

FOUR-DIMENSIONAL PSEUDO-RIEMANNIAN HOMOGENEOUS RICCI SOLITONS

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ABSTRACT. We consider four-dimensional homogeneous pseudo-Riemannian manifolds with non-trivial isotropy and completely classify the cases giving rise to non-trivial homogeneous Ricci solitons. In particular, we show the existence of non-compact homogeneous (and also invariant) pseudo-Riemannian Ricci solitons which are not isometric to solv-manifolds, and of conformally flat homogeneous pseudo-Riemannian Ricci solitons which are not symmetric.

1. INTRODUCTION

Ricci solitons were introduced by Hamilton [15] and they are a natural generalization of Einstein metrics. A pseudo-Riemannian metric g on a smooth manifold M is called a *Ricci soliton* if there exists a smooth vector field X , such that

$$(1.1) \quad \mathcal{L}_X g + \varrho = \lambda g,$$

where \mathcal{L}_X denotes the Lie derivative in the direction of X , ϱ denotes the Ricci tensor and λ is a real number. A Ricci soliton g is said to be a *shrinking*, *steady* or *expanding* according to whether $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively.

Ricci solitons are the self-similar solutions of the *Ricci flow* and are important in understanding its singularities. A survey and further references on the geometry of Ricci solitons may be found in [9]. The interest in Ricci solitons has also risen among theoretical physicists in relation with String Theory [1],[14],[18]. After their introduction in the Riemannian case, the study of pseudo-Riemannian Ricci solitons attracted a growing number of authors (see for instance [3]-[7],[10],[26]).

If $M = G/H$ is a homogeneous space, a *homogeneous Ricci soliton* on M is a G -invariant metric g for which equation (1.1) holds. In particular, by an *invariant Ricci soliton* we mean a homogeneous one, such that equation (1.1) is satisfied by an invariant vector field. It is a natural question to determine which homogeneous manifolds G/H admit a G -invariant Ricci soliton [22]. All known examples of homogeneous Riemannian Ricci soliton metrics on non-compact homogeneous manifolds are isometric to some solvsolitons, that is, to invariant Ricci solitons on a solvable Lie group ([17, Remark 1.5]).

The difference between Riemannian and pseudo-Riemannian settings lead to different results concerning the existence of homogeneous Ricci solitons. In fact, although there exist three-dimensional Riemannian homogeneous Ricci solitons [2],[22], there are

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no left-invariant Ricci solitons on three-dimensional Riemannian Lie groups [11], while left-invariant Ricci solitons on three-dimensional Lorentzian Lie groups were classified in [3]. Four-dimensional Ricci solitons on non-reductive homogeneous pseudo-Riemannian manifolds (which do not have a Riemannian counterpart, as a homogeneous Riemannian manifold is necessarily reductive) were classified in [7].

In the present article we provide a full classification of four-dimensional homogeneous pseudo-Riemannian Ricci solitons in all the cases with non-trivial isotropy. The starting point is the complete local classification of four-dimensional homogeneous pseudo-Riemannian manifolds with non-trivial isotropy obtained in [19] (see also [20]), leading to the remarkable number of 186 different forms of these spaces. Each of them admits a family of invariant pseudo-Riemannian metrics, depending of a number of real parameters varying from 1 to 4. Among them, we were able to determine 44 different examples of homogeneous spaces $M = G/H$ for which equation (1.1) holds for some vector fields $X \in \mathfrak{m}$ and some invariant metrics which are not Einstein. In particular, for 41 of these examples, equation (1.1) is satisfied by an invariant vector field. Some of these examples show that there exist homogeneous (and also invariant) pseudo-Riemannian Ricci solitons which are not isometric to solvsolitons.

In the Riemannian case, most of the known examples of non-trivial Ricci solitons are Kähler-Ricci solitons, with the exception of the homogeneous solitons on nilpotent Lie groups [21], the rotationally symmetric Bryant solitons on \mathbb{R}^n ($n > 2$), Ivey's generalization of these solutions [16] and the complete steady gradient Ricci solitons constructed in [12]. By using the results in [8] we show that none of the four-dimensional non-trivial pseudo-Riemannian homogeneous Ricci solitons is Kähler.

Conformally flat Einstein pseudo-Riemannian manifolds have constant sectional curvature. In particular, they are symmetric. Conformally flat homogeneous Riemannian manifolds are always symmetric [27]. On the other hand, some of our examples show the existence of conformally flat homogeneous pseudo-Riemannian Ricci solitons which are not symmetric.

The paper is organized in the following way. In Section 2, we shall report some basic facts on four-dimensional homogeneous pseudo-Riemannian manifolds with non-trivial isotropy. The classification of homogeneous solutions to (1.1) will be then presented in Section 3. In Section 4 we shall investigate several geometric properties of four-dimensional pseudo-Riemannian homogeneous Ricci solitons (with non-trivial isotropy). In particular, we show the non-existence of pseudo-Kähler examples, i.e., solutions corresponding to holomorphic vector fields $X \in \mathfrak{m}$, and we classify the conformally flat examples.

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2. PRELIMINARIES

Let M be a homogeneous space, $H = H_x$ the stabilizer of an arbitrary point x of M , and $(\mathfrak{g}, \mathfrak{h})$ the pair of Lie algebras corresponding to the pair (G, H) of Lie groups. As remarked in [20], the pair $(\mathfrak{g}, \mathfrak{h})$ locally uniquely defines the homogeneous space.

With regard to four-dimensional pseudo-Riemannian homogeneous spaces, the classification obtained by Komrakov [19, 20] starts with the list of subalgebras in the Lie algebras $\mathfrak{so}(4)$, $\mathfrak{so}(3,1)$ and $\mathfrak{so}(2,2)$. Up to conjugacy, Lie algebras $\mathfrak{so}(4)$, $\mathfrak{so}(3,1)$ and $\mathfrak{so}(2,2)$ respectively admit 6, 14 and 28 distinct Lie subalgebras (some of them depending on parameters).

Given a subalgebra \mathfrak{h} of either $\mathfrak{so}(4)$, $\mathfrak{so}(3,1)$ or $\mathfrak{so}(2,2)$, assuming that \mathfrak{h} acts naturally on \mathbb{R}^4 , there is a one-to-one correspondence between faithful generalized modules $(\mathfrak{h}, \mathbb{R}^4)$ and corresponding subalgebras of $\mathfrak{so}(p,q)$. This leads to determine 186 distinct models of four-dimensional homogeneous pseudo-Riemannian manifolds with non-trivial isotropy. Following the notation and the classification used in [20], the space identified by the type $\mathbf{n.m}^{\mathbf{k}.q}$ is the one corresponding to the q -th pair $(\mathfrak{g}, \mathfrak{h})$ of type $\mathbf{n.m}^{\mathbf{k}}$, where $\mathbf{n} = \dim(\mathfrak{h})$ ($= 1, \dots, 6$), \mathbf{m} is the number of the complex subalgebra $\mathfrak{h}^{\mathbb{C}}$ of $\mathfrak{so}(4, \mathbb{C})$ and \mathbf{k} is the number of the real form of $\mathfrak{h}^{\mathbb{C}}$. When the index q is omitted, one refers simultaneously to all homogeneous spaces corresponding to pairs $(\mathfrak{g}, \mathfrak{h})$ of type $\mathbf{n.m}^{\mathbf{k}}$.

Consider now an arbitrary pair $(\mathfrak{g}, \mathfrak{h})$ (that is, the corresponding homogeneous space $M = G/H$, with H connected). Let $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$ denote the four-dimensional factor space, which identifies with a subspace of \mathfrak{g} complementary to \mathfrak{h} . The different cases are explicitly described in [20], listing a basis $\{e_j\}$ of \mathfrak{h} , a basis $\{u_i\}$ of \mathfrak{m} and their Lie brackets. The pair $(\mathfrak{g}, \mathfrak{h})$ uniquely defines the isotropy representation

$$\psi : \mathfrak{h} \rightarrow \mathfrak{gl}(\mathfrak{m}), \quad \psi(x)(y) = [x, y]_{\mathfrak{m}} \quad \text{for all } x \in \mathfrak{h}, y \in \mathfrak{m}.$$

A bilinear form B on \mathfrak{m} is invariant if and only if $\psi(x)^t \circ B + B \circ \psi(x) = 0$, for all $x \in \mathfrak{h}$, where $\psi(x)^t$ denotes the transpose of $\psi(x)$. In particular, requiring that $B = g$ is symmetric and nondegenerate, this leads to the classification of all invariant pseudo-Riemannian metrics on G/H .

