphisms of $\mathbb{X}$. A structure $\mathbb{X}$ is called *ultrahomogeneous* iff for each $\varphi \in \mathrm{Pi}(\mathbb{X})$ there is $f \in \mathrm{Aut}(\mathbb{X})$ such that $\varphi \subset f$.

If $\mathbb{X}_i = \langle X_i, \rho_i \rangle$, $i \in I$, are binary structures and $X_i \cap X_j = \emptyset$, for different $i, j \in I$, then the structure $\bigcup_{i \in I} \mathbb{X}_i = \langle \bigcup_{i \in I} X_i, \bigcup_{i \in I} \rho_i \rangle$ will be called the *disjoint union* of the structures $\mathbb{X}_i$, $i \in I$.

If $\langle X, \rho \rangle$ is a binary structure, then the transitive closure ρ_{rst} of the relation $\rho_{rs} = \Delta_X \cup \rho \cup \rho^{-1}$ (given by $x \; \rho_{rst} \; y$ iff there are $n \in \mathbb{N}$ and $z_0 = x, z_1, \ldots, z_n = y$ such that $z_i \; \rho_{rs} \; z_{i+1}$, for each $i < n$) is the minimal equivalence relation on X containing ρ. For $x \in X$ the corresponding element of the quotient X/ρ_{rst} will be denoted by $[x]$ and called the *component* of $\langle X, \rho \rangle$ containing x. The structure $\langle X, \rho \rangle$ will be called *connected* iff $|X/\rho_{rst}| = 1$. It is easy to check (see Proposition 7.2 of [6]) that $\langle \bigcup_{x \in X} [x], \bigcup_{x \in X} \rho_{[x]} \rangle$ is the unique representation of $\langle X, \rho \rangle$ as a disjoint union of connected structures. Also, if $\rho^c = (X \times X) \setminus \rho$, then at least one of the structures $\langle X, \rho \rangle$ and $\langle X, \rho^c \rangle$ is connected (Proposition 7.3 of [6]). The following facts (Lemma 7.4 and Theorem 7.5 of [6]) will be used in the sequel.

Fact 2.1 Let $\langle X, \rho \rangle$ and $\langle Y, \tau \rangle$ be binary structures and $f : X \to Y$ an embedding. Then for each $x \in X$

(a) $f[[x]] \subset [f(x)]$;

(b) $f \mid [x] : [x] \to f[[x]]$ is an isomorphism;

(c) If, in addition, f is an isomorphism, then $f[[x]] = [f(x)]$.

Fact 2.2 Let κ be a cardinal, let $\mathbb{X}_\alpha = \langle X_\alpha, \rho_\alpha \rangle, \alpha < \kappa$, be disjoint connected binary structures and $\mathbb{X}$ their union. Then $C \in \mathbb{P}(\mathbb{X})$ iff there is a function $f : \kappa \to \kappa$ and there are embeddings $e_\xi : \mathbb{X}_\xi \hookrightarrow \mathbb{X}_{f(\xi)}$, $\xi < \kappa$, such that $C = \bigcup_{\xi < \kappa} e_\xi[X_\xi]$ and

$$\forall \{\xi, \zeta\} \in [\kappa]^2 \;\; \forall x \in X_\xi \;\; \forall y \in X_\zeta \;\; \neg \, e_\xi(x) \; \rho_{rs} \; e_\zeta(y). \tag{1}$$

Let $\mathbb{P} = \langle P, \leq \rangle$ be a pre-order. Then $p \in P$ is an *atom*, in notation $p \in \mathrm{At}(\mathbb{P})$, iff each $q, r \leq p$ are compatible (there is $s \leq q, r$). $\mathbb{P}$ is called *atomless* iff $\mathrm{At}(\mathbb{P}) = \emptyset$; *atomic* iff $\mathrm{At}(\mathbb{P})$ is dense in $\mathbb{P}$. If κ is a regular cardinal, $\mathbb{P}$ is called κ*-closed* iff for each $\gamma < \kappa$ each sequence $\langle p_\alpha : \alpha < \gamma \rangle$ in P, such that $\alpha < \beta \Rightarrow p_\beta \leq p_\alpha$, has a lower bound in P. Two pre-orders $\mathbb{P}$ and $\mathbb{Q}$ are called *forcing equivalent* iff they produce the same generic extensions. The following fact is folklore.

Fact 2.3 (a) The direct product of a family of κ-closed pre-orders is κ-closed.

(b) If $\kappa^{<\kappa} = \kappa$, then all atomless separative κ-closed pre-orders of size κ are forcing equivalent (for example, to the poset (Coll$(\kappa, \kappa))^+$, or to $(P(\kappa)/[\kappa]^{<\kappa})^+$).

A partial order $\mathbb{P} = \langle P, \leq \rangle$ is called *separative* iff for each $p, q \in P$ satisfying $p \not\leq q$ there is $r \leq p$ such that $r \perp q$. The *separative modification* of $\mathbb{P}$ is the separative pre-order $\mathrm{sm}(\mathbb{P}) = \langle P, \leq^* \rangle$, where $p \leq^* q \Leftrightarrow \forall r \leq p \; \exists s \leq r \;\; s \leq q$. The *separative quotient* of $\mathbb{P}$ is the separative poset $\mathrm{sq}(\mathbb{P}) = \langle P/{=^*}, \trianglelefteq \rangle$, where $p =^* q \Leftrightarrow p \leq^* q \wedge q \leq^* p$ and $[p] \trianglelefteq [q] \Leftrightarrow p \leq^* q$.

Fact 2.4 (Folklore) Let $\mathbb{P}, \mathbb{Q}$ and $\mathbb{P}_i$, $i \in I$, be partial orderings. Then

(a) $\mathbb{P}$, $\mathrm{sm}(\mathbb{P})$ and $\mathrm{sq}(\mathbb{P})$ are forcing equivalent forcing notions;
(b) $\mathbb{P}$ is atomless iff $\mathrm{sm}(\mathbb{P})$ is atomless iff $\mathrm{sq}(\mathbb{P})$ is atomless;
(c) $\mathrm{sm}(\mathbb{P})$ is κ-closed iff $\mathrm{sq}(\mathbb{P})$ is κ-closed;
(d) $\mathbb{P} \cong \mathbb{Q}$ implies that $\mathrm{sm}\,\mathbb{P} \cong \mathrm{sm}\,\mathbb{Q}$ and $\mathrm{sq}\,\mathbb{P} \cong \mathrm{sq}\,\mathbb{Q}$;
(e) $\mathrm{sm}(\prod_{i \in I} \mathbb{P}_i) = \prod_{i \in I} \mathrm{sm}\,\mathbb{P}_i$ and $\mathrm{sq}(\prod_{i \in I} \mathbb{P}_i) \cong \prod_{i \in I} \mathrm{sq}\,\mathbb{P}_i$.

3 Isomorphic and strongly connected components

A relational structure $\mathbb{X} = \langle X, \rho \rangle$ will be called *strongly connected* iff it is connected and for each $A, B \in \mathbb{P}(\mathbb{X})$ there are $a \in A$ and $b \in B$ such that $a \; \rho_{rs} \; b$. (The structures satisfying $\mathbb{P}(\mathbb{X}) = \{X\}$ have the second property, but can be disconnected.)

Example 3.1 Some strongly connected structures are: linear orders, full relations, complete graphs, etc. The binary tree $\langle {}^{<\omega}2, \subset \rangle$ is a connected, but not a strongly connected partial order.

Theorem 3.2 Let κ be a cardinal and $\mathbb{X}=\bigcup_{\alpha<\kappa}\mathbb{X}_\alpha$ the union of disjoint, isomorphic and strongly connected binary structures. Then

(a) $\langle \mathbb{P}(\mathbb{X}),\subset\rangle \cong \langle \mathbb{P}(\mathbb{X}_0),\subset\rangle^\kappa$ and $\mathrm{sq}\langle \mathbb{P}(\mathbb{X}),\subset\rangle \cong (\mathrm{sq}\langle \mathbb{P}(\mathbb{X}_0),\subset\rangle)^\kappa$, if $\kappa<\omega$;

(b) $\mathrm{sq}\langle \mathbb{P}(\mathbb{X}),\subset\rangle$ is an atomless poset, if $\kappa\geq\omega$;

(c) $\mathrm{sq}\langle \mathbb{P}(\mathbb{X}),\subset\rangle$ is a κ^+-closed poset, if $\kappa\geq\omega$ is regular;

(d) $\mathrm{sq}\langle \mathbb{P}(\mathbb{X}),\subset\rangle$ is forcing equivalent to the poset $(P(\kappa)/[\kappa]^{<\kappa})^+$, if $\kappa\geq\omega$ is regular and $|\mathbb{P}(\mathbb{X}_0)|\leq 2^\kappa=\kappa^+$. The same holds for $\langle \mathbb{P}(\mathbb{X}),\subset\rangle$.

