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Sieving the Positive Integers by Large Primes

D. A. GOLDSTON

Department of Mathematics and Computer Science, San Jose State University, San Jose, California 95192

AND

KEVIN S. MCCURLEY*

Department of Mathematics, University of Southern California, Los Angeles, California 90089-1113

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Let Q be a set of primes having relative density δ among the primes, with $0<\delta<1$, and let $\psi(x, y, Q)$ be the number of positive integers $\leqslant x$ that have no prime factors from Q exceeding y. We prove that if $y \to \infty$, then $\psi(x, y, Q) \sim x\rho_\delta(u)$, where $u = (\log x)/(\log y)$, and ρ_δ is the continuous solution of the differential delay equation $u\rho'_\delta(u) = -\delta\rho_\delta(u-1)$, $\rho_\delta(u) = 1$, $0 \leqslant u \leqslant 1$. This generalizes work by Dickman, de Bruijn, and Hildebrand, who considered the case where Q consists of all primes (and $\delta = 1$). © 1988 Academic Press, Inc.

1. INTRODUCTION

Let $\psi(x, y)$ be the number of positive integers $\leqslant x$ that have no prime factor exceeding y. Several researchers have investigated the function $\psi(x, y)$, including Dickman [D], de Bruijn [dB1, dB2], Hildebrand [H1], and Hildebrand and Tenenbaum [HT] (see [N, H1] for surveys of the previous work on this subject). If y is not too small compared to x, then the asymptotic behaviour of $\psi(x, y)$ is related to the Dickman function $\rho(u)$, which can be defined as the continuous solution of the differential

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delay equation $u\rho'(u) = -\rho(u-1)$, with $\rho(u) = 1$, $0 \leqslant u \leqslant 1$. In particular Hildebrand [H1] proved that

$$\psi(x, y) = x\rho(u)\left(1 + O\left(\frac{\log(2+u)}{\log y}\right)\right),$$

$$\text{if} \quad y \geqslant \exp((\log\log x)^{5/3+\varepsilon}), \tag{1}$$

where in this paper we will always use u to denote $(\log x)/(\log y)$. This improved on previous work of de Bruijn [dB1], whose results imply that $\psi(x, y) \sim x\rho(u)$ for $y \geqslant \exp((\log x)^{3/8-\varepsilon})$ and $\varepsilon > 0$.

Let Q be a set of primes. In this paper we wish to consider the more general problem of estimating the number of positive integers $\leqslant x$ that have no prime factors from Q exceeding y, which we denote by $\psi(x, y, Q)$. In the case that Q consists of all primes, then we obviously have $\psi(x, y, Q) = \psi(x, y)$, and in any case we have $\psi(x, y, Q) \geqslant \psi(x, y)$. The size of $\psi(x, y, Q)$ depends on the density of the set Q, and for measuring this we define, for an arbitrary set of primes A,

$$\vartheta(x, A) = \sum_{\substack{p \leqslant x \\ p \in A}} \log p.$$

In the case that A contains all primes, then $\vartheta(x, A)$ agrees with the classical Chebyshev function $\vartheta(x)$, and by the prime number we know that $\vartheta(x) = x + O(x \cdot \exp(-(\log x)^{3/5-\varepsilon})$. In this paper we shall assume that a set of primes Q has relative density δ among the primes (in a strong sense) where $0 < \delta < 1$. More precisely, define a function $\varepsilon(x)$ by the relation

$$\vartheta(x, Q^c) = (1-\delta)x + x\varepsilon(x), \qquad x \geqslant 1, \tag{2}$$

where Q^c denotes the set of primes that do not belong to Q. We shall assume that there exists a function B satisfying

$$\begin{aligned} &|\varepsilon(x)| \leqslant B(x), \\ &B(x) \text{ is nonincreasing for } x \geqslant 1, \\ &B(x) = O(\log x)^{-A}) \text{ as } x \to \infty \text{ for some constant } A > 1. \end{aligned} \tag{3}$$

We have chosen to state our assumptions on the set Q in the form (2) and (3), but we could also state equivalent conditions in terms of $\pi(x, Q)$, which we use to denote the number of primes in Q that are $\leqslant x$. In particular we can prove via standard methods in analytic number theory that (2) and (3) follow from the assumption that $\pi(x, Q) = \delta\,\mathrm{li}(x) + O(x/(\log x)^B)$ for some constant $B > 2$. There are several natural examples for which this condition is satisfied, e.g., if the set Q consists of the primes belonging to a finite union of arithmetic progressions, or if Q consists of the set of primes that divide some value of a fixed polynomial with integer coefficients.

In light of (1), it should not be surprising that the asymptotic behaviour of $\psi(x, y, Q)$ involves a solution of a certain differential delay equation. For $0<\delta\leqslant 1$, we define the modified Dickman function $\rho_\delta(u)$ by

$$\begin{aligned} \rho_\delta(u) &= 1, && 0\leqslant u\leqslant 1, \\ \rho_\delta(u) &= 1-\delta\int_0^{u-1}\frac{\rho_\delta(t)}{t+1}\,dt, && u\geqslant 1. \end{aligned} \tag{4}$$

Note that for $\delta=1$, ρ_δ is the standard Dickman function.

We are now prepared to state the main result of this paper.

THEOREM 1. *Let Q be a set of primes satisfying* (2) *and* (3), *with* $0<\delta<1$, *and let* $u=(\log x)/(\log y)$. *Then*

$$\psi(x, y, Q)=x\rho_\delta(u)\left(1+O\left(\frac{1}{\log y}\right)\right), \tag{5}$$

uniformly for $u\geqslant 1$ *and* $y\geqslant 1.5$.

In comparing Theorem 1 with (1), Theorem 1 gives an asymptotic for $\psi(x, y, Q)$ for a range of y that is much larger than (1). One reason for this is the fact that when u is very large, $\psi(x, y)$ depends very strongly on irregularities in the distribution of small primes, but these effects are not felt in $\psi(x, y, Q)$. Another reason is that if $0<\delta<1$, then the modified Dickman function decreases more slowly than the original Dickman function, allowing us to estimate the error terms that arise with greater accuracy. For $\delta=1$, de Bruijn [dB3] proved that

$$\rho_1(u)=\exp\left(-u\left(\log u+\log\log u-1-\frac{1}{\log u}\right.\right.$$
$$\left.\left.+\frac{\log\log u}{\log u}+O\left(\frac{\log\log^2 u}{\log^2 u}\right)\right)\right). \tag{6}$$

For $0<\delta<1$ it turns out that $\rho_\delta(u)$ has quite different asymptotic behaviour as $u\to\infty$, given by the following.