With regard to curvature properties, following [20], an invariant nondegenerate symmetric bilinear form g on \mathfrak{m} uniquely defines its invariant linear Levi-Civita connection ∇ , described in terms of the corresponding homomorphism of \mathfrak{h} -modules $\Lambda : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{m})$, such that $\Lambda(x)(y_{\mathfrak{m}}) = [x, y]_{\mathfrak{m}}$ for all $x \in \mathfrak{h}, y \in \mathfrak{g}$. Explicitly, one has

$$(2.1) \quad \Lambda(x)(y_{\mathfrak{m}}) = \frac{1}{2}[x, y]_{\mathfrak{m}} + v(x, y), \quad \text{for all } x, y \in \mathfrak{g},$$

where $v : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{m}$ is the \mathfrak{h} -invariant symmetric mapping uniquely determined by

$$2g(v(x, y), z_{\mathfrak{m}}) = g(x_{\mathfrak{m}}, [z, y]_{\mathfrak{m}}) + g(y_{\mathfrak{m}}, [z, x]_{\mathfrak{m}}), \quad \text{for all } x, y, z \in \mathfrak{g}.$$

The curvature tensor is then determined by the mapping $R : \mathfrak{m} \times \mathfrak{m} \rightarrow \mathfrak{gl}(\mathfrak{m})$, such that

$$(2.2) \quad R(x, y) = [\Lambda(x), \Lambda(y)] - \Lambda([x, y]),$$

for all $x, y \in \mathfrak{m}$. Finally, the Ricci tensor ϱ of g , described in terms of its components with respect to $\{u_i\}$, is given by

$$(2.3) \quad \varrho(u_i, u_j) = \sum_{r=1}^4 R_{ri}(u_r, u_j), \quad i, j = 1, \dots, 4.$$

It must be noted that whenever $X = \sum x_k u_k \in \mathfrak{m}$, equation (1.1) reads as a system of algebraic equations for the components x_k of X , namely,

$$(2.4) \quad \sum_{k=1}^4 x_k (g([u_k, u_i], u_j) + g(u_i, [u_k, u_j])) + \varrho(u_i, u_j) = \lambda g_{ij}, \quad i, j = 1, \dots, 4.$$

3. CLASSIFICATION OF FOUR-DIMENSIONAL HOMOGENEOUS RICCI SOLITONS

We shall now present the classification of invariant Ricci solitons among four-dimensional homogeneous pseudo-Riemannian manifolds $M = G/H$ with non-trivial isotropy.

To this purpose we first describe the Lie brackets, the generic invariant metrics g and their corresponding Ricci tensors ϱ , for the homogeneous spaces G/H which will appear in the classification. All the following formulae for g and ϱ are obtained applying to the different cases the technical apparatus we described in the previous Section.

1.1¹. There exists a basis $\{u_1, \dots, u_4\}$ of \mathfrak{m} and a basis $\{e_1\}$ of \mathfrak{h} , such that

$$[e_1, u_1] = u_1, \quad [e_1, u_2] = 0, \quad [e_1, u_3] = -u_3, \quad [e_1, u_4] = 0.$$

Then, with respect to $\{u_i\}$, invariant metrics are of the form

$$(3.1) \quad g = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & b & 0 & c \\ a & 0 & 0 & 0 \\ 0 & c & 0 & d \end{pmatrix},$$

for any real constants a, b, c, d satisfying $a(bd - c^2) \neq 0$. The different cases are the following.

1.1¹.1. The Lie brackets are determined by

$$[u_1, u_3] = u_2, \quad [u_2, u_4] = u_2, \quad [u_3, u_4] = u_3$$

(with $[u_i, u_j] = 0$ for the remaining indices $i < j$). With respect to $\{u_i\}$, the Ricci tensor of the metric g given by (3.1) is of the form

$$\varrho = \begin{pmatrix} 0 & 0 & -\frac{b(2a^2+c^2-bd)}{2a(bd-c^2)} & 0 \\ 0 & -\frac{b^2(4a^2+bd-c^2)}{2a^2(bd-c^2)} & 0 & -\frac{bc(4a^2+bd-c^2)}{2a^2(bd-c^2)} \\ -\frac{b(2a^2+c^2-bd)}{2a(bd-c^2)} & 0 & 0 & 0 \\ 0 & -\frac{bc(4a^2+bd-c^2)}{2a^2(bd-c^2)} & 0 & -\frac{c^2a^2+bd c^2-c^4+3ba^2d}{2a^2(bd-c^2)} \end{pmatrix}.$$

1.1¹.2. Then, the Lie brackets are determined by

$$[u_2, u_4] = pu_2, \quad [u_3, u_4] = u_3,$$

where p is a real constant. With respect to $\{u_i\}$, the Ricci tensor of the metric (3.1) is of the form

$$\varrho = \begin{pmatrix} 0 & 0 & -\frac{ab(p+1)}{2(bd-c^2)} & 0 \\ 0 & -\frac{b^2p(p+1)}{bd-c^2} & 0 & -\frac{bcp(p+1)}{bd-c^2} \\ -\frac{ab(p+1)}{2(bd-c^2)} & 0 & 0 & 0 \\ 0 & -\frac{bcp(p+1)}{bd-c^2} & 0 & -\frac{2c^2p+bd-c^2+2bdp^2}{2(bd-c^2)} \end{pmatrix}.$$

1.1^{1.5}. Then,

$$[u_1, u_3] = e_1, \quad [u_2, u_4] = u_2$$

and the Ricci tensor of the metric (3.1) is of the form

$$\varrho = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & -\frac{b^2}{bd-c^2} & 0 & -\frac{bc}{bd-c^2} \\ -1 & 0 & 0 & 0 \\ 0 & -\frac{bc}{bd-c^2} & 0 & -\frac{bd}{bd-c^2} \end{pmatrix}.$$

1.1². There exists a basis $\{u_1, \dots, u_4\}$ of \mathfrak{m} and a basis $\{e_1\}$ of \mathfrak{h} , such that

$$(3.2) \quad [e_1, u_1] = u_3, \quad [e_1, u_2] = 0, \quad [e_1, u_3] = -u_1, \quad [e_1, u_4] = 0.$$

Then, with respect to $\{u_i\}$, invariant metrics are of the form

$$(3.3) \quad g = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & c \\ 0 & 0 & a & 0 \\ 0 & c & 0 & d \end{pmatrix}$$

for any real constants a, b, c, d satisfying $a(bd - c^2) \neq 0$. The different cases are the following.

1.1^{2.1}. Then,

$$[u_1, u_3] = -u_2, \quad [u_1, u_4] = u_1, \quad [u_2, u_4] = 2u_2, \quad [u_3, u_4] = u_3$$

and the Ricci tensor of the metric g given in (3.3) is of the form

$$\varrho = \begin{pmatrix} -\frac{(8a^2+bc-d^2)b}{2a(bc-d^2)} & 0 & 0 & 0 \\ 0 & -\frac{(16a^2-bc+d^2)b^2}{2a^2(bc-d^2)} & 0 & -\frac{(16a^2-bc+d^2)bd}{2a^2(bc-d^2)} \\ 0 & 0 & -\frac{(8a^2+bc-d^2)b}{2a(bc-d^2)} & 0 \\ 0 & -\frac{(16a^2-bc+d^2)bd}{2a^2(bc-d^2)} & 0 & -\frac{12a^2bc+4a^2d^2-bcd^2+d^4}{2a^2(bc-d^2)} \end{pmatrix}$$