Proof. For $A\in[\kappa]^\kappa$ and $g\in\prod_{\alpha\in A}\mathbb{P}(\mathbb{X}_\alpha)$ let us define $C_g=\bigcup_{\alpha\in A} g(\alpha)$.

Claim 1. $\mathbb{P}(\mathbb{X})=\{C_g : A\in[\kappa]^\kappa \wedge g\in\prod_{\alpha\in A}\mathbb{P}(\mathbb{X}_\alpha)\}$.

Proof of Claim 1. ($\subset$) If $C\in\mathbb{P}(\mathbb{X})$, then, by Fact 2.2, there is a function $f:\kappa\to\kappa$ and there are embeddings $e_\xi:\mathbb{X}_\xi\hookrightarrow\mathbb{X}_{f(\xi)}, \xi<\kappa$, such that $C=\bigcup_{\xi\in\kappa} e_\xi[X_\xi]$ and that (1) is true.

Suppose that $f(\xi)=f(\zeta)$, for some different $\xi,\zeta\in\kappa$. By the assumption we have $\mathbb{X}_\xi\cong\mathbb{X}_\zeta\cong\mathbb{X}_{f(\xi)}$, which implies $\mathbb{P}(\mathbb{X}_\xi,\mathbb{X}_{f(\xi)})=\mathbb{P}(\mathbb{X}_\zeta,\mathbb{X}_{f(\xi)})=\mathbb{P}(\mathbb{X}_{f(\xi)})$. Thus $e_\xi[X_\xi], e_\zeta[X_\zeta]\in\mathbb{P}(\mathbb{X}_{f(\xi)})$ and, since the structure $\mathbb{X}_{f(\xi)}$ is strongly connected, there are $x\in X_\xi$ and $y\in X_\zeta$ such that $e_\xi(x)(\rho_{f(\xi)})_{rs}\ e_\zeta(y)$, which, since $\rho_{f(\xi)}\subset\rho$, implies $e_\xi(x)\ \rho_{rs}\ e_\zeta(y)$, which is impossible by (1). Thus f is an injection and, hence, $A=f[\kappa]\in[\kappa]^\kappa$. For $f(\xi)\in f[\kappa]$ let $g(f(\xi)):=e_\xi[X_\xi]$; then $g(f(\xi))\in\mathbb{P}(\mathbb{X}_{f(\xi)})$, for all $\xi\in\kappa$, that is $g(\alpha)\in\mathbb{P}(\mathbb{X}_\alpha)$, for all $\alpha\in A$ and, hence, $g\in\prod_{\alpha\in A}\mathbb{P}(\mathbb{X}_\alpha)$. Also $C=\bigcup_{\xi\in\kappa} g(f(\xi))=\bigcup_{\alpha\in A} g(\alpha)=C_g$ and we are done.

($\supset$) Let $A\in[\kappa]^\kappa$, $g\in\prod_{\alpha\in A}\mathbb{P}(\mathbb{X}_\alpha)$ and let $f:\kappa\to A$ be a bijection. Then for $\xi\in\kappa$ we have $g(f(\xi))\in\mathbb{P}(\mathbb{X}_{f(\xi)})=\mathbb{P}(\mathbb{X}_\xi,\mathbb{X}_{f(\xi)})$ and, hence there is an embedding $e_\xi:\mathbb{X}_\xi\hookrightarrow\mathbb{X}_{f(\xi)}$ such that $g(f(\xi))=e_\xi[X_\xi]$. Thus $C_g=\bigcup_{\alpha\in A} g(\alpha)=\bigcup_{\xi\in\kappa} g(f(\xi))=\bigcup_{\xi\in\kappa} e_\xi[X_\xi]$. If $\xi\neq\zeta\in\kappa$, $x\in X_\xi$ and $y\in X_\zeta$, then, since f is an injection, $X_{f(\xi)}$ and $X_{f(\zeta)}$ are different components of $\mathbb{X}$ containing $e_\xi(x)$ and $e_\zeta(y)$ respectively. So $\neg e_\xi(x)\rho_{rs}e_\zeta(y)$ and (1) is true. By Fact 2.2 we have $C_g\in\mathbb{P}(\mathbb{X})$. Claim 1 is proved. □

(a) By Claim 1 we have $\mathbb{P}(\mathbb{X})=\{\bigcup_{i<\kappa} C_i : \forall i<\kappa\ \ C_i\in\mathbb{P}(\mathbb{X}_i)\}$. It is easy to see that the mapping F defined by $F(\langle C_i : i<\kappa\rangle)=\bigcup_{i<\kappa} C_i$ witnesses that the posets $\prod_{i<\kappa}\langle \mathbb{P}(\mathbb{X}_i),\subset\rangle$ and $\langle \mathbb{P}(\mathbb{X}),\subset\rangle$ are isomorphic. Since isomorphic structures have isomorphic posets of copies we have $\langle \mathbb{P}(\mathbb{X}),\subset\rangle\cong\langle \mathbb{P}(\mathbb{X}_0),\subset\rangle^\kappa$ and, by Fact 2.4(d) and (e), $\mathrm{sq}\langle \mathbb{P}(\mathbb{X}),\subset\rangle\cong\mathrm{sq}(\langle \mathbb{P}(\mathbb{X}_0),\subset\rangle^\kappa)\cong(\mathrm{sq}\langle \mathbb{P}(\mathbb{X}_0),\subset\rangle)^\kappa$.

(b) Let $\kappa\geq\omega$, $\mathrm{sm}\langle \mathbb{P}(\mathbb{X}),\subset\rangle=\langle \mathbb{P}(\mathbb{X}),\leq\rangle$ and $\mathrm{sm}\langle \mathbb{P}(\mathbb{X}_\alpha),\subset\rangle=\langle \mathbb{P}(\mathbb{X}_\alpha),\leq_\alpha\rangle$, for $\alpha<\kappa$. First we prove

Claim 2. For each $f,g\in\bigcup_{A\in[\kappa]^\kappa}\prod_{\alpha\in A}\mathbb{P}(\mathbb{X}_\alpha)$ we have $C_f\leq C_g$ if and only if

$$|(\mathrm{dom}\, f\setminus\mathrm{dom}\, g)\cup\{\alpha\in\mathrm{dom}\, f\cap\mathrm{dom}\, g : \neg f(\alpha)\leq_\alpha g(\alpha)\}|<\kappa; \qquad (2)$$

Proof of Claim 2. Let $f, g, h \in \bigcup_{A \in [\kappa]^\kappa} \prod_{\alpha \in A} \mathbb{P}(\mathbb{X}_\alpha)$. Clearly we have

$$C_f \subset C_g \Leftrightarrow \operatorname{dom} f \subset \operatorname{dom} g \wedge \forall \alpha \in \operatorname{dom} f \;\; f(\alpha) \subset g(\alpha). \tag{3}$$

Let $\perp$ denote the incompatibility relation in the posets $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ and $\langle \mathbb{P}(\mathbb{X}_\alpha), \subset \rangle$, $\alpha < \kappa$. First we prove

$$C_h \perp C_g \Leftrightarrow |\{\alpha \in \operatorname{dom} h \cap \operatorname{dom} g : h(\alpha) \not\perp g(\alpha)\}| < \kappa. \tag{4}$$

If the set $A = \{\alpha \in \operatorname{dom} h \cap \operatorname{dom} g : h(\alpha) \not\perp g(\alpha)\}$ is of size κ, for each $\alpha \in A$ we choose $k(\alpha) \in \mathbb{P}(\mathbb{X}_\alpha)$ such that $k(\alpha) \subset h(\alpha) \cap g(\alpha)$. So $k \in \prod_{\alpha \in A} \mathbb{P}(\mathbb{X}_\alpha)$ and by (a) we have $C_k \in \mathbb{P}(\mathbb{X})$. By (3) we have $C_k \subset C_h \cap C_g$ thus $C_h \not\perp C_g$. Conversely, if $C_h \not\perp C_g$, then by (a) there is $C_k \in \mathbb{P}(\mathbb{X})$ such that $C_k \subset C_h \cap C_g$. Now $A := \operatorname{dom} k \in [\kappa]^\kappa$ and by (3) we have $A \subset \operatorname{dom} h \cap \operatorname{dom} g$ and $k(\alpha) \subset h(\alpha) \cap g(\alpha)$, for all $\alpha \in A$. Thus $|\{\alpha \in \operatorname{dom} h \cap \operatorname{dom} g : h(\alpha) \not\perp g(\alpha)\}| = \kappa$.

