

# Stochastic Risk Premiums, Stochastic Skewness in Currency Options, and Stochastic Discount Factors in International Economies

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# Stochastic Risk Premiums, Stochastic Skewness in Currency Options, and Stochastic Discount Factors in International Economies

## ABSTRACT

We develop dynamic models of stochastic discount factors in international economies that produce stochastic risk premiums and stochastic skewness in currency options. The source of stochastic risk premiums and stochastic skewness can be stochastic volatility in the uncertainty of the economy or stochastic market price of the uncertainty, or both. We estimate these models using the joint time-series of option prices on dollar-yen, dollar-pound, and pound-yen exchange rates, and the underlying currency returns. The estimation reveals several results about the structure of risk premiums in the international economy. First, the average risk premium in Japan is significantly larger than the risk premium in the US or the UK. Second, the risk premium on the global risk factor is both more persistent and more volatile than the risk premiums on country-specific risks. Third, investors respond to shocks differently depending on whether the origins of the shocks are global or country-specific. The risk premium increases when the global (domestic) risk factor receives a positive (negative) shock, suggesting that investors demand a risk premium when their wealth declines relative to the global portfolio. Finally, the uncertainty in each economy contains a jump component that arrives an infinite number of times within any finite interval, but only downside jumps appear to be priced.

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KEY WORDS: Stochastic discount factors; international economy; stochastic risk premium; stochastic skewness; currency options; foreign exchange rate dynamics; time-changed Lévy processes; unscented Kalman filter.

Theoretical models of foreign exchange dynamics have positive and normative implications. Positively, the models assist us in understanding the possible departures between exchange rates and fundamentals. See, for example, Dumas (1992), Mark (1995), Evans and Lyons (2002), Engel and West (2004), and Pavlova and Rigobon (2004), for recent contributions. Starting with Lucas (1982), exchange rates constitute crucial building blocks for testable multi-period equilibrium models of the international economy. More generally, the endogenously derived models help us appreciate the links between the price of forward-looking currency derivatives and the distributional properties of the exchange rate (Garman and Kohlhagen (1983), Dumas, Jennergren, and Naslund (1995), Bates (1996), and Bakshi and Chen (1997)). On the normative side, exchange rate models can be used to advocate monetary and fiscal policy rules and for prescribing central bank interventions.

In this paper, we develop dynamic models of stochastic discount factors in international economies that are consistent with three distinct, yet interrelated, phenomena observed in currency markets. First, the risk-reversal quotes, as measured by the difference in the Black and Scholes (1973) implied volatilities between out-of-the-money call and put currency options, show substantial time-variation and often switch signs. This options market feature is symptomatic of stochastic skewness in the conditional currency return distribution. Second, the butterfly spread quotes, defined as the average out-of-the-money call and put implied volatilities minus the at-the-money counterpart, are uniformly positive across different option maturities and different underlying currencies, indicating fat-tailed risk-neutral currency return distributions. Third, extant empirical studies have documented strongly time-varying currency risk premiums (Fama (1984), Bekaert and Hodrick (1992), Dumas and Solnik (1995), Engel (1996), Bansal (1997), Backus, Foresi, and Telmer (2001), and Brennan and Xia (2005)).

What are the sources of stochastic risk premiums in international economies? What minimal theoretical structures are needed to internalize the evidence from currency returns and currency options? How do the different types of risks (say, global versus country-specific) vary over time and how are they priced differently? How are the risk premium dynamics related to observed stochastic skewness and kurtosis in the conditional currency return distribution? The thrust of this research is to answer these questions from both theoretical and empirical perspectives.

In any economy precluding arbitrage, there always exists a stochastic discount factor that links future payoffs to their intrinsic values (Harrison and Kreps (1979)). In a complete market, this stochastic discount factor is unique, and the ratio of the stochastic discount factors in the two economies determines the exchange rate between them. In the setting of Lucas (1982)), for instance, the equilibrium home-currency price of a foreign currency is the ratio of the foreign-country marginal utility to the home-country counterpart. Therefore, exchange rates offer a direct unfiltered window to the stochastic

discount factors and the relative risk-taking behaviors of investors in international economies. In this paper, we propose to identify the dynamics of stochastic discount factors in international economies using the time-series of currency returns and option prices. Specifically, using the time-series of option prices on the dollar-yen, dollar-pound, and pound-yen exchange rates, and the underlying currency returns, we study the dynamic behaviors of the stochastic discount factors and stochastic risk premiums in the three economies of the US, Japan, and the UK.

We propose a class of models of stochastic discount factors that are parsimonious in terms of identification and yet flexible to accommodate both stochastic risk premiums and stochastic skewness in the currency return distribution, which we theoretically show are inherently linked. Our model specifications allow the stochastic risk premiums on global and country-specific risks to be controlled by separate dynamic processes. Through this parameterization, we can empirically study how the risk premiums of an economy react differently to shocks on different types of risks.

If economies are strictly symmetric, the global risk component cancels in the log ratio of the two stochastic discount factors. This feature renders the global risk components of the stochastic discount factors unidentifiable from the return and options data on exchange rates. Nevertheless, our estimation shows that the three economies under investigation are asymmetric. In particular, the average risk premium in Japan is about 50 percent higher than the average risk premium in the US or the UK.

Given the asymmetry between the three economies, we can identify the risk premium dynamics on both the global risk component and the country-specific components. The estimation shows that the risk premium dynamics on the two types of risks are quite different. The risk premium on the global risk factor is both more persistent and more volatile than the risk premium on the country-specific risk factors. This empirical evidence implies that the dynamics of the stochastic discount factors share a large global risk component, suggesting that the currency option markets embed a high degree of integration among the three economies.

The estimation results also reveal that investors respond to global shocks and country-specific shocks in markedly different ways. Investors increase their risk premium when the country-specific risks receive a negative shock. In contrast, the risk premium declines when the global risk component receives a negative shock. Such a dichotomy between reactions to global and country-specific risks suggests that the risk preference of investors in an economy varies with the *relative* wealth of the economy, with the global portfolio as the benchmark. Investors demand a higher risk premium when their wealth declines relative to the global portfolio. A negative shock to the country-specific risk components and a positive shock to the global risk factor both generate a negative impact on the relative wealth of the economy and is hence associated with a rise in risk premiums.

Finally, the estimation identifies a jump component in each economy that arrives an infinite number of times within any finite time intervals. This currency-options based finding contradicts with traditional compound Poisson jump specifications (Merton (1976)), but strengthens the findings from recent empirical works on equity index options (Carr, Geman, Madan, and Yor (2002), Carr and Wu (2003), Huang and Wu (2004), and Wu (2004)). More interestingly, our multi-economy estimation identifies a significant downward jump component but not an upward jump component in the stochastic discount factors, suggesting that although the economy can receive both negative and positive shocks, investors are only concerned with downside jumps as a potential source of risk. Upside jumps are not priced.

Traditional literature often studies the behavior of risk premiums through various types of expectation hypothesis regressions. Under the null hypothesis of zero or constant risk premium, the slope coefficients of these regressions should be one. Hence, the point estimates on the regression slopes reveal whether the risk premium is constant or time-varying. More recently, researchers have recognized the rich information content of option markets and started to infer the risk premium behavior from a joint analysis of options and the underlying assets. The focus of this strand of literature is on equity index and equity index options in a single economy, mainly the United States.<sup>1</sup> In this setting, the estimated stochastic discount factors are typically one-dimensional projections on the single equity index. The pricing of risks that are orthogonal to the equity index is largely missed by this projection. Furthermore, it is difficult to use a one-dimensional projection to study the multi-dimensional nature of the stochastic discount factors in international economies. We contribute to the literature by proposing to infer the multi-dimensional dynamic behaviors of stochastic discount factors in international economies using currency returns and currency options.

The paper is organized as follows. Section I outlines the connection between stochastic discount factor dynamics and nominal exchange rates. Section II proposes a class of models for stochastic discount factors that includes both a global risk factor and country-specific risk factors. Our framework allows the risk premiums on the two types of risks to follow separate dynamics. Within this general setup, we analyze what minimal structures are necessary to capture the stylized evidence in currency returns and currency options. We then derive tractable forms for option pricing and for the characteristic function of the currency returns. Section III describes the currency and currency options data set for dollar-yen, dollar-pound, and pound-yen exchange rates. Section IV presents a joint estimation

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<sup>1</sup>Prominent examples include Jackwerth and Rubinstein (1996), Pan (2002), Engle and Rosenberg (2002), Bakshi, Kapadia, and Madan (2003), Bakshi and Kapadia (2003), Broadie, Chernov, and Johannes (2004), Eraker (2004), and Bliss and Panigirtzoglou (2004). Recently, Driessen and Maenhout (2004) investigate the nature of jump and volatility risks using equity index options from three countries.

approach using the time-series of currency returns and currency option prices on the three currency pairs. Section V discusses the estimation results and Section VI concludes.

## I. Stochastic Discount Factors in International Economies and Exchange Rate Dynamics

We describe a set of  $N$  economies by fixing a filtered complete probability space  $\{\Omega, \mathcal{F}, \mathcal{P}, (\mathcal{F}_t)_{0 \leq t \leq \mathcal{T}}\}$ , with some fixed horizon  $\mathcal{T}$ . We assume no arbitrage in each economy. Therefore, for each economy, we can identify at least one strictly positive process,  $\mathcal{M}_t^h$  ( $h = 1, \dots, N$ ), which we call the state-price deflator, such that the deflated gains process associated with any admissible trading strategy is a martingale (Duffie (1992)). We further assume that  $\mathcal{M}_t^h$  itself is a semimartingale. The ratio of  $\mathcal{M}_t^h$  at two time horizons will henceforth be referred as the stochastic discount factor, or the pricing kernel.

We use  $X^h$  to summarize the aggregate uncertainty in economy  $h$  and represent the state-price deflator via the following multiplicative decomposition:

$$\mathcal{M}_t^h = \exp\left(-\int_0^t r_s^h ds\right) \mathcal{E}\left(-\int_0^t \gamma_s^h dX_s^h\right), \quad h = 1, 2, \dots, N, \quad (1)$$

where  $r_t^h$  denotes the instantaneous interest rate in economy  $h$ ,  $\gamma_t^h$  denotes the market price of risk in economy  $h$ , and  $\mathcal{E}(\cdot)$  denotes the *Doléans-Dade* exponential operator (Jacod and Shiryaev (1987)), which can be interpreted as the Radon-Nikodým derivative that takes us from the statistical measure  $\mathcal{P}$  to the economy- $h$  risk-neutral measure  $Q^h$ :

$$\frac{dQ^h}{d\mathcal{P}} \Big|_t \equiv \mathcal{E}\left(-\int_0^t \gamma_s^h dX_s^h\right). \quad (2)$$

In equation (1), both  $r_t$  and  $\gamma_t$  can be stochastic. We can think of  $X^h$  as the return shocks to the aggregate wealth in the economy. The shocks  $X^h$  can be multi-dimensional, in which case  $\gamma_t^h dX_t^h$  denotes an inner product. In representative agent economies, the stochastic discount factor can be interpreted as the ratio of the marginal utilities of consumption over two time horizons.

No-arbitrage dictates that the ratio of the stochastic discount factors between two economies determines the exchange rate dynamics between them (Lucas (1982), Dumas (1992), Bakshi and Chen

(1997), Basak and Gallmeyer (1999), and Backus, Foresi, and Telmer (2001)). Let  $S^{hf}$  denote the currency- $h$  price of currency  $f$ , with  $h$  being the home economy, we have

$$\frac{S_t^{hf}}{S_0^{hf}} = \frac{\mathcal{M}_t^f / \mathcal{M}_0^f}{\mathcal{M}_t^h / \mathcal{M}_0^h}, \quad h, f = 1, 2, \dots, N. \quad (3)$$

Equation (3) defines the formal link between the stochastic discount factors in any two economies and the exchange rate movements between them. This link tells us that the time-series of currency returns and currency option prices contain important information about the dynamics of aggregate uncertainties affecting the two economies and how the underlying risks are priced. In complete markets, the stochastic discount factor for each economy is unique. Hence, the ratio of two stochastic discount factors uniquely determines the exchange rate dynamics between the two economies. When markets are incomplete with the primary securities such as bonds and stocks, there may exist multiple stochastic discount factors that are consistent with the prices of these securities. In this case, exchange rates and exchange rate options help complete the markets by requiring equation (3) to hold between any viable stochastic discount factors in the two economies.