THEOREM 2. *Let* $0<\delta<1$, *and define a sequence* a_k *by the equation*

$$\exp\left(\delta\sum_{k=1}^{\infty}\frac{(-1)^k x^k}{k\cdot k!}\right)=\sum_{k=0}^{\infty}a_k x^k. \tag{7}$$

Then as $u \to \infty$, $\rho_\delta(u)$ *has the asymptotic expansion*

$$\rho_\delta(u) \sim e^{\gamma\delta} \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(1-\delta-k)} u^{-k-\delta}.$$

In passing we note that the coefficients a_k can be computed from the recurrence relation

$$a_0 = 1, \qquad a_{m+1} = \frac{\delta}{m+1} \sum_{j=0}^{m} \frac{(-1)^{m+1-j}}{m+1-j} a_j, \qquad m \geq 0.$$

COROLLARY. *If* Q *satisfies* (2) *and* (3) *with some fixed* $0 < \delta < 1$, *and if* $(\log x)/(\log y) \to \infty$, *then*

$$\psi(x, y, Q) \sim \frac{e^{\gamma\delta}}{\Gamma(1-\delta)} x \prod_{\substack{y < p \leq x \\ p \in Q}} \left(1 - \frac{1}{p}\right).$$

Proof. In the case that $y \to \infty$, Theorems 1 and 2 imply that

$$\psi(x, y, Q) \sim \frac{e^{\gamma\delta}}{\Gamma(1-\delta)} x \left(\frac{\log y}{\log x}\right)^{\delta}.$$

The Corollary then follows easily from the standard estimate of the product over primes in Q. If y is fixed, the result was proved previously by Wirsing [W]. ∎

The method used to prove Theorem 1 is a modification of the method of Hildebrand [H]. The estimation of $\psi(x, y, Q)$ may be regarded as a type of "sieving" problem, and in fact, the methods for estimating $\psi(x, y)$ used by de Bruijn [dB1] and Hildebrand [H1, H2] have a great deal in common with the class of techniques of [HR, H] that are usually referred to as "sieve methods." The major similarity is the use of a type of Buchstab identity, and of its continuous analogue, a differential delay equation. The Buchstab identity used by de Bruijn [dB1] for investigating $\psi(x, y)$ is

$$\psi(x, y_1) = \psi(x, y_2) - \sum_{y_1 < p \leq y_2} \psi\left(\frac{x}{p}, p\right)$$

for $y_1 < y_2$. Hildebrand [H1] used the identity

$$\psi(x, y) \log x = \int_1^x \frac{\psi(t, y)}{t} dt + \sum_{\substack{p^m \leq x \\ p \leq y}} \psi\left(\frac{x}{p^m}, y\right) \log p.$$

The generalization of Hildebrand's identity that we shall use is

$$\psi(x, y, Q)\log x = \int_1^x \frac{\psi(t, y, Q)}{t}\,dt + \sum_{\substack{p^m \leqslant x \\ p \leqslant y}} \psi\left(\frac{x}{p^m}, y, Q\right)\log p + \sum_{\substack{p^m \leqslant x \\ p > y \\ p \notin Q}} \psi\left(\frac{x}{p^m}, y, Q\right)\log p. \tag{8}$$

A proof of (8) will be given in Section 4.

There are several other methods that have been used by previous researchers for estimating $\psi(x, y)$. Rankin [R] used a simple method to produce upper estimates for $\psi(x, y)$ that was later developed more fully by de Bruijn [dB2]. This method does not seem to have any direct analogue for the estimation of $\psi(x, y, Q)$. Very recently, Hildebrand and Tenenbaum [HT] developed a method based on the Perron inversion formula and the saddle point method that gives an asymptotic result for $\psi(x, y)$ of a different nature than (1). It seems very likely that this method can be applied as well to the estimation of $\psi(x, y, Q)$, but the details may be somewhat tedious. The main advantage of the method used in this paper over the method of [HT] is its simplicity.

In addition, it appears that the methods used here can be generalized to allow sieving of other sets in place of the integers in $[1, x]$, in much the same way that the Selberg sieve or the Brun sieve can be applied to very general sets (see [HR]). In particular it should be possible to treat (with varying degrees of success) the cases of sieving the integers from an interval, the squarefree numbers $\leqslant x$, the integers in an arithmetic progression, or the integers that are relatively prime to a fixed integer.

2. The Modified Dickman Function

For $u \geqslant 0$ and $0 < \delta \leqslant 1$, define $\rho_\delta(u)$ by (4). One may easily verify that $\rho_\delta(u)$ is the continuous solution of the differential delay equation

$$\begin{aligned} u\rho_\delta'(u) &= -\delta\rho_\delta(u-1), && u \geqslant 1, \\ \rho_\delta(u) &= 1, && 0 \leqslant u \leqslant 1. \end{aligned} \tag{9}$$

We summarize some of the other properties of ρ_δ in the following lemma.

Lemma 1. *Let* $0 < \delta \leqslant 1$. *Then*

(i) $\rho_\delta(u) = 1 - \delta \log u,\ 1 \leqslant u \leqslant 2,$

(ii) $u\rho_\delta(u) = (1-\delta)\int_0^{u-1} \rho_\delta(t)\,dt + \int_{u-1}^{u} \rho_\delta(t)\,dt,\ u \geqslant 1,$

(iii) $0 < \rho_\delta(u) \leqslant 1,\ u \geqslant 0,$

(iv) $\rho_\delta(u)$ *is nonincreasing for* $u \geqslant 0$.

Proof. Part (i) follows immediately from the definition of (4). On differentiating both sides of (ii), it follows from (9) that the two sides of (ii) differ by a constant. By setting $u = 1$, we find the constant is 0. It follows from (ii) and the fact that $\rho_\delta(u)$ is continuous that $\rho_\delta(u) > 0$. From this and (9) we obtain (iv), and from (iv) it follows that $\rho_\delta(u) \leqslant 1$ for $u \geqslant 0$. ∎

Thus far the only thing to distinguish the case $\delta = 1$ from the case $0 < \delta < 1$ is in part (ii) of Lemma 1, where for $\delta = 1$ there is a term missing. As it turns out, this alone can be used to show that the asymptotic behaviour of $\rho_\delta(u)$ as $u \to \infty$ is quite different when $0 < \delta < 1$. Before proving the asymptotic estimate of Theorem 2 we first give an elementary argument to prove that

$$(1-\delta)\,u^{-\delta} \leqslant \rho_\delta(u) \leqslant u^{-\delta}, \qquad u \geqslant 1,\ 0 < \delta < 1. \tag{10}$$