1.1^{2.2}. Then,

$$[u_1, u_4] = u_1, \quad [u_2, u_4] = pu_2, \quad [u_3, u_4] = u_3,$$

where p is a real constant. With respect to $\{u_i\}$, the Ricci tensor of the metric g given in (3.3) is now of the form

$$\varrho = \begin{pmatrix} \frac{ab(p+2)}{bc-d^2} & 0 & 0 & 0 \\ 0 & -\frac{b^2p(p+1)}{bc-d^2} & 0 & -\frac{bdp(p+1)}{bc-d^2} \\ 0 & 0 & -\frac{ab(p+1)}{bc-d^2} & 0 \\ 0 & -\frac{bdp(p+1)}{bc-d^2} & 0 & -\frac{bc(p^2+1)+d^2(p-1)}{bc-d^2} \end{pmatrix}.$$

1.1².6, 1.1².7 and **1.1².8**. Then,

$$[u_1, u_3] = \varepsilon e_1 \quad [u_2, u_4] = u_2,$$

with $\varepsilon = 1$ for **1.1².6**, $\varepsilon = -1$ for **1.1².7** and $\varepsilon = 0$ for **1.1².8**. With respect to $\{u_i\}$, the Ricci tensor of the metric g given in (3.3) is of the form

$$(3.4) \quad \varrho = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & -\frac{b^3c}{(bc-d^2)^2} & 0 & -\frac{bd(2bc-d^2)}{(bc-d^2)^2} \\ 0 & 0 & \varepsilon & 0 \\ 0 & -\frac{bd(2bc-d^2)}{(bc-d^2)^2} & 0 & -\frac{b^2c^2}{(bc-d^2)^2} \end{pmatrix}.$$

1.3¹. There exists a basis $\{u_1, \dots, u_4\}$ of \mathfrak{m} and a basis $\{e_1\}$ of \mathfrak{h} , such that

$$[e_1, u_1] = 0, \quad [e_1, u_2] = 0, \quad [e_1, u_3] = u_1, \quad [e_1, u_4] = u_2.$$

Consequently, with respect to $\{u_i\}$, invariant metrics are of the form

$$(3.5) \quad g = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \\ 0 & -a & b & c \\ a & 0 & c & d \end{pmatrix},$$

for any real constants $a \neq 0, b, c, d$. The different cases are the following.

1.3¹.1. We have

$$[u_1, u_2] = -\frac{1}{2}u_2, \quad [u_1, u_3] = u_1, \quad [u_1, u_4] = \frac{1}{2}u_4, \quad [u_2, u_3] = \frac{1}{2}u_4.$$

The Ricci tensor of the invariant metric g described by (3.5) is then given by

$$\varrho = \begin{pmatrix} 0 & 0 & 0 & \frac{3d}{8a} \\ 0 & 0 & -\frac{3d}{8a} & 0 \\ 0 & -\frac{3d}{8a} & \frac{3(6bd-5c^2)}{8a^2} & \frac{3cd}{8a^2} \\ \frac{3d}{8a} & 0 & \frac{3cd}{8a^2} & \frac{3d^2}{8a^2} \end{pmatrix}.$$

1.3¹.2. In this case,

$$[u_1, u_3] = -pe_1 + (p+1)u_1 + pu_2, \quad [u_2, u_4] = u_2,$$

where p is a real constant such that $|p| \leq 1$. The Ricci tensor of the invariant metric g described by (3.5) is then given by

$$\varrho = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}p^2 - \frac{1}{2} & \frac{1}{2}p + \frac{1}{2} \\ 0 & 0 & \frac{1}{2}p + \frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

1.3^{1.3}. Then,

$$[u_1, u_3] = u_1, \quad [u_2, u_4] = u_2, \quad [u_3, u_4] = e_1.$$

The Ricci tensor of the invariant metric g described by (3.5) is then given by

$$\varrho = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

1.3^{1.5}. There exist two real parameters $p \geq 0$ and $q \neq 1$, such that

$$\begin{aligned} [u_1, u_3] &= -\frac{p^2+q}{q-1}e_1 + \frac{1+p^2}{q-1}u_2, & [u_1, u_4] &= pe_1 + u_1 + pu_2, \\ [u_2, u_3] &= pe_1 + u_1 + pu_2, & [u_2, u_4] &= -qe_1 + (q+1)u_2. \end{aligned}$$

The Ricci tensor of the invariant metric g described by (3.5) is then given by

$$\varrho = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{p^2+p^2q-2q+4}{2(q-1)} & -\frac{1}{2}pq \\ 0 & 0 & -\frac{1}{2}pq & q - \frac{1}{2}q^2 \end{pmatrix}.$$

1.3^{1.6}. We have

$$[u_1, u_3] = -u_2, \quad [u_1, u_4] = u_1, \quad [u_2, u_3] = u_1, \quad [u_2, u_4] = u_2, \quad [u_3, u_4] = e_1$$

and

$$\varrho = \text{diag}(0, 0, 2, 0),$$

that is, ϱ is diagonal with respect to $\{u_i\}$, with $\varrho(u_3, u_3) = 2$, $\varrho(u_i, u_i) = 0$ otherwise.

1.3^{1.7}. Then,

$$\begin{aligned} [u_1, u_3] &= \frac{1}{1+p}e_1 + \frac{p}{1+p}u_1 - \frac{1}{1+p}u_2, & [u_1, u_4] &= -\frac{1}{1+p}e_1 + \frac{1}{1+p}u_1 + \frac{1}{1+p}u_2, \\ [u_2, u_3] &= -\frac{1}{1+p}e_1 + \frac{1}{1+p}u_1 + \frac{1}{1+p}u_2, & [u_2, u_4] &= -\frac{p}{1+p}e_1 + \frac{p}{1+p}u_1 + \frac{1+2p}{1+p}u_2, \end{aligned}$$

where $p \neq -1$ is a real constant. The Ricci tensor is now given by

$$\varrho = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{p-1}{2(p+1)} & \frac{p}{p+1} \\ 0 & 0 & \frac{p}{p+1} & -\frac{p}{p+1} \end{pmatrix}.$$

1.3¹.8. Then,

$$[u_2, u_3] = u_1, \quad [u_2, u_4] = u_2, \quad [u_3, u_4] = -u_3$$

and the Ricci tensor of the invariant metric g described by (3.5) is now given by

$$\varrho = \text{diag} \left(0, 0, 0, -\frac{1}{2} \right).$$

1.3¹.9. Then,

$$[u_2, u_3] = pu_1, \quad [u_2, u_4] = -pe_1 + (p+1)u_2, \quad [u_3, u_4] = -pu_3,$$

where $p \neq 0$ is a real constant. The Ricci tensor is now given by

$$\varrho = \text{diag} \left(0, 0, 0, p - \frac{1}{2}p^2 - \frac{1}{2} \right).$$

1.3¹.11. Then,

$$[u_2, u_3] = -u_1, \quad [u_2, u_4] = e_1, \quad [u_3, u_4] = e_1 + u_3.$$

The Ricci tensor of the invariant metric g described by (3.5) is given by

$$\varrho = \text{diag} (0, 0, 0, -2).$$

1.3¹.12. There exist two real constants p, q , such that

$$[u_1, u_4] = u_1, \quad [u_2, u_3] = qu_1, \quad [u_2, u_4] = -pqe_1 + (p+q)u_2, \quad [u_3, u_4] = (1-q)u_3.$$

The Ricci tensor of the invariant metric g described by (3.5) is given by

$$\varrho = \text{diag} \left(0, 0, 0, \frac{5pq-2p^2-p-q^2+3q+1}{4} \right).$$

1.3¹.13. Then,

$$[u_1, u_4] = u_1, \quad [u_2, u_3] = \frac{1}{2}u_1, \quad [u_2, u_4] = -\frac{p}{2}e_1 + (p + \frac{1}{2})u_2, \quad [u_3, u_4] = e_1 + \frac{1}{2}u_3,$$

where p is a real constant. The Ricci tensor of the metric (3.5) is given by

$$\varrho = \text{diag} \left(0, 0, 0, \frac{3+4p-4p^2}{8} \right).$$

1.3¹.14. Then,

$$[u_1, u_4] = u_1, \quad [u_2, u_3] = (1-p)u_1, \quad [u_2, u_4] = p(p-1)e_1 + u_2, \quad [u_3, u_4] = e_1 + pu_3,$$

where $p \neq 1/2$ is a real constant. The Ricci tensor of the metric (3.5) is given by

$$\varrho = \text{diag} (0, 0, 0, 2p - 2p^2).$$

1.3¹.19. Then,

$$[u_1, u_4] = u_1, \quad [u_2, u_3] = u_1, \quad [u_2, u_4] = -e_1 + u_1 + 2u_2.$$