In this paper, instead of following the extant literature in using equity index options in a single country to study the stand-alone behavior of the stochastic discount factor in that economy, we exploit the link in (3) and use currency returns and currency options to study the joint dynamics of stochastic discount factors in international economies.

## II. Model with Stochastic Risk Premium and Stochastic Skewness

We propose a class of models for the stochastic discount factors that are flexible enough to generate stochastic risk premiums and stochastic skewness in currency returns. Formally, using the language of time-changed Lévy process (Carr and Wu (2004b)), we specify the stochastic discount factor for country  $h$  as,

$$\mathcal{M}_t^h = \exp(-r^h t) \exp\left(-W_{\Pi_t^h}^g - \frac{1}{2}\Pi_t^h\right) \exp\left(-\left(W_{\Lambda_t^h}^h + J_{\Lambda_t^h}^h\right) - \left(\frac{1}{2} + k_{J^h}[-1]\right)\Lambda_t^h\right), \quad (4)$$

which decomposes the stochastic discount factor into three orthogonal components. The first component captures the contribution from interest rates. Since a dominant proportion of exchange rate movements is independent of interest rate movements, we assume deterministic interest rates for simplicity and use  $r^h$  to denote the continuously compounded spot interest rate of the relevant maturity.

The second component incorporates a global diffusion risk factor  $W_{\Pi_t^g}^g$ , where  $W^g$  denotes a standard Brownian motion and  $\Pi_t^g \equiv \int_0^t \gamma_s^g ds$  defines a stochastic time change that captures the stochastic risk premium on this global risk factor. In probability, the stochastic time-changed Brownian motion  $W_{\Pi_t^g}^g$  is equivalent to the more traditional representation  $\int_0^t \sqrt{\gamma_s^g} dW_s^g$ , with  $\gamma_s^g$  being the instantaneous variance rate and hence  $\Pi_t^g$  being the integrated variance over time  $[0, t]$ . We label  $\gamma_t^g$  as the risk premium rate (per unit time) and use the superscript  $g$  on  $\gamma_t$  to indicate that the risk premium on the global risk factor can exert a differential impact across different economies.  $\frac{1}{2}\Pi_t^g$  is the convexity adjustment that makes this second component an exponential martingale.

The third component describes a country-specific jump-diffusion risk factor  $(W^h + J^h)$ , where  $W^h$  denotes another standard Brownian motion independent of the global risk component  $W^g$ , and  $J^h$  denotes a pure jump Lévy component. We apply a separate stochastic time change to this country-specific jump diffusion risk factor,  $\Lambda_t^h \equiv \int_0^t v_s^h ds$ , which captures the stochastic risk premium on the country-specific risk component with  $v_t^h$  being the risk premium rate on the country-specific risk factor. In the language of stochastic differential equation,  $v_t^h$  is the instantaneous variance rate of the Brownian component  $W^h$  and is also proportional to the arrival rate of the jump component  $J^h$ .<sup>2</sup>

In the specification of the stochastic discount factor in (4),  $k_{J^h}[s]$  denotes the generalized cumulant exponent of the Lévy jump component  $J^h$ , defined by

$$k_{J^h}[u] \equiv \frac{1}{t} \ln E \left( e^{uJ_t^h} \right), \quad u \in \mathcal{D} \subseteq \mathcal{C} \quad (5)$$

which, as a property of Lévy processes, does not depend on the the time horizon. A cumulant exponent is normally defined on the positive real line, but it is convenient to extend the definition to the complex plane,  $u \in \mathcal{D} \subseteq \mathcal{C}$ , where the exponent is well-defined. Again,  $(\frac{1}{2} + k_{J^h}[-1])\Lambda_t^h$  denotes the convexity adjustment term of  $(W_{\Lambda_t^h}^h + J_{\Lambda_t^h}^h)$  to make the last term an exponential martingale.

In principle, we can also allow a jump component in the global risk factor, but experimental estimation shows that the jump in the global risk factor is not significant. Hence, we choose a pure diffusion specification for the global factor to maintain parsimony.

In equation (4), we decompose the risk in each economy into a global risk component and a country-specific risk component. Through stochastic time changes (Carr and Wu (2004b)), we allow the risk premiums on the two components to be governed by separate dynamics. Thus, through model estimation, we can investigate how investors respond to different types of risks in international economies. We

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<sup>2</sup>We refer interested readers to Carr and Wu (2004b) and Huang and Wu (2004) for more detailed discussions on time-changed Lévy processes and their applications.



can also study the degree of international integration by estimating the relative proportion of variation in the stochastic discount factor that is driven by the global risk component.

## A. Specification of Jumps and Risk Premium Rate Dynamics

Within the general specification in (4), we consider two classes of parameterizations, under which the models can be fully identified using currency returns and options.

### A.1. Proportional Asymmetry

A parsimonious way to capture asymmetry across economies is to use a vector of scaling coefficients  $\xi = \{\xi^h\}_{h=1}^N$  to model the average difference in risk premium in different economies. Asymmetries arise when the economies have different risk magnitudes and/or when investors have different risk preferences. For identification, we normalize the scaling coefficient for the first economy to unity:  $\xi^1 = 1$ . Then, the deviations of the scaling coefficients from unity for other economies capture their average differences in risk premium from the first economy.

With these scaling coefficients, we assume that the jump component  $J^h$  in each economy is i.i.d., and we model the Lévy density  $\pi^h[x]$  of this i.i.d. jump component using an exponentially dampened power law:

$$\pi^h[x] = \begin{cases} \lambda e^{-\beta_+ x} x^{-\alpha-1} & x > 0 \\ \lambda e^{-\beta_- |x|} |x|^{-\alpha-1} & x < 0 \end{cases}, h = 1, 2, \dots, N, \quad (6)$$

with  $\alpha \in (-1, 2)$  and  $\lambda, \beta_+, \beta_- > 0$ . We adopt this specification from Carr, Geman, Madan, and Yor (2002) as Wu (2004) shows that the dampened power law jump specification can match much evidence in stocks and currencies. The cumulant exponents when  $\alpha \neq 0$  and  $\alpha \neq 1$  are,

$$k_J[u] = \Gamma[-\alpha] \lambda \left( (\beta_+ - u)^\alpha - (\beta_+)^\alpha + (\beta_- + u)^\alpha - (\beta_-)^\alpha \right) + uC[\delta]. \quad (7)$$

where  $\Gamma[-\alpha]$  denotes the incomplete Gamma function and  $C[\delta]$  is an immaterial drift term that depends on the exact form of the truncation function used in the computation of the cumulant. We can henceforth safely ignore this term in our analysis and drop this term in our representations. The Lévy density has singularities at  $\alpha = 0$  and  $\alpha = 1$ , in which cases the cumulant exponent takes on different forms:

$$\begin{aligned} k_J[u] &= -\lambda \ln(1 - u/\beta_+) - \lambda \ln(1 + u/\beta_-) && \text{when } \alpha = 0, \\ k_J[u] &= \lambda(\beta_+ - u) \ln(1 - u/\beta_+) + \lambda(\beta_- + u) \ln(1 + u/\beta_-) && \text{when } \alpha = 1. \end{aligned} \quad (8)$$

With the i.i.d. assumption on the jump component, we accommodate the average difference in the risk premium rates across different economies by applying the constant scaling coefficients to an otherwise independent and identical risk premium rate dynamics:

$$\Lambda_t^h = \xi^h \int_0^t Y_s^h ds, \quad (9)$$

where  $Y_t^h$  can be regarded as the country-specific risk premium rate factor, which is identical but independent across the  $h = 1, 2, \dots, N$  economies. We model this independent and identical dynamic process for the country-specific risk premium rate factor using the square-root process of Cox, Ingersoll, and Ross (1985),

$$dY_t^h = \kappa_Y (\theta_Y - Y_t^h) dt + \omega_Y \sqrt{Y_t^h} dW_t^{Y^h}, \quad (10)$$

where  $\rho_Y = E(dW^{Y^h} dW^h)/dt$  captures the correlation between shocks of the country-specific diffusion risk and its risk premium rate. It is important to note that the same dynamics as in (10) governs  $N$  independent processes for  $N$  economies.

For the global risk factor, we apply the same set of scaling coefficients to a global risk premium rate factor to preserve parsimony:

$$\Pi_t^h = \xi^h \Pi_t, \quad \Pi_t = \int_0^t Z_s ds. \quad (11)$$

We assume that the global risk premium rate factor,  $Z_t$ , also follows a square-root process,

$$dZ_t = \kappa_Z (\theta_Z - Z_t) dt + \omega_Z \sqrt{Z_t} dW_t^Z, \quad (12)$$

with  $\rho_Z = E(dW^Z dW^g)/dt$ . By design, the country-specific risk and the global risk are orthogonal,  $E(dW^g dW^h) = 0$ , so are the country-specific risk premium rate factors  $Y_t^h$  and the global risk premium rate factor  $Z_t$ :  $E(dW^Z dW^{Y^h}) = 0$  for all  $h = 1, 2, \dots, N$ .

We identify the model using currency options on dollar-yen, dollar-pound, and pound-yen exchange rates, and the time-series returns on the respective currencies. For the three economies, the model has one global diffusion risk component and three country-specific jump-diffusion risk components. The risk premium rate on each of the four risk components is stochastic. Thus, our estimation on the three economies identifies four risk premium rates: one global risk premium rate factor  $Z$  and three country-

specific risk premium rates ( $Y^{USD}, Y^{JPY}, Y^{GBP}$ ). For estimation, we normalize the scaling on the US economy  $\xi^{US} = 1$ . The model has 14 parameters for the three economies:

$$\Theta \equiv [\xi^{JPY}, \xi^{GBP}, \kappa_Z, \theta_Z, \omega_Z, \rho_Z, \kappa_Y, \theta_Y, \omega_Y, \rho_Y, \lambda, \beta_+, \beta_-, \alpha].$$

Within each model, we consider three special cases for the jump specification with  $\alpha$  fixed at  $-1$ ,  $0$ , and  $1$ , respectively. The three different  $\alpha$ 's generate finite activity, infinite activity with finite variation, and infinite variation jumps, respectively.

## A.2. Strict Symmetry Across Economies

Under strict symmetry, the same parameterization for the state-price deflator applies to all economies. Reality aside, this assumption not only simplifies notation and reduces the number of free parameters, but it also highlights the issue of state-price deflator identification using exchange rates. A key implication of strict symmetry is that the contribution of the global risk factor in the two economies cancels. Thus, from currency returns and currency options, we can only identify the country-specific risk component, but not the global risk component. Accordingly, we can only identify the three country-specific risk premium rates ( $Y^{USD}, Y^{JPY}, Y^{GBP}$ ), but not the global risk premium rate  $Z$ .

Symmetry can be regarded as a degenerating case of the general proportional asymmetry case with  $\xi^h = 1$  for all  $h$ . In this setting, the global risk factor dynamics ( $\kappa_Z, \theta_Z, \omega_Z, \rho_Z$ ) can no longer be identified. Therefore, we can only identify the country-specific part of model (4), which is controlled by eight parameters:  $\Theta \equiv [\kappa_Y, \theta_Y, \omega_Y, \rho_Y, \lambda, \beta_+, \beta_-, \alpha]$ .

