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The Structure of f-rpp Semigroups[1]

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Abstract

F-rpp semigroups are initially researched by Guo-Li-Shum (see, International Mathematical Forum, 1(2006), p. 1571-1585). They introduced the concept of SFR-systems and established the structure of a class of F-rpp semigroups, namely, strongly F-rpp semigroups. In this paper we define FR-systems by weakening the conditions of SFR-systems and further obtain the structure of general F-rpp semigroups in terms of FR-systems.

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1 Introduction

A semigroup S is called *rpp* if for any $a \in S$, aS^1 regarded as an S^1-system is projective. Dually, we can define *lpp semigroups*. In order to investigate well rpp semigroups, Fountain [3] and [4] introduced usual Green's $*$-relations $\mathcal{L}^*, \mathcal{R}^*, \mathcal{H}^*, \mathcal{D}^*$ and $\mathcal{J}^*$. He pointed out that a semigroup S is rpp if and only if each $\mathcal{L}^*$-class of S contains at least an idempotent. As in [4], S is called *abundant* if each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class of S contains at least an idempotent. Equivalently, a semigroup is abundant if and only if it is both rpp and lpp. Regular semigroups are abundant semigroups. Abundant semigroups have attracted due attractions.

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An inverse semigroup S is called *F-inverse* if there exists a group congruence σ on S such that each σ-class has a greatest element with respect to the natural partial order $\leq$ on S. There are many authors having been investigating (see, McFadden and O'Carrol [9] and others). It was showed that the notion of F-inverse semigroups are generalizations of residuated inverse semigroups introduced by Blyth [1]. Later on, Edwards [2] investigated regular semigroups with the same property of F-inverse semigroups and called *F-regular semigroup*. She proved that an F-regular semigroup is indeed an F-orthodox semigroups.

As analogue of F-regular semigroups, the second author in [5] defined F-abundant semigroups. By an *F-abundant semigroup*, we mean an abundant semigroup in which there exists a cancellative congruence σ such that each σ-class contains a greatest element with respect to the Lawson order $\leq$ (see, [8]). At the same reference, we established the structure of a class of F-abundant semigroups, namely, *strongly F-abundant semigroups*. In [10], Ni-Chen-Guo obtained a structure of general F-abundant semigroups by introducing the concept of pre-homomorphisms. The natural problem whether F-abundant semigroups are strongly F-abundant semigroups or not is now open.

In order to generalize F-abundant semigroups in the range of rpp semigroups, Guo-Li-Shum [6] defined F-rpp semigroups and gave a construction method of a class of F-rpp semigroups, so-called *strongly F-rpp semigroups*. But we left an open problem: whether is any F-rpp semigroup a strongly F-rpp semigroup or not? In this paper we shall establish a construction of general F-rpp semigroups by introducing the concept of FR-systems. In addition, we provide an example which can illustrate that strongly F-rpp semigroups form a proper subclass of F-rpp semigroups. In other words, there exist F-rpp semigroups which are not strongly F-rpp semigroups.

2 Results

In this section, we shall establish the structure of F-rpp semigroups. To begin with, we recall some concepts.

Let S be an rpp semigroup. As in [8], we define a relation S:

> $x \leq_\ell y$ if and only if $L^*(x) \subseteq L^*(y)$ and there exists $f \in E(S) \bigcap L^*_x$ such that $x = yf$,

where $L^*(x)$ is the left *-ideal generated by x (see Fountain [4]) and L^*_x denotes the $\mathcal{L}^*$-class of S containing x. Then $\leq_\ell$ is a partial order on S (see, [6]).

A congruence ρ on a semigroup S is called a *left cancellative monoid congruence* on S if S/ρ is a left cancellative monoid.

By an *F-rpp semigroup*, we mean an rpp semigroup in which there exists a left cancellative monoid congruence σ on S such that each σ-class of S contains a greatest element with respect to the Lawson order $\leq_\ell$. In this case, the σ is indeed the smallest left cancellative monoid congruence on S (see, [6]). We denote by m_a the greatest element of the σ-class of S containing a and by a^* an idempotent in L^*_a. An F-rpp semigroup S is called *strong* if for each $a \in S$, $E(S)m^*_a = m^*_a E(S)$, i.e. m^*_a is central in $E(S)$ (where $E(S)$ is the set of idempotents of S) (for detail, see [6]).