The Ricci tensor of the metric (3.5) is given by

$$\varrho = \text{diag} \left(0, 0, 0, \frac{1}{2} \right).$$

1.3¹.20. Then,

$$[u_2, u_3] = u_1, \quad [u_2, u_4] = -u_1 + u_2, \quad [u_3, u_4] = -u_3.$$

The Ricci tensor of the metric (3.5) is given by

$$\varrho = \text{diag} \left(0, 0, 0, -\frac{1}{2} \right).$$

1.3¹.21. Then,

$$[u_1, u_4] = u_1, \quad [u_2, u_3] = pu_1, \quad [u_2, u_4] = -pe_1 + (1-p)u_1 + (1+p)u_2, \quad [u_3, u_4] = (1-p)u_3,$$

where $p \neq 1$ is a real constant. The Ricci tensor of the metric (3.5) is given by

$$\varrho = \text{diag} \left(0, 0, 0, p - \frac{1}{2}p^2 \right).$$

1.3¹.22. Then,

$$[u_1, u_4] = u_1, \quad [u_2, u_3] = \frac{1}{2}u_1, \quad [u_2, u_4] = -\frac{1}{2}e_1 + \frac{1}{2}u_1 + \frac{3}{2}u_2, \quad [u_3, u_4] = e_1 + \frac{1}{2}u_3.$$

The Ricci tensor of the metric (3.5) is given by

$$\varrho = \text{diag} \left(0, 0, 0, \frac{3}{8} \right).$$

1.3¹.28. Then,

$$[u_1, u_3] = 2u_1, \quad [u_1, u_4] = 2u_2, \quad [u_2, u_3] = u_2, \quad [u_2, u_4] = e_1 - \frac{1}{2}u_1, \quad [u_3, u_4] = u_4$$

and

$$\varrho = \text{diag} \left(0, 0, -\frac{3}{2}, -\frac{1}{2} \right).$$

1.4¹. There exists a basis $\{u_1, \dots, u_4\}$ of \mathfrak{m} and a basis $\{e_1\}$ of \mathfrak{h} , such that

$$[e_1, u_1] = 0, \quad [e_1, u_2] = u_1, \quad [e_1, u_3] = u_2, \quad [e_1, u_4] = 0.$$

Consequently, with respect to $\{u_i\}$, invariant metrics are of the form

$$(3.6) \quad g = \begin{pmatrix} 0 & 0 & -a & 0 \\ 0 & a & 0 & 0 \\ -a & 0 & b & c \\ 0 & 0 & c & d \end{pmatrix},$$

for any real constants $a \neq 0$, b, c and $d \neq 0$. The different cases are the following.

1.4¹.2. There exist a real parameter p , such that

$$[u_1, u_4] = pu_1, \quad [u_2, u_4] = (p-1)u_2, \quad [u_3, u_4] = (p-2)u_3.$$

The Ricci tensor of the invariant metric g described by (3.6) is then given by

$$\varrho = \begin{pmatrix} 0 & 0 & \frac{3a(p-1)^2}{d} & 0 \\ 0 & -\frac{3a(p-1)^2}{d} & 0 & 0 \\ \frac{3a(p-1)^2}{d} & 0 & -\frac{b(3p^2-9p+8)}{d} & -\frac{3c(p-1)^2}{d} \\ 0 & 0 & -\frac{3c(p-1)^2}{d} & -3p^2 + 6p - 3 \end{pmatrix}.$$

1.4¹.9. There exist real parameters p, r , such that

$$[u_1, u_3] = u_1, \quad [u_2, u_3] = re_1 + u_2 + u_4, \quad [u_3, u_4] = pu_4.$$

The Ricci tensor of the invariant metric g described by (3.6) is given by

$$\varrho = \text{diag} \left(0, 0, -r - \frac{d}{2a} - p^2 - p, 0 \right).$$

1.4¹.10. There exist real parameters p, r , such that

$$[u_1, u_3] = u_1, \quad [u_2, u_3] = re_1 + u_2, \quad [u_3, u_4] = pu_4$$

and the Ricci tensor is

$$\varrho = \text{diag}(0, 0, -r - p^2 - p, 0).$$

1.4¹.11. In this case,

$$[u_1, u_3] = u_1, \quad [u_2, u_3] = re_1 + u_2 + u_4, \quad [u_3, u_4] = u_1 - u_4,$$

where r is a real constant. The Ricci tensor of the metric (3.6) is

$$\varrho = \text{diag}\left(0, 0, -r - \frac{d}{2a}, 0\right).$$

1.4¹.12. Then,

$$[u_1, u_3] = u_1, \quad [u_2, u_3] = re_1 + u_2, \quad [u_3, u_4] = u_1 - u_4,$$

where r is a real constant, and the Ricci tensor is

$$\varrho = \text{diag}(0, 0, -r, 0).$$

1.4¹.15, 1.4¹.16 and 1.4¹.17. In these cases,

$$[u_2, u_3] = \varepsilon e_1 + u_4, \quad [u_3, u_4] = u_1,$$

with $\varepsilon = 1$ for **1.4¹.15**, $\varepsilon = -1$ for **1.4¹.16** and $\varepsilon = 0$ for **1.4¹.17**. The Ricci tensor of the metric (3.6) is then given by

$$\varrho = \text{diag}\left(0, 0, -\varepsilon - \frac{d}{2a}, 0\right).$$

1.4¹.21 and 1.4¹.22. Then,

$$[u_2, u_3] = \varepsilon e_1, \quad [u_3, u_4] = u_1,$$

with $\varepsilon = 1$ for **1.4¹.21** and $\varepsilon = -1$ for **1.4¹.22**. The Ricci tensor is given by

$$\varrho = \text{diag}(0, 0, -\varepsilon, 0).$$

1.4¹.24 and 1.4¹.25. In this case,

$$[u_2, u_3] = \varepsilon e_1,$$

with $\varepsilon = 1$ for **1.4¹.24** and $\varepsilon = -1$ for **1.4¹.25**. The Ricci tensor is then given by

$$\varrho = \text{diag}(0, 0, -\varepsilon, 0).$$

2.2¹. There exists a basis $\{u_1, \dots, u_4\}$ of \mathfrak{m} and a basis $\{e_1, e_2\}$ of \mathfrak{h} , such that

$$\begin{aligned} [e_1, u_1] &= u_1, & [e_1, u_2] &= 0, & [e_1, u_3] &= -u_3, & [e_1, u_4] &= 0, \\ [e_2, u_1] &= 0, & [e_2, u_2] &= u_1, & [e_2, u_3] &= -u_4, & [e_2, u_4] &= 0. \end{aligned}$$

Then, with respect to $\{u_i\}$, invariant metrics are of the form

$$(3.7) \quad g = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & c & 0 & a \\ a & 0 & b & 0 \\ 0 & a & 0 & 0 \end{pmatrix}$$

for any real constants $a \neq 0$, b, c . The different cases are the following.

2.2¹.2. Then,

$$[u_1, u_2] = e_2, \quad [u_1, u_3] = u_4, \quad [u_2, u_3] = (p-1)u_3, \quad [u_2, u_4] = pu_4,$$

where p is a real constant. The Ricci tensor of the metric g described in (3.7) is then given by

$$\varrho = \text{diag} \left(0, \frac{p^2-4}{2}, 0, 0 \right).$$

2.2¹.3. Then,

$$[u_2, u_3] = u_3, \quad [u_2, u_4] = u_4.$$

The Ricci tensor of the metric g described in (3.7) is then given by

$$\varrho = \text{diag} \left(0, \frac{1}{2}, 0, 0 \right).$$

2.5¹. There exists a basis $\{u_1, \dots, u_4\}$ of \mathfrak{m} and a basis $\{e_1, e_2\}$ of \mathfrak{h} , such that

$$\begin{aligned} [e_1, u_1] &= 0, & [e_1, u_2] &= u_1, & [e_1, u_3] &= -u_4, & [e_1, u_4] &= 0, \\ [e_2, u_1] &= 0, & [e_2, u_2] &= 0, & [e_2, u_3] &= -u_2, & [e_2, u_4] &= u_1. \end{aligned}$$

Then, with respect to $\{u_i\}$, invariant metrics are of the form

$$(3.8) \quad g = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ a & 0 & b & 0 \\ 0 & a & 0 & 0 \end{pmatrix},$$

for any real constants $a \neq 0$ and b . The different cases are the following.