## B. Currency Return Dynamics

Under the above parameterization, the log return on the exchange rate  $s_t^{hf} \equiv \ln S_t^{hf} / S_0^{hf}$  over the horizon  $[0, t]$  is,

$$\begin{aligned} s_t^{hf} &= \ln \frac{\mathcal{M}_t^f}{\mathcal{M}_0^f} - \ln \frac{\mathcal{M}_t^h}{\mathcal{M}_0^h} \\ &= (r^h - r^f)t + \left( \sqrt{\xi^h} - \sqrt{\xi^f} \right) W_{\Pi_t}^g + \frac{1}{2} \Pi_t (\xi^h - \xi^f) \\ &\quad + \left( W_{\Lambda_t^h}^h + J_{\Lambda_t^h}^h + \left( \frac{1}{2} + k_J [-1] \right) \Lambda_t^h \right) - \left( W_{\Lambda_t^f}^f + J_{\Lambda_t^f}^f + \left( \frac{1}{2} + k_J [-1] \right) \Lambda_t^f \right). \end{aligned} \tag{13}$$

Under this specification, the exchange rate dynamics between any two economies are governed by (i) one diffusion global risk component ( $W^g$ ), (ii) two jump-diffusion country-specific risk components ( $W^h + J^h, W^f + J^f$ ), and (iii) three risk premium rates ( $Z, Y^h, Y^f$ ) that define the three stochastic time changes ( $\Pi_t, \Lambda_t^h, \Lambda_t^f$ ). Equation (13) also shows that when economies admit strictly symmetry ( $\xi^h = \xi^f$ ), the impact of the global risk factor  $W^g$  vanishes.

Consider the special case when the risk premium rates are constant at one:  $Y_t^h = Z_t = 1$ , the currency risk premium per unit time in country  $h$  becomes:

$$\begin{aligned} \text{RP}^h &\equiv E\left(S_1^{hf}/S_0^{hf}\right) - (r^h - r^f) \\ &= \left(\xi^h - \sqrt{\xi^h \xi^f}\right) + \xi^h (1 + k_J[1] + k_J[-1]), \end{aligned} \quad (14)$$

where the first term captures the contribution from the global risk factor and the second term captures the contribution from the country-specific risk factor. Under this special case, the risk premium  $\text{RP}^h$  in equation (14) is a constant. We introduce stochastic currency risk premium via the stochastic time changes  $\Lambda_t^h$  and  $\Pi_t$ , or equivalently the stochastic risk premium rates  $Y^h$  and  $Z$ .

In the absence of the stochastic time changes and hence stochastic risk premiums, the currency return is governed by three Brownian motions with constant volatilities and two jump components with constant arrival rates. The two jump components can generate distributional non-normality (skewness and kurtosis) for the currency return. Specifically, under the special case of constant risk premium rates  $Y_t^h = Z_t = 1$ , by taking successive partial derivatives of the cumulant exponent  $c_n \equiv \left. \frac{\partial^n \log \mathbb{E}[u]}{\partial u^n} \right|_{u=0}$ , we can show that the variance ( $c_2$ ) and the third ( $c_3$ ) and fourth cumulants ( $c_4$ ) for the currency return are given by,

$$\begin{aligned} c_2 &= (\xi^h + \xi^f) \lambda \Gamma[2 - \alpha] \left( (\beta_+)^{\alpha-2} + (\beta_-)^{\alpha-2} \right) + V_d, \\ c_3 &= (\xi^h - \xi^f) \lambda \Gamma[3 - \alpha] \left( (\beta_+)^{\alpha-3} - (\beta_-)^{\alpha-3} \right), \\ c_4 &= (\xi^h + \xi^f) \lambda \Gamma[4 - \alpha] \left( (\beta_+)^{\alpha-4} + (\beta_-)^{\alpha-4} \right), \end{aligned} \quad (15)$$

where  $V_d$  capture the variance contribution from the diffusion components,

$$V_d = 2 \left( \xi^h + \xi^f \right) - 2 \sqrt{\xi^h \xi^f}. \quad (16)$$

The diffusion components have zero contribution to higher-order cumulants. Thus, the currency return shows nonzero skewness or non-zero third cumulant  $c_3$  when (1) the jump component in the log state-price deflator is asymmetric:  $\beta_+ \neq \beta_-$ , and (2) the two economies are asymmetric in the average magnitudes of risk premiums:  $\xi^h \neq \xi^f$ . In fact, these conditions are necessary for the existence of non-

zero odd-order cumulants beyond three. In contrast, the fourth cumulant ( $c_4$ ) or the excess kurtosis for the currency return is strictly positive as long as the jump component in the log state-price deflator is not degenerating ( $\lambda \neq 0$ ). Positive fourth cumulant implies that the tails of the distribution are fatter compared to the normal distribution. Nevertheless, since all the cumulants in (15) are constant, a model with constant risk premiums (i.e.,  $Y_t^h = Z_t = 1$ ) cannot capture the evidence from currency option markets that the currency return skewness is stochastic (Carr and Wu (2004a)). Stochastic skewness in currency return distribution warrants stochastic risk premium.

When the risk premium rates are allowed to be stochastic as in currency dynamics (13), currency return skewness can also arise from three additional sources: (1) correlation ( $\rho_Z$ ) between  $W_t^g$  and  $Z_t$ , (2) correlation ( $\rho_Y^h$ ) between  $W_t^h$  and  $Y_t^h$ , and (3) correlation ( $\rho_Y^f$ ) between  $W_t^f$  and  $Y_t^f$ . Allowing the three risk premium rates ( $Z_t, Y_t^h, Y_t^f$ ) to be stochastic produces stochastic skewness in currency returns.

To derive the risk-neutral return dynamics, we note that the measure change from the statistical measure  $\mathcal{P}$  to the home-country risk-neutral  $Q^h$  is defined by the exponential martingale:

$$\left. \frac{dQ^h}{d\mathcal{P}} \right|_t \equiv \exp\left(-W_{\Pi_t^g}^g - \frac{1}{2}\Pi_t^h\right) \exp\left(-\left(W_{\Lambda_t^h}^h + J_{\Lambda_t^h}^h\right) - \left(\frac{1}{2} + k_J[-1]\right)\Lambda_t^h\right). \quad (17)$$

The martingale condition requires that under the home-economy risk-neutral measure  $Q^h$ ,

$$\begin{aligned} s_t^{hf} &= (r^h - r^f)t + \left(\sqrt{\xi^h} - \sqrt{\xi^f}\right)W_{\Pi_t}^g - \frac{1}{2}\left(\sqrt{\xi^h} - \sqrt{\xi^f}\right)^2\Pi_t \\ &\quad + \left(W_{\Lambda_t^h}^h + J_{\Lambda_t^h}^{hQ} - \left(\frac{1}{2} + k_J^Q[1]\right)\Lambda_t^h\right) + \left(-W_{\Lambda_t^f}^f - J_{\Lambda_t^f}^f - \left(\frac{1}{2} + k_J[-1]\right)\Lambda_t^f\right), \end{aligned} \quad (18)$$

Since  $J^f$  is independent of  $J^h$ , it remains unchanged under  $Q^h$ . The home-economy jump component changes from  $J_{\Lambda_t^h}^h$  to  $J_{\Lambda_t^h}^{hQ}$  under  $Q^h$ , where the Lévy density for  $J_{\Lambda_t^h}^{hQ}$  becomes

$$\pi^{Q^h}[x] = e^{-x}\pi^h[x] = \begin{cases} \lambda e^{-(\beta_++1)x}x^{-\alpha-1} & x > 0 \\ \lambda e^{-(\beta_- - 1)|x|}|x|^{-\alpha-1} & x < 0 \end{cases}. \quad (19)$$

Hence,  $\pi^{Q^h}[x]$  and  $\pi^h[x]$  share the same parametric form with  $\beta_+^Q = \beta_+ + 1$  and  $\beta_-^Q = \beta_- - 1$ . For  $\beta_-^Q > 0$ , we need  $\beta_- > 1$ .

Under measure  $Q^h$ , the country-specific and global risk premium rates factor dynamics change to

$$\begin{aligned} dY_t^h &= \left(\kappa_Y\theta_Y - \kappa_Y^Q Y_t^h\right)dt + \omega_Y \sqrt{Y_t^h} dW_t^{Y^h}, \\ dZ_t &= \left(\kappa_Z\theta_Z - \kappa_Z^Q Z_t\right)dt + \omega_Z \sqrt{Z_t} dW_t^Z, \end{aligned}$$

with  $\kappa_Y^Q = \kappa_Y + \sqrt{\xi^h} \omega_Y \rho_Y$  and  $\kappa_Z^Q = \kappa_Z + \sqrt{\xi^h} \omega_Z \rho_Z$ . The process for  $Y_t^f$  does not change under measure  $Q^h$  since  $W^{Y^f}$  is independent of  $W^g$  and  $W^h$ .

In light of our analysis, it is natural to ask: What minimal structures are necessary to reproduce the observed behaviors in currency returns and options? To gain some basic understanding and economic intuition on risk and pricing in international economies, consider a prototype model where risk in each economy is governed by a Brownian motion with constant volatility and constant market price of risk:

$$\mathcal{M}_t^h = \exp(-r^h t) \exp\left(-\gamma^h \sigma^h W_t^h - \frac{1}{2}(\gamma^h \sigma^h)^2 t\right), \quad h = 1, 2, \dots, N, \quad (20)$$

where  $W_t$  denotes a standard Brownian motion and  $\sigma$  and  $\gamma$  are constant scalars. Instead of separately specifying a global risk component, we allow constant correlation between the Brownian motions for any two economies  $h$  and  $f$ :  $\rho^{hf} \equiv E(W_t^h W_t^f) / t$ . It is obvious from (20) that in a pure diffusion setting, the diffusion volatility  $\sigma$  as a risk measure and the market price of risk  $\gamma$  cannot be separately identified from the stochastic discount factors. Under (20), the log return on the exchange rate is:

$$s_t^{hf} = (r^h - r^f)t + \frac{1}{2} \left( (\gamma^h \sigma^h)^2 - (\gamma^f \sigma^f)^2 \right) t + \left( \gamma^h \sigma^h W_t^h - \gamma^f \sigma^f W_t^f \right), \quad (21)$$

which implies that the currency return is normally distributed under the statistical measure  $\mathcal{P}$  with mean,  $\mu_s \equiv (r^h - r^f) + \frac{1}{2} \left( (\gamma^h \sigma^h)^2 - (\gamma^f \sigma^f)^2 \right)$  and variance  $V_s \equiv (\gamma^h \sigma^h)^2 + (\gamma^f \sigma^f)^2 - 2\gamma^h \sigma^h \gamma^f \sigma^f \rho^{hf}$ . In this economy, the annualized expected return on the exchange rate is  $\frac{1}{t} \ln E(S_t/S_0) = \mu_s + \frac{1}{2} V_s$  and the currency risk premium is a constant:  $RP^h = (\gamma^h \sigma^h)^2 - \gamma^h \sigma^h \gamma^f \sigma^f \rho^{hf}$ . The magnitude and sign of the currency risk premium depends on both the market prices of risk (determined by relative risk aversions) of the two countries  $(\gamma^h, \gamma^f)$  and on the variance and covariance of the return shocks on the aggregate wealth  $(\sigma^h, \sigma^f, \rho^{hf})$ . Clearly, this prototype model is incapable of producing any currency return non-normalities. One possible direction to generate distribution non-normality is by incorporating Lévy jumps in shocks to aggregate wealth:  $\mathcal{M}_t^h = \exp(-r^h t) \exp\left(-\gamma^h \sigma^h W_t^h - \frac{1}{2}(\gamma^h \sigma^h)^2 t - \gamma^h J_t^h - k_{J^h}[-\gamma^h] t\right)$ . However, under Lévy specification and constant market price of risk and volatility, the risk premium and risk-neutral skewness are still constant. Our model in (4) with global and country-specific risk factors and stochastic risk premium rates appears desirable from both theoretical and empirical standpoints.

