LTH458
hep-ph/9909570
Revised Version

Quasi-infra-red fixed points and renormalisation group invariant trajectories for non-holomorphic soft supersymmetry breaking

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In the MSSM the quasi-infra-red fixed point for the top-quark Yukawa coupling gives rise to specific predictions for the soft-breaking parameters. We discuss the extent to which these predictions are modified by the introduction of additional "non-holomorphic" soft-breaking terms. We also show that in a specific class of theories, there exists an RG-invariant trajectory for the "non-holomorphic" terms, which can be understood using a holomorphic spurion term.

Sept 1999

1. Introduction

The enduring popularity of the MSSM derives originally from the demonstration that it gave rise to gauge coupling unification, at a scale consistent with proton decay limits (at least with regard to contributions from dimension 6 operators). This success is predicated on (or at least consistent with) the desert hypothesis, whereby the next fundamental physics scale beyond the weak scale is far beyond it: gauge unification, a string scale, or even the Planck mass. Within this context, a "standard" picture of the origin of supersymmetry breaking has emerged: supersymmetry is broken (dynamically or spontaneously) in a distinct sector of the theory and transmitted to observable physics via a "messenger sector". At energies below a characteristic mass scale M the observable effective field theory can be expanded in powers of $1/M$; then we suppose that the breaking of supersymmetry can be parametrised by the vacuum expectation value of the F-term of a chiral superfield $\mathcal{Z}$, such that $< F_{\mathcal{Z}} > \approx M_Z M$, and it is easy to show that the following soft terms are $O(M_Z)$:

$$L_{\rm SOFT}^{(1)} = (m^2)^j{}_i \phi^i \phi_j + \left(\tfrac{1}{6} h^{ijk} \phi_i \phi_j \phi_k + \tfrac{1}{2} b^{ij} \phi_i \phi_j + \tfrac{1}{2} M\lambda\lambda + \text{h.c.}\right) \tag{1.1}$$

whereas the following further possible dimension 3 terms are suppressed by powers of M_Z/M:

$$L_{\rm SOFT}^{(2)} = \tfrac{1}{2} r_i^{jk} \phi^i \phi_j \phi_k + \tfrac{1}{2} m_F{}^{ij} \psi_i \psi_j + m_A{}^{ia} \psi_i \lambda_a + \text{h.c.} \tag{1.2}$$

The terms in Eq. (1.2) arise from non-holomorphic terms (D-terms) in the effective field theory, so we will refer to them as non-holomorphic soft terms (an abuse of terminology, in fact, inasmuch as of course the first term in Eq. (1.1) also arises from a non-holomorphic term).

In fact, if there are no gauge singlets, the terms in Eq. (1.2) are "natural" in the same sense as those of Eq. (1.1), in that they do not give rise to quadratic divergences; but in any event (within the paradigm described above) one would not exclude them even if they do give quadratic divergences, since we only require naturalness up to the scale M. This was emphasised recently by Martin[1], who also pointed out that by the same token there are dimension-4 supersymmetry-breaking contributions which (although suppressed by more powers of $1/M$) may give rise to interesting effects.

Returning to the terms shown in Eq. (1.2), however, there are two reasons why we should consider them. Firstly, their suppression compared to Eq. (1.1) is founded on a specific framework for the origin of supersymmetry breaking which may or may not be true;

secondly, even given the framework, the recent model-building trend has been away from the desert hypothesis: for example, in the suggestion of (very) large extra dimensions. It is not clear to us whether in such theories the suppression of Eq. (1.2) relative to Eq. (1.1) will necessarily be sustained. Be that as it may, we believe that there is a case for an agnostic approach to supersymmetry-breaking whereby all dimension 2 and dimension 3 terms are considered without prejudice, in theories where they do not cause quadratic divergences.

In a previous paper[2] we gave the one-loop β-functions for the parameters defined in Eq. (1.2), both in general and in the MSSM context. In this paper we extend the general results to two loops. We find (and verify through two loops) a RG-invariant relation which can be imposed between r, b, m^2 and m_A. We also investigate the consequences of Yukawa infra-red (and quasi-infra-red) fixed point structure for the MSSM, where we find that some (but not all) of the predictions founded on the MSSM survive in the presence of the non-holomorphic terms.

2. The β-functions

We begin with the one-loop β-functions for a theory with

$$L = L_{\rm SUSY} + L_{\rm SOFT}, \tag{2.1}$$

where

$$L_{\rm SOFT} = L^{(1)}_{\rm SOFT} + L^{(2)}_{\rm SOFT}, \tag{2.2}$$

and where $L_{\rm SUSY}$ is the Lagrangian for the supersymmetric gauge theory, containing the gauge multiplet $\{A_\mu, \lambda\}$ (λ being the gaugino) and a chiral superfield Φ_i with component fields $\{\phi_i, \psi_i\}$ transforming as a (in general reducible) representation R of the gauge group $\mathcal{G}$. (We give results here for a simple gauge group, though the extension to a non-simple gauge group is straightforward.) We assume a superpotential of the form

$$W = \tfrac{1}{6} Y^{ijk} \phi_i \phi_j \phi_k. \tag{2.3}$$

Note that we do not include an explicit supersymmetric μ-term in W; the usual theory containing only $L^{(1)}_{\rm SOFT}$ together with a supersymmetric μ-term can be recovered by taking in $L^{(2)}_{\rm SOFT}$

$$m_A^{ia} = 0, \quad m_F = \mu, \quad r_i^{jk} = Y^{jkl}\mu_{il} \tag{2.4}$$

and replacing $(m^2)^i{}_j$ in $L^{(1)}_{\rm SOFT}$ by $(m^2)^i{}_j + \mu^{il}\mu_{jl}$.

The one-loop results for the gauge coupling β-function β_g and for the chiral field anomalous dimension γ are:

$$16\pi^2\beta_g = g^3 Q \quad \text{and} \quad 16\pi^2\gamma^i{}_j = P^i{}_j, \tag{2.5}$$

where

$$Q = T(R) - 3C(G), \quad \text{and} \quad P^i{}_j = \tfrac{1}{2}Y^{ikl}Y_{jkl} - 2g^2 C(R)^i{}_j. \tag{2.6}$$

Here

$$T(R)\delta_{ab} = \mathrm{Tr}(R_a R_b), \quad C(G)\delta_{ab} = f_{acd}f_{bcd} \quad \text{and} \quad C(R)^i{}_j = (R_a R_a)^i{}_j, \tag{2.7}$$

and as usual $Y^*_{ijk} = Y^{ijk}$ etc. For the new soft terms from Eq. (1.2) we have[2]:

$$16\pi^2\beta_{m_{Fij}} = P^k{}_i m_{Fkj} + P^k{}_j m_{Fik}, \tag{2.8a}$$

$$16\pi^2\beta_{m_{Aia}} = P^j{}_i m_{Aja} + g^2 Q m_{Aia}, \tag{2.8b}$$

and

$$\begin{aligned}16\pi^2(\beta_r)^{jk}_i &= \tfrac{1}{2}P^l{}_i r^{jk}_l + P^k{}_l r^{jl}_i + \tfrac{1}{2}r^{mn}_i Y_{lmn}Y^{ljk} + 2r^{mj}_l Y_{imn}Y^{kln} + 2g^2 r^{jk}_l C(R)^l{}_i \\ &+ 2g^2 r^{mj}_l (R_a)^k{}_i (R_a)^l{}_m - 2m_{Flm}Y^{mnj}Y^{plk}Y_{npi} - 4g^2 m_{Fil}C(R)^l{}_m Y^{mjk} \\ &- 4g\sqrt{2}\left[g^2 C(G) m^{ja}_A (R_a)^k{}_i + (R_a)^j{}_l Y^{lmk}Y_{mni}m^{na}_A\right] \quad + (k \leftrightarrow j).\end{aligned} \tag{2.9}$$

For the original soft terms in Eq. (1.1) we have

$$16\pi^2\beta^{ijk}_h = U^{ijk} + U^{kij} + U^{jki}, \tag{2.10a}$$

$$16\pi^2\beta^{ij}_b = V^{ij} + V^{ji}, \tag{2.10b}$$

$$16\pi^2[\beta_{m^2}]^i{}_j = W^i{}_j, \tag{2.10c}$$

$$16\pi^2\beta_M = 2g^2 QM, \tag{2.10d}$$

where

$$U^{ijk} = h^{ijl}P^k{}_l + Y^{ijl}X^k{}_l, \tag{2.11a}$$

$$\begin{aligned}V^{ij} &= b^{il}P^j{}_l + r^i_{lm}h^{jlm} + r^{im}_l r^{jl}_m - m_{Fkl}Y^{ilm}m_{Fmn}Y^{jnk} \\ &+ 4g^2 M m^{ik}_F C(R)^j{}_k - 4g^2 C(G) m^{ia}_A m^{ja}_A,\end{aligned} \tag{2.11b}$$

$$\begin{aligned}W^i{}_j &= \tfrac{1}{2}Y_{jpq}Y^{pqn}(m^2)^i{}_n + \tfrac{1}{2}Y^{ipq}Y_{pqn}(m^2)^n{}_j + 2Y^{ipq}Y_{jpr}(m^2)^r{}_q + h_{jpq}h^{ipq} \\ &+ r^{kl}_j r^i_{kl} + 2r^k_{jl}r^{il}_k - 4(m^{kl}_F m_{Flm} + m_{Ama}m^{ka}_A)Y^{imn}Y_{jkn} \\ &- 8g^2(MM^*C(R)^i{}_j + m^{kl}_F m_{Fjk}C(R)^i{}_l + C(G)m^{ia}_A m_{Aja} + (R_a R_b)^i{}_j m_{Aka}m^{kb}_A) \\ &- 4\sqrt{2}g(Y^{iml}m_{Fmn}(R_a)^n{}_j m_{Ala} + Y_{jml}m^{mn}_F (R_a)^i{}_n m^{la}_A)\end{aligned} \tag{2.11c}$$

with

$$X^i{}_j = h^{ikl}Y_{jkl} + 4g^2 MC(R)^i{}_j. \tag{2.12}$$

Note that we have omitted from Eq. (2.10c) a contribution of the form $g^2(R_a)^i{}_j\mathrm{Tr}[R_a m^2]$. This term arises only for $U(1)$ and amounts to a renormalisation of the linear D-term that is allowed in that case. The two-loop β-functions are listed in the Appendix (for the case $m_F = 0$).