2.5¹.3. Then,

$$\begin{aligned} [u_1, u_3] &= u_1, & [u_2, u_3] &= e_1 + re_2 + (1-h)u_2, \\ [u_2, u_4] &= hu_1, & [u_3, u_4] &= -(r+h)e_1 + se_2 - (1+h)u_4, \end{aligned}$$

where h, r, s are real constants with $h \geq 0$ if $s \neq 0$. The Ricci tensor of the invariant metric (3.8) is then given by

$$\varrho = \text{diag} \left(0, 0, 2r + h - \frac{h^2}{2}, 0 \right).$$

2.5¹.4 In this case,

$$[u_1, u_3] = u_1, \quad [u_2, u_3] = re_2 + (1-h)u_2, \quad [u_2, u_4] = hu_1, \quad [u_3, u_4] = -(r+h)e_1 - (1+h)u_4,$$

where $h \geq 0$ and r are real constants. The Ricci tensor is then given by

$$\varrho = \text{diag} \left(0, 0, 2r + h - \frac{h^2}{2}, 0 \right).$$

2.5².2. We have

$$\begin{aligned} [e_1, u_1] &= 0, & [e_1, u_2] &= u_1, & [e_1, u_3] &= -u_2, & [e_1, u_4] &= 0, \\ [e_2, u_1] &= 0, & [e_2, u_2] &= 0, & [e_2, u_3] &= u_4, & [e_2, u_4] &= -u_1 \end{aligned}$$

and

$$\begin{aligned} [u_1, u_3] &= u_1, & [u_2, u_3] &= (p+s)e_1 + re_2 + u_2 - 2ru_4, \\ [u_2, u_4] &= 2ru_1, & [u_3, u_4] &= -re_1 + (p-s)e_2 - 2ru_2 - u_4, \end{aligned}$$

where p and $r, s \geq 0$ are real constants. The invariant metrics and corresponding Ricci tensor are then given by

$$g = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & a & 0 & 0 \\ a & 0 & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \quad \text{and} \quad \varrho = \text{diag}(0, 0, 2p - 2r, 0),$$

for any real constants $a \neq 0$ and b .

3.3¹.1. We have

$$\begin{aligned} [e_1, u_1] &= 0, & [e_1, u_2] &= u_2, & [e_1, u_3] &= 0, & [e_1, u_4] &= -u_4, \\ [e_2, u_1] &= 0, & [e_2, u_2] &= u_1, & [e_2, u_3] &= -u_4, & [e_2, u_4] &= 0 \\ [e_3, u_1] &= 0, & [e_3, u_2] &= 0, & [e_3, u_3] &= -u_2, & [e_3, u_4] &= u_1 \end{aligned}$$

and

$$[u_1, u_3] = u_1, \quad [u_2, u_3] = pe_3 + u_2, \quad [u_3, u_4] = -pe_2 - u_4,$$

where p is a real constant. The invariant metrics and corresponding Ricci tensor are then given by

$$g = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ a & 0 & b & 0 \\ 0 & a & 0 & 0 \end{pmatrix} \quad \text{and} \quad \varrho = \text{diag}(0, 0, 2p, 0),$$

for any real constants $a \neq 0$ and b .

3.3².1. In this case, we have

$$\begin{aligned} [e_1, u_1] &= 0, & [e_1, u_2] &= u_4, & [e_1, u_3] &= 0, & [e_1, u_4] &= -u_2, \\ [e_2, u_1] &= 0, & [e_2, u_2] &= u_1, & [e_2, u_3] &= -u_2, & [e_2, u_4] &= 0 \\ [e_3, u_1] &= 0, & [e_3, u_2] &= 0, & [e_3, u_3] &= u_4, & [e_3, u_4] &= -u_1 \end{aligned}$$

and

$$[u_1, u_3] = u_1, \quad [u_2, u_3] = pe_2 + u_2, \quad [u_3, u_4] = pe_3 - u_4,$$

where p is a real constant. Invariant metrics and corresponding Ricci tensor are given by

$$g = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & a & 0 & 0 \\ a & 0 & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \quad \text{and} \quad \varrho = \text{diag}(0, 0, 2p, 0),$$

for any real constants $a \neq 0$ and b .

Remark 3.1. Note that the expressions of the isotropy in cases **1.1¹**, **1.1²** and **2.2¹** given in [20] are more general than the ones we reported, but only in the cases we listed above they give rise to non-trivial homogeneous Ricci solitons.

We can now state the following classification result.

Theorem 3.2. *Let $(M = G/H, g)$ be a four-dimensional pseudo-Riemannian homogeneous non-trivial Ricci soliton, where g is an invariant pseudo-Riemannian metric, $\dim H \geq 1$ and equation (1.1) holds for some vector field $X \in \mathfrak{m}$. Then, one of the following cases occurs:*

1.1^{1.1} : g with $b = 0$, $c \neq \pm a$, λ is arbitrary and

$$X = \frac{a^2 - c^2 - 2da^2\lambda}{4a^2c}u_2 + \lambda u_4.$$

Such a vector field X is invariant.

1.1^{1.2} : one of the following cases occurs:

(1) $p = 1$, g with $b = 0$, λ is arbitrary and

$$X = x_1u_1 - \frac{2d\lambda - 1}{4c}u_2 + \lambda u_4.$$

X is invariant if and only if $x_1 = 0$.

(2) $p \neq \frac{1}{2}$, g with $b = 0$, $\lambda = 0$ and

$$X = x_1u_1 + \frac{2p - 1}{4cp}u_2.$$

X is invariant if and only if $x_1 = 0$.

1.1^{1.5} : g with $b = 0$, $\lambda = -\frac{1}{a}$ and

$$X = x_1u_1 + \frac{d}{2ac}u_2 + x_3u_3 - \frac{1}{a}u_4.$$

X is invariant if and only if $x_1 = x_3 = 0$.

1.1^{2.1} : g with $b = 0$, λ is arbitrary and

$$X = \frac{4a^2 + d^2 - 2\lambda a^2c}{8a^2d}u_2 + \frac{1}{2}\lambda u_4.$$

X is invariant.

1.1^{2.2} : $p \neq 1$, g with $b = 0$, $\lambda = 0$ and

$$X = \frac{p - 1}{2dp}u_2.$$

X is invariant.

1.1^{2.6} and **1.1^{2.7}**: one of the following cases occurs:

(1) g with $b = c = 0$, $\lambda = \frac{\varepsilon}{a}$ and

$$X = x_1u_1 + x_3u_3 + \frac{\varepsilon}{a}u_4,$$

for any real value of x_1, x_3 . X is invariant if and only if $x_1 = x_3 = 0$.

(2) g with $c = 0$, $b = \frac{\varepsilon d^2}{2a}$, $\lambda = \frac{\varepsilon}{a}$ and

$$X = x_1 u_1 + x_3 u_3 + \frac{\varepsilon}{2a} u_4,$$

for any real value of x_1, x_3 . X is invariant if and only if $x_1 = x_3 = 0$.

(3) g with $ab(bc - 2d^2) = -\varepsilon(bc - d^2)^2$, $\lambda = -\frac{b(bc - 2d^2)}{(bc - d^2)^2}$ and

$$X = x_1 u_1 - \frac{bcd}{(bc - d^2)^2} u_2 + x_3 u_3 + \frac{bd^2}{(bc - d^2)^2} u_4,$$

for any real value of x_1, x_3 . X is invariant if and only if $x_1 = x_3 = 0$.

1.1².8 : g with $c = \frac{2d^2}{b} \neq 0$, $\lambda = 0$ and

$$(3.9) \quad X = x_1 u_1 - \frac{2}{d} u_2 + x_3 u_3 + \frac{b}{d^2} u_4,$$

for any real value of x_1, x_3 . X is invariant if and only if $x_1 = x_3 = 0$.

1.3¹.1 : g with $d = 0 \neq c$, $\lambda = 0$ and

$$X = -\frac{15c}{8a^2} u_2.$$

X is invariant.