Let M be a left cancellative monoid with identity 1 and E be a band with identity ε. Denote by $End(E)$ the set of endomorphisms of E into itself. It is well known that with respect to the composition of mappings, $End(E)$ is a semigroup. Define mappings

$$\mathcal{G} : M \to E; \quad m \mapsto g_m,$$

$$\mathcal{P} : M \times M \to E; \quad (m, n) \mapsto p_{m,n}$$

and

$$\Psi : M \to End(E);\ m \mapsto \varphi_m,$$

where

$$\varphi_m : E \to E;\ \ x \mapsto x\varphi_m.$$

The above quadruple $(M, E; \mathcal{G}, \mathcal{P}; \Psi)$ is called an *FR-system* if the following conditions are satisfied:

- $(G1)$ $g_1 = \varepsilon$.
- $(G2)$ For all $m \in M, x \in E$, $g_m x g_m = g_m x$.
- $(P1)$ For all $m \in M, p_{m,1} = g_m$.
- $(P2)$ For all $m, n \in M, p_{m,n} = p_{m,n} p_{1,n} \in \omega(g_{mn})$, where
  $\omega(g_{mn}) = \{e \in E \mid e = eg_{mn} = g_{mn}e\}$.
- $(PH1)$ φ_1 is the identity mapping on E.
- $(PH2)$ For all $m \in M, x \in E, x\varphi_m \in \omega(g_m)$.
- $(FR1)$ For all $m, n \in M, x \in E, p_{m,n}(x\varphi_m\varphi_n) = (x\varphi_{mn})p_{m,n}$.
- $(FR2)$ For all $m, n \in M, x \in \omega(g_m)$ and $y, z \in \omega(g_n)$,
  $p_{m,n}(x\varphi_n)y = p_{m,n}(x\varphi_n)z \Rightarrow (x\varphi_n)y = (x\varphi_n)z$.
- $(FR3)$ For all $m, n, t \in M$, $p_{m,nt}p_{n,t} = p_{mn,t}(p_{m,n}\varphi_t)$.

Now, for a given FR-system $(M, E; \mathcal{G}, \mathcal{P}; \Psi)$, put

$$FR = FR(M, E; \mathcal{G}, \mathcal{P}; \Psi) = \{(m, x) \in M \times E : x \in \omega(g_m)\}.$$

Define a multiplication on FR by

$$(m,x)\circ(n,y)=(mn,p_{m,n}(x\varphi_n)y).$$

Note that $p_{m,n}\in\omega(g_{mn})$. We observe that $p_{m,n}(x\varphi_n)y\in\omega(g_{mn})$ by (G2), and so $(m,x)\circ(n,y)\in FR$. Thus $\circ$ is well-defined and $(FR,\circ)$ is a semigroup (see Section 3).

The following is the structure theorem of F-rpp semigroups.

Theorem 2.1 *Let $(M,E;\mathcal{G},\mathcal{P};\Psi)$ be an FR-system. Then $FR(M,E;\mathcal{G},\mathcal{P};\Psi)$ is an F-rpp semigroup. Conversely, any F-rpp semigroup can be constructed in this manner.*

Recall from Guo-Li-Shum [6] that an SFR-system is just an FR-system $(M,E;\mathcal{G},\mathcal{P};\Psi)$ in which g_m is central in E for all $m\in M$. Now, by Theorem 2.1, we can easily obtain the structure of strongly F-rpp semigroups (that is, [6, Theorem 3.10]). On the other hand, by the same arguments as [6, Theorem 4.3], we can obtain the following structure theorem for F-abundant semigroups and we omit the detail proof.

Theorem 2.2 *Let $(M,E;\mathcal{G},\mathcal{P};\Psi)$ be an FR-system. Assume that M is a cancellative monoid and for all $(m,x)\in FR=FR(M,E;\mathcal{G},\mathcal{P};\Psi)$, there exists $[m,x]\in E$ such that*

$(ABF1)$ $([m,x]\varphi_m)x=x$

$(ABF2)$ *For all $n\in M$ and $y,z\in\omega(g_n)$,*

$$p_{n,m}(y\varphi_m)x=p_{n,m}(z\varphi_m)x\Rightarrow y[m,x]=z[m,x].$$

(IC) *For some $y\in\omega(x)$, there exists $z\in E$ such that $(z\varphi_m)x=y$.*

Then $FR(M,E;\mathcal{G},\mathcal{P};\Psi)$ is an F-abundant semigroup. Conversely, any F-abundant semigroup can be constructed in this manner.