There has been much interest recently in RG-invariant relations expressing the usual soft couplings M, h^{ijk} and $(m^2)^i{}_j$ in terms of the β-functions for the unbroken theory. In Refs. [3] these relations were derived from the superconformal anomaly, while in Ref. [4] they were derived using exact results for the soft-breaking β-functions obtained using the spurion formalism. From the latter point of view, there would seem no *a priori* reason to expect such RG-invariant results for the new non-standard couplings. The reason for this is that the spurion formalism enables us to relate the renormalisation of the standard soft terms M, h^{ijk} and $(m^2)^i{}_j$ to the anomalous dimension γ of the chiral superfield. This does not carry over to, for example, the case of r_i^{jk} because the corresponding superspace interaction is $\Phi^2\Phi^*$ which is nonrenormalisable and hence leads to divergences beyond those described by γ. It is (at first sight) surprising, therefore, that it is in fact possible to develop RG-invariant expressions for the non-standard couplings. We start by writing $m_F = \mu$ in Eqs. (2.8)–(2.11), since, as we shall explain in more detail later, m_F will effectively be playing the rôle of a supersymmetric μ-term. Then firstly, the relation

$$r_i^{jk} = \sqrt{2}g\left[(R_a)^j{}_i m_A^{ka} + (R_a)^k{}_i m_A^{ja}\right] + Y^{jkl}\mu_{il} \tag{2.13}$$

defines a renormalisation-group trajectory for r_i^{jk}. If we impose Eq. (2.13) in Eq. (2.9), we find

$$\begin{aligned}(\beta_r)_i^{jk} = {}& \sqrt{2}\beta_g\left[(R_a)^j{}_i m_A^{ka} + (R_a)^k{}_i m_A^{ja}\right] + \sqrt{2}g\left[(R_a)^j{}_i\beta_{m_A}^{ka} + (R_a)^k{}_i\beta_{m_A}^{ja}\right] \\ & + \beta_Y^{jkl}\mu_{il} + Y^{jkl}\beta_{\mu il}.\end{aligned} \tag{2.14}$$

This clearly implies that Eq. (2.13) is RG-invariant. Now suppose that in the usual theory, with a supersymmetric μ term and only the soft terms contained in $L^{(1)}_{\mathrm{SOFT}}$, we have solved the RG equations, with the functions $(m_s^2)^i{}_j$ and b_s^{ij} being the solutions for $(m^2)^i{}_j$ and b^{ij}. If we additionally impose

$$b^{ij} = b_s^{ij} + 2m_A^{ai}m_A^{aj}, \tag{2.15}$$

we find, on imposing Eq. (2.15) in Eq. (2.10b),

$$\beta_b^{ij} = \mu\frac{d}{d\mu}b_s^{ij} + 2\beta_{m_A}^{ai}m_A^{aj} + 2m_A^{ai}\beta_{m_A}^{aj}, \tag{2.16}$$

which implies that Eq. (2.15) is RG-invariant. Finally, if we set

$$m^{Aia}m_{Aja} = \rho\delta^i{}_j, \qquad (m^2)^i{}_j = (m_s^2)^i{}_j + \mu^{ik}\mu_{kj} + 2\rho\delta^i{}_j \tag{2.17}$$

where ρ is an arbitrary constant, and the matter multiplet satisfies $C(R)^i{}_j = C(G)\delta^i{}_j$, then we find on substituting Eq. (2.17) into Eq. (2.10c) that

$$(\beta_{m^2})^i{}_j = \mu\frac{d}{d\mu}(m_s^2)^i{}_j + \beta_\mu^{ik}\mu_{kj} + \mu^{ik}\beta_{\mu kj} + 2\beta_{m_A}^{ai}m_{Aaj} + 2m_A^{ai}\beta_{m_A aj}, \tag{2.18}$$

demonstrating the RG-invariance of Eq. (2.17). Note that here we are including a supersymmetric μ-term. To be more explicit, another way to phrase our results is to say that in a theory with $W = \frac{1}{6}Y^{ijk}\phi_i\phi_j\phi_k + \frac{1}{2}\mu^{ij}\phi_i\phi_j$, together with $L_{\rm SOFT}$ as in Eq. (2.2)(but taking $m_F = 0$ in Eq. (1.2)), the relations

$$r_i^{jk} = \sqrt{2}g\left[(R_a)^j{}_i m_A^{ka} + (R_a)^k{}_i m_A^{ja}\right], \tag{2.19a}$$

$$b^{ij} = b_s^{ij} + 2m_A^{ai}m_A^{aj}, \tag{2.19b}$$

$$m^{Aia}m_{Aja} = \rho\delta^i{}_j, \qquad (m^2)^i{}_j = (m_s^2)^i{}_j + 2\rho\delta^i{}_j \tag{2.19c}$$

are RG-invariant (once again with the proviso that the matter multiplet satisfies $C(R)^i{}_j = C(G)\delta^i{}_j$ in the case of Eq. (2.19c)). Using the two-loop results given in the Appendix, we can show that the trajectory is also RG-invariant at two-loop order. In the special case of a one-loop finite theory (and setting $\mu = 0$) the above trajectory was described in Ref. [2].

The existence of the RG trajectory described by Eq. (2.19) can in fact be understood using spurions. [1] Consider the term

$$L_{\rm SOFT} = \sqrt{2}m_A\int\theta^\alpha W_\alpha^a\Phi^a\, d^2\theta \quad + \text{c.c.} \tag{2.20}$$

where $\Phi^a(\phi, \psi, F)$ is a chiral superfield in the adjoint representation and

$$W_\alpha^a = \lambda_\alpha^a - D^a\theta_\alpha + \cdots \tag{2.21}$$

[1] We are most grateful to the referee for indicating to us the following argument.

is the usual superspace gauge field strength. In the Wess-Zumino gauge this reduces to

$$L_{\mathrm{SOFT}} = m_A(\lambda^a \phi^a + \mathrm{c.c.}) - \sqrt{2} m_A D^a(\phi^a + \phi^{*a}). \tag{2.22}$$

When the auxiliary field D is eliminated this produces the following contributions to the Lagrangian:

$$L = m_A(\lambda^a \psi^a + \mathrm{c.c.}) + \tfrac{1}{2}\left[g\phi^* R^a \phi + \sqrt{2} m_A(\phi^a + \phi^{*a})\right]^2 \tag{2.23}$$

which, it is easy to see, precisely accounts for all the terms in Eq. (2.19). The fact that we were forced to place the chiral superfield in the adjoint representation to obtain an RG invariant trajectory is now simply understood in that for such a field we can obtain all our "non-holomorphic" soft breakings from a single *holomorphic* term, Eq. (2.20). Moreover, the fact that it *is* holomorphic means that we can immediately apply the non-renormalisation theorem to show that (on the trajectory)

$$\beta_{m_A} = \frac{\beta_g}{g} + \gamma. \tag{2.24}$$

It is easy to verify this result through two loops using Eqs. (A2), (2.8b).

3. The MSSM

Retaining only the third generation Yukawa couplings we have the superpotential

$$W = \lambda_t H_2 Q\bar{t} + \lambda_b H_1 Q\bar{b} + \lambda_\tau H_1 L\bar{\tau}, \tag{3.1}$$

and soft breaking terms

$$\begin{aligned} L_{\mathrm{SOFT}}^{(1)} = \sum_\phi m_\phi^2 \phi^* \phi + \left[m_3^2 H_1 H_2 + \sum_{i=1}^{3} \tfrac{1}{2} M_i \lambda_i \lambda_i + \mathrm{h.c.}\right] \\ + \left[A_t \lambda_t H_2 Q\bar{t} + A_b \lambda_b H_1 Q\bar{b} + A_\tau \lambda_\tau H_1 L\bar{\tau} + \mathrm{h.c.}\right] \end{aligned} \tag{3.2}$$

and

$$L_{\mathrm{SOFT}}^{(2)} = m_\psi \psi_{H_1}\psi_{H_2} + \overline{A}_t \lambda_t H_1^* Q\bar{t} + \overline{A}_b \lambda_b H_2^* Q\bar{b} + \overline{A}_\tau \lambda_\tau H_2^* L\bar{\tau} + \mathrm{h.c.} \tag{3.3}$$

If we set $m_\psi = \overline{A}_\tau = \overline{A}_b = \overline{A}_t = \mu$ and $m_{1,2}^2 \to m_{1,2}^2 + \mu^2$ then we recover the MSSM. (A note on notation: in our previous paper[2] we followed Inoue et al.[5], who used $m_\psi = m_4$, $\overline{A}_\tau = m_5$, $\overline{A}_b = m_7$, $\overline{A}_t = m_9$, and correspondingly $A_\tau = m_6$, $A_b = m_8$ and $A_t = m_{10}$.) As in Eq. (3.1) we assume 3rd. generation dominance here (this may not be true, of course).