In order to prove the upper bound in (10), observe that

$$\frac{\rho_\delta'(w)}{\rho_\delta(w)} = -\frac{\delta\rho_\delta(w-1)}{w\rho_\delta(w)} < -\frac{\delta}{w},$$

for $w \geqslant 1$, since ρ_δ is nonincreasing. Hence,

$$\log \rho_\delta(u) = \int_1^u \frac{\rho_\delta'(w)}{\rho_\delta(w)}\,dw < -\delta \int_1^u w^{-1}\,dw,$$

which gives the upper bound. The lower bound is only slightly more complicated. From Lemma 1(ii) we obtain

$$u\rho_\delta(u) \geqslant (1-\delta)\int_0^u \rho_\delta(w)\,dw. \tag{11}$$

If $u \geqslant 1$, it follows that $u\rho_\delta(u) \geqslant (1-\delta)\int_0^1 dw$, or $\rho_\delta(u) \geqslant (1-\delta)\,u^{-1}$. Substituting this into (11) we obtain

$$\begin{aligned} u\rho_\delta(u) &\geqslant (1-\delta) + (1-\delta)^2 \int_1^u w^{-1}\,dw \\ &= (1-\delta) + (1-\delta)^2 \log u, \end{aligned}$$

so that $\rho_\delta(u) \geqslant (1-\delta)\, u^{-1} + (1-\delta)^2 u^{-1} \log u$. After iterating this procedure n times we obtain

$$\rho_\delta(u) \geqslant (1-\delta)\, u^{-1} \sum_{k=0}^{n} \frac{(1-\delta)^k \log^k u}{k!},$$

from which we obtain the lower bound of (10). Further iterations will yield sharper inequalities, but we shall not pursue this here.

de Bruijn's proof of the asymptotic estimate for $\rho_1(u)$ is somewhat involved, requiring among other things two applications of the saddle point method and a result on Volterra equations. He proved that

$$\rho_1(u) \sim \frac{e^\gamma}{\sqrt{2\pi u}} \exp\left(-\xi u + \int_0^\xi \frac{e^t - 1}{t}\, dt\right), \tag{12}$$

where ξ is the real solution of $e^\xi - 1 = \xi u$. Canfield [C] has given a simpler proof, but his proof still occupies five typewritten pages. In the case $0 < \delta < 1$, we are able to give a much simpler proof of an asymptotic estimate for $\rho_\delta(u)$ based on the following Tauberian theorem of Doetsch [Do1, p. 150] (the proof of this result is given in [Do2, pp. 250–254.]).

LEMMA 2. *Let $f(t)$ have Laplace transform $F(s)$, given by $F(s) = \int_0^\infty e^{-st} f(t)\, dt$, and let $F(s)$ be given by the absolutely convergent series expansion*

$$F(s) = \sum_{k=0}^{\infty} a_k s^{\lambda_k} \qquad (-1 < \Re\lambda_0 < \Re\lambda_1 < \cdots)$$

in some sector $\{z \mid |\arg z| \leqslant \psi, z \neq 0\}$ with $\pi/2 < \psi < \pi$. Then as $t \to \infty$, $f(t)$ has the asymptotic expansion

$$f(t) \sim \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(-\lambda_k)} t^{-\lambda_k - 1}.$$

(If λ_k is a nonnegative integer, then $1/\Gamma(-\lambda_k) = 0$.)

The proof of Lemma 2 is fairly straightforward. We first represent the original function in terms of $F(s)$ using the inverse Laplace transform and then deform the contour to the boundary of the region of analyticity. The resulting integrals may then be interpreted as Laplace transforms and estimated asymptotically using an Abelian theorem.

In order to prove the asymptotic estimate of Theorem 2 for $\rho_\delta(u)$ when $0 < \delta < 1$, we proceed as in [KTP] to first obtain a formula for the Laplace transform of $\rho_\delta(u)$ in terms of well-known function. For $x > 0$, define $g(x)$ by

$$g(x) = \int_0^\infty e^{-xt} \rho_\delta(t)\, dt.$$

Note that

$$g(x) = e^x \int_1^\infty \frac{\rho_\delta(w-1)}{w}\, we^{-xw}\, dw$$

$$= -\frac{e^x}{\delta} \int_0^\infty \rho'_\delta(w)\, we^{-xw}\, dw,$$

and if we integrate by parts and simplify the result, we obtain $g(x) = \delta^{-1} e^x (g(x) + xg'(x))$. Solving this differential equation for g we obtain

$$g(x) = Cx^{-1} \exp\left(-\delta \int_x^\infty e^{-t} t^{-1}\, dt\right),$$

for some constant C. In order to evaluate C, we integrate by parts to obtain

$$xg(x) = \int_0^\infty xe^{-xt} \rho_\delta(t)\, dt$$

$$= 1 + \int_1^\infty e^{-xt} \rho'_\delta(t)\, dt$$

$$= 1 - \delta \int_1^\infty e^{-xt} \rho_\delta(t-1)\, t^{-1}\, dt$$

$$= 1 - \delta \int_x^\infty e^{-w} \rho_\delta(wx^{-1} - 1)\, w^{-1}\, dw,$$

and this last quantity tends to 1 as $x \to \infty$. Hence $C = 1$.

From [O, pp. 40] we obtain

$$\int_x^\infty e^{-t} t^{-1}\, dt = -\log x - \gamma - \sum_{k=1}^\infty \frac{(-1)^k}{kk!} x^k$$

for all complex x in the cut plane for which $\log x$ is defined. It follows that

$$g(x) = e^{\gamma\delta} \sum_{k=0}^\infty a_k x^{k+\delta-1}.$$

Theorem 2 now follows directly from Lemma 2. ∎

The first term in the asymptotic expansion of $\rho_\delta(u)$ may also be deduced from a Tauberian theorem of Hardy and Littlewood [HL]. From the asymptotic behaviour of $g(x)$ as $x \to 0$ and the fact that $\rho_\delta(u)$ is of one sign, we can deduce from the theorem of Hardy and Littlewood that

$$\int_0^u \rho_\delta(t)\,dt \sim \frac{e^{\delta\gamma}}{\Gamma(2-\delta)} u^{1-\delta}, \qquad u \to \infty.$$

From Lemma 1(ii) we now obtain

$$\begin{aligned}\rho_\delta(u) &= \frac{1-\delta}{u}\int_0^u \rho_\delta(t)\,dt + \frac{\delta}{u}\int_{u-1}^u \rho_\delta(t)\,dt\\ &\sim \frac{(1-\delta)\,e^{\delta\gamma}}{\Gamma(2-\delta)} u^{-\delta},\end{aligned}$$

which gives the first term of the expansion since $\Gamma(2-\delta) = (1-\delta)\,\Gamma(1-\delta)$.