3 Proofs

The following result shall be used in the sequel.

Lemma 3.1 [8] *Let S be an rpp semigroup and $x,y\in S$. Then $x\leq_\ell y$ if and only if for all [for some] y^*, there exists a unique $f\in\omega(y^*)$ such that $x=yf$.*

Note that the proofs [6, Lemma 3.3, Lemma 3.4, Lemma 3.5, Lemma 3.6 and Lemma 3.7] need not use the assumption that each c_m is central in E.

Then by the same argument, we can show that $FR(M, E; \mathcal{G}, \mathcal{P}; \Psi)$ is an F-rpp semigroup if $(M, E; \mathcal{G}, \mathcal{P}; \Psi)$ is an FR-system. Thus we need only prove that any F-rpp semigroup is isomorphic to some $FR(M, E; \mathcal{G}, \mathcal{P}; \Psi)$.

In the rest of this section, we always assume that S is an F-rpp semigroup with a left cancellative monoid congruence ρ on S such that each ρ-class contains a greatest element, and that E is the idempotent band of S. We denote by M the set of greatest elements in all ρ-classes of S. In general, M does not form a subsemigroup of S. Now define a multiplication $\star$ as follows:

$$m \star n = \text{the greatest element of the } \rho\text{-class of } S \text{ containing } mn.$$

It is not difficult to check that $(M, \star)$ is a semigroup isomorphic to S/ρ.

Lemma 3.2 [6] (1) *$E(S)$ is a band with identity.*

(2) *For all $m \in M$ and m^*, $m^*E(S) \subseteq E(S)m^*$.*

(3) *For all $m \in M$ and $e \in E(S)$, there exists a unique $f \in \omega(m^*)$ such that $em = mf$.*

(4) *For each $x \in S$, there exists a unique idempotent $f_x \in \omega(m_x^*)$ such that $x = m_x f_x$. In this case, $f_x \mathcal{L}^* x$.*

Denote by 1 the identity of E. It is easy to see that $1 \in M$. For $m \in M$, we let m^* be a fixed idempotent in L_m^*, where $1^* = 1$.

Define a mapping $\mathcal{G}$ by

$$M \to E; \quad m \mapsto g_m = m^*.$$

Obviously, $g_1 = 1^* = 1$ and whence $\mathcal{G}$ satisfies (G1). By Lemma 3.2 (2), for all $m \in M$ and $e \in E$, we have $m^*em^* = m^*e$. This implies that $g_m e g_m = g_m e$ for all $m \in M$. That is, $\mathcal{G}$ satisfied (G2).

Let $m, n \in M$. Note that $mn\rho m \star n$. Then by Lemma 3.1, there exists a unique $f_{mn} \in \omega((m \star n)^*)$ such that $mn = (m \star n) f_{mn}$. Now, define a mapping $\mathcal{P}$ as follows:

$$M \times M \to E; \quad (m, n) \mapsto p_{m,n} = f_{mn}.$$

Consider that $m \star 1 = m_m \star m_1 = m_{m1} = m_m = m$ and $m = mm^*$. We have that $m = (m \star 1)m^*$ and since $m = m1 = (m \star 1) f_{m1}$, then by the uniqueness of f_{mn}, we have $p_{m,1} = f_{m1} = m^* = g_m$. Thus $\mathcal{P}$ satisfies (P1). On the other hand, since $mn = (m \star n) f_{mn}$ with $f_{mn} \in \omega((m \star n)^*)$, we have $p_{m,n} \in \omega((m * n)^*)$. Compute

$$(m \star n)(f_{mn} n^*) = mnn^* = mn = (m \star n) f_{mn}.$$

Since $f_{mn} \in \omega((m \star n)^*)$ and

$$\begin{aligned} f_{mn}n^* &= (m \star n)^* f_{mn} n^* \\ &= (m \star n)^* f_{mn} n^* (m \star n)^* = f_{mn} n^* (m \star n)^* \quad \text{(by Lemma 3.2)}, \end{aligned}$$

we have $f_{mn}n^* \in \omega(g_{m\star n})$. By the uniqueness of f_{mn}, this shows that $f_{mn} = f_{mn}n^*$. That is, $p_{m,n} = p_{m,n}g_n = p_{m,n}p_{1,n} \in \omega(g_{m\star n})$. Thus $\mathcal{P}$ satisfies (P2).