In fact we neglect all mixing between the generations and all couplings associated with the first two generations throughout; for the generalisation to include these (in the absence of our non-holomorphic terms) in the quasi-fixed-point context, see Ref. [6].

The supersymmetric couplings evolve according to the well-known equations

$$\frac{d\alpha_i}{dt} = -b_i\alpha_i^2, \quad (i=1,2,3) \tag{3.4a}$$

$$\frac{dy_t}{dt} = -y_t(6y_t + y_b - \sum_i C_i^t\alpha_i), \tag{3.4b}$$

$$\frac{dy_b}{dt} = -y_b(6y_b + y_t + y_\tau - \sum_i C_i^b\alpha_i), \tag{3.4c}$$

$$\frac{dy_\tau}{dt} = -y_\tau(4y_\tau + 3y_b - \sum_i C_i^\tau\alpha_i), \tag{3.4d}$$

where $t = -\frac{1}{2\pi}\ln\mu$,

$$\alpha_i = \frac{g_i^2}{4\pi}, \qquad y_t = \frac{\lambda_t^2}{4\pi} \qquad \text{etc.} \tag{3.5}$$

and

$$\begin{aligned} b_i &= \left(\tfrac{33}{5}, 1, -3\right), \quad C_i^t = \left(\tfrac{13}{15}, 3, \tfrac{16}{3}\right), \\ C_i^b &= \left(\tfrac{7}{15}, 3, \tfrac{16}{3}\right), \quad C_i^\tau = \left(\tfrac{9}{5}, 3, 0\right), \quad i=1,2,3. \end{aligned} \tag{3.6}$$

It is straightforward to show from our results that

$$\frac{dm_\psi}{dt} = -\tfrac{1}{2}(y_\tau + 3y_b + 3y_t - 2\sum_i C_i^H\alpha_i)m_\psi, \tag{3.7a}$$

$$\frac{d\overline{A}_\tau}{dt} = -\tfrac{1}{2}(y_\tau - 3y_b + 3y_t)\overline{A}_\tau - 3y_b\overline{A}_b + (2m_\psi - \overline{A}_\tau)\sum_i C_i^H\alpha_i, \tag{3.7b}$$

$$\begin{aligned} \frac{d\overline{A}_b}{dt} &= -\tfrac{1}{2}(3y_b + 5y_t - y_\tau)\overline{A}_b - \overline{A}_\tau y_\tau + y_t(2m_\psi - \overline{A}_t) \\ &\quad + (2m_\psi - \overline{A}_b)\sum_i C_i^H\alpha_i, \end{aligned} \tag{3.7c}$$

$$\frac{d\overline{A}_t}{dt} = -\tfrac{1}{2}(y_\tau + 5y_b + 3y_t)\overline{A}_t + y_b(2m_\psi - \overline{A}_b) + (2m_\psi - \overline{A}_t)\sum_i C_i^H\alpha_i, \tag{3.7d}$$

$$\frac{dA_\tau}{dt} = -4y_\tau A_\tau - 3y_bA_b - \sum_i C_i^\tau\alpha_iM_i, \tag{3.7e}$$

$$\frac{dA_b}{dt} = -y_\tau A_\tau - 6y_bA_b - y_tA_t - \sum_i C_i^b\alpha_iM_i, \tag{3.7f}$$

$$\frac{dA_t}{dt} = -y_bA_b - 6y_tA_t - \sum_i C_i^t\alpha_iM_i, \tag{3.7g}$$

$$\frac{dm_1^2}{dt} = -y_\tau(m_1^2 + A_\tau^2 + m_L^2 + m_{\overline{\tau}}^2) - 3y_b(m_1^2 + A_b^2 + m_Q^2 + m_{\overline{b}}^2) - 3y_t\overline{A}_t^{\,2} + 2\sum_i C_i^H \alpha_i(m_\psi^2 + M_i^2), \tag{3.7h}$$

$$\frac{dm_2^2}{dt} = -3y_t(m_2^2 + A_t^2 + m_Q^2 + m_{\overline{t}}^2) - y_\tau\overline{A}_\tau^{\,2} - 3y_b\overline{A}_b^{\,2} + 2\sum_i C_i^H \alpha_i(m_\psi^2 + M_i^2), \tag{3.7i}$$

$$\frac{dm_3^2}{dt} = -\tfrac{1}{2}(y_\tau + 3y_b + 3y_t)m_3^2 - y_\tau\overline{A}_\tau A_\tau - 3y_b\overline{A}_b A_b - 3y_t\overline{A}_t A_t + \tfrac{1}{2}\sum_i C_i^H \alpha_i(m_3^2 - 2M_i m_\psi), \tag{3.7j}$$

$$\frac{dm_Q^2}{dt} = -X_b - X_t + 2\sum_i C_i^Q \alpha_i M_i^2, \tag{3.7k}$$

$$\frac{dm_{\overline{t}}^2}{dt} = -2X_t + 2\sum_i C_i^{\overline{t}} \alpha_i M_i^2, \tag{3.7l}$$

$$\frac{dm_{\overline{b}}^2}{dt} = -2X_b + 2\sum_i C_i^{\overline{b}} \alpha_i M_i^2, \tag{3.7m}$$

$$\frac{dm_L^2}{dt} = -X_\tau + 2\sum_i C_i^H \alpha_i M_i^2, \tag{3.7n}$$

$$\frac{dm_{\overline{\tau}}^2}{dt} = -2X_\tau + 2\sum_i C_i^{\overline{\tau}} \alpha_i M_i^2, \tag{3.7o}$$

$$\frac{dM_i}{dt} = -b_i M_i \alpha_i, \tag{3.7p}$$

where

$$C^Q = \left(\tfrac{1}{30}, \tfrac{3}{2}, \tfrac{8}{3}\right), \quad C^{\overline{t}} = \left(\tfrac{8}{15}, 0, \tfrac{8}{3}\right), \quad C^{\overline{b}} = \left(\tfrac{2}{15}, 0, \tfrac{8}{3}\right), \quad C^{\overline{\tau}} = \left(\tfrac{6}{5}, 0, 0\right) \quad C_i^H = \left(\tfrac{3}{10}, \tfrac{3}{2}, 0\right), \quad i = 1, 2, 3 \tag{3.8}$$

and where

$$X_t = y_t(m_Q^2 + m_{\overline{t}}^2 + m_2^2 + \overline{A}_t^{\,2} + A_t^2 - 2m_\psi^2), \quad X_b = y_b(m_Q^2 + m_{\overline{b}}^2 + m_1^2 + \overline{A}_b^{\,2} + A_b^2 - 2m_\psi^2), \quad X_\tau = y_\tau(m_L^2 + m_{\overline{\tau}}^2 + m_1^2 + \overline{A}_\tau^{\,2} + A_\tau^2 - 2m_\psi^2). \tag{3.9}$$

3.1. The small $\tan\beta$ *regime*

In the small $\tan\beta$ regime where we take $y_b = y_\tau = 0$, Eqs. $(3.4a, b)$ are easily solved to give

$$\alpha_i(t) = \frac{\alpha_0}{1 + b_i\alpha_0 t}, \tag{3.10a}$$

$$y_t(t) = y_0 f(t) H_6(t, y_0) \tag{3.10b}$$

where

$$f(t) = \prod_i [1 + b_i \alpha_0 t]^{\frac{C_i^t}{b_i}}, \tag{3.11}$$

and

$$H_6(t, y_0) = \frac{1}{1 + 6 y_0 F(t)}, \quad F(t) = \int_0^t f(\tau)\, d\tau \tag{3.12}$$

and where $y_0 = y_t(0)$ and we assume a common initial gauge coupling $\alpha_i(0) = \alpha_0$ at a unification scale M_U. We then easily solve Eqs. $(3.7a - d)$ to give

$$m_\psi(t) = H_6(t, y_0)^{\frac{1}{4}} \tilde{f}(t) m_\psi(0), \tag{3.13a}$$

$$\mathcal{A}_t(t) = 1 + \tilde{f}(t)^{-2} \left[\mathcal{A}_t(0) - 1\right], \tag{3.13b}$$

$$\mathcal{A}_b(t) = 1 + H_6(t, y_0)^{\frac{1}{6}} \tilde{f}(t)^{-2} \left[\mathcal{A}_t(0) + \mathcal{A}_b(0) - 2\right] + \tilde{f}(t)^{-2} \left[1 - \mathcal{A}_t(0)\right], \tag{3.13c}$$

$$\mathcal{A}_\tau(t) = 1 + \tilde{f}(t)^{-2} \left[\mathcal{A}_\tau(0) - 1\right], \tag{3.13d}$$

where

$$\tilde{f}(t) = \prod_i [1 + b_i \alpha_0 t]^{\frac{C_i^H}{b_i}}, \tag{3.14}$$

and

$$\mathcal{A}_t = \frac{\overline{A}_t(t)}{m_\psi(t)}, \quad \mathcal{A}_b = \frac{\overline{A}_b(t)}{m_\psi(t)}, \quad \mathcal{A}_\tau = \frac{\overline{A}_\tau(t)}{m_\psi(t)}. \tag{3.15}$$