3. Preliminary Lemmas

The proof of Theorem 1 follows closely the proof given by Hildebrand [H2] for the case $\delta = 1$, the only differences coming from the different asymptotic behaviour of $\rho_\delta(u)$ and the treatment of the extra sum in (8). We assume henceforth that Q, $\varepsilon(t)$, and $B(t)$ satisfy (3). In order to simplify the notation, we assume that $0 < \delta < 1$ is fixed, and write $\rho(u)$ in place of $\rho_\delta(u)$. In what follows, any implied O-constants are allowed to depend only on δ unless otherwise indicated.

Lemma 3. *If $y \geqslant 1.5$ and $u \geqslant 1$, then*

$$\int_0^u y^w \rho(w)\,dw \ll \frac{\rho(u)\,y^u}{\log y}.$$

Proof. By (10) we have

$$\begin{aligned}\int_0^u y^w\rho(w)\,dw &\ll y^{u/2}\int_0^{u/2}\rho(w)\,dw + \left(\frac{u}{2}\right)^{-\delta}\int_{u/2}^u y^w\,dw\\ &\ll y^{u/2}u^{1-\delta} + \frac{u^{-\delta}y^u}{\log y}\\ &\ll \frac{u^{-\delta}y^u}{\log y}.\end{aligned}$$

In light of (10), this proves the lemma. ∎

LEMMA 4. *If* $y \geqslant 1.5$ *and* $u \geqslant 1$, *then*

$$\sum_{\substack{p^m \leqslant y^u \\ m \geqslant 2}} \frac{\log p}{p^m} \rho\left(u - \frac{\log p^m}{\log y}\right) \ll \rho(u).$$

Proof. Define $\vartheta^*(t)$ by

$$\vartheta^*(t) = \sum_{\substack{p^m \leqslant t \\ m \geqslant 2}} \log p.$$

It follows from the Prime Number Theorem that $\vartheta^*(t) \ll t^{1/2}$. Hence the sum of the lemma is

$$\int_1^{y^u} \rho\left(u - \frac{\log t}{\log y}\right) \frac{d\vartheta^*(t)}{t}$$

$$= \frac{\Theta^*(y^u)}{y^u} + \int_1^{y^u} \frac{\vartheta^*(t)}{t^2} \left\{\rho\left(u - \frac{\log t}{\log y}\right) + \frac{1}{\log y} \rho'\left(u - \frac{\log t}{\log y}\right)\right\} dt$$

$$\ll y^{-u/2} + \int_1^{y^u} t^{-3/2} \rho\left(u - \frac{\log t}{\log y}\right) dt$$

$$\ll y^{-u/2} + \rho\left(\frac{u}{2}\right) \int_1^{y^{u/2}} t^{-3/2}\, dt + y^{-u/2} \int_{y^{u/2}}^{y^u} \rho\left(u - \frac{\log t}{\log y}\right) \frac{dt}{t}$$

$$\ll y^{-u/2} + \rho(u) + y^{-u/2} \log y \int_0^{u/2} \rho(w)\, dw$$

$$\ll y^{-u/2} + \rho(u) + y^{-u/2} u\rho(u) \log y,$$

and this last quantity is $\ll \rho(u)$. ∎

LEMMA 5. *If* $y \geqslant 1.5$, $u \geqslant 1$, *and* $0 < \alpha \leqslant 1$, *then*

$$\sum_{p \leqslant y^\alpha} \frac{\log p}{p} \rho\left(u - \frac{\log p}{\log y}\right) = \log y \int_{u-\alpha}^{u} \rho(w)\, dw + O(\rho(u)),$$

where the implied O-constant may depend on α.

Proof. We have

$$\sum_{p \leqslant y^\alpha} \frac{\log p}{p} \rho\left(u - \frac{\log p}{\log y}\right)$$

$$= \int_1^{y^\alpha} \rho\left(u - \frac{\log t}{\log y}\right) \frac{d\vartheta(t)}{t}$$

$$= \frac{\vartheta(y^\alpha)}{y^\alpha} \rho(u-\alpha) + \int_1^{y^\alpha} \frac{\vartheta(t)}{t^2} \left\{\rho\left(u - \frac{\log t}{\log y}\right) + \frac{1}{\log y} \rho'\left(u - \frac{\log t}{\log y}\right)\right\} dt.$$

We now use the Prime Number Theorem in the form $\vartheta(t)=t+O(t/\log^2 t)$, giving

$$\sum_{p\leqslant y^{\alpha}}\frac{\log p}{p}\rho\left(u-\frac{\log p}{\log y}\right)=T_1+T_2,$$

where

$$T_1=\rho(u-\alpha)+\int_{u-\alpha}^{u}\{\rho(w)\log y+\rho'(w)\}\,dw$$

$$=\rho(u)+\log y\int_{u-\alpha}^{u}\rho(w)\,dw,$$

and

$$T_2\ll\frac{\rho(u-\alpha)}{\alpha^2\log^2 y}+\int_1^{y^\alpha}\frac{1}{t\log^2 t}\left\{\rho\left(u-\frac{\log t}{\log y}\right)+\frac{1}{\log y}\rho'\left(u-\frac{\log t}{\log y}\right)\right\}dt.$$

Since $\rho(u-\alpha)\ll\rho(u)$ and $\rho'(w)=-\delta\rho(w-1)/w\ll\rho(w)$, it follows that

$$T_2\ll\rho(u)+\int_1^{y^\alpha}\frac{1}{t\log^2 t}\rho\left(u-\frac{\log t}{\log y}\right)dt$$

$$\ll\rho(u)+\rho(u-\alpha),$$

which proves the lemma. ∎

LEMMA 6. *If $y\geqslant 1.5$ and $1\leqslant\alpha\leqslant u$, then*

$$\sum_{\substack{y<p\leqslant y^\alpha\\ p\notin Q}}\frac{\log p}{p}\rho\left(u-\frac{\log p}{\log y}\right)=(1-\delta)\log y\int_{u-\alpha}^{u-1}\rho(w)\,dw+O(\rho(u)).$$

Proof. We have

$$\sum_{\substack{y<p\leqslant y^\alpha\\ p\notin Q}}\frac{\log p}{p}\rho\left(u-\frac{\log p}{\log y}\right)=\int_y^{y^\alpha}\rho\left(u-\frac{\log t}{\log y}\right)\frac{d\vartheta(t,Q^c)}{t}$$

$$=\frac{\vartheta(y^\alpha,Q^c)\,\rho(u-\alpha)}{y^\alpha}-\frac{\vartheta(y,Q^c)\,\rho(u-1)}{y}+\int_y^{y^\alpha}\frac{\vartheta(t,Q^c)}{t^2}\left\{\rho\left(u-\frac{\log t}{\log y}\right)+\frac{1}{\log y}\rho'\left(u-\frac{\log t}{\log y}\right)\right\}dt.$$