Now let $m \in M$ and $e \in E$. Since $(em)\rho m$ and $em \leq_\ell m$, there exists an unique $f_{em} \in \omega(m^*) = \omega(g_m)$ such that $em = mf_{em}$. Define a mapping θ_m by

$$E \to E; \quad e \mapsto e\theta_m = f_{em}.$$

If $g \in E$, then $(eg)m = mf_{egm}$. On the other hand, we have

$$mf_{egm} = (eg)m = e(gm) = e(mf_{gm}) = (em)f_{gm} = mf_{em}f_{gm}.$$

Note that $f_{egm}, f_{em}f_{gm} \in \omega(m^*)$. By the uniqueness of f_{mn}, we have $f_{egm} = f_{em}f_{gm}$. That is, $(eg)\theta_m = (e\theta_m)(g\theta_m)$. This means that θ_m is a homomorphism of E into itself.

Finally, we define a mapping Θ of M into $End(E)$ by

$$M \to End(E); \quad m \mapsto \theta_m.$$

It is clear that Θ satisfies conditions (PH1) and (PH2).

Lemma 3.3 $(M, E; \mathcal{G}, \mathcal{P}; \Theta)$ *is an FR-system.*

Proof: Let $m, n \in M$ and $x \in E$. Since

$$\begin{aligned} (m \star n)p_{m,n}(x\theta_m\theta_n) &= mn(x\theta_m\theta_n) = m(x\theta_m)n \\ &= xmn = x(m \star n)p_{m,n} = (m \star n)(x\theta_{m\star n})p_{m,n} \end{aligned}$$

and $p_{m,n}(x\theta_m\theta_n), (x\theta_{m\star n})p_{m,n} \in \omega((m\star n)^*)$, by Lemma 3.2, we have $p_{m,n}(x\theta_m\theta_n) = (x\theta_{m\star n})p_{m,n}$. That is, (FR1) holds.

For $m, n \in M$ and $x \in \omega(g_m), y, z \in \omega(g_n)$, if $p_{m,n}(x\theta_n)y = p_{m,n}(x\theta_n)z$, then we have

$$\begin{aligned} mxny &= mn(x\theta_n)y = (m \star n)p_{m,n}(x\theta_n)y \\ &= (m \star n)p_{m,n}(x\theta_n)z = mn(x\theta_n)z = mxnz \end{aligned}$$

Since $m\mathcal{L}^*g_m = m^*$, by the above formula, we have $m^*xny = m^*xnz$ and so $n(x\theta_n)y = n(x\theta_n)z$. This shows that $n^*(x\theta_n)y = n^*(x\theta_n)z$ since $n\mathcal{L}^*n^*$. Consequently, $(x\theta_n)y = (x\theta_n)z$ by the definition of θ_n, and hence (FR2) holds.

Finally, let $m, n, t \in M$. Since

$$\begin{aligned}(m \star n \star t)p_{m,n\star t}p_{n,t} &= m(n \star t)p_{n,t} = mnt \\ &= (m \star n)p_{m,n}t = (m \star n)t(p_{m,n}\theta_t) \\ &= (m \star n \star t)p_{m\star n,t}(p_{m,n}\theta_t)\end{aligned}$$

and $m \star n \star t \mathcal{L}^*(m \star n \star t)^*$, we have

$$(m \star n \star t)^* p_{m,n\star t}p_{n,t} = (m \star n \star t)^* p_{m\star n,t}(p_{m,n}\theta_t). \qquad (1)$$

Since $p_{m\star n,t}, p_{m,n\star t} \in \omega((m\star n\star t)^*)$, we have $p_{m,n\star t}p_{n,t}, p_{m\star n,t}(p_{m,n}\theta_t) \in \omega((m\star n\star t)^*)$ and, by the above equality (1), we obtain that $p_{m,n\star t}p_{n,t} = p_{m\star n,t}(p_{m,n}\theta_t)$. That is, (FR3) also holds. This completes the proof. □

Proof of Theorem 2.1. It remains to prove that $S \cong FR = FR(M, E; \mathcal{G}, \mathcal{P}; \Theta)$. We only need to prove that the mapping

$$\varphi : S \to FR = FR(M, E; \mathcal{G}, \mathcal{P}; \Theta);\ s \mapsto (m_s, f_s),$$

where m_s and f_s have the same meanings as Lemma 3.2, is an isomorphism. By Lemma 3.2 (4), φ is well-defined and we have $(mx)\varphi = (m, x)$ for all $(m, x) \in FR$. Hence φ is surjective. That φ is injective follows from Lemma 3.2.