1.3¹.2 : $p = 0$, g with $d = b \neq 0$, $\lambda = -\frac{1}{2b}$ and

$$X = x_1 u_1 + \left(x_1 - \frac{b+c}{2ab} \right) u_2 - \frac{1}{2b} u_3 - \frac{1}{2b} u_4,$$

where x_1 is a real constant. X is not invariant.

1.3¹.3 : g with $d = b \neq 0$, $\lambda = -\frac{1}{2b}$ and

$$X = x_1 u_1 + \left(x_1 - \frac{b+c}{2ab} \right) u_2 - \frac{1}{2b} u_3 - \frac{1}{2b} u_4,$$

where x_1 is a real constant. X is not invariant.

1.3¹.5 with one of the following conditions:

(1) $p = q = 0$, g with $c = 0$, $b \neq d$, $\lambda = \frac{2}{b-d}$ and

$$X = -\frac{d\lambda}{2a} u_1 + x_2 u_2 + \lambda u_4,$$

where x_2 is a real constant. X is not invariant.

(2) $q = 0 \neq p$, g is arbitrary, $\lambda = 0$ and

$$X = -\frac{p^2 + 4}{4ap} u_2.$$

X is invariant.

1.3¹.7 : $p = 0$, g is arbitrary, $\lambda = 0$ and

$$X = -\frac{1}{4a} u_2.$$

X is invariant.

1.3^{1.8} : g with $b = 0$, $c \neq 0$, $\lambda = 0$ and

$$X = x_1 u_1 + x_2 u_2 + \frac{1}{4c} u_3,$$

where x_1 and x_2 are real constants. X is not invariant.

1.3^{1.9} : $p \neq 1$, g is arbitrary, $\lambda = 0$ and

$$X = x_1 u_1 - \frac{(p-1)^2 b}{4ac} u_2 + \frac{(p-1)^2}{4cp} u_3,$$

where x_1 is a real constant. X is not invariant.

1.3^{1.11} : g with $c \neq 0$, $\lambda = 0$ and

$$X = x_1 u_1 - \frac{b}{2ac} u_2 - \frac{1}{2c} u_3,$$

where x_1 is a real constant. X is not invariant.

1.3^{1.12} with one of the following conditions:

(1) $2p^2 - 5pq + p + q^2 - 3q - 1 \neq 0$, g is arbitrary, $\lambda = 0$ and

$$X = -\frac{2p^2 - 5pq + p + q^2 - 3q - 1}{8a} u_1.$$

X is invariant.

(2) $p = -q$, g is arbitrary, $\lambda = 0$ and

$$X = -\frac{8q^2 - 4q - 1}{8a} u_1 + x_2 u_2,$$

where x_2 is a real constant. X is invariant if and only if $x_2 = 0$.

(3) $p = -\frac{1}{2}q$, g with $b = 0$ and

$$X = \frac{-8d\lambda + 8q^2 - 7q - 2}{16a} u_1 + \frac{3c\lambda}{a} u_2 + \lambda u_4.$$

X is invariant if and only if $\lambda = 0$.

(4) $p = \frac{1}{8}$, $q = -\frac{1}{4}$, g with $b = 0$ and

$$X = x_1 u_1 + \frac{3c\lambda}{a} u_2 + x_3 u_3 + \lambda u_4,$$

for any real value of x_3 , where either $\lambda = 0$ (and x_1 is arbitrary) or $x_1 = -\frac{1+16cx_3+32d\lambda}{64a}$. X is invariant if and only if $\lambda = x_3 = 0$.

(5) $q = -\frac{1}{4}$, g with $b = 0$, $\lambda = 0$ and

$$X = \frac{-32cx_3 + 32p^2 + 36p - 3}{128a} u_1 + x_3 u_3,$$

where x_3 is a real constant. X is invariant if and only if $x_3 = 0$.

(6) $p = q = 0$, g with $b = 0$, λ is arbitrary and

$$X = -\frac{4d\lambda + 1}{8a} u_1 + x_2 u_2 + \lambda u_4,$$

where x_2 is a real constant. X is invariant if and only if $\lambda = 0$.

(7) $p = -q = \frac{1}{4}$, g is arbitrary, $\lambda = 0$ and

$$X = -\frac{4cx_3 + 1}{16a}u_1 + x_2u_2 + x_3u_3,$$

where x_2, x_3 are real constants. X is invariant if and only if $x_3 = 0$.

(8) $p = -\frac{1}{3}$, $q = \frac{2}{3}$, g is arbitrary, λ is arbitrary and

$$X = -\frac{18d\lambda + 7}{36a}u_1 + \frac{3c\lambda}{a}u_2 + \lambda u_4.$$

X is invariant if and only if $\lambda = 0$.

1.3¹.13 with one of the following conditions:

(1) $4p^2 - 4p - 3 \neq 0$, g is arbitrary, $\lambda = 0$ and

$$X = \frac{4p - 4p^2 + 3}{16a}u_1.$$

X is invariant.

(2) $p = 0$, g with $b = c = 0$, λ is arbitrary and

$$X = -\frac{8d\lambda + 3}{16a}u_1 + x_2u_2 + \lambda u_4,$$

for any real value of x_2 . X is invariant if and only if $x_2 = 0$.

1.3¹.14 : $p(p - 1) \neq 0$, g is arbitrary, $\lambda = 0$ and

$$X = -\frac{p(p - 1)}{a}u_1.$$

X is invariant.

1.3¹.19 : g is arbitrary, $\lambda = 0$ and

$$X = \frac{1}{4a}u_1.$$

X is invariant.

1.3¹.20 with one of the following conditions:

(1) g is arbitrary, $\lambda = 0$ and

$$X = x_1u_1 + \frac{1}{4a}u_2,$$

for any real value of x_1 . X is invariant.

(2) g with $b = 0$, $\lambda = 0$ and

$$X = x_1u_1 - \frac{4cx_3 + 1}{4a}u_2 + x_3u_3,$$

where x_1 and x_3 are real constants. X is invariant if and only if $x_3 = 0$.

1.3¹.21 : $p(p - 2) \neq 0$, g is arbitrary, $\lambda = 0$ and

$$X = -\frac{p(p - 2)}{4a}u_1.$$

X is invariant.

1.3¹.22 : g is arbitrary, $\lambda = 0$ and

$$X = \frac{3}{16a}u_1.$$

X is invariant.

1.3¹.28 : g with $c = 0$ and $b - 6d \neq 0$, $\lambda = \frac{1}{2(b-6d)}$ and

$$X = x_1u_1 + \frac{b-9d}{2a(b-6d)}u_2 - \frac{1}{2(b-6d)}u_3,$$

where x_1 is a real constant. X is not invariant.

1.4¹.2 : $p = 1$, g is arbitrary, $\lambda = 0$ and

$$X = -\frac{1}{d}\left(\frac{c}{a}u_1 + u_4\right).$$

X is invariant.

1.4¹.9 with one of the following conditions:

(1) g with $d + 2ra + 2ap(p+1) \neq 0$, $\lambda = 0$ and

$$X = \frac{d + 2ra + 2ap(p+1)}{4a^2}u_1.$$

X is invariant.

(2) $p = 0$, g is arbitrary, $\lambda = 0$ and

$$X = \frac{2ra + d}{4a^2}u_1 + x_4u_4,$$

where x_4 is a real constant. X is invariant.

1.4¹.10 : $r + p(p+1) \neq 0$, g is arbitrary, $\lambda = 0$ and

$$X = \frac{r + p(p+1)}{2a}u_1.$$

X is invariant.

1.4¹.11 : with one of the following conditions:

(1) g with $2ra + d \neq 0$, $\lambda = 0$ and

$$X = \frac{2ra + d}{4a^2}u_1.$$

X is invariant.

(2) $p = 0 \neq r$, g is arbitrary, $\lambda = 0$ and

$$X = \frac{2(a+c)x_4 + r}{2a}u_1 + x_4u_4.$$

X is invariant.

1.4¹.15, 1.4¹.16 and **1.4¹.17** : $2\epsilon a + d \neq 0$, g is arbitrary, $\lambda = 0$ and

$$X = x_1u_1 - \frac{2\epsilon a + d}{4a^2}u_4$$

for any real value of x_1 . X is invariant.