Using the elementary solution of Eq. $(3.7p)$,

$$M_i = \frac{M_0}{1 + b_i \alpha_0 t}, \tag{3.16}$$

where we assume a common initial gaugino mass $M_i(0) = M_0$, we can also solve Eq. $(3.7g)$, giving

$$A_t(t) = \{A_t(0) + 6 y_0 M_0 [t f(t) - F(t)]\} H_6(t, y_0) - M_0 t \frac{1}{f(t)} \frac{df}{dt}. \tag{3.17}$$

It is instructive to note that the boundary condition on the gaugino masses plays a crucial rôle in determining the form of the solution. Thus if we take instead

$$M_i(0) = m_{\frac{3}{2}} b_i \alpha_0, \tag{3.18}$$

then we obtain

$$A_t = H_6\left[A_t(0) + m_{\frac{3}{2}}(6y_0 - \sum_i C_i^t\alpha_0)\right] + m_{\frac{3}{2}}\left[\sum_i C_i^t\alpha_i(t) - 6y(t)\right], \tag{3.19}$$

which, if we impose the initial condition $A_t(0) + m_{\frac{3}{2}}(6y_0 - \sum_i C_i^t\alpha_0) = 0$, is the one-loop form of the conformal anomaly solution[3][4] for A_t.

Proceeding with Eq. (3.17), we can (with more labour) solve Eqs $(3.7h, i, k-o)$, giving

$$\begin{aligned}
m_Q^2(t) &= m_Q^2(0) + M_0^2(\tfrac{8}{3}f_3(t) + \tfrac{3}{2}f_2(t) + \tfrac{1}{30}f_1(t)) + \tfrac{1}{6}\Delta(t) + Y(t),\\
m_{\overline{t}}^2(t) &= m_{\overline{t}}^2(0) + M_0^2(\tfrac{8}{3}f_3(t) + \tfrac{8}{15}f_1(t)) + \tfrac{1}{3}\Delta(t) + 2Y(t),\\
m_{\overline{b}}^2(t) &= m_{\overline{b}}^2(0) + M_0^2(\tfrac{8}{3}f_3(t) + \tfrac{2}{15}f_1(t)),\\
m_2^2(t) &= m_2^2(0) + M_0^2(\tfrac{3}{2}f_2(t) + \tfrac{3}{10}f_1(t)) + \tfrac{1}{2}\Delta(t) - 3Y(t),\\
m_1^2(t) &= m_1^2(0) + M_0^2(\tfrac{3}{2}f_2(t) + \tfrac{3}{10}f_1(t))\\
&\qquad + \{\tilde{f}(t)^2 m_\psi(0)^2 + 2m_\psi(0)[\overline{A}_t(0) - m_\psi(0)]\}H_6(t,y_0)^{\frac{1}{2}}\\
&\qquad + m_\psi(0)[m_\psi(0) - 2\overline{A}_t(0)] - 3y_0[\overline{A}_t(0) - m_\psi(0)]^2\Omega_{(6,\frac{3}{2})}(t),\\
m_L^2(t) &= m_L^2(0) + M_0^2\left(\tfrac{3}{2}f_2(t) + \tfrac{3}{10}f_1(t)\right),\\
m_{\overline{\tau}}^2(t) &= m_{\overline{\tau}}^2(0) + \tfrac{6}{5}M_0^2 f_1(t)
\end{aligned} \tag{3.20}$$

where

$$\begin{aligned}
f_i(t) &= \frac{1}{b_i}\left(1 - \frac{1}{(1+b_i\alpha_0 t)^2}\right),\\
\Delta(t) &= [\Sigma(0) - A_t(0)^2]H_6(t,y_0) + \{A_t(0) + 6M_0y_0[tf(t) - F(t)]\}^2H_6(t,y_0)^2\\
&\quad - 6y_0M_0^2H_6(t,y_0)t^2\frac{df}{dt} - \Sigma(0)\\
&\quad + \{m_\psi(0)^2\tilde{f}(t)^2 - m_\psi(0)[m_\psi(0) - \overline{A}_t(0)]\}H_6(t,y_0)^{\frac{1}{2}}\\
&\quad + \{m_\psi(0)[2\overline{A}_t(0) - 3m_\psi(0)] - 3[\overline{A}_t(0) - m_\psi(0)]^2\Omega_{(6,\frac{1}{2})}(t)\}H_6(t,y_0),
\end{aligned} \tag{3.21}$$

with $\Sigma = m_Q^2 + m_{\overline{t}}^2 + m_2^2$, and where

$$\Omega_{(a,n)}(t) = \int_0^t f(\tau)\tilde{f}(\tau)^{-2}H_a(\tau,y_0)^n d\tau \tag{3.22}$$

and

$$\begin{aligned}
Y(t) &= -\tfrac{1}{6}\{m_\psi(0)^2\tilde{f}^2 + 2m_\psi(0)[m_\psi(0) - \overline{A}_t(0)]\}H_6(t,y_0)^{\frac{1}{2}}\\
&\quad + \tfrac{1}{6}m_\psi(0)[3m_\psi(0) - 2\overline{A}_t(0)] - \tfrac{1}{2}y_0[\overline{A}_t(0) - m_\psi(0)]^2\Omega_{(6,\frac{3}{2})}(t),
\end{aligned} \tag{3.23}$$

Once again, use of the alternative boundary condition Eq. (3.18) and the corresponding solution for $A_t(t)$ leads instead (with appropriate initial conditions for the masses) to the conformal anomaly form for the m^2 terms. This we leave as an exercise for the reader.

In the special case of the MSSM, explicit solutions for the soft parameters were written down in Refs. [7]. Recently Codoban and Kazakov [8] have given an elegant derivation using the spurion formalism; their results may be obtained by setting $m_\psi = \overline{A}_\tau = \overline{A}_b = \overline{A}_t = 0$. We note that in the more general case considered here it is not possible to obtain a simple closed form for $m_3^2(t)$. However, this is not a major drawback since in typical running analyses, $m_3^2(M_Z)$ is in any case derived by minimising the effective potential.

3.2. The large $\tan\beta$ *region*

In the large $\tan\beta$ region, if we make the approximation[9] $y_b \approx y_t = y$, $y_\tau \approx 0$, the Yukawa coupling is given to a good approximation by

$$y(t) = y_0 \hat{f}(t) H_7(t, y_0), \tag{3.24}$$

where

$$\hat{f}(t) = \prod_i [1 + b_i \alpha_0 t]^{\frac{C_i^{tb}}{b_i}}, \tag{3.25}$$

with $C^{tb} = \left(\frac{2}{3}, 3, \frac{16}{3}\right)$, and

$$H_7(t, y_0) = \frac{1}{1 + 7y_0 \hat{F}(t)}, \quad \hat{F}(t) = \int_0^t \hat{f}(\tau)\, d\tau. \tag{3.26}$$

Note that $C_{2,3}^{tb} = C_{2,3}^t = C_{2,3}^b$ while we have chosen to set $C_1^{tb} = \frac{1}{2}(C_1^t + C_1^b)$. (In fact, it makes very little difference if we instead use $C^{tb} = C^t$, in which case $f = \hat{f}$ and $F = \hat{F}$.) We can then solve Eqs. $(3.7a - d)$ to obtain

$$m_\psi(t) = H_7(t, y_0)^{\frac{3}{7}} \tilde{f}(t) m_\psi(0), \tag{3.27a}$$

$$\begin{aligned} \mathcal{A}_t(t) &= 1 + \tfrac{1}{2}\tilde{f}(t)^{-2} H_7(t, y_0)^{\frac{2}{7}} \left[\mathcal{A}_t(0) + \mathcal{A}_b(0) - 2\right] \\ &\quad + \tfrac{1}{2}\tilde{f}(t)^{-2} \left[\mathcal{A}_t(0) - \mathcal{A}_b(0)\right], \end{aligned} \tag{3.27b}$$

$$\begin{aligned} \mathcal{A}_b(t) &= 1 + \tfrac{1}{2}\tilde{f}(t)^{-2} H_7(t, y_0)^{\frac{2}{7}} \left[\mathcal{A}_t(0) + \mathcal{A}_b(0) - 2\right] \\ &\quad - \tfrac{1}{2}\tilde{f}(t)^{-2} \left[\mathcal{A}_t(0) - \mathcal{A}_b(0)\right], \end{aligned} \tag{3.27c}$$

$$\begin{aligned} \mathcal{A}_\tau(t) &= 1 + H_7(t, y_0)^{-\frac{3}{7}} \tilde{f}(t)^{-2} \left[\mathcal{A}_\tau(0) + \tfrac{1}{5}\mathcal{A}_t(0) - \tfrac{4}{5}\mathcal{A}_b(0) - \tfrac{2}{5}\right] \\ &\quad + \tfrac{3}{10} H_7(t, y_0)^{\frac{2}{7}} \tilde{f}(t)^{-2} \left[\mathcal{A}_t(0) + \mathcal{A}_b(0) - 2\right] \\ &\quad - \tfrac{1}{2}\tilde{f}(t)^{-2} \left[\mathcal{A}_t(0) - \mathcal{A}_b(0)\right]. \end{aligned} \tag{3.27d}$$