Now suppose that $C_f \leq C_g$. Then for each $C_h \in \mathbb{P}(\mathbb{X})$ satisfying $C_h \subset C_f$ we have $C_h \not\perp C_g$ so, by (4) we have

$$\forall C_h \in \mathbb{P}(\mathbb{X}) \;\; (C_h \subset C_f \Rightarrow |\{\alpha \in \operatorname{dom} h \cap \operatorname{dom} g : h(\alpha) \not\perp g(\alpha)\}| = \kappa). \tag{5}$$

Suppose that the set $A := \operatorname{dom} f \setminus \operatorname{dom} g$ is of size κ. Then $h := f \upharpoonright A \in \prod_{\alpha \in A} \mathbb{P}(\mathbb{X}_\alpha)$, clearly $C_h \subset C_f$ and, by (a), $C_h \in \mathbb{P}(\mathbb{X})$. Also we have $\operatorname{dom} h \cap \operatorname{dom} g = \emptyset$, which is impossible by (5). Thus

$$|\operatorname{dom} f \setminus \operatorname{dom} g| < \kappa. \tag{6}$$

Suppose that the set $A := \{\alpha \in \operatorname{dom} f \cap \operatorname{dom} g : \neg f(\alpha) \leq_\alpha g(\alpha)\}$ is of size κ. For $\alpha \in A$ there is $C_\alpha \in \mathbb{P}(\mathbb{X}_\alpha)$ such that $C_\alpha \subset f(\alpha)$ and $C_\alpha \perp g(\alpha)$ and we define $h(\alpha) = C_\alpha$. Now $h \in \prod_{\alpha \in A} \mathbb{P}(\mathbb{X}_\alpha)$, by (a) we have $C_h \in \mathbb{P}(\mathbb{X})$ and, by (3), $C_h \subset C_f$. So by (5) there is $\alpha \in \operatorname{dom} h \cap \operatorname{dom} g = A$ such that $C_\alpha = h(\alpha) \not\perp g(\alpha)$, which is not true. Thus

$$|\{\alpha \in \operatorname{dom} f \cap \operatorname{dom} g : \neg f(\alpha) \leq_\alpha g(\alpha)\}| < \kappa. \tag{7}$$

Now from (6) and (7) we obtain (2).

Conversely, assuming (6) and (7) in order to prove $C_f \leq C_g$ we prove (5) first. Let $C_h \in \mathbb{P}(\mathbb{X})$ and $C_h \subset C_f$. Then, by (3),

$$\operatorname{dom} h \subset \operatorname{dom} f \;\wedge\; \forall \alpha \in \operatorname{dom} h \;\; h(\alpha) \subset f(\alpha), \tag{8}$$

which by (6) implies $|\operatorname{dom} h \setminus \operatorname{dom} g| < \kappa$ and, hence, $|\operatorname{dom} h \cap \operatorname{dom} g| = \kappa$. Since $\operatorname{dom} h \cap \operatorname{dom} g \subset \operatorname{dom} f \cap \operatorname{dom} g$ by (7) we have $|\{\alpha \in \operatorname{dom} h \cap \operatorname{dom} g : \neg f(\alpha) \leq_\alpha g(\alpha)\}| < \kappa$ and, hence, $B := \{\alpha \in \operatorname{dom} h \cap \operatorname{dom} g : f(\alpha) \leq_\alpha g(\alpha)\}$

is a set of size κ. By (8), for $\alpha \in B$ we have $h(\alpha) \subset f(\alpha) \leq_\alpha g(\alpha)$ which implies $h(\alpha) \not\perp g(\alpha)$. So $B \subset \{\alpha \in \operatorname{dom} h \cap \operatorname{dom} g : h(\alpha) \not\perp g(\alpha)\}$ and (5) is true. Now, by (5) and (4) we have $\forall C_h \in \mathbb{P}(\mathbb{X}) \;\; (C_h \subset C_f \Rightarrow C_h \not\perp C_g)$, that is $C_f \leq C_g$. Claim 2 is proved. $\Box$

Let A_1 and A_2 be disjoint elements of $[\kappa]^\kappa$. By Claim 1, $C_1 = \bigcup_{\alpha \in A_1} X_\alpha$ and $C_2 = \bigcup_{\alpha \in A_2} X_\alpha$ are disjoint elements of $\mathbb{P}(\mathbb{X})$ and, hence, they are incompatible in $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$. So, by Theorem 2.2(c) of [6], the poset $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ is atomless and, by Fact 2.4(b), the poset $\operatorname{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ is atomless too.

(c) Let $\kappa \geq \omega$ be a regular cardinal. By Fact 2.4(c), it is sufficient to prove that the pre-order $\operatorname{sm}\langle \mathbb{P}(\mathbb{X}), \leq \rangle$ is κ^+-closed. Let $\langle C_{f_\xi} : \xi < \kappa \rangle$ be a decreasing sequence in $\langle \mathbb{P}(\mathbb{X}), \leq \rangle$, that is

$$\forall \zeta_1, \zeta_2 < \kappa \;\; (\zeta_1 < \zeta_2 \Rightarrow C_{f_{\zeta_2}} \leq C_{f_{\zeta_1}}). \tag{9}$$

For $\zeta_1, \zeta_2 < \kappa$ let

$$K_{\zeta_2, \zeta_1} = \{\alpha \in \operatorname{dom} f_{\zeta_2} \cap \operatorname{dom} f_{\zeta_1} : \neg f_{\zeta_2}(\alpha) \leq_\alpha f_{\zeta_1}(\alpha)\}. \tag{10}$$

Then, by (9) and (c)

$$\forall \zeta_1, \zeta_2 < \kappa \;\; (\zeta_1 < \zeta_2 \Rightarrow |\operatorname{dom} f_{\zeta_2} \setminus \operatorname{dom} f_{\zeta_1}| < \kappa \wedge |K_{\zeta_2, \zeta_1}| < \kappa) \tag{11}$$

and we prove that

$$\forall \xi < \kappa \;\; |\textstyle\bigcap_{\zeta \leq \xi} \operatorname{dom} f_\zeta| = \kappa. \tag{12}$$

First $\bigcap_{\zeta \leq \xi} \operatorname{dom} f_\zeta = \bigcap_{\zeta < \xi} \operatorname{dom} f_\xi \cap \operatorname{dom} f_\zeta = \operatorname{dom} f_\xi \cap \bigcap_{\zeta < \xi} (\operatorname{dom} f_\xi^c \cup \operatorname{dom} f_\zeta)$ $= \operatorname{dom} f_\xi \setminus \bigcup_{\zeta < \xi} (\operatorname{dom} f_\xi \setminus \operatorname{dom} f_\zeta)$. By (11), $|\operatorname{dom} f_\xi \setminus \operatorname{dom} f_\zeta| < \kappa$, for all $\zeta < \xi$ and, since $|\xi| < \kappa$, by the regularity of κ we have $|\bigcup_{\zeta < \xi} (\operatorname{dom} f_\xi \setminus \operatorname{dom} f_\zeta)| < \kappa$ which, since by (a) we have $|\operatorname{dom} f_\xi| = \kappa$, implies (12).

By recursion we define a sequence $\langle \alpha_\xi : \xi < \kappa \rangle$ in κ as follows.

Let $\alpha_0 = \min \operatorname{dom} f_0$.

If $\xi < \kappa$ and $\alpha_\zeta \in \kappa$ are defined for $\zeta < \xi$, then for all $\zeta < \xi$ by (11) we have $|K_{\xi, \zeta}| < \kappa$ and, clearly, $|\alpha_\zeta + 1| < \kappa$ so, by (12) and the regularity of κ, we can define

$$\alpha_\xi = \min \Big[\Big(\textstyle\bigcap_{\zeta \leq \xi} \operatorname{dom} f_\zeta \Big) \setminus \Big(\bigcup_{\zeta < \xi} K_{\xi, \zeta} \cup \bigcup_{\zeta < \xi} (\alpha_\zeta + 1) \Big) \Big]. \tag{13}$$

By (13), $\langle \alpha_\xi : \xi < \kappa \rangle$ is an increasing sequence and, hence, $A := \{\alpha_\xi : \xi < \kappa\} \in [\kappa]^\kappa$. By (13) again, for $\xi < \kappa$ we have $\alpha_\xi \in \operatorname{dom} f_\xi$ so $f_\xi(\alpha_\xi) \in \mathbb{P}(\mathbb{X}_{\alpha_\xi})$. So, for $f \in \prod_{\alpha_\xi \in A} \mathbb{P}(\mathbb{X}_{\alpha_\xi})$, defined by $f(\alpha_\xi) = f_\xi(\alpha_\xi)$, for $\xi < \kappa$, by (a) we have $C_f \in \mathbb{P}(\mathbb{X})$.