## C. Option Pricing

Given the  $Q^h$ -dynamics of the log currency return in (18), we can derive its generalized Fourier transform as in Carr and Wu (2004b),

$$\begin{aligned}
\phi_s^Q &\equiv E^Q \left( e^{iu s_t^{hf}} \right) \\
&= E^Q \left( e^{iu \left( (r^h - r^f)t + \left( (\sqrt{\xi^h} - \sqrt{\xi^f}) W_{\Pi_t}^g - \frac{1}{2} (\sqrt{\xi^h} - \sqrt{\xi^f})^2 \Pi_t \right) \right)} \right) \\
&\quad \times E^Q \left( e^{iu \left( \left( W_{\Lambda_t^h}^h + J_{\Lambda_t^h}^{hQ} - \left( \frac{1}{2} + k_J^Q [1] \right) \Lambda_t^h \right) + \left( -W_{\Lambda_t^f}^f - J_{\Lambda_t^f}^f - \left( \frac{1}{2} + k_J [-1] \right) \Lambda_t^f \right) \right)} \right) \\
&= e^{iu(r^h - r^f)t} E^{\mathcal{N}^g} \left( e^{-\psi^g[u] \Pi_t} \right) E^{\mathcal{N}^h} \left( e^{-\psi^h[u] \Lambda_t^h} \right) E^{\mathcal{N}^f} \left( e^{-\psi^f[u] \Lambda_t^f} \right), \tag{22}
\end{aligned}$$

where  $(\psi^g[u], \psi^h[u], \psi^f[u])$  denote the characteristic exponents of the three Lévy components prior to time-change that are due to the global risk component, home country-specific risk component, and foreign country-specific risk component, respectively:

$$\begin{aligned}
\psi^g[u] &= \frac{1}{2} \left( \sqrt{\xi^h} - \sqrt{\xi^f} \right)^2 (iu + u^2), \\
\psi^h[u] &= iu \left( \frac{1}{2} + k_J^Q [1] \right) + \frac{1}{2} u^2 - k_J^Q [iu], \\
\psi^f[u] &= iu \left( \frac{1}{2} + k_J [-1] \right) + \frac{1}{2} u^2 - k_J [-iu].
\end{aligned}$$

$[\mathcal{N}^g, \mathcal{N}^h, \mathcal{N}^f]$  denote three new measures defined by the following exponential martingales,

$$\begin{aligned}
\left. \frac{d\mathcal{N}^g}{dQ} \right|_t &= \exp \left( iu \left( \left( \sqrt{\xi^h} - \sqrt{\xi^f} \right) W_{\Pi_t}^g - \frac{1}{2} \left( \sqrt{\xi^h} - \sqrt{\xi^f} \right)^2 \Pi_t \right) + \psi^g[u] \Pi_t \right), \\
\left. \frac{d\mathcal{N}^h}{dQ} \right|_t &= \exp \left( iu \left( W_{\Lambda_t^h}^h + J_{\Lambda_t^h}^{hQ} - \left( \frac{1}{2} + k_J^Q [1] \right) \Lambda_t^h \right) + \psi^h[u] \Lambda_t^h \right), \\
\left. \frac{d\mathcal{N}^f}{dQ} \right|_t &= \exp \left( iu \left( -W_{\Lambda_t^f}^f - J_{\Lambda_t^f}^f - \left( \frac{1}{2} + k_J [-1] \right) \Lambda_t^f \right) + \psi^f[u] \Lambda_t^f \right). \tag{23}
\end{aligned}$$

To take the expectation, we need the dynamics for the risk premium rates under their respective new measures:

$$dY_t^h = \left( \kappa_Y \theta_Y - \kappa_Y^{\mathcal{N}^h} Y_t^h \right) dt + \omega_Y \sqrt{Y_t^h} dW_t^{Y^h}, \tag{24}$$

$$dY_t^f = \left( \kappa_Y \theta_Y - \kappa_Y^{\mathcal{N}^f} Y_t^f \right) dt + \omega_Y \sqrt{Y_t^f} dW_t^{Y^f}, \tag{25}$$

$$dZ_t = \left( \kappa_Z \theta_Z - \kappa_Z^{\mathcal{N}^g} Z_t \right) dt + \omega_Z \sqrt{Z_t} dW_t^Z, \tag{26}$$

with

$$\begin{aligned}
\kappa_Y^{\mathcal{N}^h} &= \kappa_Y + (1 - iu) \sqrt{\xi^h} \omega_Y \rho_Y, \\
\kappa_Y^{\mathcal{N}^f} &= \kappa_Y + iu \sqrt{\xi^f} \omega_Y \rho_Y, \\
\kappa_Z^{\mathcal{N}^g} &= \kappa_Z + \sqrt{\xi^h} \omega_Z \rho_Z - iu \left( \sqrt{\xi^h} - \sqrt{\xi^f} \right) \omega_Z \rho_Z.
\end{aligned} \tag{27}$$

Since the three risk premium rates follow affine dynamics under their relevant measures  $\mathcal{N}^g$ ,  $\mathcal{N}^h$ , and  $\mathcal{N}^f$ , the expectations in (22) generate exponential-affine solutions:

$$\begin{aligned}
\phi_s^Q &= e^{iu(r^h - r^f)t} E^{\mathcal{N}^g} \left( e^{-\Psi^g[u] \Pi_t} \right) E^{\mathcal{N}^h} \left( e^{-\Psi^h[u] \Lambda_t^h} \right) E^{\mathcal{N}^f} \left( e^{-\Psi^f[u] \Lambda_t^f} \right) \\
&= e^{iu(r^h - r^f)t} e^{-b_g(t)z_0 - c_g(t) - b_h(t)y_0^h - c_h(t) - b_f(t)y_0^f - c_f(t)},
\end{aligned} \tag{28}$$

where  $(z_0, y_0^h, y_0^f)$  are the time-0 realized levels of the three risk premium rates  $(Z, Y^h, Y^f)$  and the coefficients  $[b(t), c(t)]$  on each risk premium rate take the same functional forms:

$$\begin{aligned}
b_c(t) &= \frac{2\Psi^c(1 - e^{-\eta^c t})}{2\eta^c - (\eta^c - \kappa^{\mathcal{N}^c c})(1 - e^{-\eta^c t})}, \\
c_c(t) &= \frac{\kappa^{\mathcal{N}^c c}}{\sigma_v^2} \left[ 2 \ln \left( 1 - \frac{\eta^c - \kappa^{\mathcal{N}^c c}}{2\eta^c} (1 - e^{-\eta^c t}) \right) + (\eta^c - \kappa^{\mathcal{N}^c c})t \right],
\end{aligned} \tag{29}$$

with  $\eta^c = \sqrt{(\kappa^{\mathcal{N}^c c})^2 + 2\sigma_v^2\Psi^c}$  and for  $c = g, h, f$ , respectively. Given the generalized Fourier transform, we can now follow Carr and Madan (1999) and use fast Fourier inversion to obtain option prices.

#### D. Characteristic Function under Measure $\mathcal{P}$

For estimation, we also need to derive the log likelihood function for the currency returns. We first derive the characteristic function of the log currency returns under the statistical measure  $\mathcal{P}$  and then obtain the density of the currency return via fast Fourier inversion.

Given the  $\mathcal{P}$ -dynamics for the currency return in (13), we derive its characteristic function as,

$$\begin{aligned}
\phi_s^{\mathcal{P}} &\equiv E^{\mathcal{P}} \left( e^{i u s_t^{h f}} \right), \\
&= E^{\mathcal{P}} \left( e^{i u \left( (r^h - r^f)t + \left( \left( \sqrt{\xi^h} - \sqrt{\xi^f} \right) W_{\Pi_t}^g + \frac{1}{2} \Pi_t (\xi^h - \xi^f) \right) \right)} \right) \\
&\quad \times E^{\mathcal{P}} \left( e^{i u \left( \left( W_{\Lambda_t^h}^h + J_{\Lambda_t^h}^h + \left( \frac{1}{2} + k_J[-1] \right) \Lambda_t^h \right) - \left( W_{\Lambda_t^f}^f + J_{\Lambda_t^f}^f + \left( \frac{1}{2} + k_J[-1] \right) \Lambda_t^f \right) \right)} \right) \\
&= e^{i u (r^h - r^f)t} E^{\mathcal{N}^g} \left( e^{-\Psi^g[u] \Pi_t} \right) E^{\mathcal{N}^h} \left( e^{-\Psi^h[u] \Lambda_t^h} \right) E^{\mathcal{N}^f} \left( e^{-\Psi^f[u] \Lambda_t^f} \right),
\end{aligned} \tag{30}$$



where  $(\Psi^g[u], \Psi^h[u], \Psi^f[u])$  now denote the characteristic exponents of the three Lévy components prior to time change under the statistical measure  $\mathcal{P}$ :

$$\begin{aligned}\Psi^g[u] &= -\frac{1}{2}iu(\xi^h - \xi^f) + \frac{1}{2}\left(\sqrt{\xi^h} - \sqrt{\xi^f}\right)^2 u^2, \\ \Psi^h[u] &= -iu\left(\frac{1}{2} + k_J[-1]\right) + \frac{1}{2}u^2 - k_J[iu], \\ \Psi^f[u] &= iu\left(\frac{1}{2} + k_J[-1]\right) + \frac{1}{2}u^2 - k_J[-iu].\end{aligned}$$

$[\mathcal{N}^g, \mathcal{N}^h, \mathcal{N}^f]$  denote three new measures defined by the following exponential martingales,

$$\begin{aligned}\left.\frac{d\mathcal{N}^g}{d\mathcal{P}}\right|_t &= \exp\left(iu\left(\left(\sqrt{\xi^h} - \sqrt{\xi^f}\right)W_{\Pi_t}^g + \frac{1}{2}\Pi_t(\xi^h - \xi^f)\right) + \Psi^g[u]\Pi_t\right), \\ \left.\frac{d\mathcal{N}^h}{d\mathcal{P}}\right|_t &= \exp\left(iu\left(W_{\Lambda_t^h}^h + J_{\Lambda_t^h}^h + \left(\frac{1}{2} + k_J[-1]\right)\Lambda_t^h\right) + \Psi^h[u]\Lambda_t^h\right), \\ \left.\frac{d\mathcal{N}^f}{d\mathcal{P}}\right|_t &= \exp\left(iu\left(-W_{\Lambda_t^f}^f - J_{\Lambda_t^f}^f - \left(\frac{1}{2} + k_J[-1]\right)\Lambda_t^f\right) + \Psi^f[u]\Lambda_t^f\right).\end{aligned}\quad (31)$$

The dynamics for the three risk premium rates under the new measures become,

$$\begin{aligned}dZ_t &= \left(\kappa_Z\theta_Z - \kappa_Z^{\mathcal{N}^g}Z_t\right)dt + \omega_Z\sqrt{Z_t}dW_t^Z, \\ dY_t^h &= \left(\kappa_Y\theta_Y - \kappa_Y^{\mathcal{N}^h}Y_t^h\right)dt + \omega_Y\sqrt{Y_t^h}dW_t^{Y^h}, \\ dY_t^f &= \left(\kappa_Y\theta_Y - \kappa_Y^{\mathcal{N}^f}Y_t^f\right)dt + \omega_Y\sqrt{Y_t^f}dW_t^{Y^f},\end{aligned}\quad (32)$$

with

$$\kappa_Z^{\mathcal{N}^g} = \kappa_Z - iu\left(\sqrt{\xi^h} - \sqrt{\xi^f}\right)\omega_Z\rho_Z, \quad \kappa_Y^{\mathcal{N}^h} = \kappa_Y - iu\sqrt{\xi^h}\omega_Y\rho_Y, \quad \kappa_Y^{\mathcal{N}^f} = \kappa_Y + iu\sqrt{\xi^f}\omega_Y\rho_Y. \quad (33)$$

Since the three risk premium rates follow affine dynamics under their relevant measures  $\mathcal{N}^g$ ,  $\mathcal{N}^h$ , and  $\mathcal{N}^f$ , the expectations in (30) leads to the solution below:

$$\phi_s^{\mathcal{P}} = e^{iu(r^h - r^f)t} e^{\left(-b_g(t)z_0 - c_g(t) - b_h(t)y_0^h - c_h(t) - b_f(t)y_0^f - c_f(t)\right)}, \quad (34)$$

where  $(z_0, y_0^h, y_0^f)$  are the time-0 realized levels of the three risk premium rates  $(Z, Y^h, Y^f)$  and the coefficients  $[b_c(t), c_c(t)]$  for  $c = h, f, g$  are given by the same equations as in (29), with appropriate changes in the definitions of  $\Psi^c$  and  $\kappa^{\mathcal{N}^c}$ .