We also find from Eqs. (3.7f, g) that

$$
\begin{aligned}
A_t(t) &= \{\tfrac{1}{2}(A_t(0) + A_b(0)) + 7y_0 M_0[t\hat{f}(t) - \hat{F}(t)]\}H_7(t, y_0) - M_0 t\frac{1}{\hat{f}(t)}\frac{d\hat{f}}{dt} \\
&\quad + \{\tfrac{1}{2}(A_t(0) - A_b(0)) + 5y_0 M_0[tg(t) - G(t)]\}H_5(t, y_0) - M_0 t\frac{1}{g(t)}\frac{dg}{dt}, \\
A_b(t) &= \{\tfrac{1}{2}(A_t(0) + A_b(0)) + 7y_0 M_0[t\hat{f}(t) - \hat{F}(t)]\}H_7(t, y_0) - M_0 t\frac{1}{\hat{f}(t)}\frac{d\hat{f}}{dt} \\
&\quad - \{\tfrac{1}{2}(A_t(0) - A_b(0)) + 5y_0 M_0[tg(t) - G(t)]\}H_5(t, y_0) + M_0 t\frac{1}{g(t)}\frac{dg}{dt},
\end{aligned}
\tag{3.28}
$$

where

$$
g = [1 + b_1\alpha_0 t]^{\frac{c_1}{b_1}}, \tag{3.29}
$$

with $2c_1 = C_1^t - C_1^b = \frac{1}{5}$, and

$$
\begin{aligned}
G &= \frac{1}{(c_1 + b_1)\alpha_0}\left\{[1 + b_1\alpha_0 t]^{\frac{c_1}{b_1}+1} - 1\right\}, \\
H_5(t, y_0) &= \frac{1}{1 + 5y_0\hat{F}(t)}.
\end{aligned}
\tag{3.30}
$$

With the further assumptions $\overline{A}_b(0) \approx \overline{A}_t(0)$, $A_b(0) \approx A_t(0)$, $m_1^2 \approx m_2^2$, $m_{\tilde{b}}^2 \approx m_{\tilde{t}}^2$, and using $g(t) \approx 1$ and $G(t) \approx t$, Eqs. (3.27), (3.28) simplify to

$$
m_\psi(t) = H_7(t, y_0)^{\frac{3}{7}}\tilde{f}(t)m_\psi(0), \tag{3.31a}
$$

$$
\mathcal{A}_t = \mathcal{A}_b = 1 + \tilde{f}(t)^{-2}H_7(t, y_0)^{\frac{2}{7}}\left[\mathcal{A}_t(0) - 1\right], \tag{3.31b}
$$

$$
\begin{aligned}
\mathcal{A}_\tau &= 1 + H_7(t, y_0)^{-\frac{3}{7}}\tilde{f}(t)^{-2}\left[\mathcal{A}_\tau(0) - \tfrac{3}{5}\mathcal{A}_t(0) - \tfrac{2}{5}\right] \\
&\quad + \tfrac{3}{5}H_7(t, y_0)^{\frac{2}{7}}\tilde{f}(t)^{-2}\left[\mathcal{A}_t(0) - 1\right],
\end{aligned}
\tag{3.31c}
$$

$$
A_t(t) = A_b(t) = \{A_t(0) + 7y_0 M_0[t\hat{f}(t) - \hat{F}(t)]\}H_7(t, y_0) - M_0 t\frac{1}{\hat{f}(t)}\frac{d\hat{f}}{dt}, \tag{3.31d}
$$

and we find that with these assumptions we can obtain the following explicit solutions for the soft masses:

$$
\begin{aligned}
m_Q^2(t) &= m_Q^2(0) + M_0^2(\tfrac{8}{3}f_3(t) + \tfrac{3}{2}f_2(t) + \tfrac{1}{30}f_1(t)) + \tfrac{2}{7}\tilde{\Delta}(t) + \tilde{Y}(t), \\
m_{\tilde{t}}^2(t) &= m_{\tilde{t}}^2(0) + M_0^2(\tfrac{8}{3}f_3(t) + \tfrac{8}{15}f_1(t)) + \tfrac{2}{7}\tilde{\Delta}(t) + \tilde{Y}(t), \\
m_2^2(t) &= m_2^2(0) + M_0^2(\tfrac{3}{2}f_2(t) + \tfrac{3}{10}f_1(t)) + \tfrac{3}{7}\tilde{\Delta}(t) - 2\tilde{Y}(t),
\end{aligned}
\tag{3.32}
$$

where

$$\begin{aligned}\tilde{\Delta} &= [\Sigma(0) - A_t(0)^2]H_7(t,y_0) + [A_t(0) + 7M_0y_0(t\hat{f}(t) - \hat{F}(t))]^2 H_7(t,y_0)^2 \\ &- 7y_0M_0^2H_7(t,y_0)t^2\frac{d\hat{f}}{dt} - \Sigma(0) \\ &+ \Big[m_\psi(0)^2(\tilde{f}(t)^2H_7(t,y_0)^{-\frac{1}{7}} - 1) - 7y_0[\overline{A}_t(0) - m_\psi(0)]^2\hat{\Omega}_{(7,\frac{6}{7})}(t) \\ &+ 14m_\psi(0)(\overline{A}_t(0) - m_\psi(0))(H_7(t,y_0)^{\frac{1}{7}} - 1)\Big]H_7(t,y_0),\end{aligned} \tag{3.33}$$

with

$$\tilde{Y}(t) = \tfrac{2}{7}m_\psi(0)^2\{1 - H_7(t,y_0)^{\frac{6}{7}}\tilde{f}(t)^2\}. \tag{3.34}$$

$\hat{\Omega}$ is defined like Ω in Eq. (3.22), except that $f \to \hat{f}$.

3.3. Quasi-infra red fixed points and sum rules

The possibility that the weak-scale values of various parameters in the MSSM are governed by quasi-infra-red fixed-point (QIRFP) behaviour [10] has received a good deal of attention; see for example [6][8][9][11]–[14]. In this scenario, the value of the Yukawa coupling at the weak scale is close to the value corresponding to having a Landau pole at the unification scale. It follows that this value will be obtained for a wide range of input Yukawa couplings at M_U. In the small $\tan\beta$ case, for example, we have from Eq. (3.10*b*) that when $6y_0F(t) >> 1$ then $y_t \approx f(t)/6F(t)$, independent of y_0. Moreover, since $F(M_Z) \approx 18$ it follows that there is a range of perturbatively believable values of y_0 such that the QIRFP is approached at M_Z. (For a discussion of the extent to which this scenario is preserved at higher orders, see Ref. [12].) In what follows we will investigate whether this behaviour of the Yukawa coupling causes QIRFP behaviour for the soft parameters, simply by taking the limit of large y_0, and examining whether the results are independent of the initial conditions at M_Z. Of course whether the range of y_0 corresponding to close approach to any resulting QIRFP includes perturbatively believable values will depend on the details of the solution.

Thus from Eq. (3.17) we see that for small $\tan\beta$ and large y_0,

$$A_t(t) \approx M_0\left(\frac{tf(t)}{F(t)} - 1 - \frac{t}{f(t)}\frac{df}{dt}\right). \tag{3.35}$$

In the large $\tan\beta$ case, we have from Eq. (3.24) that for large $\tan\beta$, $y \approx \hat{f}(t)/7\hat{F}(t)$, and from Eq. (3.28)

$$\begin{aligned} A_t \approx & M_0 \left(\frac{t\hat{f}(t)}{\hat{F}(t)} - \frac{t}{\hat{f}(t)} \frac{d\hat{f}}{dt} + \frac{tg(t)}{G(t)} - \frac{t}{g(t)} \frac{dg}{dt} - 2 \right) \\ A_b \approx & M_0 \left(\frac{t\hat{f}(t)}{\hat{F}(t)} - \frac{t}{\hat{f}(t)} \frac{d\hat{f}}{dt} - \frac{tg(t)}{G(t)} + \frac{t}{g(t)} \frac{dg}{dt} \right) \end{aligned} \tag{3.36}$$

Since the only difference between f and $\hat{f}$, and correspondingly F and $\hat{F}$, is the replacement of C_1^t by C_1^{tb}, and since we have $g(t) \approx 1$ and $G(t) \approx t$, we see that the QIRFP predictions for A_t and A_b for large $\tan\beta$ are in fact close to the small $\tan\beta$ prediction for A_t. To be more explicit, for small $\tan\beta$ we find

$$\frac{A_t(M_Z)}{M_3(M_Z)} \approx -0.6, \tag{3.37}$$

with less than a 1% difference in the large $\tan\beta$ case for $\frac{A_t(M_Z)}{M_3(M_Z)}$ or $\frac{A_b(M_Z)}{M_3(M_Z)}$, in agreement with Refs. [6][9][13].