Finally, let $s, t \in S$. Then we have

$$\begin{aligned}(m_s \star m_t)f_{st} &= m_{st}f_{st} = st = m_s f_s m_t f_t \\ &= m_s m_t(f_s\theta_{m_t})f_t = (m_s \star m_t)p_{m_s,m_t}(f_s\theta_{m_t})f_t.\end{aligned}$$

This implies that $f_{st} = p_{m_s,m_t}(f_s\theta_{m_t})f_t$. Thus we have

$$(st)\varphi = (m_s \star m_t, p_{m_s,m_t}(f_s\theta_{m_t})f_t) = (m_s, f_s)(m_t, f_t) = (s\varphi)(t\varphi)$$

and so φ is a homomorphism from S onto FR. Hence φ is an isomorphism, as required. □

4 An example

In this section we provide an example which illustrates that strongly F-rpp semigroups form a proper subclass of F-rpp semigroups.

Example 4.1 *Let M be a left cancellative semigroup, E be a left zero band. Let $S = M \cup E \cup \{\varepsilon\}$, where ε is neither an element of M nor one of E. Define a multiplication $\circ$ on S by*

$$a \circ b = \begin{cases} ab & \text{if } a, b \in M, \\ a & \text{if } a \in M, b \in E \cup \{\varepsilon\} \\ b & \text{if } a \in E \cup \{\varepsilon\}, b \in M \\ a & \text{if } a, b \in E \\ \varepsilon & \text{if } a = \varepsilon = b \\ b & \text{if } a = \varepsilon,\ b \in E \\ a & \text{if } a \in E,\ b = \varepsilon \end{cases}$$

Then $(S, \circ)$ is an F-rpp semigroup but not a strongly F-rpp semigroup.

Proof: A routine calculation can show that $(S, \circ)$ is a monoid with identity ε and $E(S) = E \cup \{\varepsilon\}$. We prove next that S is rpp. For this, we let $m \in M, e \in E, x, y \in S$ and $mx = my$. We consider the following two cases:

(1) One of x and y in $E \cup \{\varepsilon\}$. Without loss of generality, we assume $x \in E \cup \{\varepsilon\}$. Then $mx = m = my$ and $y \in E \cup \{\varepsilon\}$. It is clear that $ex = ey$ for all $e \in E$.

(2) $x, y \in M$. Note that M is left cancellative. Then $mx = my$ implies that $x = y$. This shows that $ex = ey$ for all $e \in E$.

However, $mx = my$ can imply that $ex = ey$, for all $e \in E$. By the definition of multiplication, $me = m$. We have now proved that $m\mathcal{L}^*e$ for all $e \in E$ and $m \in M$. Therefore S is rpp.

Define a relation ρ on S as follows:

$$\rho = \{(x, y) \in S \times S : (x, y) \in (E \cup \{\varepsilon\}) \times (E \cup \{\varepsilon\}) \text{ or } x = y \in M\}.$$

It is easy to see that $\rho = [(E \cup \{\varepsilon\}) \times (E \cup \{\varepsilon\})] \cup \Delta_M$, where Δ_M is the identity relation on M. Furthermore, from the above proof (1) and (2), it is not difficult to check that ρ is a left cancellative monoid congruence on S. Note that $\rho_x = \{x\}$ if $x \in M$ while $\rho_x = E \cup \{\varepsilon\}$ if $x \in E \cup \{\varepsilon\}$. Then each ρ-class of S contains a greatest element with respect to $\leq_\ell$ since ε is the greatest idempotent. This means that S is an F-rpp semigroup.

For $e, f \in E$, we have that $e \neq f$ implies that $ef = e \neq f = fe$, hence each element of E is not central in $E(S) = E \cup \{\varepsilon\}$, and thus S is not strong. □

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