It remains to be shown that for each $\xi_0 \in \kappa$ we have $C_f \leq C_{f_{\xi_0}}$, that is, by (c),

$$|A \setminus \operatorname{dom} f_{\xi_0}| < \kappa \quad \text{and} \tag{14}$$

$$|\{\xi < \kappa : \alpha_\xi \in \operatorname{dom} f_{\xi_0} \wedge \neg f_\xi(\alpha_\xi) \leq_{\alpha_\xi} f_{\xi_0}(\alpha_\xi)\}| < \kappa. \tag{15}$$

By (13), for each $\xi \geq \xi_0$ we have $\alpha_\xi \in \bigcap_{\zeta \leq \xi} \operatorname{dom} f_\zeta \subset \operatorname{dom} f_{\xi_0}$ and, hence, $A \setminus \operatorname{dom} f_{\xi_0} \subset \{\alpha_\xi : \xi < \xi_0\}$ and (14) is true.

For a proof of (15) it is sufficient to show that

$$\forall \xi > \xi_0 \;\; f_\xi(\alpha_\xi) \leq_{\alpha_\xi} f_{\xi_0}(\alpha_\xi). \tag{16}$$

By (13), for $\xi > \xi_0$ we have $\alpha_\xi \in \operatorname{dom} f_\xi \cap \operatorname{dom} f_{\xi_0}$ and $\alpha_\xi \not\in K_{\xi,\xi_0}$, that is $\alpha_\xi \not\in \{\alpha \in \operatorname{dom} f_\xi \cap \operatorname{dom} f_{\xi_0} : \neg f_\xi(\alpha) \leq_\alpha f_{\xi_0}(\alpha)\}$ thus $f_\xi(\alpha_\xi) \leq_{\alpha_\xi} f_{\xi_0}(\alpha_\xi)$ and (16) is true.

(d) Let $\kappa \geq \omega$ be a regular cardinal and $|\mathbb{P}(\mathbb{X}_\alpha)| \leq 2^\kappa = \kappa^+$, for all $\alpha < \kappa$. Then for $A \in [\kappa]^\kappa$ we have $|\prod_{\alpha \in A} \mathbb{P}(\mathbb{X}_\alpha)| \leq (2^\kappa)^\kappa = 2^\kappa = \kappa^+$ and, by Claim 1, $|\mathbb{P}(\mathbb{X})| \leq |\bigcup_{A \in [\kappa]^\kappa} \prod_{\alpha \in A} \mathbb{P}(\mathbb{X}_\alpha)| \leq 2^\kappa 2^\kappa = 2^\kappa = \kappa^+$, which implies $|\operatorname{sq} \mathbb{P}(\mathbb{X})| \leq \kappa^+$. By (b) and (c) $\operatorname{sq} \mathbb{P}(\mathbb{X})$ is an atomless κ^+-closed poset and, hence, it contains a copy of the reversed tree $\langle 2^{\leq \kappa}, \supset \rangle$ thus $|\operatorname{sq} \mathbb{P}(\mathbb{X})| = \kappa^+$. (Another way to prove this is to use an almost disjoint family $\mathcal{A} \subset [\kappa]^\kappa$ of size κ^+; then $\{\bigcup_{\alpha \in A} X_\alpha : A \in \mathcal{A}\} \subset \mathbb{P}(\mathbb{X})$ determines an antichain in $\operatorname{sq} \mathbb{P}(\mathbb{X})$ of size κ^+.) Since $(\kappa^+)^{<\kappa^+} = (2^\kappa)^\kappa = \kappa^+$, by Fact 2.3(b) the poset $\operatorname{sq} \mathbb{P}(\mathbb{X})$ is forcing equivalent to the poset $(P(\kappa)/[\kappa]^{<\kappa})^+$ (since it is an atomless separative κ^+-closed poset of size κ^+). By Fact 2.4(a), the same holds for $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$. □

Corrolary 3.3 If $\kappa \leq \omega$ and $\mathbb{X} = \bigcup_{n < \kappa} \mathbb{X}_n$ is the union of disjoint, isomorphic and strongly connected binary structures, then

(a) $\langle \mathbb{P}(\mathbb{X}), \subset \rangle \cong \langle \mathbb{P}(\mathbb{X}_0), \subset \rangle^\kappa$ and $\operatorname{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle \cong (\operatorname{sq}\langle \mathbb{P}(\mathbb{X}_0), \subset \rangle)^\kappa$, if $\kappa < \omega$;

(b) If $\kappa = \omega$, then $\operatorname{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ is a separative atomless and ω_1-closed poset. Under CH it is forcing equivalent to the poset $(P(\omega)/\operatorname{Fin})^+$.

The following examples show that for infinite cardinals κ the statements of Theorem 3.2 are the best possible.

Example 3.4 The posets $\operatorname{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ and $(P(\kappa)/[\kappa]^{<\kappa})^+$ are not forcing equivalent, although $\kappa \geq \omega$ is regular and $|\mathbb{P}(\mathbb{X}_\alpha)| \leq 2^\kappa$.

Let $\mathbb{X} = \bigcup_{i < \omega} \mathbb{X}_i$ be the union of countably many copies $\mathbb{X}_i = \langle X_i, <_i \rangle$ of the linear order $\langle \omega, < \rangle$. Then, since linear orders are strongly connected, by Theorem 3.2 the poset $\operatorname{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ is atomless, ω_1-closed and, clearly, of size 2^ω. If, in addition $2^\omega = \omega_1$, then $\operatorname{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ is forcing equivalent to the poset $(P(\omega)/\operatorname{Fin})^+$.

Since, in addition, the components of $\mathbb{X}$ are maximally embeddable (which means that $\mathbb{P}(\mathbb{X}_i, \mathbb{X}_j) = [\mathbb{X}_j]^{|\mathbb{X}_i|}$, for $i, j \in \omega$), by the results of [7] the poset $\mathrm{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ is isomorphic to the poset $(P(\omega \times \omega)/(\mathrm{Fin} \times \mathrm{Fin}))^+$, which is not ω_2-closed [16] and, consistently, neither $\mathfrak{t}$-closed nor $\mathfrak{h}$-distributive [5]. Thus in some models of ZFC the posets $\mathrm{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ and $(P(\omega)/\,\mathrm{Fin})^+$ are not forcing equivalent.

Example 3.5 In some models of ZFC the poset $\mathrm{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ is not κ^{++} closed, although the posets $\mathrm{sq}\langle [\kappa]^\kappa, \subset \rangle$ and $\mathrm{sq}\langle \mathbb{P}(\mathbb{X}_\alpha), \subset \rangle$, $\alpha < \kappa$ are (take $\kappa = \omega$, a model satisfying $\mathfrak{t} > \omega_1$ and $\mathbb{X}$ from Example 3.4).

Example 3.6 Statement (c) of Theorem 3.2 is not true for a singular κ. It is known that the algebra $P(\kappa)/[\kappa]^{<\kappa}$ is not ω_1-distributive and, hence, the poset $(P(\kappa)/[\kappa]^{<\kappa})^+$ is not ω_2-closed, whenever κ is a cardinal satisfying $\kappa > \mathrm{cf}(\kappa) = \omega$ (see [1], p. 377). For $\alpha < \kappa$ let $\mathbb{X}_\alpha = \langle \{\alpha\}, \emptyset \rangle$ and let $\mathbb{X} = \bigcup_{\alpha<\kappa} \mathbb{X}_\alpha$. Then it is easy to see that $\mathbb{P}(\mathbb{X}) = [\kappa]^\kappa$ and $\mathrm{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle = (P(\kappa)/[\kappa]^{<\kappa})^+$. Thus the poset $\mathrm{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ is not ω_2-closed and, since $\kappa \geq \aleph_\omega$, it is not κ^+-closed.

4 Non biconnected ultrahomogeneous structures

A binary structure $\mathbb{X} = \langle X, \rho \rangle$ is a *directed graph (digraph)* iff for each $x, y \in X$ we have $\neg x \rho x$ (ρ is irreflexive) and $\neg x\rho y \vee \neg y \rho x$ (ρ is asymmetric). If, in addition, $x\rho y \vee y \rho x$, for each different $x, y \in X$, then $\mathbb{X}$ is a *tournament*. For convenience we introduce the following notation. If $\mathbb{X} = \langle X, \rho \rangle$ is a binary structure, then its *complement*, $\langle X, \rho^c \rangle$, where $\rho^c = X^2 \setminus \rho$, will be denoted by $\mathbb{X}^c$, its *inverse*, $\langle X, \rho^{-1} \rangle$, by $\mathbb{X}^{-1}$, its *reflexification*, $\langle X, \rho \cup \Delta_X \rangle$, by $\mathbb{X}_{re}$ and its *irreflexification*, $\langle X, \rho \setminus \Delta_X \rangle$, by $\mathbb{X}_{ir}$. The binary relation ρ_e on X defined by

$$x \rho_e y \;\Leftrightarrow\; x\rho y \vee (x \neq y \wedge \neg x \rho y \wedge \neg y \rho x) \tag{17}$$

will be called the *enlargement* of ρ and the corresponding structure, $\langle X, \rho_e \rangle$, will be denoted by $\mathbb{X}_e$. A structure $\mathbb{X}$ will be called *biconnected* iff both $\mathbb{X}$ and $\mathbb{X}^c$ are connected structures. The following theorem is the main result of this section.