Given the characteristic function, we use fast Fourier inversion to obtain the conditional density of the currency return as a function of the current risk premium rates. In our estimation, we first extract the risk premium rates from the options data, and then calculate the likelihood of the time series currency returns as a function of these risk premium rates.

Before proceeding to the data and estimation, we note that by the triangular arbitrage relation, the time- $t$  yen-pound cross-currency exchange rate is completely determined by the other two primary currency pairs: dollar-yen and dollar-pound. Hence, a separate quote on the cross rate is redundant. However, it is important to realize that the two marginal distributions on the two primary rates dollar-yen and dollar-pound (whether under  $\mathcal{P}$  or  $\mathcal{Q}$ ) are insufficient to determine the marginal distribution of yen-pound cross rate. For this reason, the cross-currency option quotes are not redundant, but provide an important information source to pin down the risk-neutral marginal distribution of the cross rate yen-pound. Analogously, the fast Fourier inversion in equation (34) provides the marginal likelihoods of the dollar-yen and dollar-pound, and yen-pound returns. Thus, it must be understood that the likelihood of yen-pound returns cannot be uniquely reconstructed from the knowledge of the marginal densities of dollar-yen and dollar-pound. In this sense, including the marginal likelihood of each currency pair is useful for identifying the stochastic discount factors of US, UK, and Japan, even though the log cross rate  $\log(S_t^{JPY,GBP})$  is a linear combination of two log primary rates  $\log(S_t^{USD,JPY})$  and  $\log(S_t^{USD,GBP})$ .

### III. Data on Currency Straddles, Risk-Reversals, and Butterfly Spreads

We obtain over-the-counter quotes on currency options and spot exchange rates for three currency pairs: JPYUSD (the dollar price of one yen), GBPUSD (the dollar price of one pound), and GBPJPY (the yen price of one pound), over the sample period of November 7, 2001 to January 28, 2004. The data are sampled weekly. Options quotes are available at seven fixed time-to-maturities: one week, one, two, three, six, nine, and 12 months. At each maturity, quotes are available at five fixed moneyness. There are a total of 12,285 option quotes.

The five options at each maturity are quoted in the following forms:

- **Delta-neutral Straddle Implied Volatility (SIV):** A straddle is a sum of a call option and a put option with the same strike. The SIV market quote corresponds to a near-the-money implied volatility that makes  $\Delta_S^c + \Delta_S^p = 0$ , where  $\Delta_S^c = e^{-r^f \tau} N[d_1]$  and  $\Delta_S^p = -e^{-r^f \tau} N[-d_1]$  are the Black-Scholes delta of the call and put options in the straddle.  $N[\cdot]$  denotes the cumulative normal function, and  $d_1 = \frac{\ln(S_t/K) + (r^h - r^f)\tau}{IV\sqrt{\tau}} + \frac{1}{2}IV\sqrt{\tau}$ , with  $IV$  being the implied volatility input,  $\tau$  being the option time-to-maturity, and  $K$  being the strike price of the straddle. Since the delta-neutral restriction implies  $d_1 = 0$ , the implicit strike is close to the spot or the forward price.
- **Ten-delta Risk-Reversal, RR[10], and 25-delta Risk-Reversal, RR[25]:** The RR[10] measures the difference in Black-Scholes implied volatility between a ten-delta out-of-the-money

call option and a ten-delta out-of-the-money put option:  $RR[10] = IV^c[10] - IV^p[10]$ .  $RR[25]$  is analogously defined on 25-delta call and put options. Option traders use risk-reversal quotes to quantify the asymmetry of the implied volatility curve, which reflects the skewness in the risk-neutral currency distribution.

- **Ten-delta Butterfly Spread, BF[10], and 25-delta Butterfly Spread, BF[25]:** Butterfly spreads are defined as the average difference between out-of-the-money implied volatilities and the at-the-money implied volatility:  $BF[10] = (IV^c[10] + IV^p[10])/2 - SIV$  and  $BF[25] = (IV^c[25] + IV^p[25])/2 - SIV$ . Butterfly spread quotes capture the average curvature of the implied volatility curve, which reflects the kurtosis of the risk-neutral currency return distribution.

Based on the above definitions, we recover the underlying implied volatilities as: (i)  $IV(0) = SIV$ , (ii)  $IV^c[25] = BF[25] + SIV + RR[25]/2$ , (iii)  $IV^p[25] = BF[25] + SIV - RR[25]/2$ , (iv)  $IV^c[10] = BF[10] + SIV + RR[10]/2$ , and (v)  $IV^p[10] = BF[10] + SIV - RR[10]/2$ . For the purpose of estimation, the volatility quotes are converted into out-of-the-money option prices. In this calculation, the maturity-matched domestic and foreign interest rates are constructed using LIBOR and swap rates from Bloomberg.

Table I reports the mean, the standard deviation, and the  $t$ -statistics on the significance of the sample mean for risk-reversal and butterfly spread series, all in percentages of the corresponding at-the-money implied volatility (SIV). The  $t$ -statistics adjust serial dependence according to Newey and West (1987), with the number of lags optimally chosen according to Andrews (1991) based on an AR(1) specification.

Average butterfly spreads are uniformly positive across all maturities, implying that out-of-the-money option implied volatilities on average are significantly higher than the at-the-money implied volatility. The lowest  $t$ -statistic is 10.98. Regardless of the currency pair, the butterfly spread quotes are strongly supportive of excess kurtosis in the risk-neutral return conditional distribution.

The sign and magnitudes of risk-reversals are informative about the asymmetry of the conditional return distribution. Consider JPYUSD where the sample averages of the risk-reversals are positive, implying that out-of-the-money calls are generally more expensive than out-of-the-money puts. This evidence suggests that, on average, the JPYUSD risk-neutral conditional return distribution is right-skewed. The average risk-reversals for GBPUSD are also positive, albeit to a lesser degree. In contrast, the average magnitudes of risk-reversals are negative for GBPJPY, implying the presence of negative risk-neutral return skewness.

Figure 1 plots the time-series of ten-delta risk-reversals in the left-panels and ten-delta butterfly spreads in the right-panels, fixing maturity at one month (solid lines) and three months (dashed lines). Over the sample period, there is significant variation in both risk-reversals and butterfly spreads, but more so for risk-reversals. Indeed, the risk-reversals vary so much that the sign switches. The ten-delta risk-reversals on JPYUSD have varied from  $-20$  percent to over  $50$  percent of the at-the-money implied volatility, the risk-reversals on GBPUSD have varied from  $-10$  to  $20$  percent, and the risk-reversals on GBPJPY have varied from  $-35$  to over  $15$  percent. The evidence is broadly consistent with stochastic skewness in the conditional currency return distributions.

[Figure 1 about here.]

#### IV. Joint Maximum Likelihood Estimation

We estimate the models using the time-series of both currency returns and currency option prices on JPYUSD, GBPUSD, and GBPJPY. Since the risk premium rates are unobservable, we cast the models into a state-space form and infer the risk premium rates at each date using a filtering technique. Then, we estimate structural parameters by maximizing the joint likelihood of options and currency returns.

In the state-space form, we regard the risk premium rates in the three economies as unobservable states. For the general asymmetric models, we use  $v_t \equiv [Y_t^{USD}, Y_t^{JPY}, Y_t^{GBP}, Z_t]$  to denote the  $(4 \times 1)$  state vector. For the symmetric models, we drop the global risk premium rate  $Z_t$  from the state vector since it is no longer identifiable. We specify the state propagation equation using an Euler approximation of the risk premium rates dynamics:

$$v_t = A + \Phi v_{t-1} + \sqrt{G_t} \varepsilon_t, \quad v_t \in \mathfrak{R}^{4+} \quad (35)$$

where  $\varepsilon_t$  denotes an i.i.d. standard normal innovation vector and

$$\begin{aligned} \Phi &= \exp(-\kappa \Delta t), \quad \kappa = \langle [\kappa_Y, \kappa_Y, \kappa_Y, \kappa_Z] \rangle, \\ A &= (I - \Phi)\theta, \quad \theta = [\theta_Y, \theta_Y, \theta_Y, \theta_Z]^\top, \\ G_t &= \langle [\omega_Y^2 Y_{t-1}^{USD}, \omega_Y^2 Y_{t-1}^{JPY}, \omega_Y^2 Y_{t-1}^{GBP}, \sigma_Z^2 Z_{t-1}] \Delta t \rangle, \end{aligned} \quad (36)$$

where  $\Delta t = 7/365$  corresponds to the weekly frequency of the data and  $\langle \cdot \rangle$  denotes a diagonal matrix with the diagonal elements given by the vector inside the bracket. It has been well-established that an

Euler approximation works reasonably well for model estimation when the time series are observed at relatively high frequencies such as daily or weekly.

Measurement equations are based on the observed out-of-the-money option prices, assuming additive, normally-distributed measurement errors:

$$y_t = o[v_t; \Theta] + e_t, \quad E(e_t e_t^\top) = J, \quad y_t \in \mathfrak{R}^{105+}, \quad (37)$$

where  $y_t$  denotes the 105 observed out-of-the-money option prices scaled by Black-Scholes vega at time  $t$  for the three currency pairs (across seven maturities and five moneyness categories).  $o[v_t; \Theta]$  denotes the corresponding model-implied values as a function of the parameter set  $\Theta$  and the state vector  $v_t$ . We assume that the scaled pricing errors are i.i.d. normal with zero mean and constant variance. Hence, we can write the covariance matrix as,  $J = \sigma_r I$ , with  $\sigma_r$  being a scalar and  $I$  being an identity matrix of the relevant dimension of 105.

The objective function (37) deserves some explanation. One may choose to define the pricing error as the difference between the Black-Scholes implied volatility quote and its model-implied fair value. However, recall that our algorithm generates option prices from the return characteristic function. Converting the option prices into Black-Scholes implied volatility involves an additional minimization routine that can be inefficient when embedded in the global optimization procedure. By dividing the out-of-the-money option prices by its Black-Scholes vega  $S e^{-r\tau} \sqrt{\tau} N'[d_1]$ , we are essentially converting the option price into the implied volatility space via a linear approximation. Scaling by  $\sqrt{\tau}$  accounts for maturity effects while scaling by the normal probability density adjusts for the fact that out-of-the-money options are cheaper than at-the-money options. For the estimation, we first convert the implied volatility quotes into out-of-the-money option prices in percentages of the underlying spot. Then, we ignore the interest rate effect and apply time-homogeneous weighting on options prices at fixed delta ( $\Delta_S$ ) and time-to-maturity:  $w[\Delta_S, \tau] = \frac{1}{100 \sqrt{\tau} N'[N^{-1}[\Delta_S]]}$ .

Let  $\bar{v}_t, \bar{P}_t, \bar{y}_t, \bar{V}_t$  denote the time- $(t-1)$  ex ante forecasts of time- $t$  values of the state vector, the covariance of the state vector, the measurement series, and the covariance of the measurement series, respectively. Let  $\hat{v}_t$  and  $\hat{P}_t$  denote the ex post update, or filtering, on the state vector and its covariance at the time  $t$  based on observations ( $y_t$ ) at time  $t$ . In the case of linear measurement equations,

$$y_t = H v_t + e_t, \quad (38)$$

the Kalman-filter provides efficient updates. The ex ante predictions are,

$$\begin{aligned}
\bar{v}_t &= A + \Phi \hat{v}_{t-1}, \\
\bar{P}_t &= \Phi \hat{P}_{t-1} \Phi^\top + \mathcal{G}_{t-1}, \\
\bar{y}_t &= H \bar{v}_t, \\
\bar{V}_t &= H \bar{P}_t H^\top + \mathcal{J},
\end{aligned} \tag{39}$$

and the ex-post filtering updates are,

$$\begin{aligned}
\hat{v}_{t+1} &= \bar{v}_{t+1} + \mathcal{K}_{t+1} (y_{t+1} - \bar{y}_{t+1}), \\
\hat{P}_{t+1} &= \bar{P}_{t+1} - \mathcal{K}_{t+1} \bar{V}_{t+1} \mathcal{K}_{t+1}^\top,
\end{aligned} \tag{40}$$

where  $\mathcal{K}_{t+1}$  is the Kalman gain, given by,

$$\mathcal{K}_{t+1} = \bar{P}_{t+1} H^\top (\bar{V}_{t+1})^{-1}. \tag{41}$$

The iterative procedure defined by (39) and (40) yields a time-series of the ex-ante forecasts and ex-post updates on the mean and covariance of the state vectors and observed series.