We now use (2) to rewrite this as $T_1 + T_2$, where

$$\begin{aligned} T_1 &= (1-\delta)\,\rho(u-\alpha) - (1-\delta)\,\rho(u-1) \\ &\quad + \int_y^{y^\alpha} \frac{1-\delta}{t}\left\{\rho\left(u - \frac{\log t}{\log y}\right) + \frac{1}{\log y}\,\rho'\left(u - \frac{\log t}{\log y}\right)\right\} dt \\ &= (1-\delta)\log y \int_{u-\alpha}^{u-1} \rho(w)\,dw, \end{aligned}$$

and

$$T_2 = \rho(u-\alpha)\,\varepsilon(y^\alpha) - \rho(u-1)\,\varepsilon(y) + \int_{u-\alpha}^{u-1} \varepsilon(y^{u-w})\{\rho(w)\log y + \rho'(w)\}\,dw.$$

It now follows from (3) and the estimate $\rho'(w) \ll \rho(w)$ that

$$\begin{aligned} T_2 &\ll B(y^\alpha)\,\rho(u-\alpha) + B(y)\,\rho(u-1) + \log y \int_0^{u-1} B(y^{u-w})\,\rho(w)\,dw \\ &\ll \rho(u) + (\log y)^{1-A} \int_0^{u-1} \frac{\rho(w)\,dw}{(u-w)^A} \\ &\ll \rho(u) + (\log y)^{1-A}\left\{\frac{1}{u^A}\int_0^{(u-1)/2} \rho(w)\,dw + \rho\left(\frac{u-1}{2}\right)\int_{(u-1)/2}^{u-1} \frac{dw}{(u-w)^A}\right\} \\ &\ll \rho(u) + (\log y)^{1-A}\{u^{1-A}\rho(u) + \rho(u)\}, \end{aligned}$$

which proves the lemma. ∎

LEMMA 7. *If* $0 < \varepsilon < 0.5$ *and* $u \geqslant 1.5$, *then*

$$\sum_{\substack{x/y^\varepsilon < p^m \leqslant x \\ p > y \\ p \notin Q}} \psi\left(\frac{x}{p^m}, y, Q\right) \log p = (1-\delta)\,\varepsilon x \log y + O(x) + O\left(\frac{xy^\varepsilon}{\log^A x}\right).$$

Proof. From $xp^{-m} < y^\varepsilon < y$ it follows that in this range we have $\psi(xp^{-m}, y, Q) = [xp^{-m}]$. The contribution to the sum by the terms with $m \geqslant 2$ is bounded by

$$\sum_{\substack{x/y^\varepsilon < p^m \leqslant x \\ m \geqslant 2}} \frac{x \log p}{p^m} \ll x \cdot \frac{y^\varepsilon}{x} \sum_{\substack{p^m \leqslant x \\ m \geqslant 2}} \log p = y^\varepsilon \vartheta^*(x) \ll y^\varepsilon x^{1/2}.$$

Note that the condition $p > y$ is implied by the condition $xy^{-\varepsilon} < p$ since $\varepsilon < 0.5$. It follows that our sum is

$$\sum_{\substack{x/y^{\varepsilon} < p \leqslant x \\ p \notin Q}} \left[\frac{x}{p}\right] \log p + O(y^{\varepsilon} x^{1/2}).$$

We now have

$$\begin{aligned}
&\sum_{\substack{x/y^{\varepsilon} < p \leqslant x \\ p \notin Q}} \left[\frac{x}{p}\right] \log p \\
&= \sum_{k=1}^{[y^{\varepsilon}]-1} k \Bigg\{ \sum_{\substack{x/(k+1) < p \leqslant x/k \\ p \notin Q}} \log p \Bigg\} + \sum_{\substack{x/y^{\varepsilon} < p \leqslant x/[y^{\varepsilon}] \\ p \notin Q}} [y^{\varepsilon}] \log p \\
&= \sum_{k=1}^{[y^{\varepsilon}]-1} k \left\{ \vartheta\left(\frac{x}{k}, Q^c\right) - \vartheta\left(\frac{x}{k+1}, Q^c\right) \right\} \\
&\quad + [y^{\varepsilon}] \left\{ \vartheta\left(\frac{x}{[y^{\varepsilon}]}, Q^c\right) - \vartheta\left(\frac{x}{y^{\varepsilon}}, Q^c\right) \right\} \\
&= \sum_{k=1}^{[y^{\varepsilon}]-1} k \left\{ (1-\delta)\frac{x}{k} - (1-\delta)\frac{x}{k+1} + O\left(\frac{x}{k} B\left(\frac{x}{k}\right)\right) \right\} \\
&\quad + O\left(y^{\varepsilon} \left(\frac{x}{[y^{\varepsilon}]} - \frac{x}{y^{\varepsilon}} + \frac{x}{y^{\varepsilon}} B\left(\frac{x}{y^{\varepsilon}}\right) \right) \right) \\
&= (1-\delta)\, x \sum_{k=1}^{[y^{\varepsilon}]-1} \frac{1}{k+1} + O\left(x y^{\varepsilon} B\left(\frac{x}{y^{\varepsilon}}\right) \right) + O(x),
\end{aligned}$$

and this proves the result since $xy^{-\varepsilon} > x^{1/2}$ and

$$\sum_{k=1}^{z} \frac{1}{k+1} = \log z + O(1). \quad ∎$$

LEMMA 8. *If Q satisfies* (2) *and* (3), *then there exists a constant D such that*

$$\sum_{\substack{p \leqslant x \\ p \in Q}} \frac{1}{p} = \delta \log \log x + D + O\left(\frac{1}{\log x}\right),$$

uniformly for $x \geqslant 3$.

Proof. The proof is virtually identical with the proof for the case that Q consists of all primes. See for example [I, p. 22]. ∎

4. Proof of Theorem 1

We first give a proof of the fundamental identity (8) by evaluating a sum in two different ways. Let $P(n, Q)$ be the largest prime factor of n that belongs to Q, with $P(n, Q)=1$ if n has no prime factors from Q. Then

$$\begin{aligned}\sum_{\substack{n\leqslant x\\ P(n,Q)\leqslant y}} \log n &= \sum_{\substack{n\leqslant x\\ P(n,Q)\leqslant y}} \sum_{p^m\mid n} \log p\\ &= \sum_{\substack{p^m\leqslant x\\ p\leqslant y}} \sum_{\substack{n\leqslant x\\ p^m\mid n\\ P(n,Q)\leqslant y}} \log p + \sum_{\substack{p^m\leqslant x\\ p>y\\ p\notin Q}} \sum_{\substack{n\leqslant x\\ p^m\mid n\\ P(n,Q)\leqslant y}} \log p\\ &= \sum_{\substack{p^m\leqslant x\\ p\leqslant y}} \psi\left(\frac{x}{p^m}, y, Q\right)\log p + \sum_{\substack{p^m\leqslant x\\ p>y\\ p\notin Q}} \psi\left(\frac{x}{p^m}, y, Q\right)\log p.\end{aligned}$$

On the other hand we can integrate by parts to obtain

$$\begin{aligned}\sum_{\substack{n\leqslant x\\ P(n,Q)\leqslant y}} \log n &= \int_1^x \log t\, d\psi(t, y, \mathrm{Q})\\ &= \psi(x, y, Q)\log x - \int_1^x \frac{\psi(t, y, Q)}{t}\, dt,\end{aligned}$$

which proves (8).