Theorem 4.1 For each reflexive or irreflexive ultrahomogeneous binary structure $\mathbb{X}$ we have

- Either $\mathbb{X}$ is biconnected,

- Or there are an ultrahomogeneous digraph $\mathbb{Y}$ and a cardinal $\kappa > 1$ such that the structure $\mathbb{X}$ is isomorphic to $\bigcup_\kappa \mathbb{Y}_e$, $(\bigcup_\kappa \mathbb{Y}_e)_{re}$, $(\bigcup_\kappa \mathbb{Y}_e)^c$ or $((\bigcup_\kappa \mathbb{Y}_e)_{re})^c$.

A proof of Theorem 4.1 is given at the end of this section. It is based on the following statement concerning irreflexive structures.

Theorem 4.2 An irreflexive disconnected binary structure is ultrahomogeneous iff its components are isomorphic to the enlargement of an ultrahomogeneous digraph.

Theorem 4.2 follows from two lemmas given in the sequel. A binary structure $\mathbb{X} = \langle X, \rho \rangle$ is called *complete* (see [4], p. 393) iff

$$\forall x, y \ \ (x \neq y \Rightarrow x\rho y \vee y\rho x). \tag{18}$$

Lemma 4.3 An irreflexive disconnected binary structure $\mathbb{X}$ is ultrahomogeneous iff its components are isomorphic, ultrahomogeneous and complete.

Proof. Let $\mathbb{X} = \langle X, \rho \rangle = \bigcup_{i \in I} \mathbb{X}_i$, where $\mathbb{X}_i = \langle X_i, \rho_i \rangle$, $i \in I$, are disjoint, irreflexive and connected binary structures and $|I| > 1$.

($\Rightarrow$) Suppose that $\mathbb{X}$ is ultrahomogeneous. Then, for $i, j \in I$, $x \in X_i$ and $y \in X_j$ we have $\varphi = \{\langle x, y \rangle\} \in \mathrm{Pi}(\mathbb{X})$ and there is $f \in \mathrm{Aut}(\mathbb{X})$ such that $\varphi \subset f$. By (c) and (b) of Fact 2.1, $f|X_i : \mathbb{X}_i \to \mathbb{X}_j$ is an isomorphism. Thus $\mathbb{X}_i \cong \mathbb{X}_j$.

For $i \in I$ and $\varphi \in \mathrm{Pi}(\mathbb{X}_i)$ we have $\varphi \in \mathrm{Pi}(\mathbb{X})$ and there is $f \in \mathrm{Aut}(\mathbb{X})$ such that $\varphi \subset f$. Again, by (c) and (b) of Fact 2.1, $f|X_i : \mathbb{X}_i \to \mathbb{X}_i$ is an isomorphism, that is $f|X_i \in \mathrm{Aut}(\mathbb{X}_i)$. Thus the structure $\mathbb{X}_i$ is ultrahomogeneous.

Suppose that for some $i \in I$ there are different elements x and y of X_i satisfying $\neg x\rho y$ and $\neg y\rho x$. Let $j \in I \setminus \{i\}$ and $z \in X_j$. Then $\varphi = \{\langle x, x \rangle, \langle y, z \rangle\} \in \mathrm{Pi}(\mathbb{X})$ and there is $f \in \mathrm{Aut}(\mathbb{X})$ such that $\varphi \subset f$. But then, by Fact 2.1(c) we would have both $f[X_i] = X_i$ and $f[X_i] = X_j$, which is, clearly, impossible. Thus the structures $\mathbb{X}_i$ are complete.

($\Leftarrow$) Suppose that the components $\mathbb{X}_i$, $i \in I$, of $\mathbb{X}$ are ultrahomogeneous, isomorphic and complete. Let $\varphi \in \mathrm{Pi}(\mathbb{X})$, where $\mathrm{dom}\, \varphi = Y$ and $\varphi[Y] = Z$, let $J = \{i \in I : Y \cap X_i \neq \emptyset\}$ and, for $j \in J$, let $Y_i = Y \cap X_i$ and $Z_i = \varphi[Y_i]$. By (18), the structures $\mathbb{Y}_i = \langle Y_i, \rho_{Y_i} \rangle = \langle Y_i, (\rho_i)_{Y_i} \rangle$, $i \in J$, are connected and, clearly, disjoint, thus $\mathbb{Y} = \bigcup_{i \in J} \mathbb{Y}_i$ and $\mathbb{Y}_i$, $i \in J$, are the components of $\mathbb{Y}$. Since the restrictions $\varphi|Y_i : Y_i \to Z_i$ are isomorphisms, the structures $\mathbb{Z}_i = \langle Z_i, \rho_{Z_i} \rangle$, $i \in J$, are connected too and, since φ is a bijection, disjoint. Thus $\mathbb{Z} = \bigcup_{i \in J} \mathbb{Z}_i$ and $\mathbb{Z}_i$, $i \in J$, are the components of $\mathbb{Z}$.

Since $\varphi : \mathbb{Y} \hookrightarrow \mathbb{X}$, by Fact 2.1(a) for each $i \in J$ there is $k_i \in I$ such that $Z_i \subset X_{k_i}$. Suppose that $k_i = k_j = k$, for some different $i, j \in J$. Then, for $x \in Y_i$ and $y \in Y_j$ we would have $\neg x\rho y$ and $\neg y\rho x$ and, hence, $\neg\varphi(x)\rho\varphi(y)$ and $\neg\varphi(y)\rho\varphi(x)$, which is impossible since $\varphi(y), \varphi(x) \in X_k$ and $\mathbb{X}_k$ satisfies (18).

Thus the mapping $i \mapsto k_i$ is a bijection and there is a bijection $f : I \to I$ such that $f(i) = k_i$, for all $i \in J$. Since the structures $\mathbb{X}_i$ are isomorphic, for each $i \in I$ there is an isomorphism $g_i : \mathbb{X}_i \to \mathbb{X}_{f(i)}$.

For $i \in J$ we have $g_i^{-1} \circ (\varphi|Y_i) : \mathbb{Y}_i \hookrightarrow \mathbb{X}_i$ and, hence, $g_i^{-1} \circ (\varphi|Y_i) \in \mathrm{Pi}(\mathbb{X}_i)$. So, since the structure $\mathbb{X}_i$ is ultrahomogeneous, there is $h_i \in \mathrm{Aut}(\mathbb{X}_i)$ such that $g_i^{-1} \circ (\varphi|Y_i) \subset h_i$. Now $g_i \circ h_i : \mathbb{X}_i \to \mathbb{X}_{f(i)}$ is an isomorphism and for $x \in Y_i$ we have $g_i(h_i(x)) = g_i(g^{-1}(\varphi(x))) = \varphi(x)$, which implies

$$(g_i \circ h_i)|Y_i = \varphi|Y_i. \tag{19}$$

Now it is easy to check that $F = \bigcup_{i \in I \setminus J} g_i \cup \bigcup_{i \in J} g_i \circ h_i : \mathbb{X} \to \mathbb{X}$ is an automorphism of $\mathbb{X}$ and, by (19), $\varphi \subset F$. Thus $\mathbb{X}$ is an ultrahomogeneous structure. $\Box$

In the sequel we will use the following elementary fact.

Fact 4.4 Let $\mathbb{X} = \langle X, \rho \rangle$ be a binary structure. Then

(a) $\mathrm{Pi}(\mathbb{X}) = \mathrm{Pi}(\mathbb{X}^c) = \mathrm{Pi}(\mathbb{X}^{-1})$ and $\mathrm{Aut}(\mathbb{X}) = \mathrm{Aut}(\mathbb{X}^c) = \mathrm{Aut}(\mathbb{X}^{-1})$; hence $\mathbb{X}$ is ultrahomogeneous iff $\mathbb{X}^c$ is ultrahomogeneous iff $\mathbb{X}^{-1}$ is ultrahomogeneous. Also $\mathrm{Emb}(\mathbb{X}) = \mathrm{Emb}(\mathbb{X}^c) = \mathrm{Emb}(\mathbb{X}^{-1})$; hence $\mathbb{P}(\mathbb{X}) = \mathbb{P}(\mathbb{X}^c) = \mathbb{P}(\mathbb{X}^{-1})$.

(b) If ρ is an irreflexive relation, then $\mathrm{Pi}(\mathbb{X}) = \mathrm{Pi}(\mathbb{X}_{re})$, $\mathrm{Aut}(\mathbb{X}) = \mathrm{Aut}(\mathbb{X}_{re})$ and, hence, $\mathbb{X}$ is ultrahomogeneous iff $\mathbb{X}_{re}$ is ultrahomogeneous. Also $\mathrm{Emb}(\mathbb{X}) = \mathrm{Emb}(\mathbb{X}_{re})$; hence $\mathbb{P}(\mathbb{X}) = \mathbb{P}(\mathbb{X}_{re})$.