Turning to the soft masses, we find that for small $\tan\beta$ and large y_0

$$\Delta \approx M_0^2 \left(\frac{(tf-F)^2}{F^2} - \frac{t^2}{F} \frac{df}{dt} \right) - \Sigma(0) \tag{3.38}$$

with a similar equation for $\tilde{\Delta}$ in the large $\tan\beta$ case, but with $f \to \hat{f}$, $F \to \hat{F}$ (after setting $A_b \approx A_t$, $\overline{A}_b \approx \overline{A}_t$), so that Δ depends on the initial values of the soft masses through $\Sigma(0)$.

In the standard case where the superpotential Eq. (3.1) contains also a μ term, but the soft terms are given only by Eq. (3.2), the resulting QIRFP pattern has been discussed by previous authors. As mentioned earlier, we can reproduce this case by setting $m_\psi = \overline{A}_\tau = \overline{A}_b = \overline{A}_t = \mu$ and $m_{1,2}^2 \to m_{1,2}^2 + \mu^2$. However, for ease of presentation we start by analysing the case $m_\psi = \overline{A}_\tau = \overline{A}_b = \overline{A}_t = 0$; but it is straightforward to check that our results are still valid when we include the supersymmetric μ term as above. The most robust prediction is easily seen to be that (at small $\tan\beta$)

$$\frac{\Sigma(M_3^2)}{M_3(M_Z)^2} \approx \delta \left[\frac{(tf-F)^2}{F^2} + \frac{d}{dt}\left(\frac{t^2}{f}\frac{df}{dt} \right) - \frac{t^2}{F}\frac{df}{dt} \right] \Bigg|_{M_Z}, \tag{3.39}$$

where

$$\delta = \left(\frac{\alpha_0}{\alpha_3(M_Z)} \right)^2 \tag{3.40}$$

and we have used $\sum_i C_i^t f_i = \frac{d}{dt}\left(\frac{t^2}{f}\frac{df}{dt}\right)$. There is an analogous expression for large $\tan\beta$. So we see that for large y_0, Σ is independent of the initial values of the soft masses. The result

$$\frac{\Sigma(M_Z)}{M_3(M_Z)^2} \approx \begin{cases} 0.75 & \text{small } \tan\beta \\ 0.76 & \text{large } \tan\beta \end{cases} \tag{3.41}$$

(note the negligible difference between the large and small $\tan\beta$ cases) is in agreement with Refs. [6][13].

If we assume a universal scalar (mass)^2, m_0^2, at M_U then it is easy to see that there are similar fixed points for the following quantities:

at small $\tan\beta$:

$$\frac{m_Q^2 + m_2^2}{M_3(M_Z)^2} \approx 0.28, \tag{3.42a}$$

$$\frac{m_1^2 + 2m_2^2}{M_3(M_Z)^2} \approx -0.75, \tag{3.42b}$$

$$\frac{m_{\tilde{t}}^2}{M_3(M_Z)^2} \approx 0.47, \tag{3.42c}$$

in broad agreement with Refs. [6],[13];

while at large $\tan\beta$:

$$\frac{m_Q^2 - m_{\tilde{t}}^2}{M_3(M_Z)^2} \approx 0.05, \tag{3.43a}$$

$$\frac{2m_Q^2 + m_2^2}{M_3(M_Z)^2} \approx 0.81, \tag{3.43b}$$

which do not seem to appear explicitly in the literature, although it is easy to see that, for example, they are implied by Eqs. (20)-(23) of Ref. [9]. Note also that, writing

$$\begin{aligned} \frac{\Delta(M_Z)}{M_3^2(M_Z)} &\approx \delta\left[\frac{(tf-F)^2}{F^2} - \frac{t^2}{F}\frac{df}{dt}\right]\Bigg|_{M_Z} - \delta\frac{\Sigma(0)}{M_0^2} \\ &\approx -0.94 - 0.12\frac{\Sigma(0)}{M_0^2} \end{aligned} \tag{3.44}$$

then as long as $\frac{\Sigma(0)}{M_0^2} < 7$ then the dependence on $\Sigma(0)$ of this ratio is suppressed. The result is further QIRFP behaviour, for a *limited* range of boundary conditions at M_U for the soft masses[8]; we will not discuss this possibility further, however.

As we pointed out before, the above predictions remain valid when the non-holomorphic terms simply reproduce the supersymmetric μ-term. Let us turn now to examine the extent to which they survive the introduction of completely general non-holomorphic terms; firstly in the small $\tan\beta$ case. We see that $Y(t)$ in Eq. (3.23) still depends on $m_\psi(0)$ and $\overline{A}_t(0)$ as $y_0 \to \infty$, and this dependence in fact grows with y_0, since the integrand of $\Omega_{(6,\frac{3}{2})}$ develops a pole at $\tau = 0$ as $y_0 \to \infty$; similarly for m_1^2. Clearly, however, since Σ is independent of Y, the results Eqs. (3.39) and (3.41) survive in the general case, but not Eq. (3.42).

For large $\tan\beta$, we find that for $\tilde{Y}$ in Eq. (3.34) we have $\tilde{Y} \approx \frac{2}{7} m_\psi(0)^2$ as $y_0 \to \infty$. Σ, $m_Q^2 - m_{\tilde{t}}^2$ and $2m_Q^2 + m_2^2$ are, however, independent of $\tilde{Y}$ so we obtain

$$\frac{\Sigma(M_Z)}{M_3(M_Z)^2} \approx .76 \tag{3.45}$$

for arbitrary initial scalar masses, and

$$\begin{aligned} \frac{m_Q^2 - m_{\tilde{t}}^2}{M_3(M_Z)^2} &\approx 0.05, \\ \frac{2m_Q^2 + m_2^2}{M_3(M_Z)^2} &\approx 0.81 \end{aligned} \tag{3.46}$$

for a universal scalar (mass)^2. The fact that the latter QIRFPs are valid even for non-supersymmetric m_ψ, $\overline{A}_\tau$, $\overline{A}_b$ and $\overline{A}_t$ is rather remarkable. It is clear from Eq. (3.27) that this happens because in the limit $y_0 \to \infty$, $m_\psi, \overline{A}_{t,b,\tau}$ all approach zero. In Ref. [2] we argued that in the presence of the non-holomorphic soft terms it might be that there was no explicit supersymmetric μ-term, and we explicitly demonstrated that there were regions of parameter space corresponding to an acceptable electroweak vacuum. Unfortunately this scenario cannot be implemented here, since using $m_1^2 \approx m_2^2$ we obtain at once (using the tree minimisation conditions in the absence of a μ-term) that $m_1^2 \approx m_2^2 \approx -\frac{1}{2}M_Z^2$ which violates the well known requirement that $m_1^2 + m_2^2 > |m_3^2|$.

The new parameters themselves do exhibit QIRFP behaviour if we consider ratios of $\overline{A}_{t,b,\tau}$ to m_ψ. Starting with the small $\tan\beta$ case, we see that while $\mathcal{A}_t$, $\mathcal{A}_b$ and $\mathcal{A}_\tau$ have no individual QIRFP, we have (as $y_0 \to \infty$)

$$\mathcal{A}_t + \mathcal{A}_b \approx 2. \tag{3.47}$$

As pointed out in Ref. [2] and clearly manifested in Eqs. (3.13), the ratios of $\overline{A}_{t,b,\tau}$ to m_ψ have true infra-red fixed points (i.e. as $t \to \infty$) of 1, corresponding to the supersymmetric limit, and so $\mathcal{A}_t + \mathcal{A}_b$ has an infra-red fixed point of 2. The point is that the QIRFP behaviour occurs for finite t rather than for $t \to \infty$. In Fig. 1 we show the approach to the QIRFP for $\mathcal{A}_t + \mathcal{A}_b$ for $\tan\beta$ close to the QIRFP value. There is clear convergence towards the QIRFP, although this convergence is somewhat slowed by the power $\frac{1}{6}$ of $H_7(t, y_0)$ in Eq. (3.13c). This means that to see significant convergence we need to be at or beyond the limit of perturbative believability for y_0 (though in Ref.[12] we argued using Padé-Borel summation techniques that the domain of attraction of the QIRFP could be extended beyond the naïve perturbative region).

In Fig. 2 we show the contrasting behaviour of the individual ratio $\mathcal{A}_b$ which clearly has no QIRFP; the approach to the fixed point value $\mathcal{A}_b = 1$ is much slower than the approach to the QIRFP in Fig. 1.

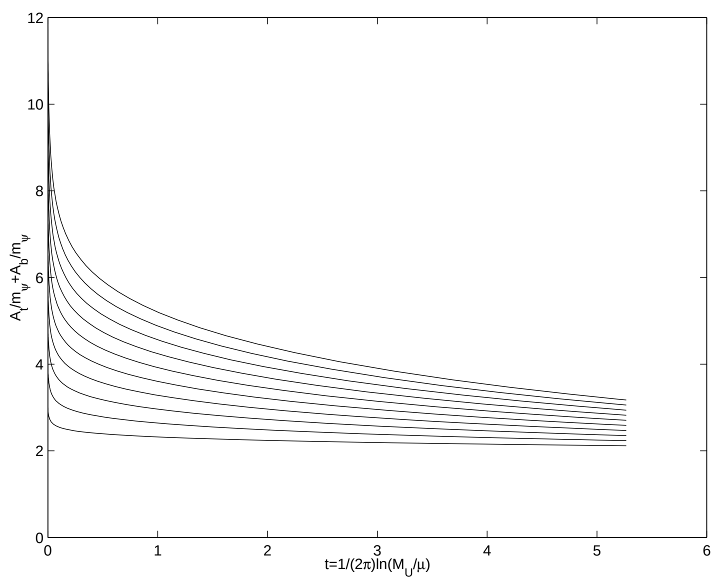

Fig. 1: A plot of $\mathcal{A}_t + \mathcal{A}_b$ against $t = \frac{1}{2\pi}\ln\left(\frac{M_U}{\mu}\right)$ for $\tan\beta \approx 1.7$, with $\mathcal{A}_b(M_U) = \mathcal{A}_\tau(M_U) = 1$, and with $2 \leq \mathcal{A}_t(M_U) \leq 11$.