In our application, the measurement equations are not linear in the state vector. We use the unscented Kalman filter to cope with this nonlinearity. The unscented Kalman filter uses a set of (sigma) points to approximate the state distribution. If we let  $\bar{k}$  denote the number of states (four in the asymmetric models and three in the symmetric models) and let  $\zeta > 0$  denote a control parameter, we generate a set of  $2\bar{k} + 1$  sigma vectors  $\chi_i$  according to the following equations,

$$\begin{aligned}
\chi_{t,0} &= \hat{v}_t, \\
\chi_{t,i} &= \hat{v}_t \pm \sqrt{(\bar{k} + \zeta)(\hat{P}_t + \mathcal{G}_t)_j}, \quad j = 1, \dots, \bar{k}; \quad i = 1, \dots, 2\bar{k},
\end{aligned}$$

with the corresponding weights  $w_i$  given by,

$$w_0 = \zeta / (\bar{k} + \zeta), \quad w_i = 1 / [2(\bar{k} + \zeta)], \quad i = 1, \dots, 2\bar{k}.$$

These sigma vectors form a discrete distribution with  $w_i$  being the corresponding probabilities. We can verify that the mean, covariance, skewness, and kurtosis of this distribution are  $\hat{v}_t$ ,  $\hat{P}_t + \mathcal{G}_t$ , 0, and  $\bar{k} + \zeta$ , respectively. Thus, we can use the control parameter  $\zeta$  to accommodate conditional non-normalities in the state propagation equation.

Given the sigma points, the prediction steps are given by:

$$\begin{aligned}
\bar{\chi}_{t,i} &= A + \Phi \chi_{t,i}, \\
\bar{v}_{t+1} &= \sum_{i=0}^{2\bar{k}} w_i \bar{\chi}_{t,i}, \\
\bar{P}_{t+1} &= \sum_{i=0}^{2\bar{k}} w_i (\bar{\chi}_{t,i} - \bar{v}_{t+1})(\bar{\chi}_{t,i} - \bar{v}_{t+1})^\top, \\
\bar{y}_{t+1} &= \sum_{i=0}^{2\bar{k}} w_i O[\bar{\chi}_{t,i}; \Theta], \\
\bar{V}_{t+1} &= \sum_{i=0}^{2\bar{k}} w_i (O[\bar{\chi}_{t,i}; \Theta] - \bar{y}_{t+1})(O[\bar{\chi}_{t,i}; \Theta] - \bar{y}_{t+1})^\top + \mathcal{J},
\end{aligned} \tag{42}$$

and the filtering updates are given by

$$\begin{aligned}
\hat{v}_{t+1} &= \bar{v}_{t+1} + \mathcal{K}_{t+1}(y_{t+1} - \bar{y}_{t+1}), \\
\hat{P}_{t+1} &= \bar{P}_{t+1} - \mathcal{K}_{t+1} \bar{V}_{t+1} \mathcal{K}_{t+1}^\top,
\end{aligned} \tag{43}$$

with

$$\mathcal{K}_{t+1} = \left[ \sum_{i=0}^{2\bar{k}} w_i (\bar{\chi}_{t,i} - \bar{v}_{t+1})(O[\bar{\chi}_{t,i}; \Theta] - \bar{y}_{t+1})^\top \right] (\bar{V}_{t+1})^{-1}. \tag{44}$$

We refer to Wan and van der Merwe (2001) for general treatments of the unscented Kalman filter.

Given the forecasted option prices  $\bar{y}$  and its conditional covariance matrix  $\bar{V}$  obtained from the filtering technique, we compute the log likelihood value for each week's observations on the option prices assuming normally distributed forecasting errors,

$$l_{t+1}[\Theta]^O = -\frac{1}{2} \log |\bar{V}_t| - \frac{1}{2} \left( (y_{t+1} - \bar{y}_{t+1})^\top (\bar{V}_{t+1})^{-1} (y_{t+1} - \bar{y}_{t+1}) \right). \tag{45}$$

Furthermore, given the extracted risk premium rates from the options data, we apply fast Fourier inversion to the characteristic function in (34) to obtain the statistical density of weekly returns on each of the three currency pair as a function of the risk premium rates. We use  $l_{t+1}[\Theta]^s$  to denote the weekly log likelihood of the currency returns on the three currency pairs.

We choose model parameters to maximize the log likelihood of the data series, which is a summation of the weekly log likelihood values on both options and currency returns,

$$\Theta \equiv \arg \max_{\Theta} \mathcal{L}[\Theta, \{y_t\}_{t=1}^T], \quad \text{with} \quad \mathcal{L}[\Theta, \{y_t\}_{t=1}^T] = \sum_{t=0}^{T-1} (l_{t+1}[\Theta]^O + l_{t+1}[\Theta]^S), \quad (46)$$

where  $T = 117$  denotes the number of weeks in our sample.

## V. Empirical Results

Building on established themes, the models with proportional asymmetry and strict symmetry are estimated using the maximum likelihood procedure in (46). For each specification, four models are estimated that allow for different parameterizations of the dampened power law jump class in (6). Specifically, we allow for unrestricted power coefficient,  $\alpha$ , and the nested special cases of  $\alpha = -1$ ,  $\alpha = 0$ , and  $\alpha = 1$ . Hence, altogether we estimate eight distinct models. The estimated model parameters and their standard errors (in parenthesis), as well as the maximized log likelihood values, are reported in Table II for the four symmetric models and in Table III for the four asymmetric models.

### A. The US, Japan, and UK Economies Are Asymmetric

The maximized likelihood values from the general asymmetric specifications (Table III) are much larger than the corresponding symmetric specifications (Table II). Likelihood ratio tests for nested models suggest that the differences are significant beyond any reasonable confidence level. The estimated variance of the pricing errors ( $\sigma_r^2$ ) of the symmetric models is almost twice as large as that of the asymmetric models. Therefore, by allowing asymmetry between the stochastic discount factors of the US, Japan, and the UK, the models capture the currency return and currency options behavior much better.

The scaling coefficient on the US economy is normalized to unity:  $\xi^{USD} = 1$ . Hence, the deviations from unity for the estimates of  $\xi^{GBP}$  and  $\xi^{JPY}$  measure the degree of asymmetry between the three economies. The estimates for the scaling coefficient on UK,  $\xi^{GBP}$ , are only slightly larger than one, but the estimates for the scaling coefficient on Japan,  $\xi^{JPY}$ , are significantly larger than unity at around 1.5. This result suggests that the Japanese economy is significantly different from the US economy and the UK economy. The average risk premium in Japan is about 50 percent larger than that in the US or the



UK. The larger risk premium can be due to either larger risk in the economy or higher risk aversion for investors in Japan.

## **B. Risk Premium Rates on the Global Risk Factor Are More Persistent and More Volatile**

Given the observed asymmetry between the three economies, we can identify the global risk factor and its risk premium. The estimates of the parameters that control the global risk factor ( $\kappa_Z, \theta_Z, \sigma_Y, \rho_Y$ ) are mostly statistically significant and are relatively stable across different parameterizations on  $\alpha$ . Comparing the estimates for the global risk factor parameters to those on the country-specific risk factors in Table III, we observe that the global risk premium rate is both more persistent and more volatile than the country-specific risk premium rates. The mean-reversion parameter estimates for the global risk factor,  $\kappa_Z$ , is not distinguishable from zero, implying near non-stationary behavior. In contrast, the estimates of mean-reversion for the country-specific factor,  $\kappa_Y$ , range from 3.053 to 5.204, implying a relatively short half life of two to three months. These estimates suggest that the variations in the country-specific risk premium rates are much more transitory than variations in the risk premium rate of the global risk factor.

Furthermore, the volatility coefficient estimates  $\omega_Z$  for the global risk premium factor are around 0.8, about five times larger than the corresponding volatility coefficients  $\omega_Y$  for the country-specific risk premium rates, which are between 0.14 to 0.18.

Our findings are consistent with Engle, Ito, and Lin (1990), who use the analogies of meteor showers versus heat waves to describe global versus country-specific shocks, respectively. Using intra-day exchange rate data, they find that volatility clustering in exchange rates is mainly driven by global shocks. Using weekly data on currency returns and currency options, we document that the risk premium rates on the global risk factor are both more persistent and more volatile than the risk premium rates on the country-specific risk factors. Our findings also suggest that the more permanent variations of the state-price deflator are mainly driven by a global risk component, indicating a high degree of international integration among the three economies. In this sense our evidence on the role of the global risk factor agrees with Brandt, Cochrane, and Santa-Clara (2005), who reason that the log stochastic discount factors must be highly correlated to explain the relative smoothness of the exchange rates.

### C. Risk Premium Increases When the Wealth Declines *Relative* to the Global Portfolio

The correlation parameter  $\rho_Z$  captures how the risk premium rate changes with the global shocks while the correlation parameter  $\rho_Y$  measures how the risk premium rate changes with the country-specific shocks. The estimates for  $\rho_Y$  are strongly negative between -0.702 and -0.999, depending on different  $\alpha$  specifications. A negative correlation implies that the risk premium increases when the economy receives a negative country-specific shock. Such a risk premium increase can come from either or both of the two sources: (1) A negative shock is associated with higher economy-wide volatility. (2) Investors become significantly more risk averse after a negative shock and demand higher compensation for the same amount of risk.

Our empirical estimates for the correlation between the risk premium rate and the global risk factor  $\rho_Z$  are positive and range between 0.52 to 0.65. Thus, investors respond quite differently to the global risk component and the country-specific risk component. Although investors demand a higher risk premium in the presence of a negative country-specific shock to the economy, they actually ask for a lower risk premium if the origin of the negative shock is global.

A possible interpretation for the different responses is that the risk premium in an economy changes with the *relative* wealth of the economy with the global portfolio serving as a benchmark. Investors demand a higher premium only when the wealth of the economy declines relative to the global portfolio. Thus, when the global risk factor receives a negative shock, the economy wealth decreases in absolute terms, but increases relative to the global portfolio, and hence the risk premium declines. In contrast, a negative shock to the country-specific risk factor decreases the economy wealth in both absolute and relative terms, and the risk premium in this economy increases unambiguously.

When studying how an economy responds to external shocks, it is important to distinguish the different possible sources of the shocks. An analysis that fails to discriminate between country-specific and global shocks can lead to misleading conclusions. It is worthwhile to mention that the extant literature often studies the behavior of stochastic discount factors and economy-wide risk premium using equity index returns and equity index options. Since the stochastic discount factors estimated from these data are projections of the pricing kernel on the equity index of a single economy, these studies do not typically distinguish between global shocks versus country-specific shocks. Our joint analysis based on a triangular pair of currency returns and currency options reveals the complex multi-dimensional feature of the stochastic discount factors in international economies and highlights the inadequacy of one-dimensional projections.

## D. Jumps Arrive Frequently, But Only Downside Jumps Are Priced

Our model for the stochastic discount factors incorporates a jump component, the arrival rate of which follows an exponentially dampened power law. Under this specification, the power coefficient  $\alpha$  controls the jump type. The model generates finite-activity jumps when  $\alpha < 0$ , under which jumps arrive only a finite number of times within any finite interval and hence can be regarded as rare events. On the other hand, when  $\alpha \geq 0$ , jumps arrive an infinite number of times within any finite interval and can therefore be used to capture more frequent movements.