(c) If ρ is a reflexive relation, then $\mathrm{Pi}(\mathbb{X}) = \mathrm{Pi}(\mathbb{X}_{ir})$, $\mathrm{Aut}(\mathbb{X}) = \mathrm{Aut}(\mathbb{X}_{ir})$ and, hence, $\mathbb{X}$ is ultrahomogeneous iff $\mathbb{X}_{ir}$ is ultrahomogeneous. Also $\mathrm{Emb}(\mathbb{X}) = \mathrm{Emb}(\mathbb{X}_{ir})$; hence $\mathbb{P}(\mathbb{X}) = \mathbb{P}(\mathbb{X}_{ir})$.

(d) If $\mathbb{X}$ is a digraph, then $\mathbb{X}_e = ((\mathbb{X}^{-1})_{re})^c$. So $\mathrm{Pi}(\mathbb{X}) = \mathrm{Pi}(\mathbb{X}_e)$, $\mathrm{Aut}(\mathbb{X}) = \mathrm{Aut}(\mathbb{X}_e)$, $\mathrm{Emb}(\mathbb{X}) = \mathrm{Emb}(\mathbb{X}_e)$ and $\mathbb{P}(\mathbb{X}) = \mathbb{P}(\mathbb{X}_e)$. Hence $\mathbb{X}$ is ultrahomogeneous iff $\mathbb{X}_e$ is.

Proof. The proofs of (a), (b) and (c) are straightforward and we prove (d). For $x, y \in X$ we have: $\langle x, y \rangle \in ((\rho^{-1})_{re})^c$ iff $\langle x, y \rangle \not\in \Delta_X \cup \rho^{-1}$ iff $x \neq y \wedge \langle y, x \rangle \not\in \rho$ iff $x \neq y \wedge \neg y\rho x \wedge (x\rho y \vee \neg x\rho y)$ iff $(x \neq y \wedge \neg y\rho x \wedge x\rho y) \vee (x \neq y \wedge \neg y\rho x \wedge \neg x\rho y)$. Since the relation ρ is irreflexive and asymmetric we have $x \neq y \wedge \neg y\rho x \wedge x\rho y$ iff $x\rho y$; thus $\langle x, y \rangle \in ((\rho^{-1})_{re})^c$ iff $x\rho y \vee (x \neq y \wedge \neg y\rho x \wedge \neg x\rho y)$ iff $\langle x, y \rangle \in \rho_e$ and the equality $\mathbb{X}_e = ((\mathbb{X}^{-1})_{re})^c$ is proved. Now applying (a) and (b) we obtain the remaining equalities. Let $\mathbb{X}$ be ultrahomogeneous and $\varphi \in \mathrm{Pi}(\mathbb{X}_e)$. Then $\varphi \in \mathrm{Pi}(\mathbb{X})$ and, hence, there is $f \in \mathrm{Aut}(\mathbb{X})$ such that $\varphi \subset f$ and, since $f \in \mathrm{Aut}(\mathbb{X}_e)$, we proved that the structure $\mathbb{X}_e$ is ultrahomogeneous. The converse has a similar proof. $\Box$

Lemma 4.5 An irreflexive binary structure $\mathbb{X}$ is ultrahomogeneous and complete iff it is isomorphic to the enlargement of an ultrahomogeneous digraph.

Proof. Let $\mathbb{X} = \langle X, \rho \rangle$ be an irreflexive binary structure.

(⇒) Assuming that $\mathbb{X}$ is ultrahomogeneous and complete we define the binary relation $\to$ on X by

$$x \to y \Leftrightarrow x\rho y \wedge \neg y\rho x. \tag{20}$$

Claim 1. For the structure $\mathbb{Y} := \langle X, \to\rangle$ we have:

(a) $\mathrm{Pi}(\mathbb{X}) = \mathrm{Pi}(\mathbb{Y})$, $\mathrm{Aut}(\mathbb{X}) = \mathrm{Aut}(\mathbb{Y})$ and $\mathrm{Emb}(\mathbb{X}) = \mathrm{Emb}(\mathbb{Y})$;

(b) $\mathbb{Y}$ is an ultrahomogeneous digraph;

(c) $\mathbb{P}(\mathbb{X}) = \mathbb{P}(\mathbb{Y})$;

(d) $\mathbb{X} = \mathbb{Y}_e$, that is, $\rho =\to_e$.

Proof of Claim 1. (a) It is sufficient to prove that for each $A \subset X$ and each injection $f : A \to X$ the following two conditions are equivalent:

$$\forall x, y \in A \ \ (x\rho y \Leftrightarrow f(x)\rho f(y)), \tag{21}$$

$$\forall x, y \in A \ \ (x \to y \Leftrightarrow f(x) \to f(y)). \tag{22}$$

Suppose that (21) holds. For $x, y \in A$, condition $x \to y$, that is $x\rho y \wedge \neg y\rho x$, is, by (21), equivalent to $f(x)\rho f(y) \wedge \neg f(y)\rho f(x)$, that is $f(x) \to f(y)$; so (22) is true.

Let (22) hold and $x, y \in A$. If $x = y$, then (21) follows from the irreflexivity of ρ. Otherwise, we have $f(x) \neq f(y)$.

Now, if $\neg f(x)\rho f(y)$, then, by (18), $f(y)\rho f(x)$ and, hence, $f(y) \to f(x)$, which by (22) implies $y \to x$ and, hence, $\neg x\rho y$. Thus $x\rho y \Rightarrow f(x)\rho f(y)$.

If $\neg x\rho y$, then by (18) we have $y\rho x$ and, hence, $y \to x$, which by (22) implies $f(y) \to f(x)$ and, hence, $\neg f(x)\rho f(y)$. Thus $f(x)\rho f(y) \Rightarrow x\rho y$ and (21) is true.

(b) If $\varphi \in \mathrm{Pi}(\mathbb{Y})$, then, by (a), $\varphi \in \mathrm{Pi}(\mathbb{X})$ and, since $\mathbb{X}$ is ultrahomogeneous, there is $f \in \mathrm{Aut}(\mathbb{X})$ such that $\varphi \subset f$. By (a) again we have $f \in \mathrm{Aut}(\mathbb{Y})$ and, thus, $\mathbb{Y}$ is an ultrahomogeneous structure. Since the relation ρ is irreflexive, $\to$ is irreflexive too and $x \to y \wedge y \to x$ would imply $x\rho y$ and $\neg x\rho y$; thus, $\to$ is an asymmetric relation and $\mathbb{Y}$ is a digraph.

(c) By (a), $\mathbb{P}(\mathbb{X}) = \{f[X] : f \in \mathrm{Emb}(\mathbb{X})\} = \{f[X] : f \in \mathrm{Emb}(\mathbb{Y})\} = \mathbb{P}(\mathbb{Y})$.

(d) We prove that for each $x, y \in X$ we have $x\rho y \Leftrightarrow x \to_e y$, that is,

$$x\rho y \Leftrightarrow x \to y \vee (x \neq y \wedge \neg x \to y \wedge \neg y \to x). \tag{23}$$

Let $x\rho y$. If $\neg y\rho x$, then $x \to y$ and, hence, $x \to_e y$. If $y\rho x$, then, since ρ is irreflexive, $x \neq y$. Also $\neg x \to y$ and $\neg y \to x$ thus $x \to_e y$ again.

Let $x \to_e y$. If $x \to y$, then $x\rho y$ and we are done. If $\neg x \to y$, then, by the assumption, $x \neq y$ and $\neg y \to x$. By (18), $\neg x\rho y$ would imply $y\rho x$ and, hence, $y \to x$, which is not true. Thus $x\rho y$ and Claim 1 is proved. □

(⇐) W.l.o.g. suppose that $\mathbb{Y} = \langle X, \to\rangle$ is an ultrahomogeneous digraph and $\mathbb{X} = \mathbb{Y}_e$ that is $\rho =\to_e$. Then for each $x, y \in X$ we have

$$x\rho y \Leftrightarrow x \to y \vee (x \neq y \wedge \neg x \to y \wedge \neg y \to x). \tag{24}$$

For a proof that $\mathbb{X}$ is complete we take different $x, y \in X$ and show that $x\rho y$ or $y\rho x$. By (24), if $x \to y$ or $y \to x$, then $x\rho y$ or $y\rho x$ and we are done. Otherwise we have $x \neq y \wedge \neg x \to y \wedge \neg y \to x$ and by (24) again we obtain $x\rho y$.