1.4¹.21 and **1.4¹.22** : g is arbitrary, $\lambda = 0$ and

$$X = x_1u_1 + x_2u_2 - \frac{\varepsilon}{2a}u_4,$$

for any real value of x_1, x_2 . X is invariant if and only if $x_2 = 0$.

1.4¹.24 and **1.4¹.25** : g is arbitrary, $\lambda = 0$ and

$$X = x_1u_1 + x_2u_2 - \frac{\varepsilon}{2a}u_4,$$

for any real value of x_1, x_2 . X is invariant if and only if $x_2 = 0$.

2.2¹.2 : $p \neq \pm 2$, g is arbitrary, $\lambda = 0$ and

$$X = -\frac{p^2 - 4}{4ap}u_4.$$

X is invariant.

2.2¹.3 with one of the following conditions:

(1) g is arbitrary, $\lambda = 0$ and

$$X = x_1u_1 - \frac{1}{4a}u_4,$$

for any real value of x_1 . X is invariant if and only if $x_1 = 0$.

(2) g with $b = 0$, λ is arbitrary and

$$X = x_1u_1 - \lambda u_2 + \frac{2c\lambda - 1}{4a}u_4,$$

for any real value of x_1 . X is invariant if and only if $\lambda = x_1 = 0$.

2.5¹.3 and **2.5¹.4** : $2r + h - \frac{1}{2}h^2 \neq 0$, g is arbitrary, $\lambda = 0$ and

$$X = \frac{4r + 2h - h^2}{4a}u_1.$$

X is invariant.

2.5².2 : $p \neq r$, g is arbitrary, $\lambda = 0$ and

$$X = \frac{p - r}{a}u_1.$$

X is invariant.

3.3¹.1 : $p \neq 0$, g is arbitrary, $\lambda = 0$ and

$$X = \frac{p}{a}u_1.$$

X is invariant.

3.3².1 : $p \neq 0$, g is arbitrary, $\lambda = 0$ and

$$X = \frac{p}{a}u_1.$$

X is invariant.

Proof. The complete classification follows from a case-by-case argument. One first excludes the case where $[u_i, u_j]_{\mathfrak{m}} = 0$ for all indices i, j . In fact, in such a case, any vector field $X \in \mathfrak{m}$ is Killing and so, equation (2.4) reduces to the Einstein equation $\varrho = \lambda g$. In the remaining cases, we computed the Levi-Civita connection and the Ricci tensor using (2.1) and (2.3) and solved (2.4). When some solutions of (2.4) were found, we excluded the Einstein cases (trivial Ricci solitons) and checked the invariancy of the solution under the isotropy action.

As an example, we report here the calculations for the case **1.1².8**, where both invariant and not invariant solutions of (2.4) occur. So, let $M = G/H$ be a four-dimensional homogeneous space, such that the Lie algebra \mathfrak{g} and the isotropy subalgebra \mathfrak{h} are determined by conditions (3.2) and $[u_2, u_4] = u_2$, $[u_i, u_j] = 0$ otherwise.

We first determine invariant pseudo-Riemannian metrics on M , which are the same for all the cases **1.1²** corresponding to non-trivial Ricci solitons. Starting from an arbitrary bilinear symmetric form g , the isotropy representation as described by (3.2) easily yields that g is invariant (that is, $\psi(e_1)^t \circ g + g \circ \psi(e_1) = 0$) if and only if g is of the form given in (3.3). We then apply (2.1) to compute $\Lambda(u_i)$ for all indices $i = 1, \dots, 4$, so describing the Levi-Civita connection of g . We get

$$\Lambda(e_1) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\Lambda(u_1) := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Lambda(u_2) := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{bd}{bc-d^2} & 0 & \frac{bc}{bc-d^2} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{b^2}{bc-d^2} & 0 & -\frac{bd}{bc-d^2} \end{pmatrix},$$

$$\Lambda(u_3) := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Lambda(u_4) := \begin{pmatrix} 0 & \frac{d^2}{bc-d^2} & 0 & \frac{cd}{bc-d^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{bd}{bc-d^2} & 0 & -\frac{d^2}{bc-d^2} \end{pmatrix}.$$

The curvature and Ricci tensors can be now deduced from the above formulas by a direct calculation applying (2.2) and (2.3). In particular, the Ricci tensor has the form described in (3.4) with $\varepsilon = 0$. Moreover, it is easily seen from (3.3) and (3.4) that for $\varepsilon = 0$ (case **1.1².08**), the manifold $(M = G/H, g)$ is Einstein (more precisely, Ricci-flat) if and only if $b = 0$.

We now consider an arbitrary vector field $X = \sum_{k=1}^4 x_k u_k \in \mathfrak{m}$ and a real constant λ and write down (2.4). We find that X and λ determine a Ricci soliton if and only if the

components x_k of X with respect to $\{u_k\}$ and λ satisfy

$$(3.10) \quad \begin{cases} a\lambda = 0, \\ \frac{ad(dx_2+cx_4)}{bc-d^2} = 0, \\ \frac{b(-2(bc-d^2)^2x_4+b^2c-2bcd^2\lambda+d^4\lambda+b^2c^2\lambda)}{(bc-d^2)^2} = 0, \\ \frac{b^2c(bc-d^2)x_2+d^3(bc-d^2)x_4+2b^2cd-bd^3-2bcd^3\lambda+d^5\lambda+b^2c^2d\lambda}{(bc-d^2)^2} = 0, \\ \frac{c(2bd(bc-d^2)x_2+2d^2(bc-d^2)x_4+b^2c-2bcd^2\lambda+d^4\lambda+b^2c^2\lambda)}{(bc-d^2)^2} = 0. \end{cases}$$

It must be noted that components x_1, x_3 do not appear in (3.10). We now solve the system (3.10). Taking into account $a(bc-d^2) \neq 0$, we find $\lambda = 0$ (and so, the Ricci solitons are steady), $bc-2d^2=0$ (which may be rewritten as $c = \frac{2d^2}{b}$, since $bc-d^2 = d^2 \neq 0$) and

$$x_2 = -\frac{2}{d}, \quad x_4 = \frac{b}{d^2},$$

for any real value of x_1, x_3 . So, X has the form (3.9). Since $b \neq 0$, no Einstein cases occur. Finally, again by (3.2) we see at once that $X \in \mathfrak{m}$ is invariant if and only if $X \in \text{Span}\{e_2, e_4\}$. Therefore, there exists a two-parameter family of non-trivial (steady) Ricci solitons on $(M = G/H, g)$ determined by vector fields $X \in \mathfrak{m}$ of the form (3.9), but among them, only

$$X = -\frac{2}{d}u_2 + \frac{b}{d^2}u_4$$

is invariant and so, determines a homogeneous Ricci soliton. \square

It is easy to check that vector fields X of any causal character occur in the list of vector fields determining homogeneous Ricci solitons in Theorem 3.2. In particular, several examples occur where X is a light-like vector field.

4. GEOMETRIC PROPERTIES OF FOUR-DIMENSIONAL HOMOGENEOUS RICCI SOLITONS

By Theorem 3.2, there exist a large number of four-dimensional pseudo-Riemannian homogeneous Ricci solitons with non-trivial isotropy. It is a natural problem to investigate the geometry of these examples, for instance whether they are gradient Ricci solitons, their curvature properties and their relationship with some other geometric structures. Readers interested in the geometry of the homogeneous Ricci soliton metrics can easily work out several interesting cases from the above classification. A few remarkable behaviours are listed below.

4.1. Gradient Ricci solitons. We recall that a gradient Ricci soliton is a pseudo-Riemannian manifold (M, g) , together with a smooth function f on M , such that equation (1.1) holds with $X = \text{Hess}(f)$. Several classification and rigidity results hold in the specific case of gradient Ricci solitons. Some examples may be found in [4],[23],[24],[25].

If X is (locally) a gradient, then the one-form ω_X dual to X must be closed. Checking this condition for the Ricci solitons listed in Theorem 3.2, we obtain the following.