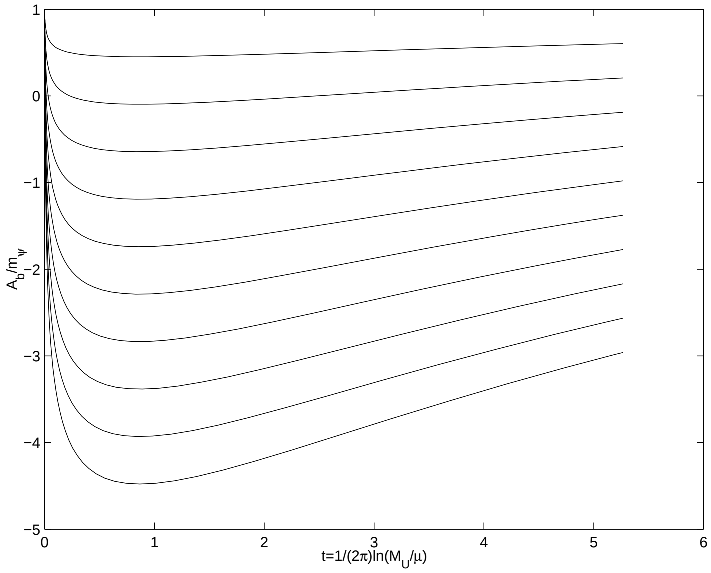

Fig. 2: A plot of $\mathcal{A}_b$ against $t = \frac{1}{2\pi}\ln\left(\frac{M_U}{\mu}\right)$ for $\tan\beta \approx 1.7$, with $\mathcal{A}_b(M_U) = \mathcal{A}_\tau(M_U) = 1$, and with $2 \leq \mathcal{A}_t(M_U) \leq 11$.

Of course for the prediction Eq. (3.47) to have experimental relevance we would need m_ψ to be non-negligible at M_Z: otherwise, the associated contributions to the squark mass matrices would be small. Since as we already remarked, in fact $m_\psi(t) \to 0$ as $y_0 \to \infty$, it follows that we would need m_ψ to be large at M_U. Therefore we cannot simultaneously have good fixed point convergence for the m^2/M_3^2 fixed points and the $\mathcal{A}$ fixed point, Eq. (3.47), *and* have the latter have experimental consequences. An exception is Eq. (3.43a), since *both* $\tilde{\Delta}$ and $\tilde{Y}$ cancel in this combination, as is easily seen from Eq. (3.32).

In the large $\tan\beta$ case, we see that if we have $\mathcal{A}_t(0) = \mathcal{A}_b(0)$ then there is a QIRFP $\mathcal{A}_t = \mathcal{A}_b = 1$, while $\mathcal{A}_\tau$ actually grows for large y_0, unless

$$5\mathcal{A}_\tau(0) + \mathcal{A}_t(0) - 4\mathcal{A}_b(0) = 2. \tag{3.48}$$

This behaviour reflects the fact that the stability matrix for the evolution of $\mathcal{A}_t$, $\mathcal{A}_b$ and $\mathcal{A}_\tau$, given in Ref. [2], has at least one negative eigenvalue in this case.

4. Summary

In this paper we have continued the study of the RG evolution of "non-holomorphic" soft terms that we began in Ref. [2]. In a special class of theories, we have shown the existence of a relation between the r-term and m^{ia} term that is RG-invariant, at least through two loops.

We have also explored the infra-red behaviour of these soft terms in the MSSM. Of course, in general we simply have a much enlarged parameter space, so we have restricted our attention to the two cases when either the top-quark Yukawa is close to its quasi-infra-red fixed point (corresponding to small $\tan\beta$) or when the top and bottom Yukawas are equal and close to a quasi-infra-red fixed point (corresponding to large $\tan\beta$)

We have shown that (for small $\tan\beta$) we obtain the predictions at M_Z (independent of the boundary conditions at M_U)

$$m_Q^2 + m_{\bar{t}}^2 + m_2^2 \approx .75M_3^2 \qquad (4.1a)$$

$$A_t \approx -0.6M_3 \qquad (4.1b)$$

$$\overline{A}_t + \overline{A}_b \approx 2m_\psi \qquad (4.1c),$$

where Eq. (4.1c) certainly holds but for Eqs.(4.1a, b) to hold it would have to be that $m_\psi << M_3$.

For large $\tan\beta$ Eq. (4.1) again holds (with the same qualification), but in addition we also have (if there is a universal m_0^2 at M_U)

$$\begin{aligned} m_Q^2 - m_{\bar{t}}^2 &\approx 0.05M_3(M_Z)^2 \\ 2m_Q^2 + m_2^2 &\approx 0.81M_3(M_Z)^2. \end{aligned} \qquad (4.2)$$

Finally we note that recently an interesting phenomenon termed "focussing" has been noticed[15]; this also confers a substantial measure of predictivity on the values of certain soft masses. In focussing, the value of some soft mass at a particular scale is independent of the soft mass scale at unification. For a certain class of boundary conditions at unification, which includes the usual "universal" case, this focus point of the RG trajectories occurs for m_2^2 and at a value close to the weak scale (for a range of moderate values of $\tan\beta$). We note that, in contrast to the QIRFP case, focussing is not driven by the behaviour of the Yukawa couplings at unification.

Acknowledgements

This work was supported in part by a Research Fellowship from the Leverhulme Trust; and part of it was done at the UK Theory Institute at Swansea. One of us (TJ) thanks Marcela Carena for a conversation, and in particular for drawing to our attention the treatment of the large $\tan\beta$ case from Ref. [9].

Appendix A.

In this appendix we list the results for the two-loop β-functions for r_i^{jk}, m_A^{ia}, b^{ij} and $(m^2)^i{}_j$, with m_F set to zero. (The two-loop β-functions for M and h^{ijk} may be found in Refs. [16][17].) We find

$$
\begin{aligned}
(16\pi^2)^2(\beta_r^{(2)})_i^{jk} = &-2g^2Y^{jpl}Y_{ipm}C(R)^m{}_n\tilde{r}_l^{kn} - 2g^2Y^{jpl}Y_{ipm}\tilde{r}_n^{km}C(R)^n{}_l \\
&- 2Y^{jlm}Y_{iln}Y_{mpq}Y^{npr}\tilde{r}_r^{kq} - 2g^4\tilde{r}_m^{jk}[C(R)^2]^m{}_i \\
&- 2g^4[C(R)^2]^j{}_l\tilde{r}_i^{kl} - 4g^4\tilde{r}_m^{jl}C(R)^k{}_lC(R)^m{}_i - 2g^2\tilde{r}_m^{jl}Y_{iln}Y^{mnq}C(R)^k{}_q \\
&- 2g^2\tilde{r}_m^{jl}Y_{lnp}Y^{kmn}C(R)^p{}_i - Y^{jkl}Y_{lmn}Y^{nqr}Y_{ipr}\tilde{r}_q^{mp} \\
&- Y^{jlm}Y_{iln}Y^{knp}Y_{mqr}\tilde{r}_p^{qr} - 2Y^{jpq}Y_{mnp}Y^{klm}Y_{ilr}\tilde{r}_q^{nr} \\
&- 3g^2Y^{jkn}Y_{lmn}\tilde{r}_p^{lm}C(R)^p{}_i - 2g^2Y^{jlm}Y_{iln}\tilde{r}_m^{np}C(R)^k{}_p \\
&+ 4g^2Y^{jlm}(R_a)^n{}_lY_{inp}\tilde{r}_m^{pq}(R_a)^k{}_q + 2g^2\tilde{r}_n^{lm}C(R)^j{}_mY^{knq}Y_{ilq} \\
&- 2g^2Y^{jkq}Y_{npq}\tilde{r}_l^{mp}(R_a)^n{}_i(R_a)^l{}_m + 4g^4\tilde{r}_i^{jm}[C(R)^2]^k{}_m \\
&+ 8g^4\tilde{r}_m^{jl}(R_aR_b)^m{}_l(R_aR_b)^k{}_i - 4g^4C(G)\tilde{r}_m^{jl}(R_a)^m{}_l(R_a)^k{}_i \\
&- g^2\left[\tilde{r}_n^{lm}Y_{lmp}Y^{jpq} + 2\tilde{r}_m^{lq}Y_{lnp}Y^{jpm} - 2g^2C(G)\tilde{r}_n^{jq}\right](R_a)^n{}_q(R_a)^k{}_i \\
&- 4g^2(R_a)^l{}_m\tilde{r}_n^{jm}P^n{}_l(R_a)^k{}_i - \tilde{r}_i^{jl}Y_{lmn}Y^{kmp}P^n{}_p \\
&- Y_{lmn}Y^{jkn}\tilde{r}_i^{mp}P^l{}_p - 2g^2\tilde{r}_i^{jl}[C(R)P]^k{}_l - g^2\tilde{r}_l^{jk}[C(R)P]^l{}_i \\
&- 2Y^{jlm}Y_{inp}P^n{}_l\tilde{r}_m^{kp} - 2Y^{jlm}Y_{iln}\tilde{r}_p^{nk}P^p{}_m - 2Y^{jlm}Y_{iln}P^n{}_p\tilde{r}_m^{kp} \\
&- g^4Q\tilde{r}_l^{jk}C(R)^l{}_i + 2g^4Q\tilde{r}_i^{jl}C(R)^k{}_l - \tfrac{1}{2}\tilde{r}_l^{jk}Y^{lmn}Y_{imp}P^p{}_n \\
&+ \sqrt{2}g\Big\{6g^4C(G)Q(R_a)^j{}_im_A^{ak} - 4g^2\mathrm{tr}[PR_aR_b](R_b)^j{}_im_A^{ak} \\
&- 2g^2C(G)(Pm_A^a)^j(R_a)^k{}_i - (R_a)^j{}_iY_{lmn}Y^{kmp}P^n{}_pm_A^{al}\Big\} \quad + j \leftrightarrow k,
\end{aligned}
\tag{A.1}
$$