Since $\mathbb{Y}$ is an ultrahomogeneous digraph, by Fact 4.4(d) the structure $\mathbb{X}$ is ultrahomogeneous as well. $\square$

Proof of Theorem 4.1. Let $\mathbb{X}$ be an ultrahomogeneous structure and first suppose that $\mathbb{X}$ is disconnected. If $\mathbb{X}$ is irreflexive, then, by Theorem 4.2, $\mathbb{X} \cong \bigcup_\kappa \mathbb{Y}_e$, for some ultrahomogeneous digraph $\mathbb{Y}$ and some $\kappa > 1$. If $\mathbb{X}$ is reflexive, then $\mathbb{X}_{ir}$ is disconnected, irreflexive and, by Fact 4.4(c), ultrahomogeneous so, by Theorem 4.2, $\mathbb{X}_{ir} \cong \bigcup_\kappa \mathbb{Y}_e$, which implies $\mathbb{X} \cong (\bigcup_\kappa \mathbb{Y}_e)_{re}$. Now, suppose that $\mathbb{X}^c$ is disconnected. By Fact 4.4(a), $\mathbb{X}^c$ is ultrahomogeneous. If $\mathbb{X}^c$ is irreflexive, by Theorem 4.2 we have $\mathbb{X}^c \cong \bigcup_\kappa \mathbb{Y}_e$, which implies $\mathbb{X} \cong (\bigcup_\kappa \mathbb{Y}_e)^c$. Finally, If $\mathbb{X}^c$ is reflexive, then $\mathbb{X}^c_{ir}$ is disconnected, irreflexive and, by Fact 4.4(c), ultrahomogeneous. So, by Theorem 4.2 again, $\mathbb{X}^c_{ir} \cong \bigcup_\kappa \mathbb{Y}_e$ which implies $\mathbb{X}^c \cong (\bigcup_\kappa \mathbb{Y}_e)_{re}$ and $\mathbb{X} \cong ((\bigcup_\kappa \mathbb{Y}_e)_{re})^c$. $\square$

5 Posets of copies of ultrahomogeneous structures

In this section we show that a classification of biconnected ultrahomogeneous digraphs, related to the properties of their posets of copies, provides the corresponding classification inside a much wider class of structures.

Theorem 5.1 Let $\mathbb{X}$ be a reflexive or irreflexive ultrahomogeneous non biconnected binary structure and let $\mathbb{Y}$ and κ be the corresponding ultrahomogeneous digraph and the cardinal from Theorem 4.1. Then

(a) $\langle \mathbb{P}(\mathbb{X}), \subset \rangle \cong \langle \mathbb{P}(\mathbb{Y}), \subset \rangle^\kappa$ and $\mathrm{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle \cong (\mathrm{sq}\langle \mathbb{P}(\mathbb{Y}), \subset \rangle)^\kappa$, if $\kappa < \omega$;

(b) $\mathrm{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ is atomless, if $\kappa \geq \omega$;

(c) $\mathrm{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ is κ^+-closed, if $\kappa \geq \omega$ is regular;

(d) $\mathrm{sq}\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ is forcing equivalent to the poset $(P(\kappa)/[\kappa]^{<\kappa})^+$, if $\kappa \geq \omega$ is regular and $|\mathbb{P}(\mathbb{Y})| \leq 2^\kappa = \kappa^+$. The same holds for $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$.

Proof. By Theorem 4.1, the structure $\mathbb{X}$ is isomorphic to $\bigcup_\kappa \mathbb{Y}_e$, $(\bigcup_\kappa \mathbb{Y}_e)_{re}$, $(\bigcup_\kappa \mathbb{Y}_e)^c$ or $((\bigcup_\kappa \mathbb{Y}_e)_{re})^c$ so, by Fact 4.4, $\mathbb{P}(\mathbb{X}) \cong \mathbb{P}(\bigcup_\kappa \mathbb{Y}_e)$. Since the structure $\mathbb{Y}_e$ is complete it is strongly connected and the statement follows from Theorem 3.2. The equality $\mathbb{P}(\mathbb{Y}_e) = \mathbb{P}(\mathbb{Y})$ is proved in Fact 4.4(d). $\square$

Theorem 5.2 Let $\mathbb{X}$ be a countable reflexive or irreflexive ultrahomogeneous binary structure. If $\mathbb{X}$ is not biconnected and $\mathbb{Y}$ and κ are the corresponding objects from Theorem 4.1, then

(i) $\mathbb{P}(\mathbb{X}) \cong \mathbb{P}(\mathbb{Z})^n$, for some biconnected ultrahomogeneous digraph $\mathbb{Z}$ and some $n \geq 2$, if $\kappa < \omega$ and $\mathbb{Y}$ has finitely many components;

(ii) $\mathrm{sq}\,\mathbb{P}(\mathbb{X})$ is an atomless and ω_1-closed poset and, under CH, forcing equivalent to the poset $(P(\omega)/\,\mathrm{Fin})^+$, if $\kappa = \omega$ or $\mathbb{Y}$ has infinitely many components.

Proof. By Theorem 4.1, $\mathbb{X}$ is isomorphic to $\bigcup_\kappa \mathbb{Y}_e$, $(\bigcup_\kappa \mathbb{Y}_e)_{re}$, $(\bigcup_\kappa \mathbb{Y}_e)^c$ or to $((\bigcup_\kappa \mathbb{Y}_e)_{re})^c$, where $\mathbb{Y}$ is an ultrahomogeneous digraph and $2 \leq \kappa \leq \omega$. So, by Fact 4.4, $\mathbb{P}(\mathbb{X}) \cong \mathbb{P}(\bigcup_\kappa \mathbb{Y}_e)$.

If $\kappa = \omega$, then (ii) follows from (b), (c) and (d) of Theorem 5.1.

If $\kappa = n < \omega$, then, by Theorem 3.2 and Fact 4.4(d), $\mathbb{P}(\mathbb{X}) \cong \mathbb{P}(\mathbb{Y}_e)^n \cong \mathbb{P}(\mathbb{Y})^n$. We have two cases.

Case 1: $\mathbb{Y}$ is connected. Then, since $\mathbb{Y}$ is a digraph, $\mathbb{Y}^c$ is a complete and, hence, a connected structure. So $\mathbb{Y}$ is biconnected and we have (i).

Case 2: $\mathbb{Y}$ is disconnected. Then, if $\mathbb{Y}$ has finitely many components, say $\mathbb{Y} = \bigcup_{i<m} \mathbb{Y}_i$, by Lemma 4.3 the structures $\mathbb{Y}_i$ are isomorphic and complete and, hence strongly connected; so by Theorem 3.2(a), $\mathbb{P}(\mathbb{Y}) \cong \mathbb{P}(\mathbb{Y}_0)^m$, which implies $\mathbb{P}(\mathbb{X}) \cong \mathbb{P}(\mathbb{Y})^n \cong \mathbb{P}(\mathbb{Y}_0)^{mn}$. Since $\mathbb{Y}_0$ is a digraph and a complete structure it is a tournament and, hence, a biconnected structure. So we have (i).

If $\mathbb{Y}$ has infinitely many components, say $\mathbb{Y} = \bigcup_{i<\omega} \mathbb{Y}_i$, then, by Lemma 4.3 the structures $\mathbb{Y}_i$ are isomorphic and complete and, hence, strongly connected. So by Theorem 3.2, the poset $\mathrm{sq}\,\mathbb{P}(\mathbb{Y})$ is atomless and ω_1-closed. Since $\mathbb{P}(\mathbb{X}) \cong \mathbb{P}(\mathbb{Y})^n$, by Fact 2.4(e) we have $\mathrm{sq}\,\mathbb{P}(\mathbb{X}) \cong (\mathrm{sq}\,\mathbb{P}(\mathbb{Y}))^n$ and, by Fact 2.3(a), the poset $\mathrm{sq}\,\mathbb{P}(\mathbb{X})$ is atomless and ω_1-closed. So we have (ii). □

The countable ultrahomogeneous digraphs have been classified by Cherlin [2, 3], see also [13]. Cherlin's list includes Schmerl's list of countable ultrahomogeneous strict partial orders [14]:

- $\mathbb{A}_\omega$, a countable antichain (that is, the empty relation on ω),
- $\mathbb{B}_n = n \times \mathbb{Q}$, for $n \in [1, \omega]$, where $\langle i_1, q_1 \rangle < \langle i_2, q_2 \rangle \Leftrightarrow i_1 = i_2 \wedge q_1 <_\mathbb{Q} q_2$,
- $\mathbb{C}_n = n \times \mathbb{Q}$, for $n \in [1, \omega]$, where $\langle i_1, q_1 \rangle < \langle i_2, q_2 \rangle \Leftrightarrow q_1 <_\mathbb{Q} q_2$,
- $\mathbb{D}$, the unique countable homogeneous universal poset (the random poset),

and Lachlan's list of ultrahomogeneous tournaments [11]:

- $\mathbb{Q}$, the rational line,
- $\mathbb{T}^\infty$, the countable universal ultrahomogeneous tournament,
- $S(2)$, the circular tournament (the local order),

and many other digraphs. Also we recall the classification of countable ultrahomogeneous graphs given by Lachlan and Woodrow [12]:

- $\mathbb{G}_{\mu,\nu}$, the union of μ disjoint copies of $\mathbb{K}_\nu$, where $\mu\nu = \omega$,
- $\mathbb{G}_{\mathrm{Rado}}$, the unique countable homogeneous universal graph, the Rado graph,
- $\mathbb{H}_n$, the unique countable homogeneous universal $\mathbb{K}_n$-free graph, for $n \geq 3$,
- the complements of these graphs.