When we estimate the general asymmetric model with  $\alpha$  as a free parameter, the estimate for  $\alpha$  is 0.227. Nevertheless, the estimate has large standard error, suggesting potential identification problems. Thus, within the general specification, we also estimate three special cases with  $\alpha$  fixed at  $-1$ ,  $0$ , and  $1$ . As shown in Table III for the asymmetric model, the model with  $\alpha = 1$  generates the highest likelihood value among the three special cases, indicating that jumps in the three economies are not rare events, but rather arrive frequently.

The relative asymmetry of jumps are controlled by the two exponential dampening coefficients  $\beta_+$  and  $\beta_-$ . A larger dampening coefficient  $\beta_+$  implies a smaller arrival rate for positive jumps and vice versa. Table III shows that the estimates for  $\beta_+$  are substantially larger than those for  $\beta_-$ , more so when  $\alpha$  is larger and hence when more frequent jumps are allowed. The large estimates for  $\beta_+$  suggest that we rarely observe positive jumps in the stochastic discount factors. In fact, the standard errors for  $\beta_+$  are also very larger, suggesting that positive jumps are so rare that we cannot accurately identify the parameter that control the positive jumps. Therefore, we can safely assume a one-sided jump structure for the log stochastic discount factor by setting the arrival rate of positive jumps to zero:  $\nu[x] = 0$  for  $x > 0$ .

To pursue this angle, Table IV reports the parameter estimates and maximized log likelihood values under this one-sided jump assumption. The estimates for most of the parameters are close to those reported in Table III under the two-sided jump parameterization. The likelihood values are also about the same. The only difference is that with the one-sided jump assumption in Table IV, the standard errors of some parameters decline, showing better identification with the more parsimonious one-sided specification. Therefore, our results support the lack of a significant pricing component for positive jumps in the stochastic discount factors.

In reality, the wealth of an economy can both jump up and jump down, with the distribution relatively symmetric. The fact that we can only detect a downside jump component in the stochastic discount factor implies that investors are only concerned with downside jumps in the economy while

ignoring upside jumps for pricing. In other words, only downside jumps are perceived as risk and are priced.

The presence of priced frequent downside jumps in the stochastic discount factors provides theoretical justification for the prevailing evidence from equity index option markets. Although the statistical return distribution for equity indexes is relatively symmetric, the risk-neutral distributions computed from option prices are highly negatively skewed (Jackwerth and Rubinstein (1996), and Bakshi, Kapadia, and Madan (2003)). Carr and Wu (2003) show that a one-sided  $\alpha$ -stable law, without exponential dampening, captures the S&P 500 index options price behavior well. When applying measure changes using exponential martingales,  $\alpha$ -stable laws are converted into exponentially dampened power laws. Hence, the dampened power law specification subsumes the  $\alpha$ -stable specification.

## VI. Conclusions

In this paper, we propose to infer the dynamic behaviors of the stochastic discount factors in international economies from currency returns and currency options. We first develop a class of models of stochastic discount factors that are sufficiently flexible to capture the observed behaviors of currency returns and currency options, especially stochastic risk premiums and stochastic skewness. We then estimate these models using the time-series of option prices on dollar-yen, dollar-pound, and pound-yen exchange rates, and the underlying currency returns. Based on the estimation results, we investigate whether investors show a differential response to country-specific risks versus global risks, and to upside-jump versus downside-jump risks. We also investigate how risk premium reacts to shocks emanating from different sources.

Our estimation results show that the average risk premium in Japan is about 50 percent larger than the average risk premium in the US or UK. The asymmetry between the three economies enables us to identify both the global risk factor and the country-specific risk factors and their associated risk premium dynamics. We find that the risk premium rate on the global risk factor is both more persistent and more volatile than the risk premium rates on the country-specific risks. Furthermore, investors react differently to shocks to the global risk factor and the country-specific risk factors. Investors demand a higher risk premium when the economy receives a negative shock that is country-specific, but demand a lower premium when the negative shock is global. Hence, the risk premium in an economy increases only when the wealth of the economy declines relative to the global portfolio. Finally, our estimation shows that jumps in each economy are not rare events, but arrive very frequently. However, investors only price downside jumps while upside jumps are not perceived as risk.

Traditional literature has studied the behavior of stochastic discount factors either through point estimates on various types of expectation hypotheses regressions, or more recently, through one-dimensional projections to equity indexes. Our study shows that the stochastic discount factors in international economies show complex multi-dimensional dynamic behaviors that cannot possibly be fully disentangled through point estimates or one-dimensional projections. Future research calls for a joint analysis of the international bond, equity, and currency markets and their options to obtain a finer distinction between the dynamics of risk and pricing.

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**Table I**  
**Risk-Reversals and Butterfly Spreads**

Each maturity has four set of volatility quotes in the form of ten-delta risk-reversal (denoted RR[10]), 25-delta risk-reversal (denoted RR[25]), ten-delta butterfly spread (denoted BF[10]), and 25-delta butterfly spread (denoted BF[25]), all as percentages of the corresponding at-the-money implied volatility (SIV). Each row represents a single maturity. The first column denotes the option maturity, with ‘w’ denoting weeks and ‘m’ denoting months. Reported are the mean, the standard deviation, and the  $t$ -statistics on the significance of the sample mean for each risk-reversal and butterfly spread series. The  $t$ -statistics adjust serial dependence according to Newey and West (1987), with the number of lags optimally chosen according to Andrews (1991) based on an AR(1) specification. Data are weekly from November 7, 2001 to January 28, 2004.

| Mat.   | RR[10] |       |       | RR[25] |       |       | BF[10] |      |       | BF[25] |      |       |
|--------|--------|-------|-------|--------|-------|-------|--------|------|-------|--------|------|-------|
| JPYUSD |        |       |       |        |       |       |        |      |       |        |      |       |
| 1w     | 11.63  | 13.81 | 3.34  | 6.45   | 7.59  | 3.35  | 13.65  | 3.84 | 11.77 | 3.40   | 0.74 | 15.50 |
| 1m     | 12.53  | 13.64 | 3.20  | 6.94   | 7.56  | 3.20  | 13.90  | 3.40 | 12.75 | 3.57   | 0.62 | 18.42 |
| 2m     | 13.91  | 14.89 | 2.85  | 7.55   | 8.08  | 2.83  | 14.49  | 2.93 | 14.81 | 3.70   | 0.53 | 21.80 |
| 3m     | 14.47  | 15.78 | 2.59  | 7.86   | 8.61  | 2.58  | 14.91  | 2.56 | 17.18 | 3.79   | 0.47 | 25.42 |
| 6m     | 15.30  | 17.98 | 2.21  | 8.23   | 9.74  | 2.20  | 15.43  | 2.20 | 19.71 | 4.02   | 0.38 | 31.52 |
| 9m     | 15.79  | 19.41 | 2.08  | 8.45   | 10.36 | 2.08  | 16.23  | 2.04 | 21.75 | 4.13   | 0.40 | 29.23 |
| 12m    | 16.19  | 20.47 | 2.00  | 8.63   | 10.94 | 2.00  | 16.55  | 2.03 | 21.78 | 4.18   | 0.43 | 27.25 |
| GBPUSD |        |       |       |        |       |       |        |      |       |        |      |       |
| 1w     | 5.86   | 8.07  | 2.93  | 3.26   | 4.42  | 2.98  | 9.74   | 2.65 | 11.11 | 2.82   | 0.59 | 15.90 |
| 1m     | 5.73   | 7.08  | 2.79  | 3.21   | 3.93  | 2.86  | 9.79   | 2.39 | 10.98 | 2.83   | 0.55 | 14.79 |
| 2m     | 5.51   | 6.32  | 2.81  | 3.19   | 3.60  | 2.94  | 9.55   | 2.12 | 11.56 | 2.76   | 0.48 | 15.91 |
| 3m     | 5.30   | 5.81  | 2.79  | 3.01   | 3.25  | 2.90  | 9.64   | 1.68 | 15.46 | 2.71   | 0.42 | 17.74 |
| 6m     | 4.87   | 5.40  | 2.25  | 2.75   | 2.97  | 2.32  | 9.53   | 1.15 | 25.83 | 2.47   | 0.46 | 13.75 |
| 9m     | 4.80   | 5.27  | 2.16  | 2.72   | 2.91  | 2.19  | 9.49   | 0.99 | 29.88 | 2.46   | 0.42 | 13.89 |
| 12m    | 4.68   | 5.30  | 2.01  | 2.67   | 2.89  | 2.09  | 9.37   | 0.91 | 32.86 | 2.42   | 0.41 | 15.14 |
| GBPJPY |        |       |       |        |       |       |        |      |       |        |      |       |
| 1w     | -5.85  | 12.08 | -1.73 | -3.18  | 6.58  | -1.72 | 11.09  | 2.56 | 17.06 | 2.95   | 0.80 | 14.38 |
| 1m     | -6.42  | 12.32 | -1.70 | -3.51  | 6.69  | -1.71 | 11.51  | 2.16 | 20.36 | 3.17   | 0.48 | 26.87 |
| 2m     | -6.32  | 12.48 | -1.62 | -3.41  | 6.68  | -1.62 | 12.02  | 2.12 | 19.55 | 3.31   | 0.45 | 28.28 |
| 3m     | -6.02  | 12.57 | -1.52 | -3.28  | 6.74  | -1.54 | 12.44  | 2.13 | 18.19 | 3.44   | 0.43 | 28.59 |
| 6m     | -5.76  | 12.62 | -1.43 | -3.12  | 6.80  | -1.43 | 13.07  | 2.00 | 18.26 | 3.54   | 0.49 | 21.35 |
| 9m     | -5.72  | 12.75 | -1.40 | -3.08  | 6.86  | -1.39 | 13.47  | 2.16 | 16.51 | 3.65   | 0.60 | 16.67 |
| 12m    | -5.70  | 13.01 | -1.35 | -3.06  | 6.98  | -1.35 | 13.64  | 2.11 | 16.83 | 3.69   | 0.63 | 15.74 |

**Table II**  
**Maximum Likelihood Estimates of Model Parameters under Strict Symmetry**

Entries report the maximum likelihood estimates of the structural parameters and their standard errors (in parentheses) for the models admitting stochastic currency risk premium and stochastic skewness under strict symmetry. Four separate models are estimated that respectively allow the power coefficient,  $\alpha$ , in the dampened power law specification for the jump component to take values of  $\alpha = -1$ ,  $\alpha = 0$ ,  $\alpha = 1$ , and  $\alpha$  unrestricted. Estimation is based on weekly currency return and currency options data from November 7, 2001 to January 28, 2004 (117 weekly observations for each series). The last row reports the maximized log likelihood value.  $\sigma_r^2$  represents the variance of the measurement error.

| $\Theta$                                     | $\alpha = -1$  | $\alpha = 0$   | $\alpha = 1$   | Free $\alpha$  |
|--|----------------|----------------|----------------|----------------|
| Country-Specific Risk Premium Rate Dynamics: |                |                |                |                |
| $\kappa_Y$                                   | 2.149 (0.108)  | 1.912 (0.096)  | 1.531 (0.053)  | 1.210 (0.081)  |
| $\theta_Y$                                   | 0.003 (0.000)  | 0.003 (0.000)  | 0.004 (0.000)  | 0.001 (0.014)  |
| $\omega_Y$                                   | 0.149 (0.010)  | 0.150 (0.009)  | 0.148 (0.008)  | 0.081 (0.486)  |
| $\rho_Y$                                     | -0.252 (0.054) | -0.321 (0.048) | -0.412 (0.046) | -0.898 (5.433) |
| Country-Specific Jump Risk Structure:        |                |                |                |                |
| $\lambda$                                    | 17.684 (1.589) | 5.255 (0.500)  | 1.184 (0.392)  | 0.747 (9.170)  |
| $\beta_-$                                    | 4.623 (0.117)  | 4.146 (0.078)  | 3.835 (1.032)  | 4.420 (4.146)  |
| $\beta_+$                                    | 43.513 (6.9e2) | 58.234 (4.4e2) | 97.645 (3.7e2) | 3.1e4 (4.5e6)  |
| $\alpha$                                     | -1 —           | 0 —            | 1 —            | 1.810 (0.403)  |
| Performance Metrics:                         |                |                |                |                |
| $\sigma_r^2$                                 | 0.336 (0.004)  | 0.334 (0.004)  | 0.329 (0.004)  | 0.324 (0.005)  |
| $\mathcal{L}/T$                              | 1.62           | 1.58           | 1.67           | 1.93           |