$$
\begin{aligned}
(16\pi^2)^2(\beta_{m_A}^{(2)})^{ai} = &-2g^2\mathrm{tr}[PR_aR_b]m_A^{ib} - Y^{ikl}Y_{kmn}P^m{}_lm_A^{na} - 2\sqrt{2}gY^{ikl}Y_{kmn}(R_a)^m{}_p\tilde{r}_l^{np} \\
&+ 2g^2C(G)[2g^2Qm_A^{ia} - (Pm_A^a)^i - \sqrt{2}g(R_a)^k{}_l\tilde{r}_k^{il}],
\end{aligned}
\tag{A.2}
$$

where $P^i{}_j$ and Q are as defined in Eq. (2.6), and

$$
\tilde{r}_i^{jk} = r_i^{jk} - \sqrt{2}g\left[(R_a)^j{}_im_A^{ka} + (R_a)^k{}_im_A^{ja}\right]. \tag{A.3}
$$

Clearly, on the RG trajectory given by Eq. (2.13) (now with $\mu = 0$) $(\beta_{m_A}^{(2)})^{ai}$ and especially

$(\beta_r^{(2)})_i^{jk}$ simplify considerably, and satisfy Eq. (2.14). We further find

$$
\begin{aligned}
(16\pi^2)^2(\beta_b^{(2)})^{ij} &= -b^{il}Y_{lmn}Y^{mpj}P^n_p - 2g^2C(R)^i{}_kV^{kj} + 2g^4b^{ik}C(R)^j{}_kQ \\
&+ 8g^4C(G)[T(R)-2C(G)]m_A^{ia}m_A^{ja} - 4g^2(R_bR_a)^i{}_kY^{jkl}Y_{lmn}m_A^{ma}m_A^{nb} \\
&+ 2g^2C(G)Y^{ijk}Y_{klm}m_A^{la}m_A^{ma} - 2Y^{ikn}Y_{klm}r^l_{np}h^{mpj} - Y_{kln}Y^{inp}r_p^{mj}r_m^{kl} \\
&- 2Y^{imn}Y_{klm}r_n^{lp}r_p^{jk} - 2r_l^{ik}r_m^{jl}P^m{}_k + 2g^2[C(R)^l{}_mr_l^{ik} - C(R)^k{}_lr_m^{il}]r_k^{jm} \\
&- Y^{ikl}X^m{}_lr^j_{km} - 2h^{ikl}P^m{}_lr^j_{km} \quad + j \leftrightarrow k,
\end{aligned}
\tag{A.4}
$$

$$
\begin{aligned}
(16\pi^2)^2(\beta_{m^2}^{(2)})^i{}_j &= \Bigg(-\Big[(m^2)_j{}^lY_{lmn}Y^{mpi} + \tfrac{1}{2}Y_{jlm}Y^{ipm}(m^2)^l{}_n + \tfrac{1}{2}Y_{jnm}Y^{ilm}(m^2)^p{}_l \\
&+ Y_{jln}Y^{irp}(m^2)^l{}_r + h_{jln}h^{ilp} \\
&+ 4g^2MM^*C(R)^i{}_n\delta^p{}_j + 2g^2(R_a)^i{}_j(R_am^2)^p{}_n\Big]P^n{}_p \\
&+ \big[2g^2M^*C(R)^p{}_j\delta^i{}_n - h_{jln}Y^{ilp}\big]X^n{}_p \\
&- \tfrac{1}{2}\big[Y_{jln}Y^{ilp} + 2g^2C(R)^p{}_j\delta^i{}_n\big]W^n{}_p + 12g^4MM^*C(R)^i{}_jQ \\
&+ 4g^4SC(R)^i{}_j + 2Y^{ikl}Y_{jmn}[P^m{}_k + g^2C(R)^m{}_k]m_{Aak}m_A^{an} \\
&+ 4g^2(R_bR_a)^i{}_jm_{Aa}Pm_A^b - 4g^2(R_bR_aP)^i{}_jm_{Aa}m_A^b \\
&+ 4Y^{ikl}Y_{jkm}(m_A^aP)_lm_A^{am} - 2g^2C(G)Y^{ikl}Y_{jkm}m_{Aal}m_A^{am} \\
&+ 8g^4Q(R_bR_a)^i{}_jm_{Aka}m_A^{kb} + 4g^4C(G)(R_bR_a)^i{}_jm_{Aka}m_A^{kb} \\
&+ 4g^4C(G)[2Q+3C(G)]m_A^{ia}m_{Aja} + 2g^2Y^{ikl}C(R)^m{}_jY_{kmn}m_{Ala}m_A^{na} \\
&- 8g^4(R_aR_b)^i{}_jm_{Ac}R_aR_bm_A^c + 2Qg^2Y^{ikl}Y_{jkm}m_{Ala}m_A^{am} \\
&+ 8g^2(R_aR_b)^i{}_kY_{jlm}Y^{kln}m_{Anb}m_A^{ma} \\
&+ 8g^2(R_a)^i{}_kY_{jlm}\left(Y^{kln}(m_{Ab}R_a)_nm_A^{bm} + (R_a)^l{}_nY^{nkp}m_{Apb}m_A^{mb}\right) \\
&+ 4g^2(R_a)^i{}_k(R_a)^l{}_jY_{lmn}Y^{kmp}m_{Apb}m_A^{nb} \\
&+ 4\sqrt{2}\left(gr_l^{ik}Y_{jmn}(R_a)^n{}_kY^{lmp}m_{Apa} + 2g^3(R_aR_b)^i{}_kr_j^{kl}\left(m_{Ab}R_a\right)_l\right) \\
&- 2Y^{ikl}Y_{lnp}r_j^{mn}r^p_{km} - Y_{jkm}Y^{mpq}r_l^{ik}r^l_{pq} \\
&- 2Y^{ikl}Y_{kmp}r_l^{pn}r^m_{jn} - 4g^2C(R)^i{}_mr_j^{kl}r^m_{kl} \\
&- 2g^2[C(R)^m{}_lr_m^{ik} - C(R)^k{}_mr_l^{im}]r^l_{kj} - 4g^2(R_a)^i{}_k(R_a)^m{}_nr_j^{kl}r^n_{ml} \\
&- r_j^{kl}r^i_{km}P^m{}_l - r_l^{ik}r^l_{jm}P^m{}_k - r_l^{ik}r^m_{jk}P^l{}_m\Bigg) + \text{h.c.},
\end{aligned}
\tag{A.5}
$$

where V^{ij}, $W^i{}_j$ and $X^i{}_j$ are as defined in Eqs. (2.11b,c), (2.12) but with $m_F = 0$, and where

$$S\delta_{ab} = (m^2)^i{}_j (R_a R_b)^i{}_j - MM^* C(G)\delta_{ab}. \tag{A.6}$$

The form of Eqs. (A.4) and (A.5) on the RG-trajectory is less clear than in the case of Eqs. (A.1) and (A.2), but nevertheless after some work we find that Eqs. (2.16) and (2.18) are satisfied at this order (with $\mu = 0$). (In the case of Eq. (A.5), we are again obliged to specialise to theories for which the matter multiplet satisfies $C(R)^i{}_j = C(G)\delta^i{}_j$, as at one loop.) *A fortiori*, we see that the same conditions which imply one-loop finiteness also guarantee two-loop finiteness, as was discovered in the case of the standard soft couplings in Ref. [16]. Although we have presented two-loop results for the case $m_F = 0$, we have checked that for $m_F = \mu$, the relations Eqs. (2.13), (2.15) and (2.17) continue to be RG-invariant–in other words, the relations Eq. (2.19) are RG-invariant in a theory with a supersymmetric μ-term together with $L_{\rm SOFT}$ as in Eq. (2.2), and with $m_F = 0$ in Eq. (1.2). As explained earlier, this is a consequence of the fact that couplings satisfying Eqs. (2.19) follow from the single holomorphic term Eq. (2.20).

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