For $u>0$ and fixed $y\geqslant 1.5$, define $\Delta(u)=\Delta(y, u)$ by the equation

$$\psi(y^u, y, Q)=y^u\rho(u)(1+\Delta(u)).$$

For fixed y and $0<\varepsilon<0.5$ we also define

$$\Delta^*(u)=\Delta^*(y, u)=\sup_{\varepsilon\leqslant u'\leqslant u} |\Delta(u')|$$

$$\Delta^{**}(u)=\Delta^{**}(y, u)=\sup_{0\leqslant u'\leqslant u} |\Delta(u')|.$$

To prove Theorem 1 it suffices to prove that

$$\Delta^*(u)\ll\frac{1}{\log y} \tag{13}$$

uniformly for $y\geqslant 1.5$ and $u\geqslant 0.5$. Without loss of generality we may assume that y is sufficiently large, for when y is finite the result is trivial. If $0\leqslant u\leqslant 1$, then $\psi(y^u, y, Q)=[y^u]$, so that $|\Delta(u)|=|([y^u]-y^u)/y^u|\leqslant y^{-u}$.

We will later choose ε to satisfy $\varepsilon > \log\log y/\log y$, so that (13) will be satisfied for $0.5 \leqslant u \leqslant 1$. It also follows trivially that $\Delta^{**}(u) \leqslant 1 + \Delta^*(u)$.

We next consider the range $1 \leqslant u \leqslant 2$. In this case we have

$$\begin{aligned}\psi(y^u, y, Q) &= [y^u] - \sum_{\substack{y < p \leqslant y^u \\ p \in Q}} \psi\left(\frac{y^u}{p}, y, Q\right) \\ &= [y^u] - \sum_{\substack{y < p \leqslant y^u \\ p \in Q}} \left[\frac{y^u}{p}\right],\end{aligned}$$

since $p > y$ implies that $y^u p^{-1} < y$. Hence

$$\begin{aligned}\psi(y^u, y, Q) &= y^u - y^u \sum_{\substack{y < p \leqslant y^u \\ p \in Q}} \frac{1}{p} + O\left(\frac{y^u}{\log y}\right) \\ &= y^u(1 - \delta \log u) + O\left(\frac{y^u}{\log y}\right),\end{aligned}$$

by Lemma 8. Hence we have $|\Delta(u)| \ll 1/\log y$, and (13) is thus proved for $1 \leqslant u \leqslant 2$.

The rest of the argument proceeds in two stages. We first use (8) to estimate $\Delta^*(u)$ in terms of $\Delta^*(\beta u)$ for some fixed number $\beta < 1$, which allows to derive through iteration an estimate for $\Delta^*(u)$ in terms of $\Delta^*(u_1)$ for some bounded number u_1. We then use (8) in a slightly different manner to estimate $\Delta^*(u_1)$ in terms of $\Delta^*(u_0)$ for some u_0 with $1 \leqslant u_0 \leqslant 2$.

If β is fixed and satisfies $0 < \beta < 1$, then by Theorem 2 we have

$$\lim_{u \to \infty} \frac{1-\delta}{u\rho(u)} \int_1^{(1-\beta)u} \rho(t)\, dt = (1-\beta)^{1-\delta}.$$

Now let β be chosen sufficiently small so that $(1-\beta)^{1-\delta} > 0.5$. Then there exists a number $u_1 = u_1(\delta)$ such that $u_1 \geqslant \max\{\beta^{-1}, (1-\beta)^{-1}\}$ and

$$\frac{1-\delta}{u\rho(u)} \int_1^{(1-\beta)u} \rho(t)\, dt \geqslant \frac{1}{2}, \qquad u \geqslant u_1.$$

For $u \geqslant u_1$ and $0 < \varepsilon < 0.5$ we define

$$\begin{aligned}\alpha_1 &= \frac{1}{u\rho(u)} \int_{u-1}^{u} \rho(t)\, dt, \\ \alpha_2 &= \frac{1-\delta}{u\rho(u)} \int_{\beta u}^{u-1} \rho(t)\, dt,\end{aligned}$$

$$\alpha_3 = \frac{1-\delta}{u\rho(u)} \int_{\varepsilon}^{\beta u} \rho(t)\,dt$$

$$\alpha_4 = \frac{(1-\delta)\varepsilon}{u\rho(u)}.$$

Note that $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1$ by Lemma 1, and $\alpha_3 \geqslant 0.5$, so that $\alpha_1 + \alpha_2 \leqslant 0.5$.

Assume now that $u \geqslant 2$. We rewrite (8) in terms of $\Delta(u)$ and divide by $\rho(u)\, y^u \log y^u$ to obtain

$$\begin{aligned}
1 + \Delta(u) &= \frac{1}{\rho(u)\, y^u \log y^u} \int_1^{y^u} \rho\left(\frac{\log t}{\log y}\right)\left(1 + \Delta\left(\frac{\log t}{\log y}\right)\right) dt \\
&\quad + \frac{1}{\rho(u) \log y^u} \sum_{\substack{p^m \leqslant y^u \\ p \leqslant y}} \frac{\log p}{p^m}\, \rho\left(u - \frac{\log p^m}{\log y}\right) \\
&\quad \times \left(1 + \Delta\left(u - \frac{\log p^m}{\log y}\right)\right) \\
&\quad + \frac{1}{\rho(u) \log y^u} \sum_{\substack{p^m \leqslant y^{u-\varepsilon} \\ p > y \\ p \notin Q}} \frac{\log p}{p^m}\, \rho\left(u - \frac{\log p^m}{\log y}\right) \\
&\quad \times \left(1 + \Delta\left(u - \frac{\log p^m}{\log y}\right)\right) \\
&\quad + \frac{1}{\rho(u)\, y^u \log y^u} \sum_{\substack{y^{u-\varepsilon} < p^m \leqslant y^u \\ p \notin Q}} \psi\left(\frac{y^u}{p^m}, y, Q\right) \log p. \qquad (14)
\end{aligned}$$