Example 5.3 By the main result of [10], for the rational line, $\mathbb{Q}$, the poset of copies $\langle \mathbb{P}(\mathbb{Q}), \subset \rangle$ is forcing equivalent to the two-step iteration $\mathbb{S} * \pi$, where $\mathbb{S}$ is the Sacks forcing and $1_{\mathbb{S}} \Vdash$ "π is a σ-closed forcing". If the equality $\mathrm{sh}(\mathbb{S}) = \aleph_1$ (implied by CH) or PFA holds in the ground model, then in the Sacks extension the second iterand is forcing equivalent to the poset $(P(\omega)/\,\mathrm{Fin})^+$.

The posets $\mathbb{B}_n$, $n \in [2, \omega]$, from the Schmerl list are disconnected ultrahomogeneous digraphs (they are disjoint unions of copies of $\mathbb{Q}$) and, by Theorem 4.2, the structures of the form $\bigcup_\kappa (\mathbb{B}_n)_e$ (or its other three variations given in Theorem 4.2) are ultrahomogeneous structures. For example, by Theorem 5.2 we have:

$\mathbb{P}(\bigcup_3 (\mathbb{B}_2)_e) \cong \mathbb{P}(\mathbb{Q})^6 \equiv_{forc} (\mathbb{S} * \pi)^6$;

$\mathbb{P}((\bigcup_\omega (\mathbb{B}_2)_e)^c)$ and $\mathbb{P}(((\bigcup_2 (\mathbb{B}_\omega)_e)_{re})^c)$ are atomless ω_1-closed posets, which are forcing equivalent to the poset $(P(\omega)/\,\mathrm{Fin})^+$ under CH.

Example 5.4 For a cardinal ν, the empty structure of size ν, $\mathbb{A}_\nu = \langle \nu, \emptyset \rangle$, can be regarded as an (empty) digraph with ν components. Then $(\mathbb{A}_\nu)_e \cong \mathbb{K}_\nu$ and for the graphs $\mathbb{G}_{\mu,\nu}$ from the Lachlan and Woodrow list we have $\mathbb{G}_{\mu,\nu} = \bigcup_\mu (\mathbb{A}_\nu)_e$. So, for $n \in \mathbb{N}$, by Theorem 5.2, $\mathbb{P}(\mathbb{G}_{\omega,n})$, $\mathbb{P}(\mathbb{G}_{n,\omega})$ and $\mathbb{P}(\mathbb{G}_{\omega,\omega})$ are atomless ω_1-closed posets, which are forcing equivalent to the poset $(P(\omega)/\,\mathrm{Fin})^+$ under CH. But, by [7] these posets are forcing equivalent to the posets $(P(\omega)/\,\mathrm{Fin})^+$, $((P(\omega)/\,\mathrm{Fin})^+)^n$ and $(P(\omega \times \omega)/(\mathrm{Fin} \times \mathrm{Fin}))^+$ respectively and in some models of ZFC the last two of them are not forcing equivalent to the poset $(P(\omega)/\,\mathrm{Fin})^+$. For the first one see [15] and for the second see Example 3.4.

Let $\mathcal{U}$ denote the class of all countable reflexive or irreflexive ultrahomogeneous binary structures and let

$\mathcal{B} = \{\mathbb{X} \in \mathcal{U} : \mathbb{X} \text{ is biconnected}\}$,

$\mathcal{D} = \{\mathbb{X} \in \mathcal{U} : \mathbb{X} \text{ is a digraph}\}$,

$\mathcal{D}_e = \{\mathbb{X}_e : \mathbb{X} \in \mathcal{D}\}$,

$\mathcal{G} = \{\mathbb{X} \in \mathcal{U} : \mathbb{X} \text{ is a graph}\}$,

$\mathcal{T} = \{\mathbb{X} \in \mathcal{U} : \mathbb{X} \text{ is a tournament}\}$.

By Lemma 5.5, the relations between these classes are displayed in Figure 1.

Lemma 5.5 Let $\mathbb{Y} \in \mathcal{D}$. Then

(a) $\mathbb{Y} \in \mathcal{B}$ iff $\mathbb{Y}$ is connected iff $\mathbb{Y}_e \in \mathcal{B}$;

(b) $\mathbb{Y} \in \mathcal{D}_e$ iff $\mathbb{Y}$ is a tournament;

(c) $\mathbb{Y} \in \mathcal{G}$ iff $\mathbb{Y} = \mathbb{A}_\omega$ iff $\mathbb{Y}_e = \mathbb{K}_\omega$ iff $\mathbb{Y}_e \in \mathcal{G}$.

Proof. The first equivalence in (a) is true since $\mathbb{Y}^c$ is connected, for each digraph $\mathbb{Y}$. Since $\mathbb{Y}_e$ is connected, by Fact 4.4(d) we have $\mathbb{Y}_e \in \mathcal{B}$ iff $(\mathbb{Y}_e)^c = (\mathbb{Y}^{-1})_{re}$ is connected iff $\mathbb{Y}^{-1}$ is connected iff $\mathbb{Y}$ is connected. The statements (b) and (c) are evident. $\square$

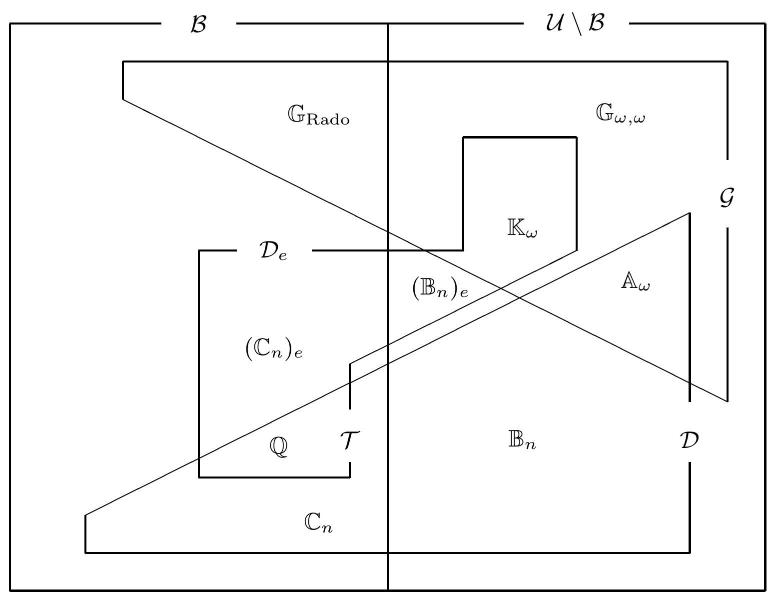

Figure 1: Countable reflexive or irreflexive ultrahomogeneous binary structures

By Theorem 4.1 the class $\mathcal{D}$ of digraphs generates all structures from $\mathcal{U} \setminus \mathcal{B}$ in a very simple way. By Theorem 5.2 and Fact 4.4(d), a forcing-related classification of the posets $\mathbb{P}(\mathbb{X})$ for the structures $\mathbb{X} \in \mathcal{D} \cap \mathcal{B}$ would provide a classification for the structures $\mathbb{X}$ belonging to a much wider class: $\mathcal{D} \cup \mathcal{D}_{re} \cup \mathcal{D}_e \cup (\mathcal{D}_e)_{re} \cup \mathcal{U} \setminus \mathcal{B}$, where for a class $\mathcal{X}$ we define $\mathcal{X}_{re} = \{\mathbb{X}_{re} : \mathbb{X} \in \mathcal{X}\}$. So, if, in addition, we obtain a corresponding classification for $\mathbb{X} \in \mathcal{G} \cap \mathcal{B}$ and hence, for $\mathcal{G} \cup \mathcal{G}_{re}$, it remains to investigate the posets $\mathbb{P}(\mathbb{X})$ for biconnected irreflexive structures $\mathbb{X}$ which are not: graphs (and, hence, $\mathbb{T}_2 \hookrightarrow \mathbb{X}$), digraphs (and, hence, $\mathbb{K}_2 \hookrightarrow \mathbb{X}$), enlarged digraphs (and, hence, $\mathbb{A}_2 \hookrightarrow \mathbb{X}$), thus they do not have forbidden substructures of size 2.

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