Proposition 4.1. *Let (M, g, X) be any of the homogeneous Ricci solitons listed in Theorem 3.2, where $(M = G/H, g)$ is a four-dimensional pseudo-Riemannian homogeneous space and $X \in \mathfrak{m}$ a solution of (1.1). The one-form ω_X dual to X is never closed in the following cases:*

1.1² : **2** and **8**;

1.3¹ : **1, 6, 7, 8, 11, 14, 19, 20, 21** and **22**;

1.1⁴ : **10, 11, 12, 15, 16** and **17**;

2.2¹.2;

2.5¹ : **3** and **4**;

2.5².2;

3.3¹.1;

3.3².1.

In the remaining cases, ω_X is closed under some restrictions on either the invariant metric g or the coefficients describing the Lie algebra.

Thus, in all the cases listed in Proposition 4.1, the homogeneous Ricci soliton cannot be a gradient one.

4.2. Conformally flat examples. In general a pseudo-Riemannian manifold (M, g) of dimension $n \geq 4$ is (locally) conformally flat if and only if

$$W = R - \frac{1}{n-2}(\varrho - \frac{\tau}{n}g) \odot g - \frac{\tau}{2n(n-1)}g \odot g = 0,$$

where W is the Weyl curvature tensor, τ the scalar curvature of (M, g) and \odot denotes the Kulkarni-Nomizu product of two symmetric two-tensors. With regard to homogeneous Ricci soliton metrics listed in Theorem 3.2, we have the following.

Theorem 4.2. *A four-dimensional homogeneous Ricci soliton (M, g) (with non-trivial isotropy), as classified in Theorem 3.2, is conformally flat if and only if it corresponds to one of the following cases:*

1.1¹.1;

1.1².1;

1.3¹.2; **1.3¹.5**, (1) with $2c = -pd$; **1.3¹.7** with $2c = -d$; **1.3¹.8**; **1.3¹.9** with $p = -1$;
1.3¹.19 with $b = 0$; **1.3¹.21** with either $p = \frac{1}{2}$ or $b = 0$; **1.3¹.28** with $b = 2d$;

1.4¹.2 with $b = 0 \neq d$; **1.4¹.9** with $p = -1/2$ and $r = -\frac{a+4d}{4a}$; **1.4¹.15** with $d = -a$;

2.2¹.2 with either $p = 0$, or $b = 0$;

2.2¹.3, (1) with $b = 0$; **2.2¹.3**, (2);

3.3¹.1;

3.3².1.

A conformally flat Einstein pseudo-Riemannian manifold is of constant curvature and so, symmetric. Moreover, a conformally flat (locally) homogeneous Riemannian manifold is again (locally) symmetric [27] (see [5] for some three-dimensional Lorentzian non-symmetric examples).

However, replacing the Einstein condition with its generalization (1.1), these rigidity results do not hold any more. In fact, checking the (local) symmetry condition $\nabla R = 0$ for the examples classified in Theorem 4.2, we obtain the following.

Corollary 4.3. *Examples 1.1¹.1, 1.1².1, 1.3¹.5, 1.3¹.7, 1.3¹.19, 1.3¹.21, 1.3¹.28, 1.4¹.9, 2.2¹.2 with $b = 0 \neq p$, 2.2¹.3, 3.3¹.1 with $r \neq 0$ and 3.3².1 with $r \neq 0$, are four-dimensional conformally flat pseudo-Riemannian homogeneous Ricci solitons which are not symmetric.*

Lorentzian conformally flat gradient Ricci solitons were classified in [4]. In particular, the ones determined by light-like vector fields are necessarily steady. Among four-dimensional pseudo-Riemannian homogeneous Ricci solitons with non-trivial isotropy, there are no examples which are at the same time Lorentzian (non-gradient), non-steady and defined by a light-like vector field $X \in \mathfrak{m}$. However, there exist such examples with metrics of neutral signature (2, 2). For example, 1.3¹.5, (1) is non-steady and always defined by a light-like vector field, and is conformally flat when $c = -\frac{1}{2}pd$.

4.3. Existence of examples which are not solvsolitons. A solvsoliton is a Ricci soliton left-invariant metric g on a solvable Lie group G . All known examples of homogeneous Riemannian Ricci soliton metrics on non-compact homogeneous manifolds are isometric to some solvsolitons ([17], Remark 1.5). On the other hand, some of the examples listed in Theorem 3.2 show that *there exist homogeneous (and also invariant) pseudo-Riemannian Ricci solitons which are not isometric to solvmanifolds*. In fact, we found some homogeneous and invariant Ricci solitons in cases 1.3¹.1 and 1.4¹.2. It is easily seen that such homogeneous pseudo-Riemannian manifolds correspond respectively to cases **B1** and **A2** in the classification of *non-reductive* four-dimensional homogeneous pseudo-Riemannian manifolds obtained in [13] (see also [7]). These spaces are diffeomorphic to \mathbb{R}^4 [13] and so, non-compact. As they do not admit any reductive decomposition, obviously they cannot be isometric to any Lie group (solvable or not).

4.4. On the form of the Ricci operator. For all the left-invariant Ricci solitons on three-dimensional Lorentzian Lie groups, the Ricci operator is not diagonalizable with a unique eigenvalue of multiplicity three [3]. A similar behaviour is found for Ricci soliton homogeneous metrics of non-reductive four-dimensional homogeneous manifolds [7]. This could suggest that the large variety of pseudo-Riemannian examples (with respect to the Riemannian ones) is related to such a special admissible form of the Ricci operator or, more generally, to the fact that the Ricci operator may be not diagonalizable.

However, for the Ricci operator of homogeneous Ricci soliton metrics classified in Theorem 3.2, different behaviours occur. Here, we limit ourselves to point out one among several examples where the Ricci operator is diagonalizable, namely, 1.1².8. The homogeneous metric g_{abcd} given by (3.3) is a Ricci soliton if and only if $c = \frac{2d^2}{b} \neq 0$. Equations (3.3) and (3.4) yield that the Ricci operator is then given by

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{b}{d^2} & 0 & -\frac{b^2}{d^3} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{2}{d} & 0 & -\frac{b}{d^2} \end{pmatrix}.$$

It is easy to check that the Ricci eigenvalues are 0 (of multiplicity two) and $-(1 \pm \sqrt{2})\frac{b}{d^2}$, and that Q is diagonalizable.

4.5. Examples which are not Kähler. In the Riemannian case, most of the known examples of non-trivial Ricci solitons correspond to Kähler metrics (*Kähler-Ricci solitons*), that is, they admit a Kähler structure and equation (1.1) holds for a holomorphic vector field X (see [9, 12] and references therein). However, *none of our examples of non-trivial pseudo-Riemannian homogeneous Ricci solitons is Kähler*. In fact, invariant Kähler structures on four-dimensional homogeneous pseudo-Riemannian manifolds with non-trivial isotropy were classified by the present authors in [8]. Comparing the classification of invariant Kähler metrics given in [8] with the one of invariant Ricci soliton metrics listed in Theorem 3.2, it turns out that the only candidates as Kähler-Ricci solitons would be **1.1²** : **1, 6, 7, 8**. However, in these cases, none of vector fields satisfying (1.1) is holomorphic.

As an example, we report the computations for **1.1².1**. By Theorem 3.2, the invariant metric g described in (3.3) is a Ricci soliton if and only if $b = 0$. This is also a Kähler metric [8]. In fact, g is compatible with the two-parameter family of complex structures

$$J_{\alpha\beta} = \pm \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & \sqrt{-\alpha\beta - 1} & 0 & \alpha \\ -1 & 0 & 0 & 0 \\ 0 & \beta & 0 & -\sqrt{-\alpha\beta - 1} \end{pmatrix},$$

for any real constants α, β such that $-\alpha\beta - 1 \geq 0$, and the pair $(g, J_{\alpha\beta})$ is pseudo-Kähler if and only if

$$c(\alpha - \beta) + d\sqrt{-\alpha\beta - 1} = 0.$$

Again by Theorem 3.2, vector fields for which g satisfies (1.1) are given by

$$X = \frac{4a^2 + d^2 - 2\lambda a^2 c}{8a^2 d} u_2 + \frac{1}{2} \lambda u_4,$$

for any real value of λ . A standard calculation yields that $\mathcal{L}_X J_{\alpha\beta} \neq 0$. Therefore, X is not holomorphic.

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