It follows that

$$\begin{aligned}
|\Delta(u)| &\leqslant (1 + \Delta^{**}(u))(R_1 + R_2) + |R_3| + |R_4| + |R_5| + |R_6| \\
&\quad + \frac{\Delta^*(u)}{\rho(u) \log y^u} \sum_{p \leqslant y} \frac{\log p}{p}\, \rho\left(u - \frac{\log p}{\log y}\right) \\
&\quad + \frac{\Delta^*(u-1)}{\rho(u) \log y^u} \sum_{\substack{y < p \leqslant y^{(1-\beta)u} \\ p \notin Q}} \frac{\log p}{p}\, \rho\left(u - \frac{\log p}{\log y}\right) \\
&\quad + \frac{\Delta^*(\beta u)}{\rho(u) \log y^u} \sum_{\substack{y^{(1-\beta)u} < p \leqslant y^{u-\varepsilon} \\ p \notin Q}} \frac{\log p}{p}\, \rho\left(u - \frac{\log p}{\log y}\right) \\
&\leqslant (1 + \Delta^{**}(u))(|R_1| + |R_2| + |R_3| + |R_4| + |R_5|) \\
&\quad + \alpha_1 \Delta^*(u) + \alpha_2 \Delta^*(u-1) + \alpha_3 \Delta^*(\beta u) + |R_6|,
\end{aligned}$$

where

$$R_1 = \frac{1}{\rho(u)\, y^u \log y^u} \int_1^{y^u} \rho\left(\frac{\log t}{\log y}\right) dt,$$

$$R_2 = \frac{1}{\rho(u) \log y^u} \sum_{\substack{p^m \leqslant y^u \\ m \geqslant 2}} \frac{\log p}{p^m}\, \rho\left(u - \frac{\log p^m}{\log y}\right),$$

$$R_3 = \frac{1}{\rho(u) \log y^u} \sum_{p \leqslant y} \frac{\log p}{p}\, \rho\left(u - \frac{\log p}{\log y}\right) - \alpha_1,$$

$$R_4 = \frac{1}{\rho(u) \log y^u} \sum_{\substack{y < p \leqslant y^{(1-\beta)u} \\ p \notin Q}} \frac{\log p}{p}\, \rho\left(u - \frac{\log p}{\log y}\right) - \alpha_2,$$

$$R_5 = \frac{1}{\rho(u) \log y^u} \sum_{\substack{y^{(1-\beta)u} < p \leqslant y^{u-\varepsilon} \\ p \notin Q}} \frac{\log p}{p}\, \rho\left(u - \frac{\log p}{\log y}\right) - \alpha_3,$$

$$R_6 = \frac{1}{\rho(u)\, y^u \log y^u} \sum_{\substack{y^{u-\varepsilon} < p^m \leqslant y^u \\ p > y \\ p \notin Q}} \psi\left(\frac{y^u}{p^m}, y, Q\right) \log p - \alpha_4.$$

The terms R_1, R_2, and R_3 are $O(1/\log y^u)$ by Lemmas 3, 4, and 5, respectively. The terms R_4 and R_5 are $O(1/\log y^u)$ by Lemma 6. Furthermore the term R_6 is

$$O\left(\frac{1}{\rho(u) \log y^u} + \frac{y^\varepsilon}{\rho(u) \log^{A+1} y^u}\right)$$

by Lemma 7. Since $\Delta^*(t)$ is nondecreasing in t, it follows that

$$|\Delta(u)| \leqslant (\alpha_1 + \alpha_2)\, \Delta^*(u) + \alpha_3\, \Delta^*(\beta u) + O\left(\frac{2 + \Delta^*(u)}{\log y^u}\right)$$
$$+ O\left(\frac{1}{\rho(u) \log y^u}\right) + O\left(\frac{y^\varepsilon}{\rho(u) \log^{A+1} y^u}\right).$$

Note that

$$\begin{aligned} 0.5(\Delta^*(u) + \Delta^*(\beta u)) &- (\alpha_1 + \alpha_2)\, \Delta^*(u) - \alpha_3\, \Delta^*(\beta u) \\ &= (0.5 - \alpha_1 - \alpha_2)(\Delta^*(u) - \Delta^*(\beta u)) + \alpha_4\, \Delta^*(\beta u) \geqslant 0, \end{aligned}$$

since $\alpha_1+\alpha_2\leqslant 0.5$ and Δ^* is nondecreasing. It follows that

$$|\Delta(u)|\leqslant 0.5(\Delta^*(u)+\Delta^*(\beta u))+O\left(\frac{2+\Delta^*(u)}{\log y^u}\right) + O\left(\frac{1}{\rho(u)\log y^u}\right)+O\left(\frac{y^\varepsilon}{\rho(u)\log^{A+1}y^u}\right). \tag{15}$$

We now show that (15) holds with $|\Delta(u)|$ replaced by $\Delta^*(u)$. If $\beta u\leqslant u'\leqslant u$, then by the monotonicity of Δ^* we can apply (15) with u replaced by u' to obtain

$$|\Delta(u')|\leqslant 0.5(\Delta^*(u)+\Delta^*(\beta u))+O\left(\frac{2+\Delta^*(u)}{\log y^u}\right) + O\left(\frac{1}{\rho(u)\log y^u}\right)+O\left(\frac{y^\varepsilon}{\rho(u)\log^{A+1}y^u}\right). \tag{16}$$

If on the other hand $\varepsilon\leqslant u'\leqslant\beta u$, then we have

$$|\Delta(u')|\leqslant\Delta^*(\beta u)\leqslant 0.5(\Delta^*(u)+\Delta^*(\beta u)).$$

Hence in any case (16) holds for $\varepsilon\leqslant u'\leqslant u$, so that we may replace Δ by Δ^* on the left side of (15) to obtain

$$\Delta^*(u)\leqslant\Delta^*(\beta u)+O\left(\frac{2+\Delta^*(u)}{\log y^u}\right) + O\left(\frac{1}{\rho(u)\log y^u}\right)+O\left(\frac{y^\varepsilon}{\rho(u)\log^{A+1}y^u}\right). \tag{17}$$

We now iterate this bound. Let k be the minimal integer so that $\beta^k u\leqslant u_1$, i.e.,

$$k=\left\lceil\frac{\log(u/u_1)}{\log(1/\beta)}\right\rceil.$$

Iterating (17) k times then gives

$$\Delta^*(u)\leqslant\Delta^*(\beta^k u)+O\left(\frac{2+\Delta^*(u)}{u\log y}\sum_{i=1}^{k}\frac{1}{\beta^i}\right) + O\left(\frac{1}{u^{1-\delta}\log y}\sum_{i=1}^{k}\frac{1}{\beta^{(1-\delta)i}}\right) + O\left(\frac{y^\varepsilon}{u^{A+1-\delta}\log^{A+1}y}\sum_{i=1}^{k}\frac{1}{\beta^{(A+1-\delta)i}}\right).$$

Each of the sums is dominated by the last term, and we obtain

$$\Delta^*(u) \leqslant \Delta^*(u_1) + O\left(\frac{2+\Delta^*(u)}{u_1 \log y}\right) + O\left(\frac{1}{u_1^{1-\delta} \log y}\right)$$
$$+ O\left(\frac{y^{\varepsilon}}{u_1^{A+1-\delta} \log^{A+1} y}\right).$$

We choose

$$\varepsilon = \frac{(A+1)\log\log y}{2\log y}.$$

For y sufficiently large, this yields

$$\Delta^*(u) \leqslant \Delta^*(u_1) + O(1/\log y).$$

Hence it now suffices to show that $\Delta^*(u_1) \ll 1/\log y$.