**Table III**  
**Maximum Likelihood Estimates of Model Parameters under Proportional Asymmetry**

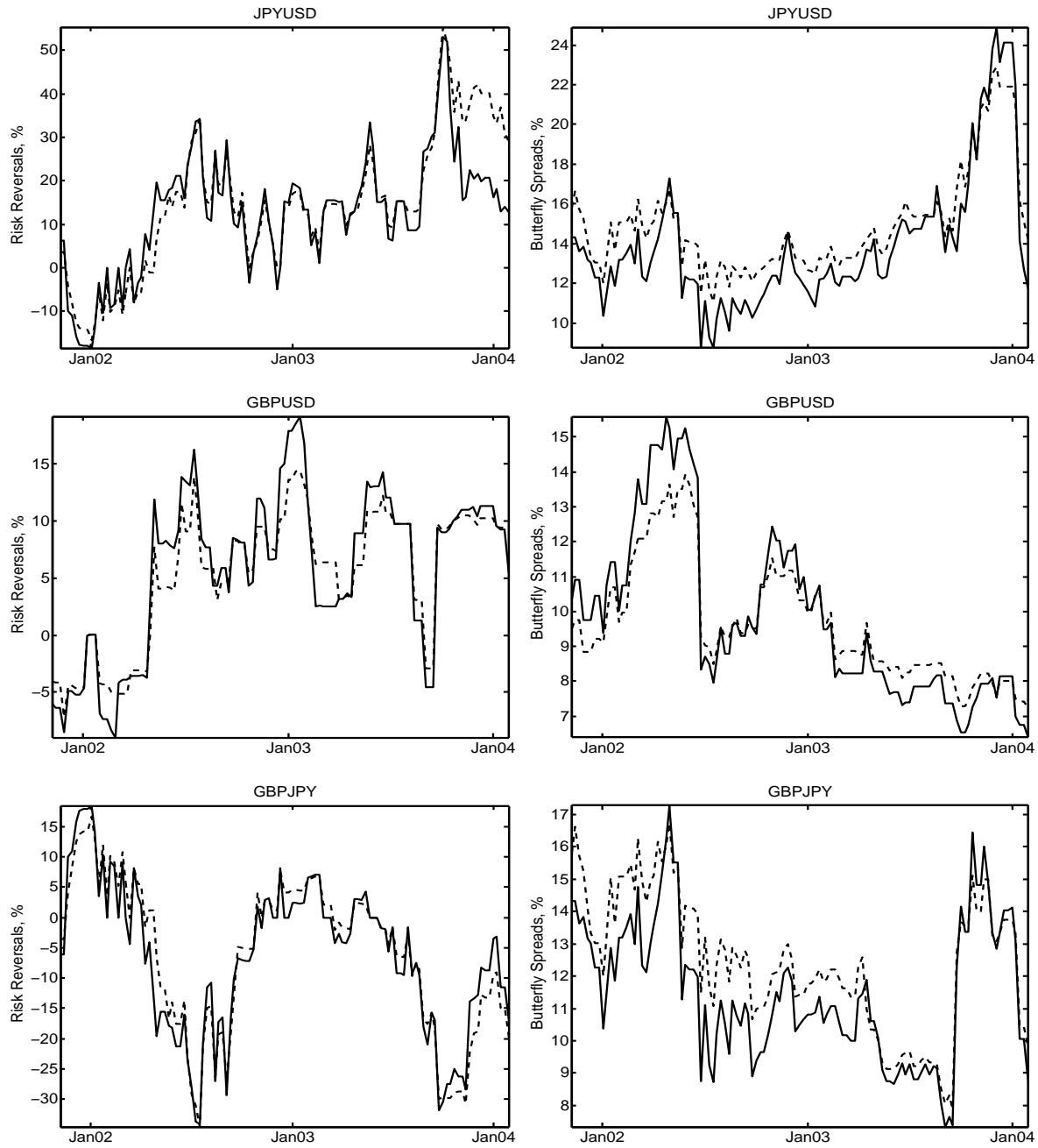
Entries report the maximum likelihood estimates of the structural parameters and their standard errors (in parentheses) for the models admitting stochastic currency risk premium and stochastic skewness under proportional asymmetry. Four separate models are estimated that respectively allow the power coefficient,  $\alpha$ , in the dampened power law specification for the jump component to take values of  $\alpha = -1$ ,  $\alpha = 0$ ,  $\alpha = 1$ , and  $\alpha$  unrestricted. Estimation is based on weekly currency return and currency options data from November 7, 2001 to January 28, 2004. The last row reports the maximized average daily log likelihood value.  $\sigma_r^2$  represents the variance of the measurement error.

| $\Theta$                                     | $\alpha = -1$  | $\alpha = 0$   | $\alpha = 1$    | Free $\alpha$   |
|--|----------------|----------------|-----------------|-----------------|
| Average Risk Premiums:                       |                |                |                 |                 |
| $\xi^{JPY}$                                  | 1.507 (0.027)  | 1.508 (0.028)  | 1.531 (0.035)   | 1.531 (0.034)   |
| $\xi^{GBP}$                                  | 1.017 (0.005)  | 1.016 (0.006)  | 1.007 (0.006)   | 1.007 (0.005)   |
| Global Risk Premium Rate Dynamics:           |                |                |                 |                 |
| $\kappa_Z$                                   | 0.000 (0.006)  | 0.000 (0.006)  | 0.000 (0.006)   | 0.000 (0.005)   |
| $\theta_Z$                                   | 0.230 (0.069)  | 0.231 (0.065)  | 0.356 (0.220)   | 0.357 (0.223)   |
| $\omega_Z$                                   | 0.807 (0.069)  | 0.797 (0.069)  | 0.815 (0.053)   | 0.813 (0.050)   |
| $\rho_Z$                                     | 0.650 (0.059)  | 0.626 (0.059)  | 0.521 (0.034)   | 0.524 (0.035)   |
| Country-Specific Risk Premium Rate Dynamics: |                |                |                 |                 |
| $\kappa_Y$                                   | 5.204 (0.190)  | 4.921 (0.210)  | 3.061 (0.059)   | 3.053 (0.061)   |
| $\theta_Y$                                   | 0.003 (0.000)  | 0.003 (0.000)  | 0.003 (0.006)   | 0.003 (0.001)   |
| $\omega_Y$                                   | 0.183 (0.006)  | 0.174 (0.006)  | 0.137 (0.163)   | 0.138 (0.016)   |
| $\rho_Y$                                     | -0.702 (0.046) | -0.713 (0.048) | -0.996 (1.185)  | -0.999 (0.115)  |
| Country-Specific Jump Risk Structure:        |                |                |                 |                 |
| $\lambda$                                    | 18.698 (9.146) | 5.659 (1.428)  | 20.489 (54.032) | 815.387 (7.8e3) |
| $\beta_-$                                    | 5.132 (0.936)  | 4.523 (0.686)  | 36.767 (9.842)  | 63.069 (59.324) |
| $\beta_+$                                    | 1.2e2 (4.3e4)  | 1.4e2 (7.8e3)  | 2.5e3 (6.9e5)   | 4.7e4 (6.9e4)   |
| $\alpha$                                     | -1 —           | 0 —            | 1 —             | 0.227 (2.205)   |
| Performance Metrics:                         |                |                |                 |                 |
| $\sigma_r^2$                                 | 0.174 (0.002)  | 0.175 (0.002)  | 0.167 (0.003)   | 0.167 (0.002)   |
| $\mathcal{L}/T$                              | 32.97          | 32.84          | 33.96           | 34.10           |

**Table IV**  
**Maximum Likelihood Estimates of Model Parameters Assuming One-Sided Jumps**

Entries report the maximum likelihood estimates of the structural parameters and their standard errors (in parentheses) for the models admitting stochastic currency risk premium and stochastic skewness under proportional asymmetry and assuming only negative jumps. Four separate models are estimated that respectively allow the power coefficient,  $\alpha$ , in the dampened power law specification for the jump component to take values of  $\alpha = -1$ ,  $\alpha = 0$ ,  $\alpha = 1$ , and  $\alpha$  unrestricted. Estimation is based on weekly currency return and currency options data from November 7, 2001 to January 28, 2004. The last row reports the maximized average daily log likelihood value.  $\sigma_r^2$  represents the variance of the measurement error.

| $\Theta$                                     | $\alpha = -1$  | $\alpha = 0$   | $\alpha = 1$    | Free $\alpha$   |
|--|----------------|----------------|-----------------|-----------------|
| Average Risk Premiums:                       |                |                |                 |                 |
| $\xi^{JPY}$                                  | 1.507 (0.026)  | 1.509 (0.027)  | 1.531 (0.034)   | 1.530 (0.034)   |
| $\xi^{GBP}$                                  | 1.017 (0.005)  | 1.016 (0.006)  | 1.007 (0.005)   | 1.008 (0.005)   |
| Global Risk Premium Rate Dynamics:           |                |                |                 |                 |
| $\kappa_Z$                                   | 0.000 (0.005)  | 0.000 (0.005)  | 0.000 (0.006)   | 0.000 (0.006)   |
| $\theta_Z$                                   | 0.230 (0.066)  | 0.231 (0.060)  | 0.357 (0.196)   | 0.348 (0.289)   |
| $\omega_Z$                                   | 0.807 (0.069)  | 0.797 (0.068)  | 0.814 (0.051)   | 0.805 (0.051)   |
| $\rho_Z$                                     | 0.650 (0.058)  | 0.626 (0.058)  | 0.521 (0.034)   | 0.529 (0.035)   |
| Country-Specific Risk Premium Rate Dynamics: |                |                |                 |                 |
| $\kappa_Y$                                   | 5.203 (0.185)  | 4.924 (0.200)  | 3.053 (0.046)   | 3.034 (0.065)   |
| $\theta_Y$                                   | 0.003 (0.000)  | 0.003 (0.000)  | 0.003 (0.000)   | 0.003 (0.001)   |
| $\omega_Y$                                   | 0.183 (0.006)  | 0.174 (0.006)  | 0.137 (0.011)   | 0.138 (0.018)   |
| $\rho_Y$                                     | -0.702 (0.042) | -0.713 (0.045) | -0.996 (0.094)  | -0.999 (0.129)  |
| Country-Specific Jump Risk Structure:        |                |                |                 |                 |
| $\lambda$                                    | 18.698 (9.137) | 5.658 (1.408)  | 21.199 (10.585) | 8.8e2 (9.4e3)   |
| $\beta_-$                                    | 5.132 (0.935)  | 4.526 (0.690)  | 37.329 (9.718)  | 66.157 (70.052) |
| $\alpha$                                     | -1 —           | 0 —            | 1 —             | 0.240 (2.428)   |
| Performance Metrics:                         |                |                |                 |                 |
| $\sigma_r^2$                                 | 0.174 (0.002)  | 0.175 (0.002)  | 0.167 (0.003)   | 0.167 (0.002)   |
| $\mathcal{L}/T$                              | 32.97          | 32.84          | 33.96           | 34.10           |



**Figure 1. Time-Variation in Risk-Reversals and Butterfly Spreads:** Left panels plot the time-series of ten-delta risk-reversals and the right panels plot the time-series of ten-delta butterfly spreads, both as a percentage of at-the-money implied volatility. The two lines correspond to distinct option maturities of one month (solid line) and three months (dashed line). Data are weekly from November 7, 2001 to January 28, 2004.