The rest of the argument is almost exactly the same as that given by Hildebrand [H1]. For $u \geqslant 2$, we define α_1, α_2, and α_3 by

$$\alpha_1 = \frac{1}{u\rho(u)} \int_{u-1/2}^{u} \rho(t)\, dt$$
$$\alpha_2 = \frac{1}{u\rho(u)} \int_{u-1}^{u-1/2} \rho(t)\, dt$$
$$\alpha_3 = \frac{1-\delta}{u\rho(u)} \int_{\varepsilon}^{u-1} \rho(t)\, dt$$
$$\alpha_4 = \frac{(1-\delta)\varepsilon}{u\rho(u)}.$$

Note that Lemma 1(ii) implies that $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1$. Hence for $u \geqslant 2$, (14) yields

$$|\Delta(u)| \leqslant (1+\Delta^{**}(u))(R_1+R_2) + |R_3| + |R_4| + |R_5| + |R_6|$$
$$+ \frac{\Delta^*(u)}{\rho(u)\log y^u} \sum_{p \leqslant \sqrt{y}} \frac{\log p}{p}\, \rho\left(u - \frac{\log p}{\log y}\right)$$
$$+ \frac{\Delta^*(u-1/2)}{\rho(u)\log y^u} \sum_{\sqrt{y} < p \leqslant y} \frac{\log p}{p}\, \rho\left(u - \frac{\log p}{\log y}\right)$$
$$+ \frac{\Delta^*(u-1)}{\rho(u)\log y^u} \sum_{\substack{y < p \leqslant y^{u-\varepsilon} \\ p \notin Q}} \frac{\log p}{p}\, \rho\left(u - \frac{\log p}{\log y}\right)$$

$$\leqslant \alpha_1 \Delta^*(u) + \alpha_2 \Delta^*(u-1/2) + \alpha_3 \Delta^*(u-1)$$

$$+ (1+\Delta^{**}(u)) \sum_{i=1}^{5} |R_i| + |R_6|,$$

where

$$R_1 = \frac{1}{\rho(u)\, y^u \log y^u} \int_1^{y^u} \rho\left(\frac{\log t}{\log y}\right) dt,$$

$$R_2 = \frac{1}{\rho(u) \log y^u} \sum_{\substack{p^m \leqslant y^u \\ m \geqslant 2}} \frac{\log p}{p^m} \rho\left(u - \frac{\log p^m}{\log y}\right),$$

$$R_3 = \frac{1}{\rho(u) \log y^u} \sum_{p \leqslant \sqrt{y}} \frac{\log p}{p} \rho\left(u - \frac{\log p}{\log y}\right) - \alpha_1,$$

$$R_4 = \frac{1}{\rho(u) \log y^u} \sum_{\sqrt{y} < p \leqslant y} \frac{\log p}{p} \rho\left(u - \frac{\log p}{\log y}\right) - \alpha_2,$$

$$R_5 = \frac{1}{\rho(u) \log y^u} \sum_{\substack{y < p \leqslant y^{u-\varepsilon} \\ p \notin Q}} \frac{\log p}{p} \rho\left(u - \frac{\log p}{\log y}\right) - \alpha_3,$$

$$R_6 = \frac{1}{\rho(u)\, y^u \log y^u} \sum_{\substack{y^{u-\varepsilon} < p^m \leqslant y^u \\ p > y \\ p \notin Q}} \psi\left(\frac{y^u}{p^m}, y, Q\right) \log p - \alpha_4.$$

The terms R_1, R_2, and R_5 are $O(1/\log y^u)$ by Lemmas 3, 4, and 6, respectively. The terms R_3 and R_4 are $O(1/\log y^u)$ by Lemma 5. Furthermore R_6 can be estimated by Lemma 7 as before. Since $\Delta^*(u)$ is nondecreasing in u, it follows that

$$|\Delta(u)| \leqslant \alpha_1 \Delta^*(u) + (\alpha_2 + \alpha_3)\, \Delta^*(u-0.5) + O\left(\frac{2+\Delta^*(u)}{\log y}\right) + O\left(\frac{y^\varepsilon}{\log^{A+1} y}\right).$$

Note that $\alpha_1 \leqslant \alpha_2$ and $\alpha_1 + \alpha_2$ and $\alpha_1 + \alpha_2 \leqslant 1$, so that $\alpha_1 \leqslant 1/2$. Hence

$$\begin{aligned} &0.5(\Delta^*(u) + \Delta^*(u-0.5)) - \alpha_1 \Delta^*(u) - (\alpha_2+\alpha_3)\, \Delta^*(u-0.5) \\ &\quad = (0.5-\alpha_1)(\Delta^*(u) - \Delta^*(u-0.5)) + \alpha_4 \Delta^*(u-0.5) \\ &\quad \geqslant 0 \end{aligned}$$

from which for $2 \leqslant u \leqslant u_1$ we obtain

$$|\Delta(u)| \leqslant 0.5(\Delta^*(u) + \Delta^*(u-0.5)) + O\left(\frac{2+\Delta^*(u)}{\log y}\right) + O\left(\frac{y^\varepsilon}{\log^{A+1} y}\right),$$

and using essentially the same argument as before, we obtain

$$\Delta^*(u) \leqslant 0.5(\Delta^*(u) + \Delta^*(u-0.5)) + O\left(\frac{2+\Delta^*(u)}{\log y}\right) + O\left(\frac{y^\varepsilon}{\log^{A+1} y}\right),$$

and thus

$$\Delta^*(u) \leqslant \Delta^*(u-0.5) + O\left(\frac{2+\Delta^*(u)}{\log y}\right) + O\left(\frac{y^\varepsilon}{\log^{A+1} y}\right),$$

for $2 \leqslant u \leqslant u_1$. Iterating a bounded number of times, we obtain

$$\Delta^*(u) \leqslant \Delta^*(u_0) + O\left(\frac{2+\Delta^*(u)}{\log y}\right) + O\left(\frac{y^\varepsilon}{\log^{A+1} y}\right),$$

for some u_0 with $1.5 \leqslant u_0 \leqslant 2$. This completes the proof.

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